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SHARP OPIAL-TYPE INEQUALITIES INVOLVING r-DERIVATIVES AND THEIR APPLICATIONS

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Abstract. In this paper we offer very general Opial-type inequalities involving higher order r-derivatives. From these inequalities we then deduce extended and improved versions of several recent results. Some applications which dwell upon the importance of the obtained inequalities are also included.

1. Introduction. Opial's inequality, in its improved form, states that if $u(t)$ is absolutely continuous on $[\alpha, \tau]$ with $u(\alpha)=0$, and $\int_\alpha^\tau (u'(t))^2 dt<\infty$, then

$$\int_\alpha^\tau |u(t)u'(t)|dt \le \frac{(\tau-\alpha)}{2}\int_\alpha^\tau (u'(t))^2 dt\,. \tag{1.1}$$

This simple inequality has motivated a large number of research papers giving its successively simpler proofs, providing various generalizations, and finding discrete analogs. (See [1], [3] and [16] for an extensive bibliography consisting of 83 articles.) Among the generalizations, there is a class of inequalities which instead of the first derivative involves the n-th ($n\ge 1$) order derivative of the given function $u(t)$. The first such result is due to Willett [27], who used this generalization to establish uniqueness results for the n-th order linear ordinary differential equations. For practical application purposes Willett's result in recent years has been improved as well as generalized in several different directions [4]–[10], [13], [17], [18], [20], [23], [28]. In this paper we shall provide very general Opial-type inequalities involving higher order r-derivatives. The obtained results are shown to be sharper and more general than several recent results. We shall also demonstrate the usefulness of our results in the field of ordinary differential equations involving r-derivatives.

2. Inequalities involving one function. Let $-\infty\le\alpha<\tau<\beta\le\infty$. Further, let $r_i(t)>0$, $i=1,\dots,n-1$ and $x(t)$ be sufficiently smooth functions on $[\alpha,\tau]$. Then, for $x(t)$ the r-derivatives are defined as follows

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$$(2.1)\qquad \begin{cases} D_r^{(0)}x = x \\ D_r^{(k)}x = r_k(D_r^{(k-1)}x)', \; k = 1, \ldots, n-1 & \left(' = \dfrac{d}{dt} = D\right) \\ D_r^{(n)}x = (D_r^{(n-1)}x)'. \end{cases}$$

Since the class of operators $D_r^{(n)}$ properly contains disconjugate linear operators $L = D^{(n)} + \sum_{i=1}^{n} a_i(t)D^{(n-i)}$, the theory of ordinary differential equations involving r-derivatives is of increasing interest, e.g., [11], [12], [21], [22], [24], [25], [26]. To obtain Opial-type inequalities involving r-derivatives we note that if $D_r^{(i)}x(\alpha) = 0$, $0 \leq k \leq i \leq n-1$ then on changing the variables several times it follows that

$$(2.2)\qquad D_r^{(k)}x(t) = \int_\alpha^t H_{n-k}(t, s)D_r^{(n)}x(s)ds, \qquad 0 \leq k \leq n-1$$

where

$$H_{n-k}(t, s) = \int_s^t \frac{dt_{k+1}}{r_{k+1}(t_{k+1})} \int_s^{t_{k+1}} \frac{dt_{k+2}}{r_{k+2}(t_{k+2})} \cdots \int_s^{t_{n-2}} \frac{dt_{n-1}}{r_{n-1}(t_{n-1})}, \qquad 0 \leq k \leq n-2$$

and

$$H_1(t, s) = 1.$$

It is clear that when $r_i(t) = 1$, $i = 1, \ldots, n-1$ for each $0 \leq k \leq n$ the r-derivative $D_r^{(k)}$ is simply the ordinary derivative $D^{(k)}$ and $H_{n-k}(t, s) = (t-s)^{n-k-1}/(n-k-1)!$, $0 \leq k \leq n-1$. Thus, in this case (2.2) reduces to the well known relation

$$(2.3)\qquad D^{(k)}x(t) = \frac{1}{(n-k-1)!} \int_\alpha^t (t-s)^{n-k-1} D^{(n)}x(s)ds.$$

Let on $[\alpha, \tau]$, $D_r^{(i)}x(t)$, $0 \leq i \leq n-1$ be continuous, $D_r^{(n-1)}x(t)$ absolutely continuous, and $D_r^{(n)}x(t)$ does not change sign. Then, in view of $H_{n-k}(t, s) \geq 0$, $\alpha \leq s \leq t \leq \tau$ the relation (2.2) implies that

$$(2.4)\qquad |D_r^{(k)}x(t)| = \int_\alpha^t H_{n-k}(t, s)|D_r^{(n)}x(s)|ds = g_k(t), \qquad 0 \leq k \leq n-1.$$

Now let the functions $p(t)$, $q(t)$ be non-negative and measurable on $[\alpha, \tau]$, and $0 \leq k \leq n-1$ be a fixed integer. Then, from Hölder's inequality with indices ρ and $\rho/(\rho-1)$ equation (2.4) gives

$$(2.5)\qquad \begin{aligned} |D_r^{(k)}x(t)| &= \int_\alpha^t H_{n-k}(t, s)(p(s))^{-1/\rho}(p(s))^{1/\rho}|D_r^{(n)}x(s)|ds \\ &\lesseqgtr [P_k(t)]^{(\rho-1)/\rho}\left[\int_\alpha^t p(s)|D_r^{(n)}x(s)|^\rho ds\right]^{1/\rho}, \end{aligned}$$

where

$$P_k(t)=\int_\alpha^t (H_{n-k}^{\rho}(t,s)p^{-1}(s))^{1/(\rho-1)}ds\,. \tag{2.6}$$

In (2.5) it is clear that $\le$ holds if $\rho>1$ and $\ge$ holds if $\rho<0$ or $0<\rho<1$. We set

$$y(t)=\int_\alpha^t p(s)|D_r^{(n)}x(s)|^\rho ds \tag{2.7}$$

so that

$$y'(t)=p(t)|D_r^{(n)}x(t)|^\rho\,,\qquad y(\alpha)=0$$

and hence for any ρ_n it follows that

$$|D_r^{(n)}x(t)|^{\rho_n}=(p(t))^{-\rho_n/\rho}(y'(t))^{\rho_n/\rho}\,. \tag{2.8}$$

Thus, if $\rho_k>0$ we have

$$q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(n)}x(t)|^{\rho_n}\lesseqgtr q(t)(P_k(t))^{\rho_k(\rho-1)/\rho}(p(t))^{-\rho_n/\rho}(y(t))^{\rho_k/\rho}(y'(t))^{\rho_n/\rho}\,, \tag{2.9}$$

where $\le$ holds if $\rho>1$ and $\ge$ holds if $\rho<0$ or $0<\rho<1$. On the other hand, if $\rho_k<0$ we have

$$q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(n)}x(t)|^{\rho_n}\gtreqless q(t)(P_k(t))^{\rho_k(\rho-1)/\rho}(p(t))^{-\rho_n/\rho}(y(t))^{\rho_k/\rho}(y'(t))^{\rho_n/\rho}\,, \tag{2.10}$$

where $\ge$ holds if $\rho>1$ and $\le$ holds if $\rho<0$ or $0<\rho<1$.

We now restrict ρ_k and ρ_n so that $((\rho_k+\rho_n)/\rho_n)>0$, and therefore $(y(\alpha))^{(\rho_k+\rho_n)/\rho_n}=0$. Next, we integrate (2.9) or (2.10) over $[\alpha,\tau]$, and apply Hölder's inequality with indices ρ/ρ_n and $\rho/(\rho-\rho_n)$, to obtain

$$\int_\alpha^\tau q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(n)}x(t)|^{\rho_n}dt\lesseqgtr C_0\left[\int_\alpha^\tau (y(t))^{\rho_k/\rho_n}y'(t)dt\right]^{\rho_n/\rho}\,, \tag{2.11}$$

where

$$\begin{aligned} C_0&=C_0(p,q,\{r_i\},\rho_k,\rho_n,\rho)\\ &=\left[\int_\alpha^\tau (q^\rho(t)p^{-\rho_n}(t))^{1/(\rho-\rho_n)}(P_k(t))^{\rho_k(\rho-1)/(\rho-\rho_n)}dt\right]^{(\rho-\rho_n)/\rho}. \end{aligned} \tag{2.12}$$

Therefore, it follows that either

$$\int_\alpha^\tau q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(n)}x(t)|^{\rho_n}dt\le C_1\left[\int_\alpha^\tau p(t)|D_r^{(n)}x(t)|^\rho dt\right]^{(\rho_k+\rho_n)/\rho} \tag{2.13}$$

or

$$\int_\alpha^\tau q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(n)}x(t)|^{\rho_n}dt \ge C_1\left[\int_\alpha^\tau p(t)|D_r^{(n)}x(t)|^{\rho}dt\right]^{(\rho_k+\rho_n)/\rho} \tag{2.14}$$

holds, where

$$C_1 = C_1(p, q, \{r_i\}, \rho_k, \rho_n, \rho) = \left(\frac{\rho_n}{\rho_k+\rho_n}\right)^{\rho_n/\rho} C_0\,. \tag{2.15}$$

Clearly, if $\le$ holds in (2.9) or (2.10) then we require $(\rho/\rho_n) > 1$ and obtain (2.13), while if $\ge$ holds in (2.9) or (2.10) we require $(\rho/\rho_n) < 0$, or $0 < (\rho/\rho_n) < 1$ and obtain (2.14).

In stating the various cases which arise we shall use the following notation.

$$PX(t) = \int_\alpha^t p(s)|D_r^{(n)}x(s)|^{\rho}ds\,,$$

$$QX = \int_\alpha^\tau q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(n)}x(t)|^{\rho_n}dt\,,$$

$$P_kQ = \int_\alpha^\tau (q^{\rho}(t)p^{-\rho_n}(t))^{1/(\rho-\rho_n)}(P_k(t))^{\rho_k(\rho-1)/(\rho-\rho_n)}dt\,.$$

Further, for $P_k(\tau)$ and $PX(\tau)$ we shall write P_k and PX.

The above analysis schows that (2.13) holds if

$$\left\{\begin{matrix} \rho>1,\ \rho_k>0,\ 0<\rho_n<\rho, \text{ or } \rho<\rho_n<0,\ \rho_k<0, \text{ or } -\rho_n<\rho_k<0, \\ 0<\rho_n<\rho<1 \text{ and } P_k(t) \text{ exists for } t\in[\alpha,\tau],\ P_kQ<\infty,\ PX<\infty. \end{matrix}\right\} \tag{2.16}$$

If $\rho < 0$, then from Hölder's inequality with indices $1-\rho$ and $(\rho-1)/\rho$, we have

$$P_k(t) \le \left(\int_\alpha^t H_{n-k}(t,s)|D_r^{(n)}x(s)|ds\right)^{\rho/(\rho-1)}(PX(t))^{1/(1-\rho)}$$

and hence the existence of $P_k(t)$ follows from that of $PX(t)$, and of $g_k(t)$. Similarly, for $0<\rho<1$, Hölder's inequality with indices $1/\rho$ and $1/(1-\rho)$ gives

$$y(t) \le \left(\int_\alpha^t H_{n-k}(t,s)|D_r^{(n)}x(s)|ds\right)^{\rho}(P_k(t))^{1-\rho}$$

and hence the existence of $y(t)$ follows from that of $P_k(t)$, and of $g_k(t)$.

With these remarks we note that (2.14) holds under any one of the following conditions:

$$\left\{\begin{matrix} \rho_k>0,\ 0<\rho<\min(\rho_n, 1), \text{ or } \rho_n<0<\rho<1,\ 0<\rho_k<-\rho_n \\ \text{and } P_k\,(t) \text{ exists for } t\in[\alpha,\tau],\ P_kQ<\infty,\ QX<\infty; \end{matrix}\right\} \tag{2.17}$$

$$\left\{\begin{array}{l}\rho_k<0,\ \rho_n<0,\ \rho>1, \text{ or } 1<\rho<\rho_n,\ -\rho_n<\rho_k<0 \text{ and} \\ P_k(t),\ PX(t) \text{ exist for } t\in[\alpha,\tau],\ P_kQ<\infty,\ QX<\infty;\end{array}\right\} \tag{2.18}$$

$$\left\{\begin{array}{l}\rho_k>0,\ \rho<0<\rho_n, \text{ or } \rho_n<\rho<0,\ 0<\rho_k<-\rho_n \text{ and} \\ PX(t) \text{ exists for } t\in[\alpha,\tau],\ P_kQ<\infty,\ QX<\infty.\end{array}\right\} \tag{2.19}$$

Thus, we have proved the following:

THEOREM 2.1. *Assume that*

(i) *$p(t)$, $q(t)$ are non-negative and measurable functions on $[\alpha, \tau]$,*

(ii) *on $[\alpha, \tau]$ functions $r_i(t)>0$, $i=1,\dots,n-1$ and $x(t)$ are sufficiently smooth so that the r-derivatives of $x(t)$ exist, $D_r^{(i)}x(t)$, $0\le i\le n-2$ are continuous, $D_r^{(n-1)}x(t)$ is absolutely continuous, and $D_r^{(n)}x(t)$ is of fixed sign,*

(iii) *for $0\le k\le n-1$ $(n\ge 1)$, but fixed, $D_r^{(i)}x(\alpha)=0$, $k\le i\le n-1$.*

Then, the inequality (2.13) *holds provided* (2.16) *holds, and* (2.14) *holds under any one of the conditions* (2.17)–(2.19).

REMARK 2.1. In those cases of (2.16) and (2.18) where $\rho>1$, Theorem 2.1 holds even if $\operatorname{sgn} D_r^{(n)}x(t)$ is not constant. Indeed, in such a case the proof is similar except that now in place of (2.4), we have

$$|D_r^{(k)}x(t)|\le\int_\alpha^t H_{n-k}(t,s)|D_r^{(n)}x(s)|ds=g_k(t)\,,\qquad 0\le k\le n-1\,. \tag{2.20}$$

REMARK 2.2. Equality holds in (2.13), (2.14) if and only if it holds in (2.5) and (2.11), i.e., if and only if

$$p(s)|D_r^{(n)}x(s)|^\rho=d_1(t)(H_{n-k}^\rho(t,s)p^{-1}(s))^{1/(\rho-1)}\,,\qquad 0\le s\le t \tag{2.21}$$

and

$$(y(t))^{\rho_k/\rho_n}y'(t)=c_1(q^\rho(t)p^{-\rho_n}(t))^{1/(\rho-\rho_n)}(P_k(t))^{\rho_k(\rho-1)/(\rho-\rho_n)}\,. \tag{2.22}$$

Equation (2.21) is the same as

$$D_r^{(n)}x(s)=d(t)(H_{n-k}(t,s)p^{-1}(s))^{1/(\rho-1)}\,,\qquad 0\le s\le t \tag{2.23}$$

and hence, from the definition of $H_{n-k}(t, s)$, unless $k=n-1$, $D_r^{(n)}x(s)=0$, and if $n=k+1$, we have

$$D_r^{(k)}x(t)=d\int_\alpha^t r_{k+1}^{-1}(s)(p(s))^{-1/(\rho-1)}ds\quad (d \text{ real})\,. \tag{2.24}$$

Further, when $n=k+1$ the condition (2.23) and the definitions of $P_k(t)$ and $y(t)$ in (2.22) give

$$q(t)=c(p(t))^{(\rho_n-1)/(\rho-1)}\left(\int_\alpha^t (p(s))^{-1/(\rho-1)}ds\right)^{\rho_k(1-\rho_n)/\rho_n} \qquad (c\geq 0)\,. \tag{2.25}$$

For $c=1$ it is easy to compute the sharp constant

$$C_1=\frac{\rho_n}{\rho_k+\rho_n}\left(\int_\alpha^\tau (p(s))^{-1/(\rho-1)}ds\right)^{((\rho_k+\rho_n)(\rho-\rho_n)/\rho\rho_n)}\,. \tag{2.26}$$

In the following three remarks we shall show how the limiting cases in inequalities (2.13) and (2.14) are meaningful.

REMARK 2.3. Hölder's inequality with indices ρ/ρ_n and $\rho/(\rho-\rho_n)$ gives

$$\int_\alpha^\tau q(t)|D_r^{(n)}x(t)|^{\rho_n}dt \lesseqgtr \left(\int_\alpha^\tau (q^\rho(t)p^{-\rho_n}(t))^{1/(\rho-\rho_n)}dt\right)^{(\rho-\rho_n)/\rho}\left(\int_\alpha^\tau p(t)|D_r^{(n)}x(t)|^\rho dt\right)^{\rho_n/\rho}$$

$$=C_1(p,q,\{r_i\},0,\rho_n,\rho)\left(\int_\alpha^\tau p(t)|D_r^{(n)}x(t)|^\rho dt\right)^{\rho_n/\rho},$$

where $\leq$ holds provided $(\rho/\rho_n)>1$ and $\leq$ holds if $(\rho/\rho_n)<0$, or $0<(\rho/\rho_n)<1$. Thus, for $\rho_k=0$ the inequality (2.13) holds provided $0<\rho_n<\rho$, or $\rho<\rho_n<0$; and the inequality (2.14) holds if $\rho<0<\rho_n$, or $\rho_n<0<\rho$, or $0<\rho<\rho_n$, or $\rho_n<\rho<0$.

REMARK 2.4. For the case $\rho=1$ inequalities (2.13) and (2.14) hold by replacing $(P_k(t))^{\rho-1}$ by $\bar{P}_k(t)=\text{ess.sup}_{s\in[\alpha,t]}[H_{n-k}(t,s)p^{-1}(s)]$, or $\tilde{P}_k(t)=\text{ess.inf}_{s\in[\alpha,t]}[H_{n-k}(t,s)p^{-1}(s)]$ appropriately. Indeed, the inequality (2.13) holds with C_1 replaced by

$$C_2=\left(\frac{\rho_n}{\rho_k+\rho_n}\right)^{\rho_n}\left[\int_\alpha^\tau (q(t)p^{-\rho_n}(t)\bar{P}_k^{\rho_k}(t))^{1/(1-\rho_n)}dt\right]^{1-\rho_n}$$

provided $\rho_k>0$, $0<\rho_n<1$; and with C_1 replaced by

$$C_3=\left(\frac{\rho_n}{\rho_k+\rho_n}\right)^{\rho_n}\left[\int_\alpha^\tau (q(t)p^{-\rho_n}(t)\tilde{P}_k^{\rho_k}(t))^{1/(1-\rho_n)}dt\right]^{1-\rho_n}$$

provided $-\rho_n<\rho_k<0<\rho_n<1$. Similarly, (2.14) holds with C_1 replaced by C_2 provided $\rho_k<0$, $\rho_n<0$, or $1<\rho_n$, $-\rho_n<\rho_k<0$; and with C_1 replaced by C_3 provided $\rho_n<0$, $0<\rho_k<-\rho_n$, or $\rho_k>0$, $1<\rho_n$.

REMARK 2.5. As in Remark 2.4 we note that for the case $\rho=\rho_n$ the inequality (2.13) holds with C_1 replaced by

$$C_4=\left(\frac{\rho_n}{\rho_k+\rho_n}\right)\text{ess.sup}_{t\in[\alpha,\tau]}[q(t)p^{-1}(t)(P_k(t))^{\rho_k(\rho_n-1)/\rho_n}]$$

provided $\rho_n>1$, $\rho_k>0$, or $\rho_n<0$, $\rho_k<0$, or $-\rho_n<\rho_k<0<\rho_n<1$. Further, the inequality (2.14) holds with C_1 replaced by

$$C_5=\left(\frac{\rho_n}{\rho_k+\rho_n}\right)\text{ess.inf}_{t\in[\alpha,\tau]}\left[q(t)p^{-1}(t)(P_k(t))^{\rho_k(\rho_n-1)/\rho_n}\right]$$

provided $\rho_k>0$, $0<\rho_n<1$, or $1<\rho_n$, $-\rho_n<\rho_k<0$, or $0<\rho_k<-\rho_n$.

REMARK 2.6. For the case $p(t)=q(t)=1$, $\rho=\rho_k+\rho_n>1$ a weaker form of the inequality (2.13) has been proved by Pachpatte [20].

REMARK 2.7. For the case $p(t)=q(t)=r_i(t)=1$, $1\le i\le n-1$ the constants C_j, $1\le j\le 5$, we rename, C_j^*, $1\le j\le 5$. The constant C_1^* can be computed and appears as

$$C_1^*=\left(\frac{\rho_n}{\rho_k+\rho_n}\right)^{\rho_n/\rho}\frac{1}{((n-k-1)!)^{\rho_k}}\left(\frac{\rho-1}{\rho(n-k)-1}\right)^{\rho_k(\rho-1)/\rho}$$
$$\times\left(\frac{\rho-\rho_n}{\rho[\rho_k(n-k)+1]-(\rho_k+\rho_n)}\right)^{(\rho-\rho_n)/\rho}(\tau-\alpha)^{[\rho_k(n-k)+1]-(\rho_k+\rho_n)/\rho}.$$

The constants $C_j^*, j=2, 4, 5$ corresponding to the limiting cases considered in Remarks 2.4 and 2.5 can be obtained from C_1^*. However, C_3^* does not exist. Thus, in particular when $\rho_k\rho_n>0$, $\rho=\rho_k+\rho_n\ge 1$ the inequality (2.13) reduces to

$$\int_\alpha^\tau |D^{(k)}x(t)|^{\rho_k}|D^{(n)}x(t)|^{\rho_n}dt\le\lambda_{n-k}\int_\alpha^\tau |D^{(n)}x(t)|^{\rho_k+\rho_n}dt, \tag{2.27}$$

where $\lambda_{n-k}=\lambda_{n-k}(\rho_k, \rho_n, \rho_k+\rho_n)=C_1(1, 1, \{1\}, \rho_k, \rho_n, \rho_k+\rho_n)$ is given by

$$\lambda_{n-k}=\nu\rho_n^{\sigma_n\nu}\left(\frac{(n-k)(1-\nu)}{n-k-\nu}\right)^{\rho_k(1-\nu)}((n-k)!)^{-\rho_k}(\tau-\alpha)^{\rho_k(n-k)},\quad \nu=(\rho_k+\rho_n)^{-1}. \tag{2.28}$$

For $k=0$ the cases $\rho_k=\rho_n=1$; $\rho_k+\rho_n>1$ and $\rho_k+\rho_n=1$ or (2.27) have been separately obtained earlier in [7].

REMARK 2.8. Let the conditions of Theorem 2.1 with $k=0$ be satisfied. Further, let $l\ge 1$, $m>0$, $\tau_i\ge 0$, $0\le i\le n-1$ with $\sum_{i=0}^{n-1}\tau_i=1$. Then, in view of the elementary inequality

$$\prod_{k=0}^{n-1}a_k^{\tau_k}\le\sum_{k=0}^{n-1}\tau_k a_k\le\left(\sum_{k=0}^{n-1}\tau_k a_k^l\right)^{1/l},\qquad (a_k\ge 0, 0\le k\le n-1) \tag{2.29}$$

it follows that

$$\int_\alpha^\tau q(t)\left(\prod_{k=0}^{n-1}|D_r^{(k)}x(t)|^{\tau_k}\right)^l|D_r^{(n)}x(t)|^m dt\le\sum_{k=0}^{n-1}\tau_k\int_\alpha^\tau q(t)|D_r^{(k)}x(t)|^l|D_r^{(n)}x(t)|^m dt. \tag{2.30}$$

Thus, if we rename $C_1(p, q, \{r_i\}, l, m, l+m)$ as λ_{n-k}^* (it depends on k as well), then a combination of (2.13) and (2.30) gives the inequality

$$\int_{\alpha}^{\tau} q(t)\left(\prod_{k=0}^{n-1} |D_r^{(k)}x(t)|^{\tau_k}\right)^l |D_r^{(n)}x(t)|^m dt \leq \sum_{k=0}^{n-1} \lambda_{n-k}^{*}\tau_k \int_{\alpha}^{\tau} p(t)|D_r^{(n)}x(t)|^{l+m} dt . \tag{2.31}$$

For $p(t)=q(t)=r_i(t)=1$, $1\leq i\leq n-1$ it is clear that $\lambda_{n-k}^{*}=\lambda_{n-k}(l, m, l+m)$. This particular case of (2.31) has been proved directly by Yang [28].

REMARK 2.9. In addition to the hypotheses in Remark 2.8 let $p(t)=q(t)$, and assume that $q(t)$ is non-increasing on $[\alpha, \tau]$. Then, it is easy to see that $C_1(q, q, \{r_i\}, l, m, l+m)\leq C_1(1, 1, \{r_i\}, l, m, l+m)=\lambda_{n-k}^{**}$ (say). Thus, in this case the following inequality holds

$$\int_{\alpha}^{\tau} q(t)\left(\prod_{k=0}^{n-1} |D_r^{(k)}x(t)|^{\tau_k}\right)^l |D_r^{(n)}x(t)|^m dt \leq \sum_{k=0}^{n-1} \lambda_{n-k}^{**}\tau_k \int_{\alpha}^{\tau} q(t)|D_r^{(n)}x(t)|^{l+m} dt . \tag{2.32}$$

As in Remark 2.8 we note that for $r_i(t)=1$, $1\leq i\leq n-1$, $\lambda_{n-k}^{**}=\lambda_{n-k}(l, m, l+m)$. This particular case of (2.32) has been obtained directly by Cheung [6].

REMARK 2.10. Once again in addition to the hypotheses in Remark 2.8 let $p(t)=q(t)$, and assume that $0<\zeta_1\leq q(t)\leq\zeta_2$, $t\in[\alpha, \tau]$. Then, it follows that $C_1(q, q, \{r_i\}, l, m, l+m)\leq(\zeta_2/\zeta_1)^{l/(l+m)}\lambda_{n-k}^{**}$. Thus, in this case the inequality (2.32) with the right side multiplied by $(\zeta_2/\zeta_1)^{l/(l+m)}$ holds. The case $r_i(t)=1$, $1\leq i\leq n-1$ of this new inequality has been proved directly by Cheung [6].

Now let the hypotheses of Theorem 2.1 with $k=0$ be satisfied. Further, let ρ_k, $0\leq k\leq n-1$, be non-negative numbers such that $\sigma=\sum_{k=0}^{n-1}\rho_k>0$, and $((\sigma+\rho_n)/\rho_n)>0$ so that $(y(\alpha))^{(\sigma+\rho_n)/\rho_n}=0$. Then, from (2.5) and (2.8) we have

$$\begin{aligned} &q(t)\prod_{k=0}^{n-1} |D_r^{(k)}x(t)|^{\rho_k} |D_r^{(n)}x(t)|^{\rho_n} \\ &\lesseqgtr q(t)(p(t))^{-\rho_n/\rho}\left[\prod_{k=0}^{n-1}(P_k(t))^{\rho_k(\rho-1)/\rho}\right](y(t))^{\sigma/\rho}(y'(t))^{\rho_n/\rho} . \end{aligned} \tag{2.33}$$

In (2.33) the $\leq$ holds if $\rho>1$ and $\geq$ holds if $\rho<0$ or $0<\rho<1$. Now, integrating (2.33) over $[\alpha, \tau]$ and applying Hölder's inequality with indices ρ/ρ_n and $\rho/(\rho-\rho_n)$, we obtain

$$\int_{\alpha}^{\tau} q(t)\prod_{k=0}^{n} |D_r^{(k)}x(t)|^{\rho_k} dt \leq C_6\left[\int_{\alpha}^{\tau} p(t)|D_r^{(n)}x(t)|^{\rho} dt\right]^{(\sigma+\rho_n)/\rho} \tag{2.34}$$

or

$$\int_{\alpha}^{\tau} q(t)\prod_{k=0}^{n} |D_r^{(k)}x(t)|^{\rho_k} dt \geq C_6\left[\int_{\alpha}^{\tau} p(t)|D_r^{(n)}x(t)|^{\rho} dt\right]^{(\sigma+\rho_n)/\rho} , \tag{2.35}$$

where

$$(2.36)\qquad C_6 = C_6(p, q, \{r_i\}, \{\rho_i\}, \rho) = \left(\frac{\rho_n}{\sigma+\rho_n}\right)^{\rho_n/\rho} \times \left[\int_\alpha^\tau (q^\rho(t)p^{-\rho_n}(t))^{1/(\rho-\rho_n)} \prod_{k=0}^{n-1} (P_k(t))^{\rho_k(\rho-1)/(\rho-\rho_n)} dt\right]^{(\rho-\rho_n)/\rho}.$$

Thus, using the notation

$$\bar{Q}X = \int_\alpha^\tau q(t) \prod_{k=0}^{n} |D_r^{(k)}x(t)|^{\rho_k} dt,$$

$$\bar{P}_k Q = \int_\alpha^\tau (q^\rho(t)p^{-\rho_n}(t))^{1/(\rho-\rho_n)} \prod_{k=0}^{n-1} (P_k(t))^{\rho_k(\rho-1)/(\rho-\rho_n)} dt,$$

we find that the inequality (2.34) holds if

$$(2.37)\qquad \left\{\begin{array}{l}\rho>1,\ \sigma>0,\ 0<\rho_n<\rho, \text{ and each } P_k(t)\\ \text{exists for } t\in[\alpha,\tau],\ PX<\infty,\ \bar{P}_kQ<\infty;\end{array}\right\}$$

and (2.35) holds under any of the following conditions

$$(2.38)\qquad \left\{\begin{array}{l}\sigma>0,\ 0<\rho<\min(\rho_n, 1), \text{ or } \rho_n<0<\rho<1,\ 0<\sigma<-\rho_n\\ \text{and each } P_k(t) \text{ exists for } t\in[\alpha,\tau],\ \bar{P}_kQ<\infty,\ \bar{Q}X<\infty;\end{array}\right\}$$

$$(2.39)\qquad \left\{\begin{array}{l}\sigma>0,\ \rho<0<\rho_n, \text{ or } \rho_n<\rho<0,\ 0<\sigma<-\rho_n \text{ and each}\\ P_k(t) \text{ exists for } t\in[\alpha,\tau],\ \bar{P}_kQ<\infty,\ \bar{Q}X<\infty.\end{array}\right\}$$

We summarize the above result in the following:

THEOREM 2.2. *Assume that the conditions of Theorem* 2.1 *with* $k=0$ *are satisfied. Further, assume that* ρ_k, $0\le k\le n-1$, *are non-negative numbers such that* $\sigma=\sum_{k=0}^{n-1}\rho_k>0$. *Then, the inequality* (2.34) *holds provided* (2.37) *holds, and* (2.35) *holds under any one of the conditions* (2.38), (2.39). *Further, when* (2.37) *holds the inequality* (2.34) *holds even when* $\operatorname{sgn} D_r^{(n)}x(t)$ *is not constant.*

REMARK 2.11. For the case $p(t)=q(t)=\rho_k=1$, $0\le k\le n$, $\rho=2$ a weaker form of the inequality (2.34) has been proved by Pachpatte [20].

REMARK 2.12. The closest to our inequality (2.34) is a result recently obtained by Li [13] for the case $r_i(t)=1$, $1\le i\le n-1$, $0<\rho_n<\rho=\sigma+\rho_n$, $\rho>1$. Indeed, in this case we have

$$(2.40)\qquad \int_\alpha^\tau q(t)\prod_{k=0}^{n} |D^{(k)}x(t)|^{\rho_k} dt \le C_7 \int_\alpha^\tau p(t)|D^{(n)}x(t)|^{\sigma+\rho_n} dt,$$

where

$$(2.41)\qquad C_7=C_6(p,q,\{1\},\{\rho_i\},\sigma+\rho_n)=\left(\frac{\rho_n}{\sigma+\rho_n}\right)^{\rho_n/(\sigma+\rho_n)}\frac{1}{\Omega}$$

$$\times\left[\int_\alpha^\tau (q(t))^{(\sigma+\rho_n)/\sigma}(p(t))^{-\rho_n/\sigma}\prod_{k=0}^{n-1}(P_k^*(t))^{\rho_k(\sigma+\rho_n-1)/\sigma}dt\right]^{\sigma/(\sigma+\rho_n)},$$

$$(2.42)\qquad P_k^*(t)=\int_\alpha^t (t-s)^{(n-k-1)(\sigma+\rho_n)/(\sigma+\rho_n-1)}(p(s))^{-1/(\sigma+\rho_n-1)}ds$$

and

$$\Omega=\prod_{k=0}^{n-1}[(n-k-1)!]^{\rho_k}.$$

Remark 2.13. Once again let $r_i(t)=1$, $1\le i\le n-1$, $0<\rho_n<\rho=\sigma+\rho_n$, $\rho>1$ and $p(t)=q(t)$, where $q(t)$ is non-increasing on $[\alpha,\tau]$. Then, it follows that

$$C_8=C_6(q,q,\{1\},\{\rho_i\},\rho),\qquad \rho=\sigma+\rho_n$$

$$\le\left(\frac{\rho_n}{\rho}\right)^{\rho_n/\rho}\prod_{k=0}^{n-1}\left[\frac{1}{(n-k-1)!}\left(\frac{\rho-1}{\rho(n-k)-1}\right)^{(\rho-1)/\rho}\right]^{\rho_k}$$

$$\times\left[\int_\alpha^\tau (t-\alpha)^{\sum_{k=0}^{n-1}\rho_k[\rho(n-k)-1]/(\rho-\rho_n)}dt\right]^{(\rho-\rho_n)/\rho}$$

$$=\left(\frac{\rho_n}{\rho}\right)^{\rho_n/\rho}\left(\frac{\rho-\rho_n}{\rho\sum_{k=0}^{n-1}\rho_k(n-k)}\right)^{(\rho-\rho_n)/\rho}$$

$$\times\prod_{k=0}^{n-1}\left[\frac{1}{(n-k-1)!}\left(\frac{\rho-1}{\rho(n-k)-1}\right)^{(\rho-1)/\rho}(\tau-\alpha)^{n-k}\right]^{\rho_k}.$$

Further, in view of (2.29), we have

$$\sum_{k=0}^{n-1}\frac{\rho_k}{\sigma}(n-k)\ge\prod_{k=0}^{n-1}(n-k)^{\rho_k/\sigma}$$

and hence

$$(2.43)\qquad C_8\le\frac{1}{\rho}(\rho_n)^{\rho_n/\rho}\prod_{k=0}^{n-1}\left[\frac{(n-k)^{-1/\rho}}{(n-k-1)!}\left(\frac{\rho-1}{\rho(n-k)-1}\right)^{(\rho-1)/\rho}(\tau-\alpha)^{n-k}\right]^{\rho_k}$$

$$=\nu_1\rho_n^{\rho_n\nu_1}\prod_{k=0}^{n-1}\left[\frac{1}{(n-k)!}\left(\frac{(n-k)(1-\nu_1)}{n-k-\nu_1}\right)^{(1-\nu_1)}(\tau-\alpha)^{n-k}\right]^{\rho_k}$$

$$=C_9,\qquad \nu_1=r^{-1}.$$

Therefore, the following inequality holds

$$\int_{\alpha}^{\tau} q(t) \prod_{k=0}^{n} |D^{(k)}x(t)|^{\rho_k} dt \leq C_9 \int_{\alpha}^{\tau} q(t) |D^{(n)}x(t)|^{\sigma+\rho_n} dt. \tag{2.44}$$

This inequality not only extends the range of ρ_k, $0 \leq k \leq n$, it is also sharper than (2.32) for the case $r_i(t)=1$, $1 \leq i \leq n-1$. To show this it suffices to note that for $\rho_k = l\tau_k$, $0 \leq k \leq n-1$, $\sum_{k=0}^{n-1} \tau_k = 1$, $\rho_n = m$, we have $\sigma = l$, $v_1 = (l+m)^{-1} = \xi$, and thus (2.29) gives

$$C_9 = \prod_{k=0}^{n-1} \left[\frac{(\xi m^{m\xi})^{1/l}}{(n-k)!} \left(\frac{(n-k)(1-\xi)}{n-k-\xi} \right)^{(1-\xi)} (\tau-\alpha)^{n-k} \right]^{l\tau_k}$$

$$= \prod_{k=0}^{n-1} \lambda_{n-k}^{\tau_k}(l, m, l+m) \leq \sum_{k=0}^{n-1} \lambda_{n-k}(l, m, l+m)\tau_k.$$

REMARK 2.14. Let $r_i(t)=1$, $1 \leq i \leq n-1$, $\rho > 1$, $\sigma > 0$, $0 < \rho_n < \rho$ and let $p(t) = q(t)$, where $q(t)$ satisfies $0 < \zeta_1 \leq q(t) \leq \zeta_2$, $t \in [\alpha, \tau]$. Then, as earlier it follows that

$$C_{10} = C_6(q, q, \{1\}, \{\rho_i\}, \rho)$$

$$\leq \left(\frac{\rho_n}{\sigma+\rho_n} \right)^{\rho_n/\rho} \frac{1}{\Omega} \zeta_1^{-\sigma/\rho} \zeta_2^{(\rho-\rho_n)/\rho} \prod_{k=0}^{n-1} \left(\frac{\rho-1}{\rho(n-k)-1} \right)^{\sigma_k(\rho-1)/\rho}$$

$$\times \left(\frac{\rho-\rho_n}{\rho \sum_{k=0}^{n-1} \rho_k(n-k) + \rho - \sigma - \rho_n} \right)^{(\rho-\rho_n)/\rho} (\tau-\alpha)^{[\rho \sum_{k=0}^{n-1} \rho_k(n-k) + \rho - \sigma - \rho_n]/\rho}$$

$$= C_{11}.$$

Therefore, the following inequality holds

$$\int_{\alpha}^{\tau} q(t) \prod_{k=0}^{n} |D^{(k)}x(t)|^{\rho_k} dt \leq C_{11} \left[\int_{\alpha}^{\tau} q(t) |D^{(n)}x(t)|^{\rho} dt \right]^{(\sigma+\rho_n)/\rho}. \tag{2.45}$$

When $\rho = \sigma + \rho_n$ the above inequality reduces to

$$\int_{\alpha}^{\tau} q(t) \prod_{k=0}^{n} |D^{(k)}x(t)|^{\rho_k} dt \leq C_9 \left(\frac{\zeta_2}{\zeta_1} \right)^{\sigma/\rho} \int_{\alpha}^{\tau} q(t) |D^{(n)}x(t)|^{\sigma+\rho_n} dt. \tag{2.46}$$

As in Remark 2.13 we note that this inequality not only extends the range of ρ_k, $0 \leq k \leq n$, it is also sharper than the corresponding result of Cheung [6] (cf. Remark 2.10).

For $\rho_k = 1$, $0 \leq k \leq n$, $\rho = 2$ and $q(t) = 1$ the inequality (2.45) reduces to

$$\int_{\alpha}^{\tau} \prod_{k=0}^{n} |D^{(k)}x(t)| dt \leq C_{12} \left[\int_{\alpha}^{\tau} |D^{(n)}x(t)|^2 dt \right]^{(n+1)/2}, \tag{2.47}$$

where C_{12} is given by

$$C_{12} = \frac{(\tau-\alpha)^{(n^2+1)/2}}{(n^2+1)(n+1)\prod_{k=0}^{n-1}(n-k-1)!} \left(\frac{(n^2+1)(n+1)}{\prod_{k=0}^{n-1}(2n-2k-1)} \right)^{1/2}. \tag{2.48}$$

This result has been directly proved by Pachpatte [18].

Next, from (2.4) we note that $g_{k+1}(t)=r_{k+1}(t)g'_k(t)$, $0\le k\le n-2$ and $g_k(\alpha)=0$, $0\le k\le n-1$. We define $g_n(t)=r_n(t)g'_{n-1}(t)$, where $r_n(t)=1$. Thus, if $((\rho_k+\rho_{k+1})/\rho_{k+1})>0$, in view of Hölder's inequality with indices $1/\rho_{k+1}$ and $1/(1-\rho_{k+1})$, it follows that

$$(2.49)\qquad \begin{aligned}&\int_\alpha^\tau q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(k+1)}x(t)|^{\rho_{k+1}}dt\\ &=\int_\alpha^\tau q(t)(r_{k+1}(t))^{\rho_{k+1}}(g_k(t))^{\rho_k}(g'_k(t))^{\rho_{k+1}}dt\\ &\lesseqgtr\left(\int_\alpha^\tau (q(t)r_{k+1}^{\sigma_{k+1}}(t))^{1/(1-\rho_{k+1})}dt\right)^{1-\rho_{k+1}}\left(\int_\alpha^\tau (g_k(t))^{\rho_k/\rho_{k+1}}g'_k(t)dt\right)^{\rho_{k+1}}\\ &=\left(\frac{\rho_{k+1}}{\rho_k+\rho_{k+1}}\right)^{\rho_{k+1}}\left(\int_\alpha^\tau (q(t)r_{k+1}^{\rho_{k+1}}(t))^{1/(1-\rho_{k+1})}dt\right)^{1-\rho_{k+1}}(g_k(\tau))^{\rho_k+\rho_{k+1}}.\end{aligned}$$

In (2.49) the inequality $\le$ holds if $(1/\rho_{k+1})>1$, and $\ge$ holds if $(1/\rho_{k+1})<0$, or $0<(1/\rho_{k+1})<1$.

From Hölder's inequality with indices ρ and $\rho/(1-\rho)$, we also have

$$(2.50)\qquad g_k(\tau)\lesseqgtr P_k^{(\rho-1)/\rho}\left[\int_\alpha^\tau p(t)|D_r^{(n)}x(t)|^\rho dt\right]^{1/\rho},$$

where $\le$ holds if $\rho>1$ and $\ge$ holds if $\rho<0$, or $0<\rho<1$.

Hence, either

$$(2.51)\qquad \int_\alpha^\tau q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(k+1)}x(t)|^{\rho_{k+1}}dt\le C_{13}\left[\int_\alpha^\tau p(t)|D_r^{(n)}x(t)|^\rho dt\right]^{(\rho_k+\rho_{k+1})/\rho}$$

or

$$(2.52)\qquad \int_\alpha^\tau q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(k+1)}x(t)|^{\rho_{k+1}}dt\ge C_{13}\left[\int_\alpha^\tau p(t)|D_r^{(n)}x(t)|^\rho dt\right]^{(\rho_k+\rho_{k+1})/\rho}$$

holds, where the constant $C_{13}=C_{13}(p, q, \{r_i\}, \rho_k, \rho_{k+1}, \rho)$ is

$$(2.53)\qquad C_{13}=\left(\frac{\rho_{k+1}}{\rho_k+\rho_{k+1}}\right)^{\rho_{k+1}}\left(\int_\alpha^\tau (q(t)r_{k+1}^{\rho_{k+1}}(t))^{1/(1-\rho_{k+1})}dt\right)^{1-\rho_{k+1}}P_k^{[(\rho-1)(\rho_k+\rho_{k+1})]/\rho}.$$

Thus, using the notation

$$Q=\int_\alpha^\tau (q(t)r_{k+1}^{\rho_{k+1}}(t))^{1/(1-\rho_{k+1})}dt,$$

$$\tilde{Q}X=\int_\alpha^\tau q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(k+1)}x(t)|^{\rho_{k+1}}dt,$$

we find that the inequality (2.51) holds if

$$\begin{Bmatrix} \rho>1>\rho_{k+1}>0,\ \rho_k+\rho_{k+1}>0, \text{ and} \\ P_k<\infty,\ Q<\infty,\ PX<\infty, \end{Bmatrix} \tag{2.54}$$

and (2.52) holds under any of the following conditions

$$\begin{Bmatrix} \rho>1,\ \rho_{k+1}<0,\ \rho_k+\rho_{k+1}<0, \text{ and} \\ P_k<\infty,\ Q<\infty,\ \tilde{Q}X<\infty; \end{Bmatrix} \tag{2.55}$$

$$\begin{Bmatrix} \rho<0, \text{ or } 0<\rho<1,\ \rho_k+\rho_{k+1}>0,\ \rho_{k+1}>1, \\ \text{and } P_k<\infty,\ Q<\infty,\ \tilde{Q}X<\infty. \end{Bmatrix} \tag{2.56}$$

We summarize the above consideration in the following:

THEOREM 2.3. *Assume that the conditions of Theorem* 2.1 *are satisfied. Then, the inequality* (2.51) *holds provided* (2.54) *holds, and* (2.52) *holds under any one of the conditions* (2.55), (2.56). *Further, when* $\operatorname{sgn} D_r^{(n)}x(t)$ *is not constant the inequality* (2.51) *remains valid provided in addition to* (2.54), $\rho_k>0$; *and the inequality* (2.52) *holds provided* (2.55) *holds with* $\rho_k<0$.

REMARK 2.15. Following the methods of Remarks 2.3–2.5 the limiting cases in the inequalities (2.34), (2.35), (2.51) and (2.52) can be discussed by replacing appropriate quantities by their ess.sup or ess.inf.

REMARK 2.16. For $\rho_k=\rho_{k+1}=p(t)=q(t)=r_i(t)=1$, $1\le i\le n-1$ the inequality (2.51) reduces to

$$\int_\alpha^\tau |D^{(k)}x(t)D^{(k+1)}x(t)|\,dt\le C_{14}\left[\int_\alpha^\tau |D^{(n)}x(t)|^\rho dt\right]^{2/\rho}, \tag{2.57}$$

where the constant $C_{14}=C_{13}(1, 1, \{1\}, 1, 1, \rho)$ is

$$C_{14}=\frac{(\tau-\alpha)^{2n-2k-2/\rho}}{2((n-k-1)!)^2[(n-k-1)\rho'+1]^{2/\rho'}}, \qquad \rho'=\rho/(\rho-1). \tag{2.58}$$

The inequality (2.57) is a recent contribution of Fink [9].

REMARK 2.17. The only overlapping case of Theorems 2.1 and 2.3 is when $k=n-1$. Further, in this case both the inequalities (2.13) and (2.51) with $\rho_k=\rho_{k+1}=p(t)=q(t)=r_i(t)=1$, $1\le i\le n-1$ $u(t)=x^{(n-1)}(t)$ reduce to (1.1).

REMARK 2.18. Let $r_i(t)>0, i=1,\ldots,n-1$ and $x(t)$ be sufficiently smooth functions on $[\tau, \beta]$ so that for $x(t)$ the r-derivatives exist. If $D_r^{(i)}x(\beta)=0$, $0\le k\le i\le n-1$ then it follows that

$$D_r^{(k)}x(t)=(-1)^{n-k}\int_t^\beta G_{n-k}(t, s)D_r^{(n)}x(s)\,ds, \qquad 0\le k\le n-1 \tag{2.59}$$

where

$$G_{n-k}(t, s)=\int_t^s \frac{dt_{k+1}}{r_{k+1}(t_{k+1})} \int_{t_{k+1}}^s \frac{dt_{k+2}}{r_{k+2}(t_{k+2})} \cdots \int_{t_{n-2}}^s \frac{dt_{n-1}}{r_{n-1}(t_{n-1})}, \qquad 0 \leq k \leq n-2$$

and

$$G_1(t, s)=1 .$$

We note that when $r_i(t)=1$, $i=1, \ldots, n-1$ for each $0 \leq k \leq n$ the r-derivative $D_r^{(k)}$ is simply the ordinary derivative $D^{(k)}$ and $G_{n-k}(t, s)=(s-t)^{n-k-1}/(n-k-1)!$, $0 \leq k \leq n-1$. Thus, in this case (2.59) reduces to the known relation

$$D^{(k)} x(t)=\frac{(-1)^{n-k}}{(n-k-1)!} \int_t^\beta (s-t)^{n-k-1} D^{(n)}(s) ds . \tag{2.60}$$

Let on $[\tau, \beta]$, $D_r^{(i)} x(t)$, $0 \leq i \leq n-1$ be continuous, $D_r^{(n-1)} x(t)$ absolutely continuous, and $D_r^{(n)} x(t)$ does not change sign. Then, in view of $G_{n-k}(t, s) \geq 0$, $\tau \leq t \leq s \leq \beta$ the relation (2.59) implies that

$$|D_r^{(k)} x(t)|=\int_t^\beta G_{n-k}(t, s)|D_r^{(n)} x(s)| ds=h_k(t), \qquad 0 \leq k \leq n-1 . \tag{2.61}$$

Further, let the functions $p(t)$, $q(t)$ be non-negative and measurable on $[\tau, \beta]$, and $0 \leq k \leq n-1$ be a fixed integer. Then, from (2.61) it is clear that all the above results remain valid provided in the hypotheses the interval $[\alpha, \tau]$, the integral $\int_\alpha^\tau$, and the term $H_{n-k}(t, s)$ are respectively replaced by $[\tau, \beta]$, $\int_\tau^\beta$, and $G_{n-k}(t, s)$. In particular, with such a replacement the inequalities (2.13), (2.14), (2.34), (2.35), (2.51) and (2.52) take the following form

$$\int_\tau^\beta q(t)|D_r^{(k)} x(t)|^{\rho_k}|D_r^{(n)} x(t)|^{\rho_n} dt \leq \bar{C}_1\left[\int_\tau^\beta p(t)|D_r^{(n)} x(t)|^\rho dt\right]^{(\rho_k+\rho_n)/\rho} ; \tag{2.31)'$$

$$\int_\tau^\beta q(t)|D_r^{(k)} x(t)|^{\rho_k}|D_r^{(n)} x(t)|^{\rho_n} dt \geq \bar{C}_1\left[\int_\tau^\beta p(t)|D_r^{(n)} x(t)|^\rho dt\right]^{(\rho_k+\rho_n)/\rho} , \tag{2.14)'$$

where

$$\bar{C}_1=\left(\frac{\rho_n}{\rho_k+\rho_n}\right)^{\rho_n/\rho}\left[\int_\tau^\beta (q^\rho(t) p^{-\rho_n}(t))^{1/(\rho-\rho_n)}(Q_k(t))^{\rho_k(\rho-1)/(\rho-\rho_n)} dt\right]^{(\rho-\rho_n)/\rho} \tag{2.62}$$

and

$$Q_k(t)=\int_t^\beta (G_{n-k}^\rho(t, s) p^{-1}(s))^{1/(\rho-1)} ds ; \tag{2.63}$$

$$(2.34)' \qquad \int_\tau^\beta q(t) \prod_{k=0}^{n} | D_r^{(k)} x(t) |^{\rho_k} dt \le \bar{C}_6 \left[\int_\tau^\beta p(t) | D_r^{(n)} x(t) |^\rho dt \right]^{(\sigma+\rho_n)/\rho};$$

$$(2.35)' \qquad \int_\tau^\beta q(t) \prod_{k=0}^{n} | D_r^{(k)} x(t) |^{\rho_k} dt \ge \bar{C}_6 \left[\int_\tau^\beta p(t) | D_r^{(n)} x(t) |^\rho dt \right]^{(\sigma+\rho_n)/\rho},$$

where

$$(2.64) \quad \bar{C}_6 = \left(\frac{\rho_n}{\sigma+\rho_n} \right)^{\rho_n/\rho} \left[\int_\tau^\beta (q^\rho(t) p^{-\rho_n}(t))^{1/(\rho-\rho_n)} \prod_{k=0}^{n-1} (Q_k(t))^{\rho_k(\rho-1)/(\rho-\rho_n)} dt \right]^{(\rho-\rho_n)/\rho};$$

$$(2.51)' \quad \int_\tau^\beta q(t) | D_r^{(k)} x(t) |^{\rho_k} | D_r^{(k+1)} x(t) |^{\rho_{k+1}} dt \le \bar{C}_{13} \left[\int_\tau^\beta p(t) | D_r^{(n)} x(t) |^\rho dt \right]^{(\rho_k+\rho_{k+1})/\rho};$$

$$(2.52)' \quad \int_\tau^\beta q(t) | D_r^{(k)} x(t) |^{\rho_k} | D_r^{(k+1)} x(t) |^{\rho_{k+1}} dt \ge \bar{C}_{13} \left[\int_\tau^\beta p(t) | D_r^{(n)} x(t) |^\rho dt \right]^{(\rho_k+\rho_{k+1})/\rho}$$

where

$$(2.65) \quad \bar{C}_{13} = \left(\frac{\rho_{k+1}}{\rho_k+\rho_{k+1}} \right)^{\rho_{k+1}} \left(\int_\tau^\beta (q(t) r_{k+1}^{\rho_{k+1}}(t))^{1/(1-\rho_{k+1})} dt \right)^{1-\rho_{k+1}} Q_k^{[(\rho-1)(\rho_k+\rho_{k+1})]/\rho}$$

and $Q_k(\tau) = Q_k$.

Finally, as in Remark 2.1 and Theorems 2.2 and 2.3 we note that in the inequalities (2.13)′, (2.14)′, (2.34)′, (2.35)′, (2.51)′ and (2.52)′ we can separate the cases which do not require the condition that $\operatorname{sgn} D_r^{(n)} x(t)$ is constant.

REMARK 2.19. Let the functions $p(t)$, $q(t)$ be non-negative and measurable on the interval $[\alpha, \beta]$, and the numbers ρ_k, ρ_n, ρ satisfy the condition (2.16). Let $r_i(t) > 0$, $i = 1, \dots, n-1$ and $x(t)$ be sufficiently smooth functions on $[\alpha, \beta]$ so that for $x(t)$ the r-derivatives exist. Let on $[\alpha, \beta]$, $D_r^{(i)} x(t)$, $0 \le i \le n-1$ be continuous, $D_r^{(n-1)} x(t)$ absolutely continuous, and $D_r^{(n)} x(t)$ does not change sign. Further, let $D_r^{(i)} x(\alpha) = D_r^{(i)} x(\beta) = 0$, $0 \le k \le i \le n-1$ $(n \ge 1)$. Then, if we denote $C_1 = C_1(\tau)$ and $\bar{C}_1 = \bar{C}_1(\tau)$, the inequalities (2.13) and (2.13)′ can be added to obtain

$$(2.66) \quad \int_\alpha^\beta q(t) | D_r^{(k)} x(t) |^{\rho_k} | D_r^{(n)} x(t) |^{\rho_n} dt$$
$$\le C_1(\tau_0) \left(\left[\int_\alpha^\tau p(t) | D_r^{(n)} x(t) |^\rho dt \right]^{(\rho_k+\rho_n)/\rho} + \left[\int_\tau^\beta p(t) | D_r^{(n)} x(t) |^\rho dt \right]^{(\rho_k+\rho_n)/\rho} \right),$$

where τ_0 is the unique solution of the equation $C_1(\tau) = \bar{C}_1(\tau)$.

In (2.66) we can use the well-known inequalities

$$(2.67) \qquad a^\lambda + b^\lambda \le (a+b)^\lambda \le 2^{\lambda-1}(a^\lambda + b^\lambda), \quad a, b \ge 0, \quad \lambda \ge 1;$$

(2.68) $$2^{\lambda-1}(a^\lambda+b^\lambda)\le(a+b)^\lambda\le a^\lambda+b^\lambda\,, \qquad a,b\ge 0\,, \quad 0\le\lambda\le 1\,;$$

(2.69) $$a^\lambda+b^\lambda\ge 2^{1-\lambda}(a+b)^\lambda\,, \qquad a,b>0\,, \quad \lambda<0\,,$$

to obtain

(2.70) $$\int_\alpha^\beta q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(n)}x(t)|^{\rho_n}dt\le LC_1(\tau_0)\left(\int_\alpha^\beta p(t)|D_r^{(n)}x(t)|^\rho dt\right)^{(\rho_k+\rho_n)/\rho},$$

where $L=1$ if $((\rho_k+\rho_n)/\rho)\ge 1$, and $L=2^{(\rho-\rho_k-\rho_n)/\rho}$ if $0<((\rho_k+\rho_n)/\rho)\le 1$.

Similarly, if ρ_k, ρ_n, ρ satisfy any one of the conditions (2.17)–(2.19), then the inequalities (2.14) and (2.14)′ can be added to obtain

(2.71) $$\int_\alpha^\beta q(t)|D_r^{(k)}x(t)|^{\rho_k}|D_r^{(n)}x(t)|^{\rho_n}dt\ge JC_1(\tau_0)\left(\int_\alpha^\beta p(t)|D_r^{(n)}x(t)|^\rho dt\right)^{(\rho_k+\rho_n)/\rho},$$

where $J=1$ if $0<((\rho_k+\rho_n)/\rho)\le 1$, and $J=2^{(\rho-\rho_k-\rho_n)/\rho}$ if $((\rho_k+\rho_n)/\rho)<0$, or $((\rho_k+\rho_n)/\rho)\ge 1$.

We also note that the inequalities (2.34), (2.34)′, (2.35), (2.35)′; (2.51), (2.51)′, and (2.52), (2.52)′ can be added to obtain inequalities analogous to (2.70) and (2.71). This leads to several new inequalities as well, and some of these extend and improve the corresponding known results of Cheung [6], Das [7] and Yang [28].

3. Inequalities involving two functions. Let the functions $p(t)$, $q(t)$, $r_i(t)>0$, $i=1,\dots,n-1$ and $x=x_1(t)$ and $x=x_2(t)$ satisfy the conditions of Theorem 2.1 except that $D_r^{(n)}x_1(t)$ and $D_r^{(n)}x_2(t)$ need not be of fixed sign. Then, in view of Remark 2.1 it is clear that for $i=1$, 2 and $0\le k\le n-1$

(3.1) $$|D_r^{(k)}x_i(t)|\le\int_\alpha^t H_{n-k}(t,s)|D_r^{(n)}x_i(s)|ds=g_{i,k}(t)\,.$$

Hence, as earlier from Hölder's inequality with indices $\rho>1$ and $\rho/(\rho-1)$, we have

(3.2) $$|D_r^{(k)}x_i(t)|\le[P_k(t)]^{(\rho-1)/\rho}[y_i(t)]^{1/\rho}\,,$$

where $P_k(t)$ is the same as in (2.6), and

(3.3) $$y_i(t)=\int_\alpha^t p(s)|D_r^{(n)}x_i(s)|^\rho ds$$

so that

(3.4) $$y_i'(t)=p(t)|D_r^{(n)}x_i(t)|^\rho\,, \qquad y_i(\alpha)=0\,.$$

Thus, for $k\le\mu\le\nu\le n-1$, but fixed, and ρ_μ, $\rho_\nu\ge 0$ it follows that

(3.5) $$q(t)|D_r^{(\mu)}x_1(t)|^{\rho_\mu}|D_r^{(\nu)}x_2(t)|^{\rho_\nu}|D_r^{(n)}x_1(t)|^{\rho_n}\le f(t)(y_1(t))^{\rho_\mu/\rho}(y_2(t))^{\rho_\nu/\rho}(y_1'(t))^{\rho_n/\rho}\,,$$

where

$$f(t)=q(t)(P_\mu(t))^{\rho_\mu(\rho-1)/\rho}(P_v(t))^{\rho_v(\rho-1)/\rho}(p(t))^{-\rho_n/\rho}. \tag{3.6}$$

Next, we integrate (3.5) over $[\alpha, \tau]$, and apply Hölder's inequality with indices $(\rho/\rho_n)>1$ and $\rho/(\rho-\rho_n)$, to obtain

$$\int_\alpha^\tau q(t)|D_r^{(\mu)}x_1(t)|^{\rho_\mu}|D_r^{(v)}x_2(t)|^{\rho_v}|D_r^{(n)}x_1(t)|^{\rho_n}dt \tag{3.7}$$
$$\leq K_0\left[\int_\alpha^\tau (y_1(t))^{\rho_\mu/\rho_n}(y_2(t))^{\rho_v/\rho_n}y_1'(t)dt\right]^{\rho_n/\rho},$$

where

$$K_0=K_0(p, q, \{r_i\}, \rho_\mu, \rho_v, \rho_n, \rho)=\left[\int_\alpha^\tau (f(t))^{\rho/(\rho-\rho_n)}dt\right]^{(\rho-\rho_n)/\rho}. \tag{3.8}$$

Similarly, we find that

$$\int_\alpha^\tau q(t)|D_r^{(v)}x_1(t)|^{\rho_v}|D_r^{(\mu)}x_2(t)|^{\rho_\mu}|D_r^{(n)}x_2(t)|^{\rho_n}dt \tag{3.9}$$
$$\leq K_0\left[\int_\alpha^\tau (y_1(t))^{\rho_v/\rho_n}(y_2(t))^{\rho_\mu/\rho_n}y_2'(t)dt\right]^{\rho_n/\rho}.$$

From (3.7) and (3.9) we shall obtain a number of interesting results. For this we will repeatedly use the inequalities (2.67) and (2.68).

When $\rho_v=0$ an addition of (3.7) and (3.9) in view of (2.67) and (2.68) gives

$$\int_\alpha^\tau q(t)[|D_r^{(\mu)}x_1(t)|^{\rho_\mu}|D_r^{(n)}x_1(t)|^{\rho_n}+|D_r^{(\mu)}x_2(t)|^{\rho_\mu}|D_r^{(n)}x_2(t)|^{\rho_n}]dt \tag{3.10}$$
$$\leq K_0(p, q, \{r_i\}, \rho_\mu, 0, \rho_n, \rho)\left(\frac{\rho_n}{\rho_\mu+\rho_n}\right)^{\rho_n/\rho}[(y_1(\tau))^{(\rho_\mu+\rho_n)/\rho}+(y_2(\tau))^{(\rho_\mu+\rho_n)/\rho}]$$
$$\leq K_1\left[\int_\alpha^\tau p(t)[|D_r^{(n)}x_1(t)|^\rho+|D_r^{(n)}x_2(t)|^\rho]dt\right]^{(\rho_\mu+\rho_n)/\rho},$$

where

$$K_1=K_1(p, q, \{r_i\}, \rho_\mu, \rho_n, \rho)=\theta_1\left(\frac{\rho_n}{\rho_\mu+\rho_n}\right)^{\rho_n/\rho} \tag{3.11}$$
$$\times\left[\int_\alpha^\tau (q^\rho(t)p^{-\rho_n}(t))^{1/(\rho-\rho_n)}(P_k(t))^{\rho_\mu(\rho-1)/(\rho-\rho_n)}dt\right]^{(\rho-\rho_n)/\rho}$$

and

$$\theta_1=\theta_1(\rho_\mu, \rho_n, \rho)=\begin{cases}2^{1-((\rho_\mu+\rho_n)/\rho)}, & \rho_\mu+\rho_n\le\rho\\ 1, & \rho_\mu+\rho_n\ge\rho .\end{cases} \tag{3.12}$$

We summarize this case in the following:

THEOREM 3.1. *Assume that*

(i) $\rho_\mu\ge 0$, $\rho_n>0$, $\rho>1$, $\rho>\rho_n$ *are given numbers,*

(ii) $p(t)$, $q(t)$ *are non-negative and measurable functions on* $[\alpha, \tau]$,

(iii) *on* $[\alpha, \tau]$ *functions* $r_i(t)>0$, $i=1,\dots,n-1$ *and* $x_1(t)$, $x_2(t)$ *are sufficiently smooth so that the r-derivatives of* $x_1(t)$, $x_2(t)$ *exist,* $D_r^{(i)}x_1(t)$, $D_r^{(i)}x_2(t)$, $0\le i\le n-2$ *are continuous,* $D_r^{(n-1)}x_1(t)$, $D_r^{(n-1)}x_2(t)$ *are absolutely continuous, and the integrals* $\int_\alpha^\tau p(t)|D_r^{(n)}x_1(t)|^\rho dt$, $\int_\alpha^\tau p(t)|D_r^{(n)}x_2(t)|^\rho dt$ *exist,*

(iv) *for* $0\le\mu\le n-1$ $(n\ge 1)$, *but fixed,* $D_r^{(i)}x_1(\alpha)=D_r^{(i)}x_2(\alpha)=0$, $\mu\le i\le n-1$.
Then, the inequality (3.10) *holds.*

When $\rho_\mu=0$ an addition of (3.7) and (3.9) in view of (2.67) provides

$$\int_\alpha^\tau q(t)[|D_r^{(\nu)}x_2(t)|^{\rho_\nu}|D_r^{(n)}x_1(t)|^{\rho_n}+|D_r^{(\nu)}x_1(t)|^{\rho_\nu}|D_r^{(n)}x_2(t)|^{\rho_n}]dt \tag{3.13}$$
$$\le 2^{(\rho-\rho_n)/\rho}K_0(p, q, \{r_i\}, 0, \rho_\nu, \rho_n, \rho)I^{\rho_n/\rho},$$

where

$$I=\int_\alpha^\tau \{(y_2(t))^{\rho_\nu/\rho_n}y_1'(t)+(y_1(t))^{\rho_\nu/\rho_n}y_2'(t)\}dt .$$

Let

$$\theta_2=\theta_2(\rho_\nu, \rho_n)=\begin{cases}1, & \rho_\nu\ge\rho_n\\ 2^{1-(\rho_\nu/\rho_n)}, & \rho_\nu\le\rho_n .\end{cases}$$

Then, from (2.67) and (2.68) we have

$$\begin{aligned}I&=\int_\alpha^\tau [(y_1(t))^{\rho_\nu/\rho_n}+(y_2(t))^{\rho_\nu/\rho_n}](y_1'(t)+y_2'(t))dt\\
&\quad-\int_\alpha^\tau [(y_1(t))^{\rho_\nu/\rho_n}y_1'(t)+(y_2(t))^{\rho_\nu/\rho_n}y_2'(t)]dt\\
&\le\theta_2\int_\alpha^\tau (y_1(t)+y_2(t))^{\rho_\nu/\rho_n}(y_1(t)+y_2(t))'dt\\
&\quad-\frac{\rho_n}{\rho_\nu+\rho_n}[(y_1(\tau))^{(\rho_\nu+\rho_n)/\rho_n}+(y_2(\tau))^{(\rho_\nu+\rho_n)/\rho_n}]\end{aligned}$$

$$= \frac{\rho_n}{\rho_v + \rho_n} [\theta_2(y_1(\tau) + y_2(\tau))^{(\rho_v + \rho_n)/\rho_n} - (y_1(\tau))^{(\rho_v + \rho_n)/\rho_n} - (y_2(\tau))^{(\rho_v + \rho_n)/\rho_n}]$$

$$\leq \frac{(\theta_2 2^{\rho_v/\rho_n} - 1)\rho_n}{\rho_v + \rho_n} [(y_1(\tau))^{(\rho_v + \rho_n)/\rho_n} + (y_2(\tau))^{(\rho_v + \rho_n)/\rho_n}] \tag{3.14}$$

$$\leq \frac{\theta_3 \rho_n}{\rho_v + \rho_n} [y_1(\tau) + y_2(\tau)]^{(\rho_v + \rho_n)/\rho_n}, \tag{3.15}$$

where

$$\theta_3 = \theta_3(\rho_v, \rho_n) = \begin{cases} 2^{\rho_v/\rho_n} - 1, & \rho_v \geq \rho_n \\ 1, & \rho_v \leq \rho_n. \end{cases} \tag{3.16}$$

On combining (3.13) and (3.15), we obtain the inequality

$$\int_\alpha^\tau q(t)[|D_r^{(v)}x_2(t)|^{\rho_v}|D_r^{(n)}x_1(t)|^{\rho_n} + |D_r^{(v)}x_1(t)|^{\rho_v}|D_r^{(n)}x_2(t)|^{\rho_n}]dt$$
$$\leq K_2 \left[\int_\alpha^\tau p(t)[|D_r^{(n)}x_1(t)|^\rho + |D_r^{(n)}x_2(t)|^\rho]dt\right]^{(\rho_v + \rho_n)/\rho}, \tag{3.17}$$

where

$$K_2 = K_2(p, q, \{r_i\}, \rho_v, \rho_n, \rho) = 2^{(\rho - \rho_n)/\rho}\left(\frac{\theta_3 \rho_n}{\rho_v + \rho_n}\right)^{\rho_n/\rho}$$
$$\times \left[\int_\alpha^\tau (q^\rho(t)p^{-\rho_n}(t))^{1/(\rho - \rho_n)}(P_v(t))^{\rho_v(\rho - 1)/(\rho - \rho_n)}dt\right]^{(\rho - \rho_n)/\rho}. \tag{3.18}$$

Thus, we have proved the following:

THEOREM 3.2. *Assume that in Theorem* 3.1 *the integer μ and the number ρ_μ are replaced by v and ρ_v, respectively. Then, the inequality* (3.17) *holds.*

REMARK 3.1. For the case $p(t) = q(t) = \rho_v = \rho_n = 1$, $\rho = 2$ a weaker form of the inequality (3.17) has been proved by Pachpatte [20].

Theorems 3.1 and 3.2 can be combined. For this, we use the arithmetic-geometric means inequality and (2.67) in the right side of (3.7), to obtain

$$\int_\alpha^\tau q(t)|D_r^{(\mu)}x_1(t)|^{\rho_\mu}|D_r^{(v)}x_2(t)|^{\rho_v}|D_r^{(n)}x_1(t)|^{\rho_n}dt \tag{3.19}$$
$$\leq K_0 \left[\int_\alpha^\tau \left(\frac{\rho_\mu}{\rho_\mu + \rho_v}(y_1(t))^{(\rho_\mu + \rho_v)/\rho_n} + \frac{\rho_v}{\rho_\mu + \rho_v}(y_2(t))^{(\rho_\mu + \rho_v)/\rho_n}\right)y_1'(t)dt\right]^{\rho_n/\rho}$$

$$\leq K_0\left[\left(\frac{\rho_\mu\rho_n}{(\rho_\mu+\rho_\nu)(\rho_\mu+\rho_\nu+\rho_n)}\right)^{\rho_n/\rho}(y_1(\tau))^{(\rho_\mu+\rho_\nu+\rho_n)/\rho}\right.$$
$$\left.+\left(\frac{\rho_\nu}{\rho_\mu+\rho_\nu}\right)^{\rho_n/\rho}\left(\int_\alpha^\tau (y_2(t))^{(\rho_\mu+\rho_\nu)/\rho_n}y_1'(t)dt\right)^{\rho_n/\rho}\right],$$

and similarly, from (3.9) we find

$$\int_\alpha^\tau q(t)|D_r^{(\nu)}x_1(t)|^{\rho_\nu}|D_r^{(\mu)}x_2(t)|^{\rho_\mu}|D_r^{(n)}x_2(t)|^{\rho_n}dt \tag{3.20}$$
$$\leq K_0\left[\left(\frac{\rho_\mu\rho_n}{(\rho_\mu+\rho_\nu)(\rho_\mu+\rho_\nu+\rho_n)}\right)^{\rho_n/\rho}(y_2(\tau))^{(\rho_\mu+\rho_\nu+\rho_n)/\rho}\right.$$
$$\left.+\left(\frac{\rho_\nu}{\rho_\mu+\rho_\nu}\right)^{\rho_n/\rho}\left(\int_\alpha^\tau (y_1(t))^{(\rho_\mu+\rho_\nu)/\rho_n}y_2'(t)dt\right)^{\rho_n/\rho}\right].$$

An addition of (3.19) and (3.20) in view of (2.67) gives

$$S=\int_\alpha^\tau q(t)[|D_r^{(\mu)}x_1(t)|^{\rho_\mu}|D_r^{(\nu)}x_2(t)|^{\rho_\nu}|D_r^{(n)}x_1(t)|^{\rho_n} \tag{3.21}$$
$$+|D_r^{(\nu)}x_1(t)|^{\rho_\nu}|D_r^{(\mu)}x_2(t)|^{\rho_\mu}|D_r^{(n)}x_2(t)|^{\rho_n}]dt$$
$$\leq K_0\left[\left(\frac{\rho_\mu\rho_n}{(\rho_\mu+\rho_\nu)(\rho_\mu+\rho_\nu+\rho_n)}\right)^{\rho_n/\rho}((y_1(\tau))^{(\rho_\mu+\rho_\nu+\rho_n)/\rho}\right.$$
$$+(y_2(\tau))^{(\rho_\mu+\rho_\nu+\rho_n)/\rho})+2^{(\rho-\rho_n)/\rho}\left(\frac{\rho_\nu}{\rho_\mu+\rho_\nu}\right)^{\rho_n/\rho}$$
$$\left.\times\left(\int_\alpha^\tau\{(y_2(t))^{(\rho_\mu+\rho_\nu)/\rho_n}y_1'(t)+(y_1(t))^{(\rho_\mu+\rho_\nu)/\rho_n}y_2'(t)\}dt\right)^{\rho_n/\rho}\right].$$

Now following as in (3.10) and (3.17), we get

$$S\leq K_3\left[\int_\alpha^\tau p(t)[|D_r^{(n)}x_1(t)|^\rho+|D_r^{(n)}x_2(t)|^\rho]dt\right]^{(\rho_\mu+\rho_\nu+\rho_n)/\rho}, \tag{3.22}$$

where $K_3=K_0M_0(\rho)$,

$$M_0(\rho)=\left(\frac{\rho_n}{(\rho_\mu+\rho_\nu)(\rho_\mu+\rho_\nu+\rho_n)}\right)^{\rho_n/\rho}[\rho_\mu^{\rho_n/\rho}\theta_4+2^{(\rho-\rho_n)/\rho}(\rho_\nu\theta_5)^{\rho_n/\rho}] \tag{3.23}$$

and $\theta_4=\theta_1(\rho_\mu+\rho_\nu,\rho_n,\rho)$, $\theta_5=\theta_3(\rho_\mu+\rho_\nu,\rho_n)$.

Thus, a result which unifies Theorems 3.1 and 3.2 is the following:

THEOREM 3.3. *Assume that* $0\leq\mu\leq\nu\leq n-1$ $(n\geq 1)$ *and* $\rho_\mu,\rho_\nu\geq 0$, $\rho_n>0$, $\rho>1$,

$\rho > \rho_n$ are given numbers. Further, assume that the functions $p(t)$, $q(t)$, $x_1(t)$, $x_2(t)$, $r_i(t)$, $1 \le i \le n-1$ are as in Theorem 3.1. Then, the inequality (3.22) holds.

REMARK 3.2. *From the arithmetic-geometric means inequality we note that*

$$S \le \frac{\rho_\mu}{\rho_\mu+\rho_\nu}\int_\alpha^\tau q(t)[|D_r^{(\mu)}x_1(t)|^{\rho_\mu+\rho_\nu}|D_r^{(n)}x_1(t)|^{\rho_n}+|D_r^{(\mu)}x_2(t)|^{\rho_\mu+\rho_\nu}|D_r^{(n)}x_2(t)|^{\rho_n}]dt$$

$$+\frac{\rho_\nu}{\rho_\mu+\rho_\nu}\int_\alpha^\tau q(t)[|D_r^{(\nu)}x_2(t)|^{\rho_\mu+\rho_\nu}|D_r^{(n)}x_1(t)|^{\rho_n}+|D_r^{(\nu)}x_1(t)|^{\rho_\mu+\rho_\nu}|D_r^{(n)}x_2(t)|^{\rho_n}]dt\,.$$

Thus, from Theorems 3.1 and 3.2 it follows that in the inequality (3.22) the constant K_3 can be replaced by K_4, where

$$K_4=\frac{\rho_\mu}{\rho_\mu+\rho_\nu}K_1(p,q,\{r_i\},\rho_\mu+\rho_\nu,\rho_n,\rho)+\frac{\rho_\nu}{\rho_\mu+\rho_\nu}K_2(p,q,\{r_i\},\rho_\mu+\rho_\nu,\rho_n,\rho)\,.$$

From the above considerations (cf. (3.7) with $\rho_\mu=\gamma\ge 0$, $\rho_\nu=\gamma+\eta$, $\rho_n=\eta>0$, and a similar inequality), and (2.67), we have

$$\begin{aligned}
S_1 &= \int_\alpha^\tau q(t)|D_r^{(\mu)}x_1(t)D_r^{(\nu)}x_2(t)|^\gamma[|D_r^{(\nu)}x_2(t)D_r^{(n)}x_1(t)|^\eta \\
&\qquad +|D_r^{(\nu)}x_1(t)D_r^{(n)}x_2(t)|^\eta]dt \\
&\le 2^{(\rho-\eta)/\rho}K_0\left[\int_\alpha^\tau (y_1(t)y_2(t))^{\gamma/\eta}(y_2(t)y_1'(t)+y_1(t)y_2'(t))dt\right]^{\eta/\rho} \\
&= 2^{(\rho-\eta)/\rho}K_0\left[\frac{\eta}{\gamma+\eta}(y_1(\tau)y_2(\tau))^{(\gamma+\eta)/\eta}\right]^{\eta/\rho} \qquad (3.24)\\
&\le 2^{(\rho-\eta)/\rho}K_0\left(\frac{\eta}{\gamma+\eta}\right)^{\eta/\rho}\left(\frac{1}{2}\right)^{2(\gamma+\eta)/\rho}[y_1(\tau)+y_2(\tau)]^{2(\gamma+\eta)/\rho}\,.
\end{aligned}$$

Thus, the following inequality holds

$$S_1 \le K_5\left[\int_\alpha^\tau p(t)[|D_r^{(n)}x_1(t)|^\rho+|D_r^{(n)}x_2(t)|^\rho]dt\right]^{2(\gamma+\eta)/\rho}, \tag{3.25}$$

where

$$K_5=2^{(\rho-2\gamma-3\eta)/\rho}\left(\frac{\eta}{\gamma+\eta}\right)^{\eta/\rho}K_0(p,q,\{r_i\},\gamma,\gamma+\eta,\eta,\rho)\,. \tag{3.26}$$

We present the above case in the following:

THEOREM 3.4. *Assume that $0\le\mu\le\nu\le n-1$ ($n\ge 1$) and $\gamma\ge 0$, $\eta>0$, $\rho>1$, $\rho>\eta$ are given numbers. Further, assume that the functions $p(t)$, $q(t)$, $x_1(t)$, $x_2(t)$, $r_i(t)$, $1\le i\le n-1$*

are as in Theorem 3.1. *Then, the inequality* (3.25) *holds.*

All the above results require that $\rho_n>0$. To prove a result without such a requirement, from (3.1) we note that $g_{i,k+1}(t)=r_{k+1}(t)g'_{i,k}(t)$, $0\le k\le n-2$ and $g_{i,k}(\alpha)=0$, $0\le k\le n-1$. We define $g_{i,n}(t)=r_n(t)g'_{n-1}(t)$, where $r_n(t)=1$. Thus, if $\rho_k\ge 0$, $0<\rho_{k+1}<1$, in view of Hölder's inequality with indices $1/\rho_{k+1}$ and $1/(1-\rho_{k+1})$, it follows that

$$\begin{aligned}(3.27)\qquad &\int_\alpha^\tau q(t)|D_r^{(k)}x_1(t)|^{\rho_k}|D_r^{(k+1)}x_2(t)|^{\rho_{k+1}}dt\\ &\le\int_\alpha^\tau q(t)(r_{k+1}(t))^{\rho_{k+1}}(g_{1,k}(t))^{\rho_k}(g'_{2,k}(t))^{\rho_{k+1}}dt\\ &\le\left(\int_\alpha^\tau (q(t)r_{k+1}^{\rho_{k+1}}(t))^{1/(1-\rho_{k+1})}dt\right)^{1-\rho_{k+1}}\left(\int_\alpha^\tau (g_{1,k}(t))^{\rho_k/\rho_{k+1}}g'_{2,k}(t)dt\right)^{\rho_{k+1}}.\end{aligned}$$

Similarly, we find that

$$\begin{aligned}(3.28)\qquad &\int_\alpha^\tau q(t)|D_r^{(k)}x_2(t)|^{\rho_k}|D_r^{(k+1)}x_1(t)|^{\rho_{k+1}}dt\\ &\le\left(\int_\alpha^\tau (q(t)r_{k+1}^{\rho_{k+1}}(t))^{1/(1-\rho_{k+1})}dt\right)^{1-\rho_{k+1}}\left(\int_\alpha^\tau (g_{2,k}(t))^{\rho_k/\rho_{k+1}}g'_{1,k}(t)dt\right)^{\rho_{k+1}}.\end{aligned}$$

Now an addition of (3.27) and (3.28) in view of (2.67) gives

$$\begin{aligned}S_3&=\int_\alpha^\tau q(t)[|D_r^{(k)}x_1(t)|^{\rho_k}|D_r^{(k+1)}x_2(t)|^{\rho_{k+1}}+|D_r^{(k)}x_2(t)|^{\rho_k}|D_r^{(k+1)}x_1(t)|^{\rho_{k+1}}]dt\\ &\le\left(2\int_\alpha^\tau (q(t)r_{k+1}^{\rho_{k+1}}(t))^{1/(1-\rho_{k+1})}dt\right)^{1-\rho_{k+1}}\\ &\quad\times\left(\int_\alpha^\tau \{(g_{1,k}(t))^{\rho_k/\rho_{k+1}}g'_{2,k}(t)+(g_{2,k}(t))^{\rho_k/\rho_{k+1}}g'_{1,k}(t)\}dt\right)^{\rho_{k+1}}.\end{aligned}$$

Thus, from (3.14) with $\rho_v=\rho_k$, $\rho_n=\rho_{k+1}$ and (2.68) it follows that

$$\begin{aligned}&S_3\le K_6[(g_{1,k}(\tau))^{(\rho_k+\rho_{k+1})/\rho_{k+1}}+(g_{2,k}(\tau))^{(\rho_k+\rho_{k+1})/\rho_{k+1}}]^{\rho_{k+1}}\\ (3.29)\qquad &\quad\le K_6[(g_{1,k}(\tau))^{\rho_k+\rho_{k+1}}+(g_{2,k}(\tau))^{\rho_k+\rho_{k+1}}],\end{aligned}$$

where

$$(3.30)\qquad K_6=\left(2\int_\alpha^\tau (q(t)r_{k+1}^{\rho_{k+1}}(t))^{1/(1-\rho_{k+1})}dt\right)^{1-\rho_{k+1}}\left(\frac{\theta_6\rho_{k+1}}{\rho_k+\rho_{k+1}}\right)^{\rho_{k+1}}$$

and $\theta_6=\theta_3(\rho_k,\rho_{k+1})$.

Since from (3.1) and (3.2) for $\rho>1$ we have

$$g_{i,k}(\tau) \le [P_k(\tau)]^{(\rho-1)/\rho}[y_i(\tau)]^{1/\rho}, \qquad i=1,2 \tag{3.31}$$

from (3.29) and (3.31), we obtain

$$S_3 \le K_7[(y_1(\tau))^{(\rho_k+\rho_{k+1})/\rho} + (y_2(\tau))^{(\rho_k+\rho_{k+1})/\rho}], \tag{3.32}$$

where

$$K_7 = K_6(P_k(\tau))^{(\rho_k+\rho_{k+1})(\rho-1)/\rho}. \tag{3.33}$$

Finally, an application of (2.67) and (2.68) in (3.32) gives the inequality

$$S_3 \le K_8\left[\int_\alpha^\tau p(t)[|D_r^{(n)}x_1(t)|^\rho + |D_r^{(n)}x_2(t)|^\rho]dt\right]^{(\rho_k+\rho_{k+1})/\rho}, \tag{3.34}$$

where $K_8 = K_7\theta_1(\rho_k, \rho_{k+1}, \rho)$.

Thus, we have established the following:

THEOREM 3.5. *Assume that* $0 \le k \le n-1$ $(n \ge 1)$ *and* $\rho_k \ge 0$, $0 < \rho_{k+1} < 1$, $\rho > 1$ *are given numbers. Further, assume that* $p(t)$, $q(t)$, $x_1(t)$, $x_2(t)$, $r_i(t)$, $1 \le i \le n-1$ *are as in Theorem* 3.1. *Then, the inequality* (3.34) *holds.*

4. Some applications. The inequalities obtained in Sections 2 and 3 will be used here to study some qualitative properties of solutions of ordinary differential equations involving r-derivatives.

Uniqueness of initial value problems: Consider the differential equation

$$D_r^{(n)}u = f(t, D_r^{(0)}u, \ldots, D_r^{(n-1)}u) \tag{4.1}$$

together with the initial conditions

$$D_r^{(i)}u(\alpha) = \omega_i, \qquad 0 \le i \le n-1 \tag{4.2}$$

where the function f is continuous on $[\alpha, \tau] \times R^n$. As an application of the inequality (2.13) we shall show that the problem (4.1), (4.2) has at most one solution on $[\alpha, \tau]$. For this, we assume that the function f satisfies the Lipschitz condition on $[\alpha, \tau] \times R^n$, i.e., for all $(t, u_0, \ldots, u_{n-1})$, $(t, \bar{u}_0, \ldots, \bar{u}_{n-1}) \in [\alpha, \tau] \times R^n$,

$$|f(t, u_0, \ldots, u_{n-1}) - f(t, \bar{u}_0, \ldots, \bar{u}_{n-1})| \le \sum_{k=0}^{n-1} q_k(t)|u_k - \bar{u}_k|, \tag{4.3}$$

where the functions $q_k(t) \ge 0$, $0 \le k \le n-1$ are continuous on $[\alpha, \tau]$. If $u(t)$ and $\bar{u}(t)$ are two solutions of (4.1), (4.2) then the function $x(t) = u(t) - \bar{u}(t)$ is a solution of the problem

$$D_r^{(n)}x(t) = f(t, D_r^{(0)}u(t), \ldots, D_r^{(n-1)}u(t)) - f(t, D_r^{(0)}\bar{u}(t), \ldots, D_r^{(n-1)}\bar{u}(t)) \tag{4.4}$$

$$D_r^{(i)}x(\alpha) = 0, \qquad 0 \le i \le n-1. \tag{4.5}$$

Multiplying (4.4) by $D_r^{(n)}x(t)$, using (4.3), and then integrating the resulting inequality from α to t, we obtain

$$\int_\alpha^t |D_r^{(n)}x(s)|^2 ds \le \sum_{k=0}^{n-1} \int_\alpha^t q_k(s)|D_r^{(k)}x(s)||D_r^{(n)}x(s)|ds. \tag{4.6}$$

For each term on the right side of (4.6) we apply the inequality (2.13) with $p(t)=1$, $\rho_k=1$, $\rho_n=1$, $\rho=2$, to get

$$\int_\alpha^t |D_r^{(n)}x(s)|^2 ds \le \left(\sum_{k=0}^{n-1} C_k^*(t)\right) \int_\alpha^t |D_r^{(n)}x(s)|^2 ds, \tag{4.7}$$

where $C_k^*(t)=C_1(1, q_k, \{r_i\}, 1, 1, 2)|_{\tau=t}$. Since $C_k^*(\alpha)=0$, $0\le k\le n-1$ there exists a point $t_1>\alpha$ such that $\max_{\alpha\le t\le t_1}(\sum_{k=0}^{n-1} C_k^*(t))<1$. Thus, (4.7) implies that $D_r^{(n)}x(t)=0$, $t\in[\alpha, t_1]$ almost everywhere. However, this in view of (4.5) leads to $D_r^{(i)}x(t)=0$, $0\le i\le n-1$, $t\in[\alpha, t_1]$. If $t_1<\tau$ we can repeat the above arguments to obtain $x(t)=0$, $t\in[\alpha, \tau]$. Hence, it follows that $u(t)=\bar{u}(t)$, $t\in[\alpha, \tau]$.

The above uniqueness criterion is not available in the collection [2].

Next, we shall consider the following system of differential equations

$$D_r^{(n)}u_j=f_j(t, D_r^{(0)}u_1, \ldots, D_r^{(n-1)}u_1, D_r^{(0)}u_2, \ldots, D_r^{(n-1)}u_2), \qquad j=1, 2 \tag{4.8}$$

together with the initial conditions

$$D_r^{(i)}u_j(\alpha)=\omega_{j,i}, \qquad j=1, 2, \quad 0\le i\le n-1 \tag{4.9}$$

where the functions f_j are continuous on $[\alpha, \tau]\times R^n\times R^n$ and satisfy the Lipschitz condition

$$\begin{aligned} &|f_j(t, u_{1,0}, \ldots, u_{1,n-1}, u_{2,0}, \ldots, u_{2,n-1})-f_j(t, \bar{u}_{1,0}, \ldots, \bar{u}_{1,n-1}, \bar{u}_{2,0}, \ldots, \bar{u}_{2,n-1})| \\ &\qquad \le \sum_{k=0}^{n-1} [q_{1,j,k}(t)|u_{1,k}-\bar{u}_{1,k}|+q_{2,j,k}(t)|u_{2,k}-\bar{u}_{2,k}|]. \end{aligned} \tag{4.10}$$

If the problem (4.8), (4.9) has two solutions $(u_1(t), u_2(t))$, $(\bar{u}_1(t), \bar{u}_2(t))$ then for the functions $x_j(t)=u_j(t)-\bar{u}_j(t)$, $j=1, 2$ it follows that

$$|D_r^{(n)}x_j(t)|^2 \le \sum_{k=0}^{n-1} [q_{1,j,k}(t)|D_r^{(k)}x_1(t)||D_r^{(n)}x_j(t)|+q_{2,j,k}(t)|D_r^{(k)}x_2(t)||D_r^{(n)}x_j(t)|].$$

Summing these two inequalities, and integrating from α to t, we obtain

$$\begin{aligned} &\int_\alpha^t [|D_r^{(n)}x_1(s)|^2+|D_r^{(n)}x_2(s)|^2]ds \\ &\quad \le \sum_{k=0}^{n-1} \int_\alpha^t \bar{q}_k(s)[|D_r^{(k)}x_1(s)||D_r^{(n)}x_1(s)|+|D_r^{(k)}x_2(s)||D_r^{(n)}x_2(s)|]ds \\ &\qquad + \sum_{k=0}^{n-1} \int_\alpha^t q_k^*(s)[|D_r^{(k)}x_2(s)||D_r^{(n)}x_1(s)|+|D_r^{(k)}x_1(s)||D_r^{(n)}x_2(s)|]ds, \end{aligned} \tag{4.11}$$

where $\bar{q}_k(t)=\max_{\alpha\le s\le t}(q_{1,1,k}(s), q_{2,2,k}(s))$ and $q_k^*(t)=\max_{\alpha\le s\le t}(q_{2,1,k}(s), q_{1,2,k}(s))$.

For each $0\le k\le n-1$ in the first summation on the right side of (4.11) we can use (3.10), whereas in the second summation (3.17), to obtain the inequality

$$\int_\alpha^t [|D_r^{(n)}x_1(s)|^2+|D_r^{(n)}x_2(s)|^2]ds\le K(t)\int_\alpha^t [|D_r^{(n)}x_1(s)|^2+|D_r^{(n)}x_2(s)|^2]ds\,,$$

where $K(t)$ is a function with the property that $K(\alpha)=0$. The above inequality implies that $u_1(t)=\bar{u}_1(t)$, $u_2(t)=\bar{u}_2(t)$, $t\in[\alpha, \tau]$. Hence, the solutions of the problem (4.8), (4.9) are unique.

Upper bounds on the solutions: Consider the differential equation

$$(D_r^{(n)}x)'=f(t, D_r^{(0)}x, \ldots, D_r^{(n)}x) \tag{4.12}$$

together with the initial conditions

$$D_r^{(i)}x(\alpha)=0,\ 0\le i\le n-1\,, \qquad D_r^{(n)}(\alpha)=\omega \tag{4.13}$$

where the function f is continuous on $[\alpha, \tau]\times R^{n+1}$, and satisfies the growth condition

$$|f(t, x_0, \ldots, x_n)|\le\sum_{k=0}^{n-1} q_k(t)|x_k|\,, \tag{4.14}$$

where again the functions $q_k(t)\ge 0$, $0\le k\le n-1$ are continuous on $[\alpha, \tau]$. Let $x(t)$ be a solution of (4.12), (4.13). Multiplying (4.12) by $D_r^{(n)}x(t)$, integrating the resulting equation from α to t, and then using (4.14), we obtain

$$|D_r^{(n)}x(t)|^2\le|\omega|^2+2\sum_{k=0}^{n-1}\int_\alpha^t q_k(s)|D_r^{(k)}x(s)||D_r^{(n)}x(s)|ds\,. \tag{4.15}$$

As earlier, for each $0\le k\le n-1$ in the summation on the right side of (4.15) we apply the inequality (2.13) with $p(t)=1$, $\rho_k=1$, $\rho_n=1$, $\rho=2$, to get

$$|D_r^{(n)}x(t)|^2\le|\omega|^2+C^{**}(t)\int_\alpha^t |D_r^{(n)}x(s)|^2ds\,, \tag{4.16}$$

where $C^{**}(t)=2\sum_{k=0}^{n-1}C_k^*(t)$. Thus, in view of Gronwall's inequality it follows that

$$|D_r^{(n)}x(t)|^2\le|\omega|\left[1+C^{**}(t)\int_\alpha^t \exp\left(\int_\alpha^t C^{**}(s_1)ds_1\right)ds\right]^{1/2}=\phi(t)\,. \tag{4.17}$$

This estimate we can use in the right side of (2.20), to find

$$|D_r^{(k)}x(t)|\le\int_\alpha^t H_{n-k}(t, s)\phi(s)ds \qquad 0\le k\le n-1\,. \tag{4.18}$$

Hence, as an application of the inequality (2.13) we obtain upper bounds on all the r-derivatives of the solutions of (4.12), (4.13).

If instead of (4.14) the function f satisfies the growth condition

$$|f(t, x_0, \ldots, x_n)| \le q(t)|x_k|^{\rho_k}|x_{k+1}|^{\rho_{k+1}}, \qquad 0 \le k \le n-1 \text{ (but fixed)}$$

where the function $q(t) \ge 0$ is continuous on $[\alpha, \tau]$, then for any solution of (4.12), (4.13) we have

$$|D_r^{(n)}x(t)| \le |\omega| + \int_\alpha^t q(s)|D_r^{(k)}x(s)|^{\rho_k}|D_r^{(k+1)}x(s)|^{\rho_{k+1}}ds\,. \tag{4.19}$$

In the right side of (4.19) we apply the inequality (2.51) with $p(t)=1$, $\rho=\rho_k+\rho_{k+1}$, to get

$$|D_r^{(n)}x(t)| \le |\omega| + \bar{C}(t)\int_\alpha^t |D_r^{(n)}x(s)|^{\rho_k+\rho_{k+1}}ds\,, \tag{4.20}$$

where $\bar{C}(t) = C_{13}(1, q, \{r_i\}, \rho_k, \rho_{k+1}, \rho_k+\rho_{k+1})\big|_{\tau=t}$.

From (4.20) an upper bound on $|D_r^{(n)}x(t)|$ can be obtained rather easily. Once again this bound we can use in the right side of (2.20). Thus, this time as an application of the inequality (2.51) we obtain upper bounds on all the r-derivatives of the solutions of (4.12), (4.13).

Finally, we remark that the bounds obtained above can be used to study the asymptotic behavior of the solutions of (4.12), (4.13).

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