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THE SIMPLE INVARIANT AND DIFFERENTIABLE STRUCTURES ON THE HORIKAWA SURFACE

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Abstract. We calculate Donaldson's simple invariant for a surface of general type known as the Horikawa surface and for regular elliptic surfaces, by using Kronheimer's method. As a corollary, it is shown that there exist infinitely many differentiable structures on these surfaces and a torus sum of them.

1. Introduction. Donaldson [4] has developed a gauge theory on closed 4-manifolds and obtained new invariants from the moduli space of anti-self-dual (ASD, for short) connections. At the same time, he also obtained a vanishing theorem for the connected sum, which follows by counting the dimension of the ASD moduli space as one shrinks the neck. Such a differential-geometric method could be also used to calculate non-zero values of the invariant. In particular, as for the simple invariant, which is defined by counting signed points in the zero-dimensional ASD moduli space, Kronheimer [14] found out a way to calculate this invariant by collapsing (-2)-curves, and applied it to $K3$ surfaces and to some homotopy $K3$ surfaces.

In this paper we calculate the simple invariant for a surface of general type using Kronheimer's method. Precisely we consider a minimal surface with $c_1^2=p_g=4$, which is known to admit a unique differentiable structure by Horikawa [11]. We know a simple model as follows: Take a non-singular curve Π_2 of genus 2 and choose a $\boldsymbol{Z}_2$-action with six fixed points. Then the quotient $(\Pi_2\times\Pi_2)/\boldsymbol{Z}_2$ with respect to the diagonal action has thirty-six double points. By resolving all double points, we obtain a simply-connected minimal surface of general type. From this construction similar to that for Kummer surfaces, we apply a method by Kronheimer [14] to deduce:

THEOREM. *The simple invariant of the Horikawa surface with* $c_1^2=p_g=4$ *is* 4 *for a certain* $SO(3)$-*bundle.*

Moreover, we can apply Gompf-Mrowka's technique [10] to construct infinitely many homotopy equivalent 4-manifolds which are distinguished by the simple invariant. Thus we have:

COROLLARY 1. *There are infinitely many differentiable structures on this Horikawa surface.*

In [12] we have obtained a torus sum formula for the simple invariant and calcu-

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lated it for regular elliptic surfaces. Using this formula, we can also calculate it for a torus sum of regular elliptic surfaces and the Horikawa surface (see Section 5 for the definition). Thus we also obtain:

COROLLARY 2. *For integers k, l with $k \geq 0$, $l > 0$, the connected sum* $(2k+10l-1)\boldsymbol{CP}^2 \# (10k+46l-1)\overline{\boldsymbol{CP}^2}$ *has infinitely many differentiable structures.*

On homotopy regular elliptic surfaces, it was shown that there exist infinitely many differentiable structures (cf. [7], [8], [10], [13], [18]). All of the above results are proved in Sections 4 and 5. In Section 4 we also calculate the simple invariant for regular elliptic surfaces, which was carried out by another method in [12]. In Sections 2 and 3, we give a characterization of orbifold $SO(3)$-bundles, which gives a formula to calculate the simple invariant.

After having written this manuscript, the author found out in Donaldson's survey [5] that Lisca [15], and Fintushel-Stern have found infinitely many differentiable structures on surfaces of general type.

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2. A characterization of orbifold $SO(3)$-bundles. We first recall the simple invariant γ defined from the moduli spaces of ASD connections on certain principal bundles by Donaldson [4], [6, Chapter 9]; let X be an oriented, closed, simply-connected, smooth 4-manifold with the condition that $b_+(X) \geq 3$ odd, where $b_+(X)$ denotes the dimension of a maximal positive subspace for the cup product on $H^2(X)$. Let $P \to X$ be a principal $SO(3)$-bundle. Then for a generic metric on X, the moduli space $\mathcal{M}_X(P)$ of ASD connections on P is a finite set of irreducible regular connections if $\dim \mathcal{M}_X(P) = -2p_1(P) - 3(1+b_+(X)) = 0$ and $w_2(P) \neq 0$. Furthermore, given an integral lift of $w_2(P)$ and the orientation of the vector space $H^+(X)$, we can attach a sign ± 1 to each point of $\mathcal{M}_X(P)$. Then the number of points in $\mathcal{M}_X(P)$ counted with sign is independent of the choice of metric and called the simple invariant. Since $SO(3)$-boundles over X are classified by the characteristic classes (cf. [2]), the set of $SO(3)$-bundles defining the simple invariant corresponds bijectively to the set $C_X = \{\eta \in H^2(X; \boldsymbol{Z}_2) \mid \eta \neq 0, \eta^2 \equiv -2(1+b_+(X))/3 \pmod 4\}$. Thus we denote the simple invariant by $\gamma_X(\eta)$. The absolute value $|\gamma_X(\eta)|$ is determined only by $\eta \in C_X$.

Kronheimer [14] has given a natural extension of the simple invariant to 4-orbifolds with singularities modeled on $\boldsymbol{R}^4/\boldsymbol{Z}_2$ and found out that this invariant is equal to that for the 4-orbifold obtained by resolving double points, via an identification of the moduli spaces. As a preliminary step to apply his idea to our case, we first give a characterization of oribifold $SO(3)$-bundles over a 4-orbifold with double points. Such a 4-orbifold $\check{X}$ is obtained by gluing a compact 4-manifold X with boundaries $\partial_i X = S^3/\boldsymbol{Z}_2$ $(1 \leq i \leq n)$ to 4-disks D_i^4 $(1 \leq i \leq n)$ with a linear $\boldsymbol{Z}_2$-action. Over the orbifold $D^4/\boldsymbol{Z}_2$, there are exactly two orbifold $SO(3)$-bundles, which are constructed from the trivial and a faithful representation of $\boldsymbol{Z}_2$ to $SO(3)$, and which are classified by the Stiefel-Whitney class of the $SO(3)$-bundle on the boundary $S^3/\boldsymbol{Z}_2$. In the case of non-trivial representations, the primary difference $\mathfrak{d}=\mathfrak{d}(1,-1)$ (cf. [17, §36]) between the identity and the other non-trivial flat automorphism -1 defines the generator of the cohomology $H^1(S^3/\boldsymbol{Z}_2; \boldsymbol{Z}_2)\cong \boldsymbol{Z}_2$. To save notation, we denote either bundle by $(D^4\times SO(3))/\boldsymbol{Z}_2\to D^4/\boldsymbol{Z}_2$ and its restriction to $S^3/\boldsymbol{Z}_2$ by $R\to S^3/\boldsymbol{Z}_2$.

Next we consider an orbifold $SO(3)$-bundle $\check{P}$ over $\check{X}=X\cup(\bigcup D_i^4/\boldsymbol{Z}_2)$. Let $\sigma=\coprod_{1\le i\le n}\sigma_i\colon \partial X=\coprod_{1\le i\le n}\partial_i X\to X$ be the inclusion. Then we can write $\check{P}=P\cup(\bigcup_{\iota_i}(D_i^4\times SO(3))/\boldsymbol{Z}_2)$, where P is an $SO(3)$-bundle over X and $\iota_i\colon \sigma_i^*(P)\to R (1\le i\le n)$ are bundle isomorphisms.

Now we need one example: The linear involution of $\overline{\boldsymbol{CP}^2}$ with one fixed point and a hyperplane (say S^2) lifts to the canonical $SO(2)$-bundle so as to act as -1 on the fixed point and 1 on S^2. Then the quotient defines an orbifold $SO(2)$-bundle, hence an orbifold $SO(3)$-bundle $\check{Q}$ over an orbifold $\check{W}$ by an embedding $SO(2)\subset SO(3)$. We write $\check{W}=W\cup(D^4/\boldsymbol{Z}_2)$ and $\check{Q}=Q\cup(D^4\times SO(3))/\boldsymbol{Z}_2$. We extend the trivial connection over D^4 to an $SO(2)$-reducible connection Θ_0 on $\check{Q}$. Then the Pontrjagin charge of the Θ_0 is $-1/2$. In the following we also use the trivial $SO(3)$-bundles over W, $\check{W}$ and the trivial connection over $\check{W}$. Again to save notation, we also denote them by Q, $\check{Q}$ and Θ_0, respectively.

DEFINITION 2.1. An element ξ in $H^2(X; \boldsymbol{Z}_2)$ is *admissible* if, for any even number of generators $\mathfrak{d}_i\in H^1(\partial X_i; \boldsymbol{Z}_2)$ for i with $\sigma_i^*(\xi)=0$, there is a set of generators $\mathfrak{d}_i\in H^1(\partial X_i; \boldsymbol{Z}_2)$ for i with $\sigma_i^*(\xi)\neq 0$ such that the sum of all the generators in these two sets is an element of the image of the map $\sigma^*\colon H^1(X; \boldsymbol{Z}_2)\to H^1(\partial X; \boldsymbol{Z}_2)$.

In the example above $w_2(Q)\in H^2(W; \boldsymbol{Z}_2)$ is obviously admissible regardless of whether Q is trivial or not.

LEMMA 2.2. *Let ι_i, $\iota_i'\colon \sigma_i^*(P)\to R_i$ $(1\le i\le n)$ be bundle isomorphisms between $\sigma_i^*(P)$ and a copy R_i of the $SO(3)$-bundle R. We consider $SO(3)$-bundles $\check{P}=P\cup(\bigcup_{\iota_i}(D_i^4\times SO(3))/\boldsymbol{Z}_2)$, $\check{P}'=P\cup(\bigcup_{\iota_i'}(D_i^4\times SO(3))/\boldsymbol{Z}_2)$, and smooth connections A_0, A_0' on $\check{P}$, $\check{P}'$ which are equal to the trivial connection over D_i^4 $(1\le i\le n)$. If $\xi=w_2(P)$ is admissible, then the following two conditions are equivalent*:

(1) *There exists $\varepsilon_i\in\{\pm 1\}$ for each i with $\sigma_i^*(\xi)\neq 0$ such that the automorphism $(\coprod_{\sigma_i^*(\xi)=0}(\iota_i')^{-1}\iota_i)\coprod(\coprod_{\sigma_i^*(\xi)\neq 0}(\iota_i')^{-1}\varepsilon_i\iota_i)$ over ∂X can be extended all over X and that $\sum_{\sigma_i^*(\xi)\neq 0}\mathfrak{d}(1,(\iota_i')^{-1}\varepsilon_i\iota_i)=0$.*

(2) $(-1/4\pi^2)\int_{\check{X}}\mathrm{Tr}(F_{A_0}\wedge F_{A_0})=(-1/4\pi^2)\int_{\check{X}}\mathrm{Tr}(F_{A_0'}\wedge F_{A_0'})$.

Regardless of whether ξ is admissible or not, if $\iota_i=\iota_i'$ except for at most one i (say i_0), then the condition (1) *with $\varepsilon_i=1$ except for $i=i_0$ is equivalent to the condition* (2).

PROOF. We consider $SO(3)$-bundles $P^*=P\cup(\bigcup_{\iota_i}Q_i)$ and $(P')^*=P\cup(\bigcup_{\iota_i'}Q_i)$ over $X\cup(\bigcup_i W_i)$, where Q_i and W_i are copies of Q and W. We extend the connections A_0 and A_0' to P^* and $(P')^*$ using the Θ_0 on Q_i, respectively. The condition (2) follows from

(1) by the Chern-Weil formula. Conversely, if (2) holds, then clearly $p_1(P^*)=p_1((P')^*)$, so $w_2(P^*)^2 \equiv w_2((P')^*)^2 \pmod 4$. Now we show that there are constant gauges $\varepsilon_i \in \{\pm 1\}$ over $S^3/\boldsymbol{Z}_2$ such that $w_2(P^*)=w_2((P')^*)$ for modified automorphisms $\varepsilon_i \iota_i$. By the exact sequence

$$H^1(X;\boldsymbol{Z}_2) \xrightarrow{\sigma^*} H^1(\partial X;\boldsymbol{Z}_2) \xrightarrow{\delta^*} H^2(X\cup(\textstyle\bigcup_i W_i);\boldsymbol{Z}_2) \longrightarrow H^2(X;\boldsymbol{Z}_2)\oplus H^2(\textstyle\bigcup_i W_i;\boldsymbol{Z}_2)\,,$$

we can write $w_2(P^*)=w_2((P')^*)+\delta^*(\sum_{1\le i\le n}\mathfrak{d}(1,(\iota_i')^{-1}\iota_i))$. Since $\delta^*(\mathfrak{d}_i)$ is the mod 2 Poincaré dual of the 2-sphere $[S_i^2]$, we obtain $w_2(P^*)^2\equiv w_2((P')^*)^2-2N \pmod 4$, where N is the cardinality of i with $\sigma_i^*(\xi)=0$ and $\mathfrak{d}(1,(\iota_i')^{-1}\iota_i)=\mathfrak{d}_i$. So N must be even.

Suppose that ξ is admissible. Then there exist $\varepsilon_i\in\{\pm 1\}$ for all i with $\sigma_i^*(\xi)\neq 0$ such that, if we replace ι_i by $\varepsilon_i\iota_i$ in the definition of P^*, we have $w_2(P^*)-w_2((P')^*)=\delta^*(\sum_{\sigma_i(\xi)\neq 0}\mathfrak{d}(1,(\iota_i')^{-1}\varepsilon_i\iota_i))$. Furthermore, for all i with $\sigma_i(\xi)\neq 0$, we replace ε_i so that $\mathfrak{d}(1,(\iota_i')^{-1}\varepsilon_i\iota_i)=0$. Then we have $w_2(P^*)=w_2((P')^*)$.

Suppose that $\iota_i=\iota_i'$ except for $i=i_0$ and that ξ is not necessarily admissible. If $\sigma_{i_0}^*(\xi)=0$, then $N=0$ and so $\mathfrak{d}(1,(\iota_{i_0}')^{-1}\iota_{i_0}))=0$, that is, $w_2(P^*)=w_2((P')^*)$. If $\sigma_{i_0}^*(\xi)\neq 0$, then $\mathfrak{d}(1,(\iota_{i_0}')^{-1}\varepsilon_{i_0}\iota_{i_0})=0$ for an $\varepsilon_{i_0}\in\{\pm 1\}$. So replacing ι_{i_0} by $\varepsilon_{i_0}\iota_{i_0}$ we also have $w_2(P^*)=w_2((P')^*)$.

Now by the Dold-Whitney theorem [2], there are bundle isomorphisms $f\colon P\to P$ and $h_i\colon Q_i\to Q_i$ with $\iota_i' f=h_i\varepsilon_i\iota_i$ $(1\le i\le n)$. Since $\pi_2(SO(3))=1$, $h_i|_{S_i^2}$ is homotopic to the identity and so is h_i. ∎

We put $w_2(\check{P})=w_2(\check{P}|_X)$ and $p_1(\check{P})=(-1/4\pi^2)\int_{\check{X}}\mathrm{Tr}(F_{A_0}\wedge F_{A_0})$. If $w_2(\check{P})$ is admissible, then these classes $w_2(\check{P})$ and $p_1(\check{P})$ uniquely determine an $SO(3)$-bundle $\check{X}$ by Lemma 2.2. In the definition of $p_1(\check{P})$, we may choose any smooth connection A_0 on $\check{P}$ without changing the integral, so Lemma 2.2 holds for any smooth connections A_0 on $\check{P}$ and A_0' on $\check{P}'$.

We let $\mathscr{A}_{\check{P}}$ be the space of connections on $\check{P}$ and $\mathscr{G}_{\check{P}}$ the gauge group on $\check{P}$. Let x_i be the center of D_i^4. For any $u\in\mathscr{G}_{\check{P}}$ and i with $\sigma_i^*(\xi)\neq 0$, define $u(x_i)\in O(2)$ by using the lift to $D_i^4\times SO(3)$. Let $\mathscr{G}_{\check{P}}'$ denote the closed subgroup

$$\{u\in\mathscr{G}_{\check{P}}\mid u(x_i)\in SO(2)\ \text{if}\ \sigma_i^*(\xi)\neq 0\}$$

and $\mathscr{B}_{\check{P}}'=\mathscr{A}_{\check{P}}/\mathscr{G}_{\check{P}}'$ the quotient. We let $\mathscr{A}_{\check{P}}^{**}\subset\mathscr{A}_{\check{P}}$ denote the subset of connections whose isotropy group does not have any non-trivial element in $\mathscr{G}_{\check{P}}'$. Let $\mathscr{B}_{\check{P}}^{**}=\mathscr{A}_{\check{P}}^{**}/\mathscr{G}_{\check{P}}'$.

LEMMA 2.3. *$\mathscr{G}_{\check{P}}'$ is connected.*

PROOF. It suffices to prove that any element u in $C^0(\mathrm{Aut}\,P)$ with $u|_{\partial X}=1$ is homotopic to the identity. Since $H_3(X,\coprod_{2\le i\le n}\partial_i X;\boldsymbol{Z})\cong\mathrm{Hom}(H_1(X,\partial_1 X;\boldsymbol{Z}),\boldsymbol{Z})$ vanishes, X is retractable to $\bar{X}=(\coprod_{2\le i\le n}\partial_i X)\cup X^{(2)}$, where $X^{(2)}$ is a union of some 2-simplexes in X. So it is enough to construct a homotopy on $\bar{X}$. We remove 3-disks D_i^3 from $\partial_i X=S^3/\boldsymbol{Z}_2$ $(2\le i\le n)$. Then the complement $\bar{X}\setminus(\coprod_{2\le i\le n}D_i^3)$ is retractable to a union of some 2-simplexes, say $\bar{X}^{(2)}$. On the other hand, $\sigma^*: H^1(X; \boldsymbol{Z}_2) \to H^1(\partial X; \boldsymbol{Z}_2)$ is injective, so we have $\mathfrak{d}(1, u)=0$. Therefore, if we denote by $\bar{X}^{(1)}$ the union of all 1-simplexes of $\bar{X}^{(2)}$, $u|_{\bar{X}^{(1)}}$ is homotopic to the identity. We extend this homotopy all over $\bar{X}^{(2)}$, using the collar of 1-simplexes in $\bar{X}^{(2)}$. Since $\pi_2(SO(3))=\{1\}$, we can modify the extended homotopy into a homotopy H_t $(0 \le t \le 1)$ from $u|_{\bar{X}^{(2)}}$ to the identity. We then extend H_t $(0 \le t \le 1)$ all over $\bar{X}$ using the collar of ∂D_i^3 $(2 \le i \le n)$. Then

$$\int_{D_i^3} \operatorname{Tr}(d_{\Gamma_i} H_1 H_1^{-1} \wedge d_{\Gamma_i} H_1 H_1^{-1} \wedge d_{\Gamma_i} H_1 H_1^{-1})$$

$$= \int_{\partial_i X} \operatorname{Tr}(d_{\Gamma_i} H_1 H_1^{-1} \wedge d_{\Gamma_i} H_1 H_1^{-1} \wedge d_{\Gamma_i} H_1 H_1^{-1}) = 0\,,$$

since the last integral is a homotopy invariant and $u|_{\partial_i X}=1$. This implies that $H_1|_{D_i^3}$ is zero in $[(D_i^3, \partial D_i^3), (SO(3), 1)] = \pi_3(SO(3)) \cong \boldsymbol{Z}$. Hence we can modify H_t to a homotopy from $u|_{\bar{X}}$ to the identity. This is a desired homotopy. ∎

One can easily see that the above proof can be applied to any smooth $SO(3)$-bundle over smooth 4-manifolds X with $H_1(X; \boldsymbol{Z}_2)=0$, by removing a ball in X. So we have:

COROLLARY 2.4. *The gauge group $\mathscr{G}_P$ of any $SO(3)$-bundle P over smooth 4-manifolds X with $H_1(X; \boldsymbol{Z}_2)=0$ is connected.*

LEMMA 2.5. *The assignment of the primary difference $\mathfrak{d}(1, u|_X)$ in $H^1(X; \boldsymbol{Z}_2)$ to each element u in $\mathscr{G}_{\check{P}}$ sets up a one-to-one correspondence between $\mathscr{G}_{\check{P}}/\mathscr{G}'_{\check{P}}$ and $\{\alpha \in H^1(X; \boldsymbol{Z}_2) \mid \sigma_i^*(\alpha)=0 \text{ if } \sigma_i^*(\xi)=0\}$.*

PROOF. We can prove the injectivity in the same way as in Lemma 2.3. We will prove the surjectivity. For any element α in $H^1(X; \boldsymbol{Z}_2)$ satisfying the condition, we define a section u on $\operatorname{Aut} \check{P}|_{\partial_i X}$ to be 1 for i with $\sigma_i^*(\alpha)=0$ and to be -1 for i with $\sigma_i^*(\alpha) \neq 0$. Clearly $\mathfrak{d}(1, u|_{\partial_i X}) = \sigma_i^*(\xi)$. So by the homotopy extension theorem (cf. [17, §37]) we get an extension of u to $\bar{X}$, where $\bar{X}$ is as in Lemma 2.3. Pulling the extension back to X, we obtain a section of $\operatorname{Aut} \check{P}$. ∎

Let $\Lambda'_{\check{P}} \to \mathscr{B}_{\check{P}}^{**}$ be the orientation bundle defined as in the case of closed manifolds (cf. [3], [6, Chapter 5]). Then by Lemma 2.3 and the homotopy exact sequence

$$1 = \pi_1(\mathscr{A}_{\check{P}}^{**}) \to \pi_1(\mathscr{B}_{\check{P}}^{**}) \to \pi_0(\mathscr{G}'_{\check{P}}) \to \pi_0(\mathscr{A}_{\check{P}}^{**}) = 1$$

we have $\pi_1(\mathscr{B}_{\check{P}}^{**})=1$. In particular, we obtain:

PROPOSITION 2.6. *The line bundle $\Lambda'_{\check{P}} \to \mathscr{B}_{\check{P}}^{**}$ is topologically trivial.*

3. Kronheimer's main result. Let $\check{X}$ be a 4-orbifold whose interior compact 4-manifold X satisfies the conditions (1) and (2) in Section 2. Then Kronheimer [14] has naturally extended the simple invariant to orbifold $SO(3)$-bundles $\check{P}$ over $\check{X}$. In our notation it can be defined as follows: For a generic metric on $\check{X}$, the moduli space $\mathcal{M}'_{\check{X}}(\check{P})\subset\mathcal{B}'_{\check{P}}$ of ASD connections is a finite set consisting of irreducible regular connections (so in $\mathcal{B}^{**}_{\check{P}}$) if $\check{P}$ satisfies the following assumption:

ASSUMPTION 3.1. (1) $w_2(\check{P})\neq 0$ and $\dim\mathcal{M}'_{\check{X}}(\check{P})=-2p_1(\check{P})-3(1+b_+(\check{X}))+\tau(\check{P})=0$, where $\tau(\check{P})$ is the cardinality of the twisted double points of $\check{P}$,

(2) if $p_1(\check{P})<0$, no $SO(3)$-bundle $\check{P}'$ with $w_2(\check{P})=w_2(\check{P}')$ admits a flat connection,

(3) if $p_1(\check{P})=0$, then for any flat connection A on $\check{P}$, the cohomology group of the Atiyah-Hitchen-Singer (AHS, for short) complex [1] over $\check{X}$ vanishes and A is in $\mathcal{A}^{**}_{\check{P}}$.

Moreover, if one gives an orientation of $H^+(X)$ and an integral lift of $w_2(\check{P})$, we can attach a sign to each point of $\mathcal{M}'_{\check{X}}(\check{P})$ (see also Proposition 2.6). Then the number of points in $\mathcal{M}'_{\check{X}}(\check{P})$ counted with signs is independent of the choice of metric. This number is the simple invariant $\gamma_{\check{X}}(\check{P})$ for $\check{X}$. (If $\check{X}$ is nonsingular, then clearly $\gamma_{\check{X}}(\check{P})=\gamma_{\check{X}}(w_2(\check{P}))$.)

Here we give a formula to calculate the invariant $\gamma_{\check{X}}(\check{P})$ when $p_1(\check{P})=0$. The flat moduli $\mathcal{X}'_{\check{X}}(\check{P})\subset\mathcal{B}'_{\check{P}}$ is not generally identified with the representation space $\mathcal{R}_{\check{X}}(\check{P})=\{\rho:\pi_1(X)\to SO(3)\mid w_2(\xi_\rho)=w_2(\check{P})\}/\mathrm{Ad}$, where ξ_ρ is the $\boldsymbol{R}^3$-bundle associated to the representation ρ. However, if $w_2(\check{P})$ is admissible, Lemma 2.2 shows that the honest flat moduli $\mathcal{X}_{\check{X}}(\check{P})=\{[A]\in\mathcal{A}_{\check{P}}/\mathcal{G}_{\check{P}}\mid F_A=0\}$ can be identified with $\mathcal{R}_{\check{X}}(\check{P})$ via the holonomy representation, and Lemma 2.5 gives the difference between $\mathcal{X}'_{\check{X}}(\check{P})$ and $\mathcal{X}_{\check{X}}(\check{P})$. To compute the simple invariant defined by $\mathcal{X}'_{\check{X}}(\check{P})$, we need to determine the sign of each point in $\mathcal{X}'_{\check{X}}(\check{P})$. When $\check{X}$ has a Kähler structure, however, it follows from the argument of [3], and [6, 6.4] that the bundle $\Lambda'_{\check{P}}$ has a canonical orientation coming from the complex structure and that the sign of each point is $+1$ in this orientation. Hence we obtain:

PROPOSITION 3.2. *For a Kähler orbifold $\check{X}$, if $\xi=w_2(\check{P})\in H^2(X;\boldsymbol{Z}_2)$ is admissible and satisfies Assumptions* 3.1, *then*

$$\gamma_{\check{X}}(\check{P})=\sum_{[\rho]\in\mathcal{R}_{\check{X}}(\check{P})}\frac{\#\{\alpha\in H^1(X;\boldsymbol{Z}_2)\mid\sigma_i^*(\alpha)=0\ \textit{if}\ \sigma_i^*(\xi)=0\}}{\#\{\textit{the isotropy of}\ \rho\}}\,.$$

Now we state Kronheimer's main theorem [14]: Let X be a 4-orbifold obtained from a compact 4-manifold satisfying (1), (2) in Section 2, by gluing the $D^4/\boldsymbol{Z}_2$ to all except one of the connected components of the boundary. Let $\check{X}=X\cup(D^4/\boldsymbol{Z}_2)$ and $\check{X}_1=X\cup W$ be the 4-orbifolds obtained from X by gluing $D^4/\boldsymbol{Z}_2$ or W to the boundary. We let P be an $SO(3)$-bundle over X. For a bundle isomorphism $\iota\colon P|_{S^3/\boldsymbol{Z}_2}\to R$, let $\check{P}=P\cup_\iota(D^4\times SO(3))/\boldsymbol{Z}_2$ and $\check{P}_1=P\cup_\iota Q$.

THEOREM 3.3 (Kronheimer [14]). *If $\check{P}$ satisfies Assumption* 3.1, *then*

$$|\gamma_{\check{X}_1}(\check{P}_1)|=|\gamma_{\check{X}}(\check{P})|\,.$$

REMARKS. 1. One easily sees that if $\check{P}$ satisfies Assumption 3.1, then the

$SO(3)$-bundle $\check{P}_1$ over $\check{X}_1$ also satisfies Assumption 3.1. So one can repeat the above formula till one resolves all double points. In particular, one may use Proposition 3.2 and Theorem 3.3 to calculate the simple invariant for an $SO(3)$-bundle obtained from a flat bundle with an admissible Stiefel-Whitney class.

2. Given an integral lift of $w_2(\check{P}_1)$ and an orientation of $H^+(\check{X}_1; \boldsymbol{R})$, we choose an integral lift of $w_2(\check{P})$ and an orientation of $H^+(\check{X})$ by the exact sequence

$$0 \to H^2(\check{X}_1; \boldsymbol{Z}) \to H^2(\check{X}; \boldsymbol{Z}) \oplus H^2(W; \boldsymbol{Z}) \to H^2(\boldsymbol{RP}^3; \boldsymbol{Z}) \cong \boldsymbol{Z}_2$$

and that tensored with $\boldsymbol{R}$. Then the above equation holds with the sign.

4. Application to the Horikawa surface. We calculate the simple invariant for the Horikawa surface and regular elliptic surfaces using Theorem 3.3 and Proposition 3.2. We first recall the definition of our model of the Horikawa surface: Let Π_2 be a non-singular curve of genus 2. Since Π_2 is necessarily hyperelliptic, it admits an analytic $\boldsymbol{Z}_2$-action with six fixed points. Then the orbifold $\check{X} = (\Pi_2 \times \Pi_2)/\boldsymbol{Z}_2$ with respect to the diagonal action has a Kähler structure inherited from that on Π_2. The minimal resolution at the thirty-six double points yields the Horikawa surface Y. Topologically we can write $Y = X \cup (\bigcup_i W_i)$, where X is a compact smooth manifold obtained by removing small cones about the thirty-six double points in $\check{X}$. Then we can easily verify that Y is simply-connected and that $b_+(Y) = 9$.

THEOREM 4.1. *$|\gamma_Y(\eta)| = 4$ for some element $\eta \in C_Y$.*

PROOF. We first give a representation of $\pi_1(X)$ in $SO(3)$. Let $y_1, \dots, y_4$ be generators of $\pi_1(\Pi_2)$ as shown in the Figure. They satisfy one relation

$$y_1 \cdots y_4 y_1^{-1} \cdots y_4^{-1} = 1 .$$

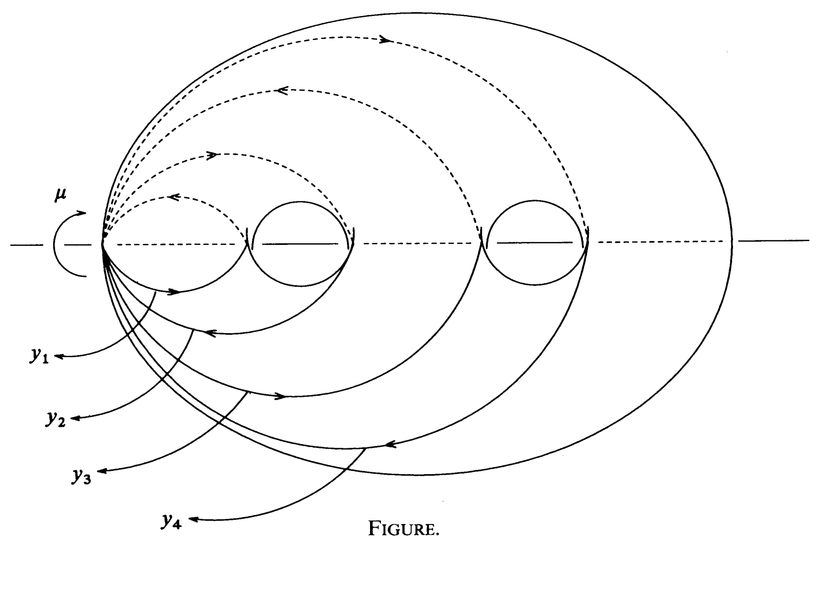

FIGURE.

Let $y_5, \dots, y_8$ be a copy of $y_1, \dots, y_4$. If we denote the generator of $\boldsymbol{Z}_2$ by μ, then $\pi_1(X)$ is represented by

$$\pi_1(X)=\left\{\begin{array}{c|c} \begin{array}{c} \mu\,, \\ y_1,\dots,y_4\,, \\ y_5,\dots,y_8 \end{array} & \begin{array}{c} \mu^2=1\,, \\ y_1\cdots y_4 y_1^{-1}\cdots y_4^{-1}=1\,, \\ y_5\cdots y_8 y_5^{-1}\cdots y_8^{-1}=1\,, \\ \mu y_i \mu=y_i^{-1}\ (1\le i\le 8)\,, \\ y_i y_j=y_j y_i\ (1\le i\le 4,\, 5\le j\le 8) \end{array} \end{array}\right\}.$$

Let $\{1, a, b, c\}$ be a copy of Klein's four-group in $SO(3)$. We define a representation $\rho: \pi_1(X)\to SO(3)$ by

$$\mu\to 1\,,$$
$$y_1\to a\,,\quad y_2\to a\,,\quad y_3\to b\,,\quad y_4\to c\,,$$
$$y_5\to b\,,\quad y_6\to 1\,,\quad y_7\to c\,,\quad y_8\to b\,.$$

Let $\check{P}\to\check{X}$ be the flat $SO(3)$-bundle corresponding to this representation and $\tilde{A}$ the flat connection. We will find out around which of the thirty-six double points the bundle is twisted. The fundamental group $\pi_1(\partial X)$ is generated by the following elements of order 2:

$$\mu\,,\quad \mu y_i\,(1\le i\le 8)\,,\quad \mu y_i y_j\,(1\le i\le 4,\, 5\le j\le 8)\,,$$
$$\mu y_1y_2y_3y_4\,,\quad \mu y_5y_6y_7y_8\,,\quad \mu y_i y_5y_6y_7y_8\,(1\le i\le 4)\,,\quad \mu y_1y_2y_3y_4y_j\,(5\le j\le 8)\,,$$
$$\mu y_1y_2y_3y_4y_5y_6y_7y_8\,.$$

It is easy to verify that there are six untwisted points corresponding to

$$\mu\,,\quad \mu y_6\,,\quad \mu y_3y_5\,,\quad \mu y_3y_8\,,\quad \mu y_4y_7\,,\quad \mu y_4y_5y_6y_7y_8\,,$$

and that all the other thirty points are twisted. So the $SO(3)$-bundle $\check{P}$ satisfies Assumption 3.1.1. We show that $w_2(\check{P})$ is admissible. Suppose that $f\in H^1(\partial X; \boldsymbol{Z}_2)\cong \mathrm{Hom}(H_1(\partial X; \boldsymbol{Z}), \boldsymbol{Z}_2)$ takes the non-zero value $1\in \boldsymbol{Z}_2=\{0, 1\}$ on two or four elements of the above six generators. We must find $g\in H^1(X; \boldsymbol{Z}_2)\cong \mathrm{Hom}(H_1(X; \boldsymbol{Z}), \boldsymbol{Z}_2)$ such that $\sigma^*(g)=g\cdot\sigma_*=f$ at the six generators. These generators satisfy only one relation in $H_1(X; \boldsymbol{Z})$:

$$\mu+(\mu+y_6)+(\mu+y_3+y_5)+(\mu+y_3+y_8)+(\mu+y_4+y_7)=\mu+y_4+y_5+y_6+y_7+y_8\,.$$

So the condition on g is equivalent to the linear relation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} g(\mu) \\ g(y_6) \\ g(y_3) \\ g(y_5) \\ g(y_8) \\ g(y_4) \\ g(y_7) \end{bmatrix} \begin{bmatrix} f(\mu) \\ f(\mu+y_6) \\ f(\mu+y_3+y_5) \\ f(\mu+y_3+y_8) \\ f(\mu+y_4+y_7) \end{bmatrix}.$$

Since the above matrix has full rank, we can find a solution. Therefore $w_2(\check{P})$ is admissible. Next we will show that $\mathscr{R}_X(\check{P})=[\rho]$.

LEMMA 4.2. *Let $\{A_1, \dots, A_m\}$ $(m \geq 2)$ and $\{B_1, \dots, B_n\}$ $(n \geq 2)$ be finite sets in $SO(3)$. Suppose that each of them has no axis common to all elements and that $[A_i, B_j]=1$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$. Then there is an element $C \in SO(3)$ such that*

$$C^{-1}A_iC \in \{1, a, b, c\}, \quad \textit{and} \quad C^{-1}B_jC \in \{1, a, b, c\}$$

for all $1 \leq i \leq m$, $1 \leq j \leq n$.

This follows from an elementary linear algebra: Rotations which commute with a fixed rotation have a common axis if the angle of the fixed rotation is neither 0 nor π.

Suppose that there were another representation $\rho' : \pi_1(X) \to SO(3)$ with the same second Stiefel-Whitney class. Pulling it back to $\pi_1(\Pi_2 \times \Pi_2)=\pi_1(\Pi_2) \oplus \pi_1(\Pi_2)$, we can apply Lemma 4.2 to deduce that, after conjugation with respect to $SO(3)$, the image of ρ' is in $\{1, a, b, c\}$. Since ρ' has the same Stiefel-Whitney class as ρ, it must send the loops around the six untwisted points to 1. So we have $\rho'(\mu)=\rho'(y_6)=1$, $\rho'(y_3)=\rho'(y_5)=\rho'(y_8)$ and $\rho'(y_4)=\rho'(y_7)$. Since ρ' should not send the loops around the thirty twisted points to 1, we can directly verify that ρ' is conjugate to ρ.

Finally we will show that $\tilde{A}$ is regular. Since the image of ρ has finite stabilizer, $H^0_{\tilde{A}}$ is zero. So it suffices to show that $H^1_{\tilde{A}}$ is zero because the index of the AHS complex is zero. Since the pull back of $\tilde{A}$ to $\Pi_2 \times \Pi_2$ splits into three real flat connections by Lemma 4.2, the vanishing of $H^1_{\tilde{A}}$ reduces to that of the $\boldsymbol{Z}_2$-invariant part of the first cohomology group on the real flat bundles over Π_2 by the Künneth formula. We split Π_2 into two $\boldsymbol{Z}_2$-invariant punctured 2-tori Π_1. Then from the Mayer-Vietoris exact sequence

$$0 \to H^1(\Pi_2)^{\boldsymbol{Z}_2} \to H^1(\Pi_1)^{\boldsymbol{Z}_2} \oplus H^1(\Pi_1)^{\boldsymbol{Z}_2}$$

together with the Künneth formula and the Poincaré duality, it suffices to show the vanishing of $H^0(S^1)^{\boldsymbol{Z}_2}$. If the real line bundle is untwisted, then $\boldsymbol{Z}_2$ acts on S^1 by reflection, so it vanishes. If twisted, then a Möbius band on S^1 has no non-zero constant section, so it also vanishes.

Since ρ has stabilizer $\{1, a, b, c\}$, we see that the flat $SO(3)$-bundle $\check{P} \to \check{X}$ satisfies

3.1.3. Clearly the dimension of $H^1(X; \boldsymbol{Z}_2)$ is 9. So by Theorem 3.3 and Proposition 3.2, we get $|\gamma_Y(\eta)| = 2^{9-5-2} = 4$ for an $\eta \in C_Y$. ■

Orbifold $SO(3)$-bundles with the same non-admissible Stiefel-Whitney class. Now we digress to construct two orbifold $SO(3)$-bundles which have the same second Stiefel-Whitney class and the same Pontrjagin charge (hence the same monodromy at double points), but which are *not* isomorphic to each other.

Let $\check{Z}$ be a simply-connected orbifold obtained from Y by resolving all except two of the double points (these two points being easily detected from the generators of $\pi_1(\partial X)$). Then we write $\check{Z} = Z \bigcup_{i=1,2} D_i^4/\boldsymbol{Z}_2$ and $Y = Z \bigcup_{i=1,2} W_i$.

We let P_1^* be an $SO(3)$-bundle over Y. We suppose that it restricts to the trivial bundle over W_i. Let P_2^* be the $SO(3)$-bundle over Y with $w_2(P_2^*) = w_2(P_1^*) + \delta^*(\mathfrak{d}_1 + \mathfrak{d}_2)$ and $p_1(P_2^*) = p_1(P_1^*)$, where $\delta^* : H^1(\partial Z; \boldsymbol{Z}_2) \to H^2(Y; \boldsymbol{Z}_2)$ is the coboundary map in the Mayer-Vietoris sequence for the pair $(Z, W_1 \coprod W_2)$. Then P_1^* and P_2^* can be written as $P_1^* = P \cup (\bigcup_{\iota_i} W_i \times SO(3))$ and $P_2^* = P \cup (\bigcup_{\iota_i'} W_i \times SO(3))$ using an $SO(3)$-bundle P over Z. So $\check{P}_1 = P \cup (\bigcup_{\iota_i} D_i^4/\boldsymbol{Z}_2 \times SO(3))$ and $\check{P}_2 = P \cup (\bigcup_{\iota_i'} D_i^4/\boldsymbol{Z}_2 \times SO(3))$ are $SO(3)$-bundles over $\check{Z}$ with $w_2(\check{P}_1) = w_2(\check{P}_2)$. By construction, they admit smooth connections A_1 and A_2 which are equal to the trivial connection over D_i^4 and $\int_{\check{Z}} \mathrm{Tr}(F_{A_1} \wedge F_{A_1}) = \int_{\check{Z}} \mathrm{Tr}(F_{A_2} \wedge F_{A_2})$.

We show that $\check{P}_1$ and $\check{P}_2$ are not isomorphic to each other. Suppose, on the contrary, that there were a bundle isomorphism $h : \check{P}_1 \to \check{P}_2$. Then $h|_{S^3/\boldsymbol{Z}_2}$ is homotopic to a constant map, since it lifts to S^3 and extends to D^4 by definition. So we may suppose that the automorphism $h|_Z$ is equal to $(\iota_i')^{-1}\iota_i$ over ∂Z. By the homotopy extension theorem, there exists an element $v \in H^1(Z; \boldsymbol{Z}_2)$ such that $\sigma^*(v) = \mathfrak{d}(1, (\iota_i')^{-1}\iota_i) = \mathfrak{d}_1 + \mathfrak{d}_2 \neq 0$, where $\sigma : \partial Z \to Z$ is the inclusion, a contradiction to $\pi_1(Z) = \{1\}$.

For example, we take P_1^* to be the trivial bundle and P_2^* to be an $SO(3)$-bundle with $w_2(P_2^*) = \mathfrak{d}_1 + \mathfrak{d}_2$ and $p_1(P_2^*) = 0$. Then the flat moduli of $\check{P}_1$ consists of the trivial connection, but that of $\check{P}_2$ is empty.

Next we will calculate the simple invariant for regular elliptic surfaces without multiple fibers. Let Π_k be a hyperelliptic curve of genus k. We consider a $\boldsymbol{Z}_2$-action on Π_k and Π_1 in the same way as on Π_2. Then the quotient $\check{S}_k = (\Pi_{k-1} \times \Pi_1)/\boldsymbol{Z}_2$ with respect to the diagonal action is a 4-orbifold with double points. Resolving the $8k$ double points in $\check{S}_k$, we obtain a regular elliptic surface $\pi : S_k \to \boldsymbol{CP}^1$, where π is induced from the projection of $\Pi_{k-1} \times \Pi_1$ to the first factor. It is well-known that S_k is simply-connected and $b_+(S_k) = 2k-1$. Let f be a general fiber in S_k.

PROPOSITION 4.3. *For some $\eta_k \in C_{S_k}$ with $\langle \eta_k, [f] \rangle \equiv 1 \pmod 2$, we have $|\gamma_{S_k}(\eta)| = 1$.*

PROOF. We take generators $y_1, \dots, y_{2k-2}$ of Π_{k-1} and generators y_{2k-1}, y_{2k} of Π_1 in the same way as in the Figure. Define a representation $\rho_k : \pi_1(\check{S}_k) \to SO(3)$ by

$$\mu \to 1\,,$$

$$\begin{cases} y_1 \to 1, y_2 \to a, y_3 \to b, y_4 \to c, \ldots, y_{2k-3} \to b, y_{2k-2} \to c, & \text{if } k \text{ is even}\,, \\ y_1 \to 1, y_2 \to a, y_3 \to b, y_4 \to c, \ldots, y_{2k-3} \to 1, y_{2k-2} \to a, & \text{if } k \text{ is odd}\,, \end{cases}$$

$$y_{2k-1} \to b,\ y_{2k} \to c\,.$$

Then it is easy to see that its isotropy group is $\{1, a, b, c\}$ and that there are $2k$ double points around which ρ_k is sent to 1. Moreover, we can verify that Assumption 3.1 is satisfied and that it is the unique representation in the same way as in Theorem 4.1. Hence we get $|\gamma_{S_k}(\eta_k)| = 2^{(2k+1)-(2k-1)-2} = 1$ for some $\eta_k \in C_{S_k}$ with $\langle \eta_k, [f] \rangle \equiv 1$. ∎

REMARK. The author [12] has calculated the simple invariant for wider classes in C_{S_k}: For *any* $\eta \in C_{S_k}$ with $\langle \eta, [f] \rangle \equiv 1 \pmod 2$, we have proved that $|\gamma_{S_k}(\eta)| = 1$.

5. The proof of corollaries. We first construct a torus sum of two copies of Y by choosing two embedded 2-tori with $\pi_1(Y \setminus T^2) = \{1\}$ and $\langle \eta, T^2 \rangle \equiv 1 \pmod 2$. One way to choose such 2-tori is as follows: We slide the loops y_1 and $y_3^{-1} y_1^{-1} y_2$ into the interior of a fundamental domain of the involution. At the same time, we take $\boldsymbol{Z}_2$-invariant dual circles z_1, z_2 which intersect y_1, $y_3^{-1} y_1^{-1} y_2$ at one point, respectively, and which are disjoint to each other. We also perform the same procedure for y_5, $y_7^{-1} y_5^{-1} y_6$, and take dual $\boldsymbol{Z}_2$-circles z_3, z_4 in the same way. Then $T_1 = y_1 \times y_5$, $T_2 = (y_3^{-1} y_1^{-1} y_2) \times (y_7^{-1} y_5^{-1} y_6)$ in $\Pi_2 \times \Pi_2$ descend to 2-tori in Y (also denoted by T_1, T_2). Since the 2-tori $z_1 \times z_3$ and $z_2 \times z_4$ descend to dual 2-spheres of T_1, T_2 in Y (in the sense that they intersect the 2-tori at one point), the complements $Y \setminus T_1$, $Y \setminus T_2$ are simply-connected. Moreover, since the ρ sends y_1, $y_7^{-1} y_5^{-1} y_6$ to a and y_5, $y_3^{-1} y_1^{-1} y_2$ to b, it follows that $\langle \eta, T_1 \rangle \equiv \langle \eta, T_2 \rangle \equiv 1 \pmod 2$. Now we take a copy of Y and identify a tubular neighborhood of T_1 in each of them and remove its interior. Then we obtain a smooth oriented 4-manifold $Y \natural Y$ and an element $\eta \natural \eta \in C_{Y \natural Y}$ from the $\eta \in C_Y$. Repeating this procedure by using T_1 and T_2, we obtain a closed oriented 4-manifold $Y_l = Y \natural \cdots \natural Y$ and an element $\eta_l = \eta \natural \cdots \natural \eta \in C_{Y_l}$ from l copies of Y and η. We call this multiplication '*torus sum*' in [12]. Next we will glue Y_l to the regular elliptic surfaces S_k. We remove a tubular neighborhood of the remaining T_2 in Y_l and that of f in S_k. Then the same procedure as before yields a closed oriented 4-manifold $S_k \natural Y_l$ and a class $\eta_k \natural \eta_l \in C_{S_k \natural Y_l}$.

PROPOSITION 5.1. $|\gamma_{S_k \natural Y_l}(\eta_k \natural \eta_l)| = 4^l$ *for integers* k, l *with* $0 \le k \le 1$, $l \ge 1$ *or* $k \ge 2$, $l \ge 0$.

PROOF. If $k \ge 2$ or 0, then we can apply [12, Theorem 2.3] to deduce that

$$|\gamma_{S_k \natural Y_l}(\eta_k \natural \eta_l)| = |\gamma_{S_k}(\eta_k)|\,|\gamma_Y(\eta)|^l = 4^l\,.$$

If $k = 1$, we consider a closed 4-manifold $S_2 \natural S_1 \natural Y_l = S_3 \natural Y_l$, where $S_2 \natural S_1$ is the fiber sum of S_2 and S_1. Then we again apply [12, Theorem 2.3] to get

$$|\gamma_{S_3 \natural Y_l}(\eta_3 \natural \eta_l)| = |\gamma_{S_2}(\eta_2)|\,|\gamma_{S_1 \natural Y_l}(\eta_1 \natural \eta_l)| = |\gamma_{S_1 \natural Y_l}(\eta_1 \natural \eta_l)| = 4^l\,.$$

■

By the last remark in Section 4, we can replace η_k by all $\eta \in C_{S_k}$ with $\langle \eta, [f] \rangle \equiv 1 \pmod 2$ in order to obtain the same conclusion as above. For example we can choose

$$\eta_k = \begin{cases} \text{P.D.}[\Sigma_k] & \text{if } k \text{ is even} \\ \text{P.D.}([\Sigma_k] + [f]) & \text{if } k \text{ is odd}\,, \end{cases}$$

where Σ_k: $\boldsymbol{CP}^1 \to S_k$ is a section and P.D. means the mod 2 Poincaré dual.

Next we consider the effect of the topological logarithmic transformation performed on a general fiber in a nucleus, in the sense of Gompf [9]. Here the *nucleus* is a regular neighborhood of the bouquet of a section and a cusp fiber in a regular elliptic surface. We find a nucleus in the surface Y. If we choose y_1 and $y_7^{-1} y_5^{-1} y_6$ to be a $\boldsymbol{Z}_2$-equivalent circle, the preimage of the 2-sphere $(y_1 \times (y_7^{-1} y_5^{-1} y_6))/\boldsymbol{Z}_2$ by the map $Y \to \check{Y}$ is identified with a singular fiber of type I_0^*. Then the preimage of the sphere $(z_1 \times z_4)/\boldsymbol{Z}_2$ can be thought of as a section. By Moishezon [16], the singular fiber of type I_0^* is deformable to four cusp fibers. Then a regular neighborhood of one cusp fiber together with this section is a nucleus in Y.

Now performing a logarithmic trransformation on a fiber with multiplicity p, we get a new manifold $Y(p)$. Since $\langle \eta, F \rangle \equiv 0 \pmod 2$, we can apply the argument of [10, IV] to deduce:

PROPOSITION 5.2. *If we write* $Y_l(p) = Y \natural \cdots \natural Y \natural Y(p)$, *then*

$$|\gamma_{S_k \natural Y_l(p)}(\eta_l)| = p\,|\gamma_{S_k \natural Y_l}(\eta_l)| = 4^l p\,.$$

Since $S_k \natural Y_l(p)$ is homotopy equivalent to $(2k+10l-1)\boldsymbol{CP}^2 \# (10k+46l-1)\overline{\boldsymbol{CP}^2}$ and $C_{S_k \natural Y_l(p)}$ is finite, $S_k \natural Y_l(p)$ has infinitely many diffeomorphism types of $S_k \natural Y_l$.

Moreover, when p is odd, we can apply the argument of [10, IV] to all $\xi \in C_{S_k \natural Y_l(p)}$ to deduce the following: If $\langle \xi, F \rangle \equiv 0 \pmod 2$, then $|\gamma_{S_k \natural Y_l(p)}(\xi)| = p\,|\gamma_{S_k \natural Y_l}(\xi)|$. If $\langle \xi, F \rangle \equiv 1 \pmod 2$, then $|\gamma_{S_k \natural Y_l(p)}(\xi)| = |\gamma_{S_k \natural Y_l}(\xi)|$. Then multiplying all non-zero values, we obtain a diffeomorphic invariant. Comparing the values of this invariant, we get:

THEOREM 5.3. *For odd prime integers* p, p', *if* $S_k \natural Y_l(p)$ *is diffeomorphic to* $S_k \natural Y_l(p')$, *then* $p = p'$.

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