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SINGULAR MODULI AND THE ARAKELOV INTERSECTION

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Abstract. The values of the modular j-function at imaginary quadratic arguments in the upper half plane are usually called singular moduli. In this paper, we use the Arakelov intersection to give the prime factorizations of a certain combination of singular moduli, coming from the Hecke correspondence. Such a result may be considered as a degenerate one of Gross and Zagier on Heegner points and derivatives of L-series, and is parellel to the result of Gross and Zagier on singular moduli.

In this paper, we will give a result on singular moduli by means of the Arakelov intersection on the modular curve $X_0(1)$, which may be considered as a degenerate case of the results in the style of Gross and Zagier on Heegner points and derivatives of L-series (cf. [GZ1], [GZ2]). Although [GZ1] asserts that such a special case of $X_0(1)$ was treated in [GZ2] on singular moduli, one finds that it is not exactly the case. In fact, the basic difference of the two papers is that for [GZ1], they assume that Heegner points are associated to the *same* imaginary quadratic field, while in [GZ2], they only deal with the case in which Heegner points (for different variables) come from *strictly different* imaginary quadratic fields, e.g., the associated discriminants are relatively prime to each other. Besides this, some of the techniques in these two papers are also different: For example, when they try to write G_s^m as a series of natural numbers, they use different strategies.

The difficulty for giving the degenerate result for $N=1$ is that originally Gross and Zagier [GZ1] used only the Néron pairing. Yet, we may equally use the Arakelov intersection pairing by a result of Faltings and Hriljac [La]. As an application of such a consideration, we will give the precise values and the prime factorizations for a certain combination of some singular moduli. More precisely, we have the following two theorems.

THEOREM 1. *Let K be an imaginary quadratic field with discriminant D, let $\mathscr{A}$ be an ideal class of K, and let $\sigma_{\mathscr{A}}$ be the element in the Galois group of the Hilbert class field H of K which corresponds to $\mathscr{A}$ by the Artin isomorphism. Define $J(\mathscr{A})$ by*

$$J(\mathscr{A}) := \prod_{\substack{\tau\ \mathrm{Heegner\ point} \\ \mathrm{disc}\,\tau = D}} \phi_m(j(\tau), j(\tau^{\sigma_{\mathscr{A}}}))\,,$$

where

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$$\phi_m(j(z_1), j(z_2)) := \prod_{\substack{\det\gamma=m \\ \bmod SL_2(\mathbf{Z})}} (j(z_1)-j(\gamma z_2))^2 .$$

Then $J(\mathscr{A})$ is a rational integer, and the prime factorization of $J(\mathscr{A})$ is given by

$$J(\mathscr{A}) = \pm \prod_{1\le n\le m|D|} \left(\prod_{p|n} p^{a_p(n)}\right)^{u^2 r_{\mathscr{A}}(m|D|-n)} .$$

Here u denotes a half of the number of units in K, $a_p(n)$ is defined by

$$a_p(n) := \begin{cases} 0, & \text{if } \varepsilon(p)=1, \\ (\operatorname{ord}_p(n)+1)\delta(n)R_{\{\mathscr{A}\kappa\}}\left(\dfrac{n}{p}\right), & \text{if } \varepsilon(p)=-1, \\ \operatorname{ord}_p(n)\delta(n)R_{\{\mathscr{A}\kappa\}}\left(\dfrac{n}{p}\right), & \text{if } \varepsilon(p)=0, \end{cases}$$

in which, ε is the Dirichlet character associated with K, i.e., $\varepsilon(n)=\left(\dfrac{D}{n}\right)$, $\delta(n):=\prod_{p|(n,D)}2$, and $r_{\mathscr{A}}(a)$ denotes the number of integral ideals of norm a in the ideal class $\mathscr{A}$. $R_{\{\kappa\}}(a)$ denotes the number of integral ideals of norm a in the genus $\{\kappa\}$, and $\{\kappa\}$ is the genus of any integral ideal with its norm satisfying $N(\kappa)\equiv -p(\operatorname{mod} D)$. In particular, we see that the prime factors in $J(\mathscr{A})$ is not greater than $m|D|$.

THEOREM 2. *In the notation as above, if $r_{\mathscr{A}}(m)=0$, we have the following identity:*

$$\lim_{s\to 1}\left[2u^2\sum_{n=1}^{\infty}\delta(n)R_{\{\mathscr{A}\}}(n)r_{\mathscr{A}}(n+m|D|)Q_{s-1}\left(1+\frac{2n}{m|D|}\right)\right.$$
$$\left.-4\pi\sigma_1(m)[2^{-s+1}|D|^{s/2}u\zeta(2s)^{-1}\zeta_K(s)-\phi(s)h_K]\right]$$
$$=-\left(24\sigma_1(m)+12\sum_{d|m}d\log\frac{m}{d^2}\right)h_K-u^2\sum_{1\le n\le m|D|}\sigma'_{\mathscr{A}}(n)r_{\mathscr{A}}(m|D|-n).$$

Here Q_{s-1} denotes the Legendre function of the second kind, and $\sigma'_{\mathscr{A}}(n):=\sum_{p|n}a_p(n)\log p$.

Basic ideas for this paper are something behind the two interesting papers of Gross and Zagier [GZ1], [GZ2], hence this paper may be regarded as a footnote to them. The reader, who is interested in the Arakelov geometry, may also consider this as a good example towards general theory. We take Lang's book [La] as the reference to the Arakelov theory used in this paper.

I. Proof of the first theorem.

I.1. Global Arakelov intersection pairing. To begin with, let K be an imaginary quadratic field, and x a Heegner point on $X_0(N)$ associated with K. Let $\sigma\in\operatorname{Gal}(H/K)$, for the Hilbert class field H of K. By the Artin isomorphism, we assume that σ corresponds to an ideal class $\mathscr{A}$ of K. Denote by h_K the class number of K and $w=2u$ the number of units of K. Then there are two distinguished cusps, 0 and ∞, on $X_0(N)$. If $N>1$, we know that $0 \neq \infty$. So if σ is not trivial, the Néron pairing $\langle x-0, (x-\infty)^\sigma \rangle$ makes sense. Furthermore, if we apply the Hecke operator T_m to the second component, we see that $\langle x-0, T_m(x-\infty)^\sigma \rangle$ has a very good interpretation. In fact, a part of the main theorem of Gross and Zagier [GZ1] may be read as follows:

THEOREM (cf. [GZ1]). *The series* $g_{\mathscr{A}}(z) := \sum_{m \geq 1} \langle x-0, T_m(x-\infty)^\sigma \rangle e^{2\pi i m z}$ *is a cusp form of weight* 2 *and level N. Moreover,* $g_{\mathscr{A}}$ *is closely related with the derivative of a certain L-series.*

REMARK. For more details, refer to Theorem I.6.1 of [GZ1], since we omit the most important part of that theorem here to emphasize our point.

On the other hand, in the degenerate case of $N=1$, we know that the two cusps 0 and ∞ coincide, if we use the same notation as in [GZ1]. So it does not make any sense to talk about the classical Néron pairing locally, and hence, we cannot directly apply the original technique of [GZ1]. (In fact, when $N=1$, some of the series introduced in [GZ1] are also divergent.) Nevertheless, if we consider the problem on the arithmetic surface $X_0(1)$ over H, by using the Arakelov intersection pairing, we see that $\langle x-\infty, T_m(x-\infty)^\sigma \rangle$ remains to make sense.

From now on, for simplicity, we assume that $r_{\mathscr{A}}(m)=0$, where $r_{\mathscr{A}}(m)$ denotes the number of integral ideals of norm m in the class of $\mathscr{A}$, since otherwise, the supports of x and $T_m x$ are not disjoint. At the end of this paper, we give the result for the cases $r_{\mathscr{A}}(m) \neq 0$ by some modifications. From the definition, note that we consider the Arakelov intersection over the Hilbert class field H of K, we see that

$$\begin{aligned}\langle x-\infty, T_m(x-\infty)^\sigma \rangle &= \langle x-\infty, T_m(x^\sigma) - T_m(\infty) \rangle \\ &= \langle x, T_m(x^\sigma) \rangle - \sigma_1(m) \langle x, \infty \rangle - \langle T_m(x^\sigma), \infty \rangle - \sigma_1(m) h_K ,\end{aligned}$$

since $\langle \infty, \infty \rangle = -h_K$, if we use the standard convention. So we only need to study

$$\langle x, T_m(x^\sigma) \rangle - \sigma_1(m) \langle x, \infty \rangle - \langle T_m(x^\sigma), \infty \rangle .$$

Now from the definition, we may separate this combination into two parts: the finite part and the infinite part. Hence, we need to consider

$$\langle x, T_m(x^\sigma) \rangle_{\mathrm{fin}} - \sigma_1(m) \langle x, \infty \rangle_{\mathrm{fin}} - \langle T_m(x^\sigma), \infty \rangle_{\mathrm{fin}}$$

and

$$\langle x, T_m(x^\sigma) \rangle_{\mathrm{inf}} - \sigma_1(m) \langle x, \infty \rangle_{\mathrm{inf}} - \langle T_m(x^\sigma), \infty \rangle_{\mathrm{inf}} .$$

I.2. Local intersection pairing at finite places. We in this subsection study local intersection pairing at finite places. In this case, note that $X_0(1)$ is just $\boldsymbol{P}^1$, so all the special fibers are the projective lines, and the cusp ∞, which is defined over rational numbers, corresponds to the infinity in the usual sense. Now by the fact that Heegner points have potentially good reduction, we may use the moduli interpretation to see that Heegner points and the cusp for $X_0(1)$ never meet at any finite place. Hence, the contribution to the Arakelov intersection pairing is zero. Therefore, to understand the contribution of the finite place to the pairing, we only need to study $\langle x, T_m(x^\sigma)\rangle_{\mathrm{fin}}$. So we need to know the multiplicity $(x, T_m(x^\sigma))_v$ of x and $T_m(x^\sigma)$ at finite places v of H.

In order to give $(x, T_m x^\sigma)_v$, we need some preparations. Let S be a complete local ring with algebraically closed residue field k, and a, b two S-valued point of $X_0(1)$ over S. Assume that the points a and b have non-cuspidal reduction, and consider $\mathrm{Hom}_S(a, b)$: First, it is a left (resp. right) module over the ring $\mathrm{End}_S(a)$ (resp. $\mathrm{End}_S(b)$). Second, the ring $\mathrm{End}_S(a)$ is either $\boldsymbol{Z}$, or an order in an imaginary quadratic field, or an order of a definite quaternion algebra of prime discriminant over $\boldsymbol{Q}$. Let $\mathrm{Hom}_S(a, b)_{\deg m}$ be the set of elements f in $\mathrm{Hom}_S(a, b)$ of degree m. We know that $\mathrm{Hom}_S(a, b)_{\deg m}$ is a finite set and admits a faithful action by the finite group $\mathrm{Aut}_S(b)$.

Let x be a Heegner point of discriminant D on $X_0(1)$ over H, and $a(x)$ the corresponding set of $X_0(1)$ over Λ_v, where Λ_v is the ring of integers in the completion H_v and has a parameter π. Then to calculate $(a(x), a(T_m(x^\sigma)))_v$, we need to extend the scalar to $X \otimes_{\Lambda_v} W$, where X is the arithmetic model of $X_0(1)$ over H, and W is the completion of the maximal unramified extension of Λ_v.

FACT 1 (cf. [GZ1]). $(a(x), a(T_m(x^\sigma)))_v = \sum_{n \geq 1} h_n(a(x^\sigma), a(x))_{\deg m}$, where

$$h_n(a, b)_{\deg m} = \frac{1}{2} \sharp \mathrm{Hom}_{W/\pi^n}(a, b)_{\deg m} .$$

The proof proceeds as follows: First, if a and b are two sections which intersect on X over W and reduce to regular, non-cuspidal points in special fibres, then we have

$$(a, b) = \sum_{n \geq 1} h_n(a, b)_{\deg 1} ,$$

which proves the assertion for $m=1$. In general, we let $m = p^r q$ with $p^r \| m$. Then, if p splits in K, we have $(a(x), a(T_m(x^\sigma))) = 0$, since in this case

$$\mathrm{Hom}_{W/\pi^n}(a(x^\sigma), a(x)) = \mathrm{Hom}_W(a(x^\sigma), a(x))$$

for all $n \geq 1$. But $r_{\mathscr{A}}(m) = 0$ implies that $\mathrm{How}_W(a(x^\sigma), a(x))$ contains no element of degree m. So we only need to assume that p has a unique prime factor in K. But in that case, $a(x)$ and $a(x^\sigma)$ have supersingular reduction $(\mathrm{mod}\, \pi)$, and $\mathrm{End}_{W/\pi}(a(x)) = R$ is an order in the quaternion algebra B over $\boldsymbol{Q}$, which is ramified at ∞ and p. We then distinguish two cases: p is inert in K, or p is ramified in K. For more details, see Sections III.6, 7, 9 of [GZ1]. With this, if we consider the summation

$$\langle x, T_m(x^\sigma)\rangle_p := \sum_{v \mid p} \langle x, T_m(x^\sigma)\rangle_v ,$$

over all the places over p, we get the following:

PROPOSITION 1 (cf. [GZ1]). (1) *If p is split in K, then*

$$\langle x, T_m(x^\sigma)\rangle_p = 0 .$$

(2) *If p is inert in K, then $B = K + Kj$ with $j^2 = -pq$ and we have a factorization $(q) = \kappa\bar{\kappa}$ in K. Furthermore,*

$$\langle x, T_m(x^\sigma)\rangle_p = u^2 \log p \sum_{\substack{0 < n < m|D| \\ n \equiv 0 (\mathrm{mod}\, p)}} \mathrm{ord}_p(pn) r_{\mathscr{A}}(m|D| - n)\delta(n) R_{\{\mathscr{A}\kappa\}}\left(\frac{n}{p}\right).$$

Here $\delta(n) := \prod_{p|(n,D)} 2$ and $R_{\{\cdot\}}$ denotes the number of integral ideals of norm n in the genus $\{\cdot\}$.

(3) *If p is ramified in K, then $B = K + Kj$ with $j^2 = -q$ and we have a factorization $(q) = \kappa\bar{\kappa}$ in K. Furthermore,*

$$\langle x, T_m(x^\sigma)\rangle_p = u^2 \log p \sum_{\substack{0 < n < m|D| \\ n \equiv 0 (\mathrm{mod}\, p)}} \mathrm{ord}_p(n) r_{\mathscr{A}}(m|D| - n)\delta(n) R_{\{\mathscr{A}\kappa\}}\left(\frac{n}{p}\right).$$

In particular, putting all these together, we have the following:

PROPOSITION 2 (see also Prop. IV.4.6, [GZ1]). *In the notation as above, if $r_{\mathscr{A}}(m) = 0$, we have*

$$\langle x, T_m(x^\sigma)\rangle_{\mathrm{fin}} - \sigma_1(m)\langle x, \infty\rangle_{\mathrm{fin}} - \langle T_m(x), \infty\rangle_{\mathrm{fin}} = u^2 \sum_{1 \le n \le m|D|} \sigma'_{\mathscr{A}}(n) r_{\mathscr{A}}(m|D| - n).$$

Here

$$\sigma'_{\mathscr{A}}(n) = \sum_{p|n} a_p(n) \log p ,$$

with

$$a_p(n) := \begin{cases} 0, & \text{if } \varepsilon(p) = 1, \\ (\mathrm{ord}_p(n) + 1)\delta(n) R_{\{\mathscr{A}\kappa\}}\left(\frac{n}{p}\right), & \text{if } \varepsilon(p) = -1, \\ \mathrm{ord}_p(n)\delta(n) R_{\{\mathscr{A}\kappa\}}\left(\frac{n}{p}\right), & \text{if } \varepsilon(p) = 0, \end{cases}$$

in which, ε is the Dirichlet character associated with K, i.e., $\varepsilon(n) = \left(\frac{D}{n}\right)$, and $\{\kappa\}$ is the genus of any integral ideal with norm $N(\kappa) \equiv -p (\mathrm{mod}\, D)$.

I.3. Local intersection pairing at infinite places. Next, let us discuss the contribution at infinite places.

Since

$$g(x,y)-g(x,\infty)-g(y,\infty)=-\log\frac{e|x-y|^2}{(1+|x|^2)(1+|y|^2)}+\log\frac{e}{1+|x|^2}+\log\frac{e}{1+|y|^2}$$
$$=1-\log|x-y|^2,$$

we have

$$g(j(\tau_1),j(\tau_2))-g(j(\tau_1),j(\infty))-g(j(\tau_2),j(\infty))=1-\log|j(\tau_1)-j(\tau_2)|^2.$$

So to evaluate the contribution to the Arakelov intersection from the infinite part, we need to understand $\log|j(x)-j(y)|^2$. But then by definition,

$$(*)\qquad \phi_m(j(z_1),j(z_2))=\prod_{\substack{\det\gamma=m\\ \mathrm{mod}\,SL_2(\boldsymbol{Z})}}(j(z_1)-j(\gamma z_2)).$$

So we see that

$$\langle x,T_m(x^\sigma)\rangle_{\rm inf}-\sigma_1(m)\langle x,\infty\rangle_{\rm inf}-\langle T_m(x^\sigma),\infty\rangle_{\rm inf}$$
$$=\sigma_1(m)h_K+\sum_{\substack{\mathscr{A}_1,\mathscr{A}_2\in\mathrm{Cl}_K\\ \mathscr{A}_1\mathscr{A}_2^{-1}=\mathscr{A}}}\log|\phi_m(j(\tau_{\mathscr{A}_1}),j(\tau_{\mathscr{A}_2}))|^2,$$

since we have to take all Archimedean places of the Hilbert class field H of K. Here $\tau_{\mathscr{A}_i}$ denotes the Heegner point on $X_0(1)$ corresponding to the ideal class $\mathscr{A}_i$. Hence, we have

$$\langle x,T_m(x^\sigma)\rangle_{\rm inf}-\sigma_1(m)\langle x,\infty\rangle_{\rm inf}-\langle T_m(x^\sigma),\infty\rangle_{\rm inf}=\sigma_1(m)h_K-|J(\mathscr{A})|.$$

I.4. The proof of Theorem 1. First, since $X_0(1)$ is just $\boldsymbol{P}^1$, so, up to a certain multiple infinite fibers, any two algebraic points (with the same degree) are Arakelov rationally equivalent. Hence, we can find a rational function over H, say f_x, such that

$$\mathrm{div}_{\rm Ar}(f_x)=x-\infty+\sum_{v\ \mathrm{Archimedean}}a_vF_v,$$

where $a_v\in\boldsymbol{R}$ and F_v denotes the fiber over v. Therefore, by the fact that $T_m(x-\infty)^\sigma$ is of degree zero at generic fibre, we see that

$$\left\langle\sum_{v\ \mathrm{Archimedean}}a_vF_v,\ T_m(x-\infty)^\sigma\right\rangle=0.$$

On the other hand, by definition, we have

$$\langle\mathrm{div}_{\rm Ar}(f_x),T_m(x-\infty)^\sigma\rangle=0.$$

In particular, we see that the intersection $\langle x-\infty,T_m(x-\infty)^\sigma\rangle$ is zero.

We may write the intersection as

$$\langle x-\infty,T_m(x-\infty)^\sigma\rangle=\langle x,T_m(x^\sigma)\rangle-\sigma_1(m)\langle x,\infty\rangle-\langle T_m(x),\infty\rangle-h_K\sigma_1(m).$$

But then, from I.2, we see that

$$\begin{aligned}&\langle x, T_m(x^\sigma)\rangle - \sigma_1(m)\langle x, \infty\rangle - \langle T_m(x), \infty\rangle \\ &\quad = \langle x, T_m(x^\sigma)\rangle_{\mathrm{fin}} - \sigma_1(m)\langle x, \infty\rangle_{\mathrm{fin}} - \langle T_m(x), \infty\rangle_{\mathrm{fin}} \\ &\qquad + \langle x, T_m(x^\sigma)\rangle_{\mathrm{inf}} - \sigma_1(m)\langle x, \infty\rangle_{\mathrm{inf}} - \langle T_m(x), \infty\rangle_{\mathrm{inf}} \\ &\quad = \langle x, T_m(x^\sigma)\rangle_{\mathrm{fin}} + \langle x, T_m(x^\sigma)\rangle_{\mathrm{inf}} - \sigma_1(m)\langle x, \infty\rangle_{\mathrm{inf}} - \langle T_m(x), \infty\rangle_{\mathrm{inf}}\,.\end{aligned}$$

Thus, by I.3, we see that

$$\langle x, T_m(x^\sigma)\rangle_{\mathrm{fin}} - h_K\sigma_1(m) - |J(\mathscr{A})| + h_K\sigma_1(m) = 0\,.$$

Therefore,

$$|J(\mathscr{A})| = \langle x, T_m(x^\sigma)\rangle_{\mathrm{fin}}\,.$$

With this, by Proposition 2, we complete the proof of Theorem 1.

II. Proof of Theorem 2. To prove Theorem 2, we need to know the precise expression for the Green function of the projective line $X_0(1)$ over $\boldsymbol{C}$ in terms of the hyperbolic parametrization.

For $s \in \boldsymbol{C}$ with $\mathrm{Re}(s) > 0$, let Q_{s-1} be the Legendre function of the second kind defined by

$$Q_{s-1}(t) := \int_0^\infty (t + \sqrt{t^2-1}\cosh v)^{-s} dv\,, \qquad (t > 1)$$

and define

$$g_s(\tau_1, \tau_2) := -2Q_{s-1}(\cosh d(\tau_1, \tau_2))$$

with $d(\tau_1, \tau_2)$ the hyperbolic distance of τ_1 and τ_2. Then the function $G_s(\tau_1, \tau_2)$ is defined by the absolutely convergent series

$$G_s(\tau_1, \tau_2) = \sum_{\gamma \in \Gamma_0(1)} g_s(\tau_1, \gamma\tau_2)\,.$$

PROPOSITION 3. *In terms of the hyperbolic parametrization, we have the following formula for the Green function of $\boldsymbol{P}^1(\boldsymbol{C})$ with respect to the normalized volume form $d\mu := (\sqrt{-1}/2\pi)dz \wedge d\bar{z}/(1+|z|^2)^2$*

$$\begin{aligned} g(j(\tau_1), j(\tau_2)) = -\lim_{s\to 1}\Bigg[&G_s(\tau_1, \tau_2) - \int_{\boldsymbol{P}^1(\boldsymbol{C})} G_s(\tau_1, \tau_2)d\mu(1) - \int_{\boldsymbol{P}^1(\boldsymbol{C})} G_s(\tau_1, \tau_2)d\mu(2) \\ &+ \int_{\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})} G_s(\tau_1, \tau_2)d\mu(1)d\mu(2)\Bigg]\,.\end{aligned}$$

Here $d\mu(i)$ means that we consider G_s as a function for the i-th variable, and the integration *is taken with respect to* $j(\tau_i)$.

PROOF. The point is that the Green function of $\boldsymbol{P}^1(\boldsymbol{C})$ with respect to $d\mu$ can be uniquely characterized by the following axioms. First, it has the logarithmic singularity when τ_1 approaches τ_2; secondly, dd^c of the Green function is nothing but $d\mu$; thirdly, the integration of the Green function with respect to $d\mu$ is zero. Obviously, the third condition is satisfied automatically by definition. On the other hand, the first one may be obtained from the definition of G_s. The second one comes from the fact that $\lim_{s\to 1}(G_s(\tau_1, \tau_2)+4\pi E(\tau_1, s))$ is a harmonic function, where $E(\tau, s)$ is the Eisenstein series defined by

$$E(\tau, s)=\frac{1}{2}\sum_{\substack{c,d\in\boldsymbol{Z}\\(c,d)=1}}\frac{(\operatorname{Im}\tau)^s}{|c\tau+d|^{2s}}.$$

In fact, one may use the following result of Gross and Zargier to obtain the assertion.

LEMMA (cf. [GZ2]). *For two points* τ_1, τ_2 *of* $\mathscr{H}$, *not equivalent under* $\Gamma_0(1)$, *we have the relation*

$$\log|j(\tau_1)-j(\tau_2)|^2=\lim_{s\to 1}(G_s(\tau_1, \tau_2)+4\pi E(\tau_1, s)+4\pi E(\tau_2, s)-4\pi\phi(s))-24.$$

Here

$$\phi(s):=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

To prove the proposition, we proceed more precisely as follows: Since

$$\int_{\boldsymbol{P}^1(\boldsymbol{C})} d\mu=1$$

and

$$\int_{\boldsymbol{P}^1(\boldsymbol{C})}\log|az+b|^2 d\mu=\log(|a|^2+|b|^2),$$

we see that

$$\begin{aligned}g(j(\tau_1), j(\tau_2))= &-\lim_{s\to 1}(G_s(\tau_1, \tau_2)+4\pi E(\tau_1, s)+4\pi E(\tau_2, s)-4\pi\phi(s))-24\\ &+\int_{\boldsymbol{P}^1(\boldsymbol{C})}[\lim_{s\to 1}(G_s(\tau_1, \tau_2)+4\pi E(\tau_1, s)+4\pi E(\tau_2, s)-4\pi\phi(s))-24]\,d\mu(1)\\ &+\int_{\boldsymbol{P}^1(\boldsymbol{C})}[\lim_{s\to 1}(G_s(\tau_1, \tau_2)+4\pi E(\tau_1, s)+4\pi E(\tau_2, s)-4\pi\phi(s))-24]\,d\mu(2)-1.\end{aligned}$$

So up to constant, we see that $g(j(\tau_1), j(\tau_2))$ is nothing but

$$-\lim_{s\to 1}\left(G_s(\tau_1,\tau_2)-\int_{P^1(C)} G_s(\tau_1,\tau_2)d\mu(1)-\int_{P^1(C)} G_s(\tau_1,\tau_2)d\mu(2)+C(s)\right).$$

Now by the third axiom, we may fix this constant $C(s)$.

REMARK. One may prove the lemma of Gross and Zagier by using the following facts: First, both sides are $\Gamma_0(1)$-invariant; secondly, both sides are continuous except for a singularity $\log|\tau_1-\tau_2|^2+O(1)$; thirdly, both sides are harmonic and have the same decay when τ goes to infinity. A function having these three properties is unique up to constant.

Next, by applying the Hecke operator T_m to the second component on both sides of $(*)$ and by the fact that $E(\tau, s)$ is an eigenfunction of T_m with eigenvalue $m^s\sigma_{1-2s}(m)$, we find

$$\begin{aligned}\log|\phi_m(j(\tau_1), j(\tau_2))|^2 &= \lim_{s\to 1}(G_s^m(\tau_1,\tau_2)+4\pi\sigma_1(m)E(\tau_1,s)+4\pi m^s\sigma_{1-2s}E(\tau_2,s)-4\pi\sigma_1(m)\phi(s))-24\sigma_1(m)\\ &= \lim_{s\to 1}(G_s^m(\tau_1,\tau_2)+4\pi\sigma_1(m)[E(\tau_1,s)+E(\tau_2,s)-\phi(s)])-24\sigma_1(m)-12\sum_{d|m} d\log\frac{m}{d^2}\end{aligned}$$

with

$$G_s^m(\tau_1,\tau_2)=\frac{1}{2}\sum_{\substack{a,b,c,d\in Z\\ ad-bc=m}} g_s\left(\tau_1, \frac{a\tau_2+b}{c\tau_2+d}\right).$$

So we need to evaluate G_s^m first. To do so, we first write G_s^m for the Heegner points in terms of a series of natural numbers, by the properties of the Heegner points, and get the following:

PROPOSITION 4 (cf. [GZ1]). *Let $\mathscr{A}_1, \mathscr{A}_2$ be ideal classes of K, and a_i integral ideals in $\mathscr{A}_i$ with $N(a_i)=A_i$. Then, for $m\in N$, $r_{\mathscr{A}_1\mathscr{A}_2^{-1}}(m)=0$, we have*

$$G_s^m(\tau_{\mathscr{A}_1},\tau_{\mathscr{A}_2})=-2\sum_{n=1}^{\infty}\rho^m(n)Q_{s-1}\left(1+\frac{2n}{m|D|}\right).$$

Here

$$\rho^m(n)=\#\left\{(\alpha,\beta)\in a_1^{-1}\bar{a}_2^{-1}\times a_1^{-1}a_2^{-1}/\{\pm\}: N(\alpha)=\frac{n+m|D|}{A_1A_2},\right.$$
$$\left. N(\beta)=\frac{n}{A_1A_2},\ A_1A_2(\alpha-\beta)\equiv 0 (\mathrm{mod}\ \delta)\right\}.$$

For the proof, see [GZ1].

Now by the fact that the infinite part of the local intersection for our problem is the summation over all Archimedean places of the Hilbert class field H of K, we get the following:

PORPOSITION 5. *In the same notation as above, assume that* $r_{\mathscr{A}}(m)=0$. *Then*

$$\begin{aligned}\langle x, T_m(x^\sigma)\rangle_{\inf}-\sigma_1(m)\langle x, \infty\rangle_{\inf}-\langle T_m(x), \infty\rangle_{\inf} \\ =\lim_{s\to 1}\Bigg[2u^2\sum_{n=1}^{\infty}\delta(n)R_{\{\mathscr{A}\}}(n)r_{\mathscr{A}}(n+m|D|)Q_{s-1}\left(1+\frac{2n}{m|D|}\right) \\ -4\pi\sigma_1(m)[2^{-s+1}|D|^{s/2}u\zeta(2s)^{-1}\zeta_K(s)-\phi(s)h_K]\Bigg] \\ +\left(24\sigma_1(m)+12\sum_{d|m}d\log\frac{m}{d^2}\right)h_K.\end{aligned}$$

PROOF. We see that

$$\begin{aligned}\log|\phi_m(j(\tau_{\mathscr{A}_1}), j(\tau_{\mathscr{A}_2}))|^2=\lim_{s\to 1}\Bigg[-2\sum_{n=1}^{\infty}\rho(n)^m Q_{s-1}\left(1+\frac{2n}{m|D|}\right) \\ +4\pi\sigma_1(m)[2^{-s}|D|^{s/2}u\zeta(2s)^{-1}(\zeta_K(\mathscr{A}_1, s)+\zeta_K(\mathscr{A}_2, s))-\phi(s)]\Bigg] \\ -24\sigma_1(m)-12\sum_{d|m}d\log\frac{m}{d^2}.\end{aligned}$$

Here we use the fact that

$$E(\tau_{\mathscr{A}}, s)=2^{-s}|D|^{s/2}u\zeta(2s)^{-1}\zeta_K(\mathscr{A}, s),$$

where $\tau_{\mathscr{A}}$ denotes the Heegner points in $X_0(1)$ associated with $\mathscr{A}$, and

$$\zeta_K(\mathscr{A}, s):=\sum_{\substack{a \text{ integral} \\ [a]=\mathscr{A}}}\frac{1}{N(a)^s}.$$

On the other hand, we have the following:

FACT 2 (cf. [GZ1]).

$$\sum_{\substack{\mathscr{A}_1,\mathscr{A}_2\in \mathrm{Cl}_K \\ \mathscr{A}_1\mathscr{A}_2^{-1}=\mathscr{A} \\ \mathscr{A}_1\mathscr{A}_2=\mathscr{B}}}\rho^m_{\mathscr{A}_1,\mathscr{A}_2}(n)=\begin{cases}u^2\delta(n)r_{\mathscr{A}}(n+m|D|)r_{\mathscr{B}}(n), & \text{if } \{\mathscr{A}\}=\{\mathscr{B}\}, \\ 0, & \text{otherwise}.\end{cases}$$

Hence, by taking the summation with respect to $\mathscr{A}_1\mathscr{A}_2=\mathscr{B}$, we get

$$\sum_{\substack{\mathscr{A}_1,\mathscr{A}_2\in \mathrm{Cl}_K\\ \mathscr{A}_1\mathscr{A}_2^{-1}=\mathscr{A}\\ \mathscr{A}_1\mathscr{A}_2=\mathscr{B}}} \log|\phi_m(j(\tau_{\mathscr{A}_1}), j(\tau_{\mathscr{A}_2}))|^2$$

$$=\lim\nolimits_{s\to 1}\Bigg[-2u^2\sum_{n=1}^{\infty}\delta(n)r_{\mathscr{A}}(n+m|D|)r_{\mathscr{B}}(n)Q_{s-1}\left(1+\frac{2n}{m|D|}\right)\delta_{\mathscr{A},\mathscr{B}}$$

$$+4\pi\sigma_1(m)\Bigg[2^{-s}|D|^{s/2}u\zeta(2s)^{-1}\Bigg(\sum_{\substack{\mathscr{A}_1,\mathscr{A}_2\in \mathrm{Cl}_K\\ \mathscr{A}_1\mathscr{A}_2^{-1}=\mathscr{A}\\ \mathscr{A}_1\mathscr{A}_2=\mathscr{B}}}(\zeta_K(\mathscr{A}_1,s)+\zeta_K(\mathscr{A}_2,s))\Bigg)$$

$$-\phi(s)\sum_{\substack{\mathscr{A}_1,\mathscr{A}_2\in \mathrm{Cl}_K\\ \mathscr{A}_1\mathscr{A}_2^{-1}=\mathscr{A}\\ \mathscr{A}_1\mathscr{A}_2=\mathscr{B}}}1\Bigg]\Bigg]-\left(24\sigma_1(m)+12\sum_{d|m}d\log\frac{m}{d^2}\right)\sum_{\substack{\mathscr{A}_1,\mathscr{A}_2\in \mathrm{Cl}_K\\ \mathscr{A}_1\mathscr{A}_2^{-1}=\mathscr{A}\\ \mathscr{A}_1\mathscr{A}_2=\mathscr{B}}}1\,.$$

Here $\delta_{\mathscr{A},\mathscr{B}}$ is the generalized Kronecker symbol. Now if we take the summation with respect to $\mathscr{B}$, we complete the proof by using the fact that

$$\sum_{\mathscr{A}}\zeta_K(\mathscr{A},s)=\zeta_K(s)\,.$$

In particular, we get the following:

COROLLARY. *In the notation as above, if* $r_{\mathscr{A}}(m)=0$, *then the Arakelov intersection pairing on the arithmetic model of* $X_0(1)$ *over the Hilbert class field* H *of* K *is*

$$\langle\tau-\infty, T_m(\tau^\sigma-\infty)\rangle=\lim\nolimits_{s\to 1}\Bigg[2u^2\sum_{n=1}^{\infty}\delta(n)R_{\{\mathscr{A}\}}(n)r_{\mathscr{A}}(n+m|D|)Q_{s-1}\left(1+\frac{2n}{m|D|}\right)$$

$$-4\pi\sigma_1(m)[2^{-s+1}|D|^{s/2}u\zeta(2s)^{-1}\zeta_K(s)-\phi(s)h_K]\Bigg]$$

$$+\left(24\sigma_1(m)+12\sum_{d|m}d\log\frac{m}{d^2}\right)h_K+u^2\sum_{1\le n\le m|D|}\sigma'_{\mathscr{A}}(n)r_{\mathscr{A}}(m|D|-n)\,.$$

With this, Theorem 2 is a direct consequence of the fact that, in the corollary above, the Arakelov intersection pairing on the left hand side is zero as we proved in I.4. In fact, one may also write down the limit for the terms involving $\zeta_K(s)$ and $\phi(s)$, by giving their Laurent expansion at $s=1$ up to degree 1 terms.

REMARKS 1. In [GZ1] and [GZ2], Gross and Zagier use the technique of J. Sturm on the holomorphic projection to show that the value given above is related with the derivative of L-series.

2. If $r_{\mathscr{A}}(m)\neq 0$, we may also have the result. But now we need to give a few more terms containing $r_{\mathscr{A}}(m)$. For example, we see that if p splits in K then the contribution to the intersection is $uhr_{\mathscr{A}}(m)\operatorname{ord}_p(m)\log p$. For a more detailed formula in each case,

see the formulas IV 9.2, 9.7 and 9.11 of [GZ1].

3. A similar strategy may also give the prime factorization of some other combinations of certain algebraic integers, which happen to be the difference of j-invariants at some points, e.g., the main theorem of [GZ2].

4. The stategy used in this paper may also be applied to higher dimensional cases, say, the projective plane. We discuss this aspect elsewhere.

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