Tôhoku Math. J.
47 (1995), 521–540

THE LOCAL THETA CORRESPONDENCE OF IRREDUCIBLE TYPE 2 DUAL REDUCTIVE PAIRS

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(Received May 9, 1994, revised April 6, 1995)

Abstract. The explicit Howe duality correspondence is partially solved in the case of irreducible type 2 dual reductive pairs defined over a non-Archimedean local field.

Introduction. Let (GL_n, GL_m) be an irreducible type 2 dual reductive pair defined over a non-Archimedean local field F. The Weil representation $\omega_{n,m}$ of $GL_n(F)\times GL_m(F)$ on the Schwartz-Bruhat space $\mathscr{S}(M_{n,m}(F))$ is given by

$$\omega_{n,m}(h, g)f(x)=|\det h|^{-m/2}|\det g|^{n/2}f(h^{-1}xg) \qquad (h\in GL_n(F),\ g\in GL_m(F)) .$$

Then a problem on the (explicit) Howe correspondence for (GL_n, GL_m) is stated as follows. For a given irreducible admissible representation σ of $GL_n(F)$, determine an irreducible admissible representation σ' of $GL_m(F)$ such that $\mathrm{Hom}_{GL_n(F)\times GL_m(F)}(\omega_{n,m}, \sigma\otimes\sigma')\neq 0$. The purpose of this paper is to study this problem in the case where $m=n+1$ and σ is generic.

Our starting point is a global theta series lifting of a cusp form on the adele group $GL_n(A)$. For a cusp form φ on $GL_n(A)$ and a Schwartz-Bruhat function $f\in\mathscr{S}(M_{n,m}(A))$, we define a theta series lifting φ_f^s, where s is a complex parameter with $\mathrm{Re}(s)\ll 0$. This φ_f^s is an automorphic form on $GL_m(A)$. In Section 1, we calculate a Whittaker function $W_{\varphi_f^s}$ of φ_f^s and prove that $W_{\varphi_f^s}$ is identically zero if $m\neq n, n+1$. In the case where $m=n$ or $m=n+1$, the function $W_{\varphi_f^s}$ is represented by a convolution of the Whittaker function W_φ of φ and a certain function $\Phi_m(f)$ related to f. More precisely, we have a formula of the form

$$W_{\varphi_f^s}(g)=\int_{U_n(A)\backslash GL_n(A)} W_\varphi(h)|\det h|_A^s\Phi_m(\omega_{n,m}(g)f)(h)dh\,, \qquad (m=n, n+1)\,.$$

On the basis of this formula, we can define a local theta series lifting of a local Whittaker function. This is the reason why we study the Howe correspondence in the case where $m=n+1$ and σ is generic. The case $m=n$ will be investigated in another paper [17].

We state the results of this paper. Let σ be an irreducible generic representation of $GL_n(F)$. By using a local analogue of the formula mentioned above, one can construct

Partly supported by the Grants-in-Aid for Encouragement of Young Scientists, The Ministry of Education, Science and Culture, Japan.

1991 *Mathematics Subject Classification*. Primary 11F70; Secondary 11F27, 22E50.

a local theta series lifting $\tilde{\sigma}$ of σ. This $\tilde{\sigma}$ is an admissible representation of $GL_{n+1}(F)$ realized in the space of Whittaker functions and satisfies the following:

(0.1) $\mathrm{Hom}_{GL_n(F)\times GL_{n+1}(F)}(\omega_{n,n+1}, \sigma^\vee \otimes \tilde{\sigma}) \neq 0$, where $\sigma^\vee$ denotes the contragradient representation of σ.

To describe the properties of $\tilde{\sigma}$, we denote by σ_1 the normalized induced representation $\mathrm{Ind}_{Q_{n+1}(F)}^{GL_{n+1}(F)} \sigma \otimes 1$ of $GL_{n+1}(F)$, where $Q_{n+1}(F)$ denotes the standard upper triangular parabolic subgroup of $GL_{n+1}(F)$ with Levi factor $GL_n(F) \times GL_1(F)$. In this introduction, we assume σ_1 to be irreducible for simplicity. (If σ_1 is not irreducible, we must modify the definition of σ_1 as will be mentioned in Section 2.) Then we show the following:

(0.2) σ_1 is a unique irreducible subrepresentation of $\tilde{\sigma}$. Furthermore, the quotient representation $\tilde{\sigma}/\sigma_1$ has no nonzero vectors fixed by the closed subgroup

$$\left\{ \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \middle| k \in GL_n(\mathcal{O}) \right\},$$

where $GL_n(\mathcal{O})$ is the maximal compact subgroup of $GL_n(F)$ consisting of integral matrices.

(0.3) $\tilde{\sigma}$ is of Whittaker type in the sense of Jacquet, Piatetski-Shapiro and Shalika [8, (2.1)].

By (0.3), one can define the gamma factor $\gamma(s, \tilde{\sigma} \times \tau, \psi)$ (cf. [8, (3.1)]) for each irreducible generic representation τ of $GL_m(F)$. Then (0.2) implies that

$$\gamma(s, \tilde{\sigma} \times \tau, \psi) = \gamma(s, \sigma_1 \times \tau, \psi)$$

for all irreducible generic representations τ of $GL_m(F)$. In light of these results, one can expect that $\tilde{\sigma} = \sigma_1$ for any generic σ. If σ is a generic spherical representation, we really have $\tilde{\sigma} = \sigma_1$.

Prasad [14, (4.6.5)] stated a conjectural form of an irreducible admissible representation σ' of $GL_{n+1}(F)$ corresponding to σ by the Howe duality. Since $\sigma_1 \neq \sigma'$, this conjectural form σ' is not consistent with the Howe correspondence if σ is a generic spherical representation.

NOTATION. For an associative ring R with the identity element, we denote by $R^\times$ the group of all invertible elements of R and by $M_{n,m}(R)$ the set of all $n \times m$ matrices with entries in R. If $n = m$, we write $M_n(R)$ for $M_{n,n}(R)$. For $A \in M_{n,m}(R)$, ${}^tA$ stands for its transpose. For $A \in M_n(R)$, $\det A$ stands for its determinant. The identity matrix in $M_n(R)$ is denoted by 1_n.

When a base field F is given, we set $G_n = GL(n, F)$. If $m < n$, we will regard G_m as a subgroup of G_n by the embedding

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1_{n-m} \end{pmatrix}.$$

We define algebraic subgroups of G_n as

B_n the set of upper triangular matrices ,
U_n the unipotent radical of B_n ,
T_n the set of diagonal matrices ,
Z_n the center of G_n ,
$P_n = \left\{ \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix},\ g \in G_{n-1},\ u \in M_{n-1,1}(F) \right\}$,
$Q_n = Z_n P_n$.

The Weyl group of G_n will be identified with the symmetric group S_n of degree n.

If G is a locally compact abelian group, then $\mathscr{S}(G)$ denotes the space of Schwartz-Bruhat functions on G.

1. The global theta lifting. In this section, let k be a global field and A the adele ring of k. For a k-subgroup G of $G_n = GL(n, k)$, $G(A)$ denotes the corresponding adele group. We fix a nontrivial additive character ψ of $k \backslash A$ and define the character ψ_n of $U_n(A)$ by

$$\psi_n(u) = \psi(u_{12} + u_{23} + \cdots + u_{n-1n}) \qquad (u = (u_{ij}) \in U_n(A)) .$$

The Weil representation $(\omega_{n,m}, \mathscr{S}(M_{n,m}(A)))$ of $G_n(A) \times G_m(A)$ is defined as follows: for $f \in \mathscr{S}(M_{n,m}(A))$, $h \in G_n(A)$ and $g \in G_m(A)$,

$$\omega_{n,m}(h, g) f(x) = |\det h|_A^{-m/2} |\det g|_A^{n/2} f(h^{-1} x g) .$$

Let μ be a character of $Z_n \backslash Z_n(A)$. For $f \in \mathscr{S}(M_{n,m}(A))$ and $s \in \mathbf{C}$, we define a modified theta series $\theta(s, \mu, f)$ as

$$\theta(s, \mu, f) = \int_{Z_n \backslash Z_n(A)} \mu(z) |\det z|_A^{s-m/2} \sum_{\substack{x \in M_{n,m}(k) \\ x \neq 0}} f(z^{-1} x) dz .$$

From [4, Lemmas 11.5 and 11.6], it follows that the integral on the right-hand side is absolutely convergent for $\mathrm{Re}(s) < -m/2$ and the function $(h, g) \mapsto \theta(s, \mu, \omega_{n,m}(h, g) f)$ is slowly increasing on $(G_n \backslash G_n(A)) \times (G_m \backslash G_m(A))$. Let φ be a cusp form on $G_n(A)$ satisfying $\varphi(zg) = \mu(z)\varphi(g)$ for any $z \in Z_n(A)$. Then we define a modified theta lifting φ_f^s of φ by

$$\varphi_f^s(g) = \int_{G_n \backslash G_n(A)} \varphi(h) |\det h|_A^s \sum_{\substack{x \in M_{n,m}(k) \\ x \neq 0}} \omega_{n,m}(h, g) f(x) dh$$

$$=\int_{Z_n(A)G_n\backslash G_n(A)}\varphi(h)|\det h|_A^s\theta(s,\mu,\omega_{n,m}(h,g)f)dh\,.$$

Since $\varphi(h)$ is rapidly decreasing on $Z_n(A)G_n\backslash G_n(A)$, this integral is absolutely convergent for $\mathrm{Re}(s)<-m/2$, and hence φ_f^s defines an automorphic form on $G_m(A)$. The purpose of this section is to calculate a Whittaker function of φ_f^s. Namely we compute the integral

$$W_{\varphi_f^s}(g)=\int_{U_m\backslash U_m(A)}\psi_m(u)^{-1}\varphi_f^s(ug)du\,.$$

We set

$$W_\varphi(h)=\int_{U_n\backslash U_n(A)}\psi_n(u)^{-1}\varphi(uh)du\,.$$

If $m\geq n$, we define the function $\Phi_m(f)$ on $G_n(A)$ by

$$\Phi_m(f)(h)=\int_{U_m(A)}\psi_m(u)^{-1}\omega_{n,m}(h,u)f(\varepsilon_{n,m})du\,,$$

where we put $\varepsilon_{n,m}=(1_n,0)\in M_{n,m}(k)$.

PROPOSITION 1. *Let φ be a cusp form on $G_n(A)$, $f\in\mathcal{S}(M_{n,m}(A))$ and $s\in\mathbf{C}$ with* $\mathrm{Re}(s)<-m/2$.

(1) *If $m\leq n-1$ or $n+2\leq m$, then $W_{\varphi_f^s}$ is identically zero.*

(2) *If $m=n$, then*

$$W_{\varphi_f^s}(g)=\int_{U_n(A)\backslash G_n(A)}W_\varphi(h)|\det h|_A^s\Phi_n(\omega_{n,n}(g)f)(h)dh$$

provided that the integral on the right-hand side is absolutely convergent.

(3) *If $m=n+1$, then*

$$W_{\varphi_f^s}(g)=\int_{U_n(A)\backslash G_n(A)}W_\varphi(h)|\det h|_A^s\Phi_{n+1}(\omega_{n,n+1}(g)f)(h)dh\,.$$

Here the integral on the right-hand side always converges absolutely.

PROOF. For a matrix $x\in M_{n,m}(k)$, let x_j denote the j-th column vector of x. We write $(x_1,x_2,\dots,x_m)$ for x. We define the subset Y_j of $M_{n,m}(k)$ as

$$Y_0=\{x\in M_{n,m}(k)-\{0\}\,|\,x_1=0\}$$

$$Y_j=\{x\in M_{n,m}(k)\,|\,\mathrm{rank}(x_1,\dots,x_j)=\mathrm{rank}(x_1,\dots,x_j,x_{j+1})=j\}\,,$$

for $1\leq j\leq\min(n,m-1)$. If $m\leq n$, we also set

$$Y_m=\{x\in M_{n,m}(k)\,|\,\mathrm{rank}(x_1,\dots,x_m)=m\}\,.$$

Then $M_{n,m}(k)-\{0\}$ is a disjoint union of Y_j, $0\leq j\leq \min(n, m)$, and each Y_j is left G_n- and right U_m-invariant. Let Y_j be a complete set of representatives for Y_j/U_m. If $0\leq j\leq \min(n, m-1)$, then we can take Y_j so that each $x\in Y_j$ has $x_{j+1}=0$. In the following, for $x\in M_{n,m}(k)$, $Z(x, U_m)$ stands for the stabilizer of x in U_m. Then $W_{\varphi_f^s}(g)$ equals

$$\int_{G_n\backslash G_n(A)}\varphi(h)|\det h|_A^s\int_{U_m\backslash U_m(A)}\psi_m(u)^{-1}\sum_{\substack{x\in M_{n,m}(k)\\ x\neq 0}}\omega_{n,m}(h, ug)f(x)dudh$$

$$=\int_{G_n\backslash G_n(A)}\varphi(h)|\det h|_A^s\int_{U_m\backslash U_m(A)}\psi_m(u)^{-1}\sum_{j=0}^{\min(n,m)}\sum_{x\in Y_j}\sum_{\gamma\in Z(x,U_m)\backslash U_m}\omega(h, ug)f(x\gamma)dudh$$

$$=\sum_{j=0}^{\min(n,m)} I_j^m ,$$

where we set

$$I_j^m=\int_{G_n\backslash G_n(A)}\varphi(h)|\det h|_A^s\sum_{x\in Y_j}\left(\int_{Z(x,U_m)\backslash Z(x,U_m(A))}\psi_m(u')du'\right)$$
$$\times\int_{Z(x,U_m(A))\backslash U_m(A)}\psi_m(u)^{-1}\omega(h, ug)f(x)dudh .$$

By our choice of representatives, ψ_m is nontrivial on $Z(x, U_m(A))$ if $x\in Y_j$ and $0\leq j\leq \min(n, m-2)$. This implies $I_j^m=0$ for $0\leq j\leq \min(n, m-2)$. Therefore we have

$$W_{\varphi_f^s}(g)=\begin{cases} I_{m-1}^m+I_m^m & \text{if } m\leq n\\ I_{m-1}^m & \text{if } m=n+1\\ 0 & \text{if } m\geq n+2 .\end{cases}$$

We consider the case $m\leq n$. We regard U_m as a subgroup of G_n. Let M_n^m be the stabilizer of the matrix ${}^t\varepsilon_{m,n}={}^t(1_m, 0)\in M_{n,m}(k)$ in G_n, i.e.

$$M_n^m=\left\{\begin{pmatrix} 1_m & u\\ 0 & g\end{pmatrix}\middle|\, g\in G_{n-m}, u\in M_{m,n-m}(k)\right\} .$$

Since $Y_m=G_n{}^t\varepsilon_{m,n}$, I_m^m equals

$$\int_{G_n\backslash G_n(A)}\varphi(h)|\det h|_A^s\int_{U_m\backslash U_m(A)}\psi_m(u)^{-1}\sum_{\gamma\in M_n^m\backslash G_n}\omega_{n,m}(rh, ug)f({}^t\varepsilon_{m,n})dudh$$

$$=\int_{M_n^m\backslash G_n(A)}\varphi(h)|\det h|_A^s\int_{U_m\backslash U_m(A)}\psi_m(u)^{-1}\omega_{n,m}(u^{-1}h, g)f({}^t\varepsilon_{m,n})dudh$$

$$=\int_{M_n^m(A)\backslash G_n(A)}\left(\int_{M_n^m\backslash M_n^m(A)}\varphi(h_0h)|\det h_0h|_A^s dh_0\right)$$

$$\times \int_{U_m \backslash U_m(A)} \psi_m(u)^{-1} \omega_{n,m}(u^{-1}h, g) f({}^t\varepsilon_{m,n}) du dh .$$

If $m<n$, the cuspidality condition of φ implies $I_m^m=0$. If $m=n$, by formal computation, we have

$$\begin{aligned} I_m^m &= \int_{G_n(A)} \left(\int_{U_n \backslash U_n(A)} \psi_n(u)^{-1} \varphi(uh) du \right) |\det h|_A^s \omega_{n,n}(h, g) f(1_n) dh \\ &= \int_{U_n(A) \backslash G_n(A)} W_\varphi(h) |\det h|_A^s \Phi_n(\omega_{n,n}(g) f)(h) dh . \end{aligned}$$

Next, let M_n^{m-1} denote the stabilizer ${}^t\varepsilon_{m-1,n} = {}^t(1_{m-1}, 0) \in M_{n,m-1}(k)$ in G_n. Then we have

$$Y_{m-1} = \{ \gamma^{-1}({}^t\varepsilon_{m-1,n}, x_m) \mid \gamma \in M_n^{m-1} \backslash G_n, \operatorname{rank}({}^t\varepsilon_{m-1,n}, x_m) = m-1 \} .$$

Therefore, I_{m-1}^m equals

$$\begin{aligned} &\int_{G_n \backslash G_n(A)} \varphi(h) |\det h|_A^s \int_{U_m \backslash U_m(A)} \psi_m(u)^{-1} \\ &\quad + \sum_{\gamma \in M_n^{m-1} \backslash G_n} \sum_{\substack{x_m \in M_{n,1}(k) \\ \operatorname{rank}({}^t\varepsilon_{m-1,n}, x_m) = m-1}} \omega_{n,m}(\gamma h, ug) f(({}^t\varepsilon_{m-1,n}, x_m)) du dh \\ &= \int_{M_n^{m-1}(A) \backslash G_n(A)} \left(\int_{M_n^{m-1} \backslash M_n^{m-1}(A)} \varphi(h_0 h) |\det h_0 h|_A^s dh_0 \right) \int_{U_m \backslash U_m(A)} \psi_m(u)^{-1} \\ &\quad \times \sum_{\substack{x_m \in M_{n,1}(k) \\ \operatorname{rank}({}^t\varepsilon_{m-1,n}, x_m) = m-1}} \omega_{n,m}(h, ug) f(({}^t\varepsilon_{m-1,n}, x_m)) du dh . \end{aligned}$$

The cuspidality of φ implies $I_{m-1}^m = 0$ for all $m \le n$. This completes the proof of the statements (1) and (2).

We consider the case $m=n+1$. By calculation similar to that above, we have

$$I_{m-1}^m = \int_{U_n(A) \backslash G_n(A)} W_\varphi(h) |\det h|_A^s \Phi_{n+1}(\omega_{n,n+1}(g) f)(h) dh .$$

We prove that the integral on the right-hand side converges absolutely. It is sufficient to show that the integral

$$\int_{T_n(A)} W_\varphi(t) |\det t|_A^s \Phi_{n+1}(f)(t) \delta_n(t)^{-1} dt$$

converges absolutely, where δ_n denotes the modular character of $B_n(A)$. By definition, for $t = \operatorname{diag}(a_1, \ldots, a_n) \in T_n(A)$,

$$\Phi_{n+1}(f)(t)=|\det t|^{-(n+1)/2}\int_{U_{n+1}\backslash U_{n+1}(A)}\varphi(u_{12}+\cdots+u_{nn+1})^{-1}$$
$$\times f\left(\begin{pmatrix} a_1^{-1} & a_1^{-1}u_{12} & \cdots & a_1^{-1}u_{1n} & a_1^{-1}u_{1n+1} \\ 0 & a_2^{-1} & \cdots & a_2^{-1}u_{2n} & a_2^{-1}u_{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n^{-1} & a_n^{-1}u_{nn+1}\end{pmatrix}\right) du_{12}\cdots+du_{nn+1}\,.$$

This integral is regarded as a partial Fourier transform of f. Hence there exists a function $\Phi\in\mathscr{S}(A^n\oplus A^n)$ such that

$$|\Phi_{n+1}(f)(t)|\leq\Phi(t,t^{-1})\,.$$

Furthermore $W_\varphi(t)$ is majorized by a gauge function ξ on $G_n(A)$ (cf. [6, Proposition 2.3.6, Lemma 8.3.3 and (12.1)]), i.e.

$$|W_\varphi(t)|\leq\xi(t)\,,\qquad (t\in T(A))\,.$$

Then it is easy to see that the integral

$$\int_{T_n(A)}\xi(t)|\det t|_A^s\Phi(t,t^{-1})\delta_n(t)^{-1}dt$$

is convergent. This completes the proof.

2. The local theta lifting. From now on, we fix a local non-Archimedean field F. In this section, we define a local theta lifting from the set of generic representations of $G_n=GL(n,F)$ to the set of smooth representations of G_{n+1} by using an integral analogous to that in Proposition 1 (3).

First we define some notation and recall certain notions. Let $\mathcal{O}$ denote the ring of integers of F, ϖ a prime element of F and q the order of $\mathcal{O}/\varpi\mathcal{O}$. The absolute valuation of F is denoted by $|\cdot|_F$, which is normalized as $|\varpi|_F=q^{-1}$. We fix, once and for all, a nontrivial additive character ψ of F with the conductor $\mathcal{O}$. We denote by K_n the maximal compact subgroup $GL(n,\mathcal{O})$ of G_n and by $\mathscr{H}_n$ the convolution algebra consisting of all locally constant and compactly supported functions on G_n. The character ψ_n of U_n is defined to be

$$\psi_n(u)=\psi(u_{12}+u_{23}+\cdots+u_{n-1n})$$

for $u=(u_{ij})\in U_n$.

Let $\boldsymbol{W}(\psi_n)$ be the space of all locally constant functions W on G_n satisfying $W(ug)=\psi_n(u)W(g)$ for any $u\in U_n$ and $g\in G_n$, i.e. $\boldsymbol{W}(\psi_n)=\mathrm{Ind}_{U_n}^{G_n}\psi_n$. Then $g\in G_n$ acts on $\boldsymbol{W}(\psi_n)$ by right translation: $\rho(g)W(g_0)=W(g_0g)$. An admissible representation σ of G_n is said to be of Whittaker type if σ is finitely generated and $\dim\mathrm{Hom}_{G_n}(\sigma,\boldsymbol{W}(\psi_n))=1$. If σ is of Whittaker type, we denote by $\boldsymbol{W}(\sigma,\psi_n)$ the image of the unique (up to constant) nonzero G_n-morphism from σ to $W(\psi_n)$. Note that the representation $(\rho, W(\sigma, \psi_n))$ need not be isomorphic to σ. If σ is of Whittaker type and irreducible, then σ is said to be generic.

A classification of irreducible generic representations of G_n is known by Bernstein and Zelevinsky [18]. In the following, for a given smooth representation π of G_n and a complex number z, we denote by $\pi[z]$ the twist of π by $|\cdot|^z$, i.e. $\pi[z](g)=|\det g|^z\pi(g)$. Let Q be a standard upper triangular parabolic subgroup of G_n with Levi factor $G_{n_1}\times G_{n_2}\times\cdots\times G_{n_k}$, $n_1+\cdots+n_k=n$. Let π^i, $1\le i\le k$, be an irreducible tempered representation of G_{n_i} and $r_1>r_2>\cdots>r_k$ real numbers. We set

$$\mathrm{Ind}_Q^{G_n}(\pi^1[r_1]\otimes\pi^2[r_2]\otimes\cdots\otimes\pi^k[r_k])\,. \tag{2.1}$$

Bernstein and Zelevinsky proved that if σ is an irreducible generic representation of G_n, then σ must be equivalent to a representation of the form (2.1), where the parabolic subgroup Q, the tempered representations $\pi^1,\dots,\pi^k$ and the real numbers $r_1>\cdots>r_k$ are uniquely determined by σ (see also [9]). We note that, by a theorem of Jacquet [11], the irreducible tempered representation π^i of G_{n_i} must be equivalent to a representation of the form

$$\mathrm{Ind}_{R_i}^{G_{n_i}}(\pi^{i,1}\otimes\pi^{i,2}\otimes\cdots\otimes\pi^{i,p_i})\,,$$

where R_i denotes a standard upper triangular parabolic subgroup of G_{n_i} with Levi factor $G_{n_{i1}}\times G_{n_{i2}}\times\cdots\times G_{n_{ip_i}}$, $n_{i1}+\cdots+n_{ip_i}=n_i$ an $\pi^{i,j}$ and irreducible square integrable representation of $G_{n_{ij}}$ for each $1\le j\le p_i$.

Let σ be an irreducible generic representation of the form (2.1). We define the representation σ_1 of G_{n+1} as follows. Assume that $r_1>r_2>\cdots>r_j\ge 0>r_{j+1}>\cdots>r_k$. Let Q' be a standard upper triangular parabolic subgroup of G_{n+1} with Levi factor $G_{n_1}\times\cdots\times G_{n_j}\times G_1\times G_{n_{j+1}}\times\cdots\times G_{n_k}$. Then we set

$$\sigma_1=\mathrm{Ind}_{Q'}^{G_{n+1}}(\pi^1[r_1]\otimes\cdots\otimes\pi^j[r_j]\otimes 1\otimes\pi^{j+1}[r_{j+1}]\otimes\cdots\otimes\pi^k[r_k])\,. \tag{2.2}$$

This σ_1 has the following properties (cf. [9, Proposition 3.2] and [18, Theorem 4.2]).

LEMMA 1. (1) *σ_1 has a unique irreducible quotient representation.*

(2) *The representation $(\rho, W(\sigma_1, \psi_{n+1}))$ is isomorphic to σ_1 itself even if σ_1 is not irreducible.*

(3) *σ_1 is reducible if and only if there exists at least one $\pi^{i,j}[r_i]$ such that $\pi^{i,j}[r_i]=\mathrm{St}_{n_{ij}}[\pm(n_{ij}+1)/2]$, where $\mathrm{St}_{n_{ij}}$ denotes the Steinberg representation of $G_{n_{ij}}$* (cf. [4, Theorem 7.11]).

It is known by [2, Theorem 2.9] that the induced representation $\mathrm{Ind}_{Q_{n+1}}^{G_{n+1}}\sigma\otimes 1$ given in the Introduction has the same composition factors as that of σ_1. However, if $\mathrm{Ind}_{Q_{n+1}}^{G_{n+1}}\sigma\otimes 1$ is not irreducible, it does not always satisfy the properties (1) and (2) above, and, furthermore, we cannot apply [8, Proposition (9.4)] to this representation. ([8, Proposition (9.4)] will be used in Section 3, (3.3) below.) This is the reason why we define σ_1 not by $\mathrm{Ind}_{Q_{n+1}}^{G_{n+1}} \sigma \otimes 1$ but by (2.2).

The local Weil representation $(\omega_{n,m}, \mathscr{S}(M_{n,m}(F)))$ of $G_n \times G_m$ is defined as follows: for $f \in \mathscr{S}(M_{n,m}(F))$, $h \in G_n$ and $g \in G_m$,

$$\omega_{n,m}(h, g)f(x) = |\det h|_F^{-m/2} |\det g|_F^{n/2} f(h^{-1}xg) .$$

We write simply ω for $\omega_{n,n+1}$ and ω_1 for $\omega_{n,n}$. For $f \in \mathscr{S}(M_{n,n+1}(F))$, we define the function $\Phi(f)$ on G_n by

$$\Phi(f)(h) = \int_{U_{n+1}} \psi_{n+1}(u)^{-1} \omega(h, u) f(\varepsilon_n) du ,$$

where we put $\varepsilon_n = (1_n, 0) \in M_{n,n+1}(F)$. For each $W \in \boldsymbol{W}(\psi_n)$, we set

$$V_{(W,f)}(g) = \int_{U_n \backslash G_n} W(h) \Phi(\omega(g)f)(h) dh \qquad (g \in G_{n+1}) .$$

Since $\Phi(f)$ has compact support in G_n modulo U_n, the integral on the right-hand side reduces to a finite sum. Furthermore, as a function in $g \in G_{n+1}$, $V_{(W,f)}$ is contained in $\boldsymbol{W}(\psi_{n+1})$. Therefore we have a correspondence

$$\boldsymbol{W}(\psi_n) \times \mathscr{S}(M_{n,n+1}(F)) \to \boldsymbol{W}(\psi_{n+1}) ,$$

which satisfies the relation

$$\rho(g) V_{(\rho(h)W, f)} = V_{(W, \omega(h^{-1}, g)f)} \qquad (h \in G_n, g \in G_{n+1}) . \tag{2.3}$$

Let σ be an irreducible generic representation of G_n. We set

$$\boldsymbol{V}(\sigma, \psi_{n+1}) = \{ V_{(W,f)} \mid W \in \boldsymbol{W}(\sigma, \psi_n), \quad f \in \mathscr{S}(M_{n,n+1}(F)) \} .$$

Then the theta lift $\tilde{\sigma}$ of σ is defined to be a smooth subrepresentation $(\rho, \boldsymbol{V}(\sigma, \psi_{n+1}))$ of $\boldsymbol{W}(\psi_{n+1})$. It is known by [13, Chapter 3, Section III, Corollary 3] that $\tilde{\sigma}$ is of finite length. Thus, by [1, Theorem 4.1], $\tilde{\sigma}$ is admissible and finitely generated.

Let $\xi_n \in \mathscr{H}_n$ be the characteristic function of K_n. We define the action of ξ_n on $\boldsymbol{W}(\psi_{n+1})$ by

$$\rho(\xi_n) W_1(g) = \int_{K_n} W_1\left(g \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\right) dk \qquad (W_1 \in \boldsymbol{W}(\psi_{n+1}), g \in G_{n+1}) .$$

The main result of this paper is stated as follows.

THEOREM 1. *Let σ be an irreducible generic representation of G_n. Then the intersection $\rho(\xi_n)\boldsymbol{V}(\sigma, \psi_{n+1}) \cap \rho(\xi_n)\boldsymbol{W}(\sigma_1, \psi_{n+1})$ contains a nonzero element. Furthermore, if σ_1 is irreducible, we have*

$$\rho(\xi_n)\boldsymbol{V}(\sigma, \psi_{n+1}) = \rho(\xi_n)\boldsymbol{W}(\sigma_1, \psi_{n+1}) .$$

This theorem will be proved in Section 5. We note that $\rho(\xi_n)\boldsymbol{W}(\sigma_1, \psi_{n+1})$ has infinite dimension.

REMARK. When z is a complex number with $\mathrm{Re}(z) \ll 0$, the representation $\sigma[z]_1$ is just the irreducible representation $\mathrm{Ind}_{Q'}^{GL_{n+1}} 1 \otimes \sigma[z]$. Here Q' denotes the standard upper triangular parabolic subgroup with Levi factor $GL_1 \times GL_n$. Then, Jacquet, Piatetski-Shapiro and Shalika [8, Proposition (6.1)] essentially proved that, for any irreducible generic representation σ,

$$(2.4) \qquad \boldsymbol{W}(\sigma[z]_1, \psi_{n+1}) \subset \boldsymbol{V}(\sigma[z], \psi_{n+1})$$

if $\mathrm{Re}(z) \ll 0$. This fact is derived as follows. Let $\mathscr{S}'(M_{n,n+1}(F))$ be the subset consisting of functions $f \in \mathscr{S}(M_{n,n+1}(F))$ with $\mathrm{supp} f \subset \varepsilon_n G_{n+1}$. The subspace $\mathscr{S}'(M_{n,n+1}(F))$ is $\omega(G_n \times G_{n+1})$-invariant. For each $f \in \mathscr{S}'(M_{n,n+1}(F))$ and $W \in \boldsymbol{W}(\sigma, \psi_n)$, the integral

$$(2.5) \quad \phi_{(W,f)}(g\,; m) = |\det m|_F^z \int_{G_n} \omega(h, g) f(\varepsilon_n) W(mh) |\det h|_F^z dh \qquad (g \in G_{n+1}, m \in G_n)$$

is convergent, and as a function in $g \in G_{n+1}$, $\phi_{(W,f)}$ is an element of $\mathrm{Ind}_{Q''}^{G_{n+1}}(\sigma[z] \otimes 1)$ (cf. loc. cit. p. 430), where Q'' denotes the standard lower parabolic subgroup with Levi factor $GL_n \times GL_1$. We note that the representation $\mathrm{Ind}_{Q''}^{G_{n+1}}(\sigma[z] \otimes 1)$ is isomorphic to $\sigma[z]_1$. Since $\sigma[z]_1$ is irreducible, the correspondence $(W, f) \mapsto \phi_{(W,f)}$ is a surjection from $\boldsymbol{W}(\sigma, \psi_n) \times \mathscr{S}'(M_{n,n+1}(F))$ onto $\mathrm{Ind}_{Q''}^{G_{n+1}}(\sigma[z] \otimes 1)$. Furthermore, for $\phi = \phi_{(W,f)}$, the integral

$$W'_\phi(g) = \int_{M_{n,1}(F)} \phi_{(W,f)}\left(\begin{pmatrix} 1_n & x \\ 0 & 1 \end{pmatrix} g\,; 1_n\right) \psi_{n+1}\left(\begin{pmatrix} 1_n & x \\ 0 & 1 \end{pmatrix}\right)^{-1} dx$$

is absolutely convergent by the assumption $\mathrm{Re}(z) \ll 0$. Then the space $\{W'_\phi \mid \phi \in \mathrm{Ind}_{Q''}^{G_{n+1}}(\sigma[z] \otimes 1)\}$ gives a Whittaker model of $\sigma[z]_1$. By replacing $\phi_{(W,f)}$ by its expression (2.5) and changing order of integrations, we obtain $W'_\phi = V_{(W \otimes |\cdot|^z, f)}$. Therefore we have (2.4).

3. Some results of Jacquet, Piatetski-Shapiro and Shalika.

First, we recall class 1 Whittaker functions of G_n by Shintani [16]. For an n-tuple $\boldsymbol{k} = (k_1, \ldots, k_n) \in \boldsymbol{Z}^n$ of rational integers, we denote by $t_{\boldsymbol{k}}$ the diagonal matrix in T_n whose i-th diagonal entry is ϖ^{k_i} for $1 \le i \le n$. We set

$$\Lambda_n = \{\boldsymbol{k} \in \boldsymbol{Z}^n \mid k_1 \ge k_2 \ge \cdots \ge k_n\}.$$

Let $\boldsymbol{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_n, X_n^{-1}]$ be the Laurent polynomial ring in indeterminates $X_1, \ldots, X_n$ and Δ_n the subalgebra consisting of the elements in $\boldsymbol{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_n, X_n^{-1}]$ which are invariant by permutations of indeterminates. We define the function $W(\,\cdot\,; X_1, \ldots, X_n; \psi_n^{-1})$ on G_n with values in Δ_n as follows: for $u \in U_n$, $t_{\boldsymbol{k}} \in T_n$ and $k \in K_n$,

$$W(ut_k k;X_1,\dots,X_n;\psi_n^{-1})$$
$$=\psi_n(u)^{-1}\times\begin{cases}\delta_n(t_k)^{1/2}\prod_{1\le i<j\le n}(X_i-X_j)^{-1}\sum_{\tau\in S_n}\operatorname{sgn}\tau\prod_{i=1}^{n}X_{\tau(i)}^{k_i+n-i} & \text{if } k\in\Lambda_n\\ 0 & \text{if } k\notin\Lambda_n,\end{cases}$$

where δ_n denotes the modular character of B_n, i.e. $\delta_n(t_k)=\prod_{i=1}^n|\varpi|_F^{(n+1-2i)k_i}$. Then, for each n-tuple $z=(z_1,\dots,z_n)\in(\boldsymbol{C}^\times)^n$ of complex numbers, the class 1 Whittaker function W_z on G_n is given by a specialization of $W(\cdot;X_1,\dots,X_n;\psi_n^{-1})$ at $(z_1,\dots,z_n)$, i.e. $W_z(h)=W(h;z_1,\dots,z_n;\psi_n^{-1})$. We denote by $\boldsymbol{W}_z(\psi_n^{-1})$ the submodule of $\boldsymbol{W}(\psi_n^{-1})$ generated by W_z, i.e. $\boldsymbol{W}_z(\psi_n^{-1})=\rho(\mathscr{H}_n)W_z$.

For $z\in(\boldsymbol{C}^\times)^n$, we define an unramified character χ_z of B_n by

$$\chi_z(t_k t'u)=z_1^{k_1}z_2^{k_2}\cdots z_n^{k_n},\qquad(k\in\boldsymbol{Z}^n,\ t'\in T_n\cap K_n,\ u\in U_n).$$

Let $\operatorname{Ind}_{B_n}^{G_n}\chi_z$ be the representation of G_n induced from χ_z. Then $\operatorname{Ind}_{B_n}^{G_n}\chi_z$ is of Whittaker type (cf. [3]). We take a $\tau\in S_n$ so that $\tau(z)=(z_{\tau(1)},z_{\tau(2)},\dots,z_{\tau(n)})$ satisfies $|z_{\tau(1)}|\le|z_{\tau(2)}|\le\cdots\le|z_{\tau(n)}|$. It is known by the argument in the proof of [12, Theorem 2.2] that the space $\boldsymbol{W}(\operatorname{Ind}_{B_n}^{G_n}\chi_{\tau(z)},\psi_n^{-1})$ coincides with $\boldsymbol{W}_z(\psi_n^{-1})$. We denote the representation $(\rho,\boldsymbol{W}_z(\psi_n^{-1}))$ by π_z. Since π_z is isomorphic to a quotient representation of $\operatorname{Ind}_{B_n}^{G_n}\chi_{\tau(z)}$, it is also of Whittaker type.

Next, we recall results of Jacquet, Piatetski-Shapiro and Shalika, which will play an essential role in our proof of Theorem 1. We set

$$w_n=\begin{pmatrix}0&\cdots&1\\ \vdots&1&\vdots\\ 1&\cdots&0\end{pmatrix}\in G_n,\qquad\eta_n={}^t(0,\dots,0,1)\in M_{n,1}(F).$$

If $h\in G_n$, we denote by h^ι the inverse transpose of h, i.e. $h^\iota={}^th^{-1}$. Let σ be an admissible representation of G_n which is of Whittaker type. We define the representation σ^ι of G_n by $\sigma^\iota(h)=\sigma(h^\iota)$ for $h\in G_n$. For $W\in\boldsymbol{W}(\sigma,\psi_n)$, we also define the function $\tilde{W}$ on G_n by

$$\tilde{W}(h)=W(w_n h^\iota).$$

Then the set of $\tilde{W}$ with $W\in\boldsymbol{W}(\sigma,\psi_n)$ coincides with the space $\boldsymbol{W}(\sigma^\iota,\psi_n^{-1})$.

For the moment, we fix an irreducible generic representation σ of G_n and the representation σ_1 of G_{n+1} defined in Section 2. Let π be another irreducible generic representation of G_n. We denote by ω_π the central character of π. Furthermore, the local factor and the epsilon factor of π given by Godement and Jacquet [4, Theorem 3.3] is denoted by $L(s,\pi)$ and $\varepsilon(s,\pi,\psi)$, respectively. For $W\in\boldsymbol{W}(\sigma,\psi_n)$, $W'\in\boldsymbol{W}(\pi,\psi_n^{-1})$ and $\varphi\in\mathscr{S}(M_{n,1}(F))$, we set

$$\Psi(s,W,W';\varphi)=\int_{U_n\backslash G_n}W(h)W'(h)\varphi({}^th\eta_n)|\det h|_F^s dh.$$

In a similar fashion, for $W_1 \in W(\sigma_1, \psi_{n+1})$, $W' \in W(\pi, \psi_n^{-1})$, we set

$$\Psi(s, W_1, W') = \int_{U_n \backslash G_n} W_1\left(\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right) W'(h) |\det h|_F^{s-1/2} dh .$$

Then Jacquet, Piatetski-Shapiro and Shalika [8, Theorems (2.7), (3.1) and Proposition (9.4)] proved the following. Each of the integrals $\Psi(s, W, W'; \varphi)$ and $\Psi(s, W_1, W')$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$ and they are rational functions of q^{-s}. The integrals $\Psi(s, W, W'; \varphi)$ span a fractional ideal $C[q^s, q^{-s}]L(s, \sigma \times \pi)$ of the ring $C[q^s, q^{-s}]$. The factor $L(s, \sigma \times \pi)$ has the form

$$L(s, \sigma \times \pi) = P_{\sigma \times \pi}(q^{-s})^{-1}, \qquad P_{\sigma \times \pi} \in C[X], \quad P_{\sigma \times \pi}(0) = 1 . \tag{3.1}$$

Furthermore, there is a factor $\varepsilon(s, \sigma \times \pi, \psi)$ of the form cq^{-ms} such that

$$\frac{\Psi(1-s, \tilde{W}, \tilde{W}'; \hat{\varphi})}{L(1-s, \sigma^{\iota} \times \pi^{\iota})} = \omega_\pi(-1)^{n-1} \varepsilon(s, \sigma \times \pi, \psi) \frac{\Psi(s, W, W'; \varphi)}{L(s, \sigma \times \pi)} , \tag{3.2}$$

where $\hat{\varphi}$ denotes the Fourier transform of φ, that is,

$$\hat{\varphi}(x) = \int_{M_{n,1}(F)} \psi({}^t xy) \varphi(y) dy .$$

Similarly, the integrals $\Psi(s, W_1, W')$ span a fractional ideal $C[q^s, q^{-s}]L(s, \sigma_1 \times \pi)$. Here the factor $L(s, \sigma_1 \times \pi)$ has the form

$$L(s, \sigma_1 \times \pi) = L(s, \sigma \times \pi) L(s, \pi) . \tag{3.3}$$

There is a functional equation

$$\frac{\Psi(1-s, \tilde{W}_1, \tilde{W}')}{L(1-s, \sigma_1^{\iota} \times \pi^{\iota})} = \omega_\pi(-1)^n \varepsilon(s, \sigma \times \pi, \psi) \varepsilon(s, \pi, \psi) \frac{\Psi(s, W_1, W')}{L(s, \sigma_1 \times \pi)} . \tag{3.4}$$

We set

$$\gamma(s, \sigma_1 \times \pi, \psi) = \varepsilon(s, \sigma \times \pi, \psi) \varepsilon(s, \pi, \psi) \frac{L(1-s, \sigma_1^{\iota} \times \pi^{\iota})}{L(s, \sigma_1 \times \pi)} .$$

4. The gamma factor of $\tilde{\sigma} \times \pi$. We fix two irreducible generic representations σ and π of G_n. The purpose of this section is to calculate the integral

$$\Psi(s, V, W') = \int_{U_n \backslash G_n} V\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) W'(g) |\det g|_F^{s-1/2} dg$$

for $V \in V(\sigma, \psi_{n+1})$ and $W' \in W(\pi, \psi_n^{-1})$. Since we do not yet know whether $\tilde{\sigma}$ is of Whittaker type, we cannot apply the results of Jacquet, Piatetski-Shapiro and Shalika. However we can compute this integral directly in the same way as in [8, (6.3)]. In the following, we explain briefly this method. See [8, (6.3)] for details.

We may assume V to be the form $V=V_{(W,f_1\otimes f_2)}$, where $W\in \boldsymbol{W}(\sigma,\psi_n)$, $f_1\in\mathscr{S}(M_n(F))$ and $f_2\in\mathscr{S}(M_{n,1}(F))$. We set

$$\Phi_1(f_1)(h,g)=\int_{U_n}\psi_n(u)^{-1}\omega_1(h,g)f_1(u)du\,.$$

From easy calculation, it follows that

$$\Phi\left(\omega\left(\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}\right)(f_1\otimes f_2)\right)(h)=|\det h|_F^{1/2}\hat{f}_2({}^th\eta_n)\Phi_1(f_1)(h,g)\,. \tag{4.1}$$

We regard $\Psi(s,V,W')$ as a formal Laurent series in $X=q^{-s}$. Thus we write $\Psi(s,V,W')$ as

$$\sum_{m=-\infty}^{\infty}\Psi_m(V,W')X^m\,. \tag{4.2}$$

This Laurent series has only finitely many nonzero negative terms. Each coefficient $\Psi_m(V,W')$ is given by

$$\Psi_m(V,W')=\int_{U_n\backslash G_n^m}V\left(\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}\right)W'(g)|\det g|_F^{-1/2}dg\,,$$

where G_n^m denotes the set of $g\in G_n$ with $|\det g|_F=q^{-m}$. By (4.1), $\Psi_m(V,W')$ equals

$$\int_{U_n\backslash G_n^m}\left\{\int_{U_n\backslash G_n}W(h)\Phi_1(f_1)(h,g)\hat{f}_2({}^th\eta_n)|\det h|_F^{1/2}dh\right\}W'(g)|\det g|_F^{-1/2}dg$$

$$=\int_{U_n\backslash G_n^m}\left\{\int_{G_n}W(h)f_1(h^{-1}g)\hat{f}_2({}^th\eta_n)|\det h|_F^{1/2-n/2}dh\right\}W'(g)|\det g|_F^{n/2-1/2}dg$$

$$=\int_{U_n\backslash G_n^m}W'(g)\int_{G_n}W(gh^{-1})f_1(h)\hat{f}_2(h^{\prime t}g\eta_n)|\det h|_F^{n/2-1/2}dhdg\,.$$

This double integral is abolutely convergent. By changing g to gh, we obtain

$$\int_{U_n\backslash G_n}W(g)\hat{f}_2({}^tg\eta_n)\int_{G_n}\chi_m(gh)W'(gh)f_1(h)|\det h|_F^{n/2-1/2}dhdg\,, \tag{4.3}$$

where χ_m denotes the characteristic function of G_n^m. We take an open compact subgroup Ω of G_n such that ${}^t\Omega=\Omega$ and $f_1(\omega h)=f_1(h)$ for $\omega\in\Omega$. Let $\boldsymbol{W}(\pi,\psi_n^{-1})^\Omega$ be the subspace of $\boldsymbol{W}(\pi,\psi_n^{-1})$ consisting of all elements fixed by Ω and $\{W_1',\dots,W_p'\}$ a basis of $\boldsymbol{W}(\pi,\psi_n^{-1})^\Omega$. Then there exist matrix coefficients $\phi_1,\dots,\phi_p$ of π such that

$$\int_\Omega W'(g\omega h)d\omega=\sum_{1\le j\le p}W_j'(g)\phi_j(h)\,. \tag{4.4}$$

Thus we have

$$\int_{G_n} \chi_m(gh)W'(gh)f_1(h)|\det h|_F^{n/2-1/2}dh = \sum_{j=1}^{p} W'_j(g)\int_{G_n} \chi_m(gh)\phi_j(h)f_1(h)|\det h|_F^{n/2-1/2}dh\,.$$

Furthermore, we have

$$\chi_m(gh) = \sum_{m'+m''=m} \chi_{m'}(g)\chi_{m''}(h)\,, \tag{4.5}$$

where m' and m'' are bounded from below. Consequently, by (4.2), (4.3), (4.4) and (4.5), we obtain

$$\Psi(s, V_{(W, f_1\otimes f_2)}, W') = \sum_{j=1}^{p} \Psi(s, W, W'_j; \hat{f}_2)\int_{G_n} f_1(h)\phi_j(h)|\det h|_F^{s+n/2-1/2}dh\,. \tag{4.6}$$

Here we note that the integral

$$Z(f_1, s+n/2-1/2, \phi_j) = \int_{G_n} f_1(h)\phi_j(h)|\det h|_F^{s+n/2-1/2}dh$$

is a zeta-integral defined by Godement and Jacquet [4] and there is a functional equation

$$\frac{Z(\hat{f}_1, 1-s+n/2-1/2, \phi_j^{\iota})}{L(1-s, \pi^{\iota})} = \varepsilon(s, \pi, \psi)\frac{Z(f_1, s+n/2-1/2, \phi_j)}{L(s, \pi)}\,, \tag{4.7}$$

where ϕ_j^{ι} is a matrix coefficient of π^{ι} given by $\phi_j^{\iota}(h) = \phi_j(h^{\iota})$. Similarly, by using [8, Proposition 6.2], we obtain

(4.8)

$$\Psi(1-s, \tilde{V}_{(W, f_1\otimes f_2)}, \tilde{W}') = \sum_{j=1}^{p} \Psi(1-s, \tilde{W}, \tilde{W}'_j; f_2)\omega_\pi(-1)Z(\hat{f}_1, 1-s+n/2-1/2, \phi_j^{\iota})\,.$$

Therefore, by (4.6), (4.7), (4.8) and (3.2), we have

$$\frac{\Psi(1-s, \tilde{V}, \tilde{W}')}{L(1-s, \sigma^{\iota}\times\pi^{\iota})L(1-s, \pi^{\iota})} = \omega_\pi(-1)^n\varepsilon(s, \sigma\times\pi, \psi)\varepsilon(s, \pi, \psi)\frac{\Psi(s, V, W')}{L(s, \sigma\times\pi)L(s, \pi)} \tag{4.9}$$

as a formal Laurent series. However, (4.9) itself implies that both sides are polynomials in (X, X^{-1}) (cf. [8, (4.4)]). Thus (4.9) may be regarded as an identity of analytic functions. As a consequence, we obtain the following result.

PROPOSITION 2. *For each* $V \in \mathcal{V}(\sigma, \psi_{n+1})$ *and* $W' \in \mathcal{W}(\pi, \psi_n^{-1})$, *the integral* $\Psi(s, V, W')$ *is absolutely convergent for* $\mathrm{Re}(s) \gg 0$ *and* $L(s, \sigma_1\times\pi)^{-1}\Psi(s, V, W')$ *is an element of* $\mathbf{C}[q^s, q^{-s}]$. *Moreover, there is an equation*

$$\Psi(1-s, \tilde{V}, \tilde{W}') = \omega_\pi(-1)^n\gamma(s, \sigma_1\times\pi, \psi)\Psi(s, V, W')\,.$$

If π is a spherical representation π_z, then we can prove the assertion of Proposition

2 in another way. Namely, we can calculate the integral $\Psi(s, V, W_z)$ more directly. Since the integral $\Psi(s, V, W_z)$ will be used in Section 5, we explain this calculation in the rest of this section. We may assume again that V is of the form $V_{(W, f_1 \otimes f_2)}$. We set

$$J(s, f_1, W_z)(h) = \int_{U_n \backslash G_n} \Phi_1(f_1)(h, g) W_z(g) |\det g|_F^{s-1/2} dg .$$

By (4.1), we have formally

$$\Psi(s, V_{(W, f_1 \otimes f_2)}, W_z) = \int_{U_n \backslash G_n} W(h) J(s, f_1, W_z)(h) \hat{f}_2({}^t h \eta_n) |\det h|_F^{1/2} dh . \tag{4.10}$$

Let $f_1^0 \in \mathcal{S}(M_n(F))$ be the characteristic function of $M_n(\mathcal{O})$. Then $\omega_1(1_n, \xi_n)$ is a projection from $\mathcal{S}(M_n(F))$ to the subspace $\mathcal{S}(M_n(F))^{\omega_1(1_n, K_n)}$ consisting of functions invariant by $\omega_1(1_n, K_n)$. By Howe [5, Theorem 10.2], the space $\mathcal{S}(M_n(F))^{\omega_1(1_n, K_n)}$ coincides with the space $\omega_1(\mathcal{H}_n, 1_n) f_1^0$. Thus, corresponding to f_1, there exists $\varphi_1 \in \mathcal{H}_n$ such that

$$\omega_1(1_n, \xi_n) f_1 = \omega_1(\varphi_1, 1_n) f_1^0 . \tag{4.11}$$

Then we have

$$J(s, f_1, W_z) = J(s, f_1, \rho(\xi_n) W_z) = J(s, \omega_1(\varphi_1, 1_n) f_1^0, W_z) = \rho(\varphi_1) J(s, f_1^0, W_z) .$$

We compute the integral $J(s, f_1^0, W_z)(h)$. The next lemma follows from simple calculation.

LEMMA 2. *Let* $\boldsymbol{k} = (k_1, \dots, k_n) \in \boldsymbol{Z}^n$ *and* $\boldsymbol{p} = (p_1, \dots, p_n) \in \boldsymbol{Z}^n$. *If* $p_1 \geq k_1 \geq p_2 \geq k_2 \geq \cdots \geq p_n \geq k_n$, *then we have*

$$\Phi_1(f_1^0)(t_{\boldsymbol{k}}, t_{\boldsymbol{p}}) = |\det t_{\boldsymbol{k}}|_F^{-1/2} \delta_n(t_{\boldsymbol{k}})^{1/2} |\det t_{\boldsymbol{p}}|_F^{1/2} \delta_n(t_{\boldsymbol{p}})^{1/2} .$$

Otherwise, $\Phi_1(f_1^0)(t_{\boldsymbol{k}}, t_{\boldsymbol{p}})$ *is zero.*

LEMMA 3. *The integral* $J(s, f_1^0, W_z)$ *absolutely converges if* $\mathrm{Re}(s)$ *is sufficiently large, and we have*

$$J(s, f_1^0, W_z)(h) = \left(\prod_{i=1}^{n} (1 - q^{-s} z_i) \right)^{-1} |\det h|_F^{s-1/2} W_z(h) .$$

PROOF. By Lemma 2 and an explicit formula for W_z, $J(s, f_1^0, W_z)(t_{\boldsymbol{k}})$ equals

$$\sum_{\boldsymbol{p} \in \Lambda_n} \Phi_1(f_1^0)(t_{\boldsymbol{k}}, t_{\boldsymbol{p}}) W_z(t_{\boldsymbol{p}}) |\det t_{\boldsymbol{p}}|_F^{s-1/2} \delta_n(t_{\boldsymbol{p}})^{-1}$$

$$= |\det t_{\boldsymbol{k}}|_F^{-1/2} \delta_n(t_{\boldsymbol{k}})^{1/2} \sum_{\substack{\boldsymbol{p} \in \Lambda_n \\ k_{i-1} \geq p_i \geq k_i \, (1 \leq i \leq n)}} |\det t_{\boldsymbol{p}}|_F^{s} W_z(t_{\boldsymbol{p}}) \delta_n(t_{\boldsymbol{p}})^{-1/2}$$

$$= |\det t_{\boldsymbol{k}}|_F^{-1/2} \delta_n(t_{\boldsymbol{k}})^{1/2} \left(\prod_{1 \leq i < j \leq n} (z_i - z_j) \right)^{-1}$$

$$\times \sum_{\tau\in S_n} \operatorname{sgn}\tau \prod_{i=1}^{n} z_{\tau(i)}^{n-i}\left\{\sum_{k_{i-1}\geq p_i\geq k_i} (q^{-s}z_{\tau(i)})^{p_i}\right\}.$$

Here we put $k_0=+\infty$ for convenience. The sum of $(q^{-s}z_{\tau(i)})^{p_1}$ over $+\infty>p_1\geq k_1$ absolutely converges if $\mathrm{Re}(s)>\max_{1\leq i\leq n}(\log_q|z_i|)$. Then, by calculation of determinants, we obtain

$$\sum_{\tau\in S_n} \operatorname{sgn}\tau \prod_{i=1}^{n} z_{\tau(i)}^{n-i}\left\{\sum_{k_{i-1}\geq p_i\geq k_i} (q^{-s}z_{\tau(i)})^{p_i}\right\}=\left(\prod_{i=1}^{n}(1-q^{-s}z_i)\right)^{-1}\sum_{\tau\in S_n}\operatorname{sgn}\tau\prod_{i=1}^{n} z_{\tau(i)}^{n-i}(q^{-s}z_{\tau(i)})^{k_i}.$$

Thus implies the assertion.

For $\varphi_1\in\mathscr{H}_n$ satisfying (4.11), we define $\varphi_1^s\in\mathscr{H}_n$ by

$$\varphi_1^s(h)=|\det h|_F^{s-1/2}\varphi_1(h)\,. \tag{4.12}$$

By Lemma 3 and (4.10), we have a relation

$$\Psi(s, V_{(W,f_1\otimes f_2)}, W_z)=\left(\prod_{i=1}^{n}(1-q^{-s}z_i)\right)^{-1}\Psi(s, W, \rho(\varphi_1^s)W_z\,;\hat{f}_2)\,. \tag{4.13}$$

Next, we compute the integral

$$\Psi(1-s, \tilde{V}_{(W,f_1\otimes f_2)}, \tilde{W}_z)=\int_{U_n\backslash G_n} V_{(W,f_1\otimes f_2)}\left(w_{n+1}\begin{pmatrix} g^{\iota} & 0\\ 0 & 1\end{pmatrix}\right)W_z(w_n g^{\iota})|\det g|_F^{1/2-s}dg\,.$$

By changing g to $w_n g^{\iota} w_n$, this integral equals

$$\begin{aligned}
&\int_{U_n\backslash G_n} V_{(W,f_1\otimes f_2)}\left(w_{n+1}\begin{pmatrix} w_n & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}\right)W_z(g)|\det g|_F^{s-1/2}dg\\
&=\int_{U_n\backslash G_n}\left\{\int_{U_n\backslash G_n} W(h)\Phi\left(\omega\left(\begin{pmatrix} 0 & 1\\ g & 0\end{pmatrix}\right)(f_1\otimes f_2)\right)(h)dh\right\}W_z(g)|\det g|_F^{s-1/2}dg\\
&=\int_{U_n\backslash G_n} W(h)\left\{\int_{U_n\backslash G_n}\Phi\left(\omega\left(\begin{pmatrix} 0 & 1\\ g & 0\end{pmatrix}\right)(f_1\otimes f_2)\right)(h)W_z(g)|\det g|_F^{s-1/2}dg\right\}dh\,.
\end{aligned}$$

For $u\in U_{n+1}$, we denote by u_1 the $n\times n$ matrix obtained by eliminating the first column vector and the $(n+1)$-st row vector from u. Then

$$\begin{aligned}
&\Phi\left(\omega\left(\begin{pmatrix} 0 & 1\\ g & 0\end{pmatrix}\right)(f_1\otimes f_2)\right)(h)\\
&\qquad=|\det h|_F^{-1/2}f_2\left(h^{-1}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}\right)\int_{U_{n+1}}\psi_{n+1}(u)^{-1}\omega_1(h,g)f_1(u_1)du\,.
\end{aligned}$$

We take $\varphi_1^s \in \mathscr{H}_n$ to be the same as (4.12) for given $f_1 \in \mathscr{S}(M_n(F))$. It follows from calculation similar to that in the proof of Lemma 3 that

$$\int_{U_n \backslash G_n} \left\{ \int_{U_{n+1}} \psi_{n+1}(u)^{-1} \omega_1(h, g) f_1(u_1) du \right\} W_z(g) |\det g|_F^{s-1/2} dg$$
$$= \left(\prod_{i=1}^{n} \frac{-q^{1-s} z_i}{1-q^{1-s} z_i} \right) |\det h|_F^{s-1/2} \rho(\varphi_1^s) W_z(h),$$

and hence

$$\Psi(1-s, \tilde{V}_{(W, f_1 \otimes f_2)}, \tilde{W}_z)$$
$$= \left(\prod_{i=1}^{n} \frac{-q^{1-s} z_i}{1-q^{1-s} z_i} \right) \int_{U_n \backslash G_n} W(h) \rho(\varphi_1^s) W_z(h) f_2 \left(h^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) |\det h|_F^{s-1} dh .$$

By the change of variable $h \mapsto w_n h^\iota w_n$, we obtain

$$\Psi(1-s, \tilde{V}_{(W, f_1 \otimes f_2)}, \tilde{W}_z) = \left(\prod_{i=1}^{n} \frac{-q^{1-s} z_i}{1-q^{1-s} z_i} \right) \Psi(1-s, \tilde{W}, \widetilde{\rho(\varphi_1^s) W_z}; f_2) . \tag{4.14}$$

Therefore, by (4.13), (4.14) and (3.2), we have

$$\frac{\Psi(1-s, \tilde{V}_{(W, f_1 \otimes f_2)}, \tilde{W}_z)}{L(1-s, \sigma_1^\iota \times \pi_z^\iota)} = \varepsilon(s, \sigma \times \pi_z, \psi) \frac{\Psi(s, V_{(W, f_1 \otimes f_2)}, W_z)}{L(s, \sigma_1 \times \pi_z)} . \tag{4.15}$$

5. Proof of Theorem 1. Let σ be an irreducible generic representation of G_n. We set $\boldsymbol{U} = \boldsymbol{V}(\sigma, \psi_{n+1}) + \boldsymbol{W}(\sigma_1, \psi_{n+1})$. By (3.4) and Proposition 2, each $V \in \boldsymbol{U}$ satisfies the functional equations

$$\Psi(1-s, \tilde{V}, \tilde{W}') = \omega_\pi(-1)^n \gamma(s, \sigma_1 \times \pi, \psi) \Psi(s, V, W') , \tag{5.1}$$

for all irreducible generic representations π of G_n and $W' \in \boldsymbol{W}(\pi, \psi_n^{-1})$.

LEMMA 4. *Let R be the restriction map $V \mapsto V|_{P_{n+1}}$ from $\boldsymbol{U}$ to $\mathrm{Ind}_{U_{n+1}}^{P_{n+1}} \psi_{n+1}$. Then R is injective.*

PROOF. We denote by $\tilde{\boldsymbol{U}}$ the space of functions $\tilde{V}$ with $V \in \boldsymbol{U}$ and define the P_{n+1}-morphism $\tilde{R}: \tilde{\boldsymbol{U}} \to \mathrm{Ind}_{U_{n+1}}^{P_{n+1}} \psi_{n+1}^{-1}$ by the restriction $\tilde{V} \mapsto \tilde{V}|_{P_{n+1}}$. If $V \in \mathrm{Ker}\, R$, then we have $\Psi(s, V, W') = 0$, and hence, by (5.1), $\Psi(1-s, \tilde{V}, \tilde{W}') = 0$ for all irreducible generic representations π of G_n and $W' \in \boldsymbol{W}(\pi, \psi_n^{-1})$. Then, by [10, Lemma (3.2)] (cf. [7, Lemma (3.5)]), we have $\tilde{V} \in \mathrm{Ker}\, \tilde{R}$. Similarly, if $\tilde{V} \in \mathrm{Ker}\, \tilde{R}$, then $V \in \mathrm{Ker}\, R$. Therefore, $V \in \mathrm{Ker}\, R$ is equivalent to $\tilde{V} \in \mathrm{Ker}\, \tilde{R}$. Let $V \in \mathrm{Ker}\, R$ and $p^\iota = {}^t p^{-1} \in P_{n+1}^\iota$. Since $\widetilde{\rho(p^\iota) V} = \rho(p) \tilde{V}$ and $\mathrm{Ker}\, \tilde{R}$ is P_{n+1}-invariant, we have $\widetilde{\rho(p^\iota) V} \in \mathrm{Ker}\, \tilde{R}$, and hence $\rho(p^\iota) V \in \mathrm{Ker}\, R$. As a result, $\mathrm{Ker}\, R$ is P_{n+1}^ι-invariant. Since the action of the center Z_{n+1} on $\boldsymbol{U}$ is through the scalar multiplication by the central character of σ, $\operatorname{Ker} R$ is both Q_{n+1}- and Q_{n+1}^{ι}-invariant. Consequently, $\operatorname{Ker} R$ is G_{n+1}-invariant. Thus we have $V(g)=\rho(g)V(1_n)=0$ for $V\in\operatorname{Ker} R$. ∎

Next we consider the integral $\Psi(s, V, W_z)$ for $V\in\rho(\xi_n)U$. By (3.4) and (4.15), there is an equation

$$\frac{\Psi(1-s, \tilde{V}, \tilde{W}_z)}{L(1-s, \sigma_1^{\iota}\times\pi_z^{\iota})}=\varepsilon(s, \sigma\times\pi_z, \psi)\frac{\Psi(s, V, W_z)}{L(s, \sigma_1\times\pi_z)} \tag{5.2}$$

for each $V\in\rho(\xi_n)U$ and $z\in(\boldsymbol{C}^\times)^n$. We replace the parameter $(z_1, \dots, z_n)$ by indeterminates $X_1, \dots, X_n$. Namely, we consider the "integral"

$$\Psi(X, V, X_1, \dots, X_n)=\int_{U_n\backslash G_n} V\left(\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}\right)W(g, X_1, \dots, X_n; \psi_n^{-1})|\det g|_F^{s-1/2}dg\,,$$

where we put $X=q^{-s}$. This $\Psi(X, V, X_1, \dots, X_n)$ is regarded as an element in the ring $\Delta_n[[X, X^{-1}]]$ of formal Laurent series with coefficients in Δ_n. Then (5.2) and the argument in the proof of [7, Theorem 4.1] implies that each $V\in\rho(\xi_n)U$ satisfies the equation

$$\begin{aligned}\Psi(q^{-1}X^{-1}, \tilde{V}, X_1^{-1}, \dots, X_n^{-1})\prod_{i=1}^{n} P_{\sigma^{\iota}}(q^{-1}X^{-1}X_i^{-1})(1-q^{-1}X^{-1}X_i^{-1})\\ =\prod_{i=1}^{n}\varepsilon_\sigma(XX_i, \psi)\Psi(X, V, X_1, \dots, X_n)\prod_{i=1}^{n}P_\sigma(XX_i)(1-XX_i)\,.\end{aligned} \tag{5.3}$$

Here polynomials $P_\sigma(X)$ and $\varepsilon_\sigma(X, \psi)$ are given by

$$L(s, \sigma)=P_\sigma(q^{-s})^{-1}\,, \qquad \varepsilon(s, \sigma, \psi)=\varepsilon_\sigma(q^{-s}, \psi)\,.$$

From (5.3) and the fact that $\Psi(X, V, X_1, \dots, X_n)$ has a finite number of nonzero terms with negative exponents in X (cf. [7, Section 3]), it follows that $\Psi(X, V, X_1, \dots, X_n)$ is contained in the polynomial ring $\Delta_n[X, X^{-1}]$ and there exists an element $\Xi(V, X_1, \dots, X_n)\in\Delta_n$ such that

$$\Xi(V, XX_1, \dots, XX_n)=\Psi(X, V, X_1, \dots, X_n)\prod_{i=1}^{n}P_\sigma(XX_i)(1-XX_i)\,. \tag{5.4}$$

LEMMA 5. *Let $V\in\rho(\xi_n)U$. If $\Xi(V, X_1, \dots, X_n)=0$, then we have $V=0$.*

PROOF. If $\Xi(V, X_1, \dots, X_n)=0$, then $\Psi(X, V, X_1, \dots, X_n)=0$. By [7, Lemma (3.5)], we have $R(V)=0$. Therefore, by Lemma 4, we have $V=0$. ∎

PROOF OF THEOREM 1. We denote by $I(\tilde{\sigma})$ (resp. $I(\sigma_1)$) the subset of Δ_n consisting of $\Xi(V, X_1, \dots, X_n)$ with $V\in\rho(\xi_n)\boldsymbol{V}(\sigma, \psi_{n+1})$ (resp. $V\in\rho(\xi_n)\boldsymbol{W}(\sigma_1, \psi_{n+1})$). Then, by the same argument as in the proof of [7, Theorem (4.1)], both $I(\tilde{\sigma})$ and $I(\sigma_1)$ are ideals of Δ_n and there exist elements $V_1\in\rho(\xi_n)\boldsymbol{V}(\sigma, \psi_{n+1})$ and $W_1\in\rho(\xi_n)\boldsymbol{W}(\sigma_1, \psi_{n+1})$ such that

$$\Xi(V_1, X_1, \dots, X_n) = \Xi(W_1, X_1, \dots, X_n) = \prod_{i=1}^{n} P_\sigma(X_i)(1 - X_i) .$$

By Lemma 5, we have $V_1 = W_1$. Thus $\rho(\xi_n)V(\sigma, \psi_{n+1})$ and $\rho(\xi_n)W(\sigma_1, \psi_{n+1})$ have a nonzero intersection. Furthermore, if σ_1 is irreducible, then we have

$$I(\tilde{\sigma}) = I(\sigma_1) = \Delta_n$$

(cf. [7, Theorem (4.1)]). This implies $\rho(\xi_n)V(\sigma, \psi_{n+1}) = \rho(\xi_n)W(\sigma, \psi_{n+1})$. We complete the proof of Theorem 1.

We note that if σ_1 is reducible, then the assertion analogous to [7, Proposition (2.1)] for σ_1 is false. Thus we cannot conclude that $I(\sigma_1) = \Delta_n$ in this case.

PROPOSITION 3. *Let σ be an irreducible generic representation of G_n. Then $\tilde{\sigma}$ never has a supercuspidal subquotient representation.*

PROOF. Suppose $\tilde{\sigma}$ has an irreducible supercuspidal subquotient $\tilde{\sigma}_c$. Then, by [1, Proposition 3.30], $\tilde{\sigma}_c$ is realized as a subrepresentation of $\tilde{\sigma}$. The representation space $V(\sigma, \psi_{n+1})_c$ of $\tilde{\sigma}_c$ in $V(\sigma, \psi_{n+1})$ is a Whittaker model of $\tilde{\sigma}_c$. We set

$$I(\tilde{\sigma}_c) = \{\Xi(V, X_1, \dots, X_n) \mid V \in \rho(\xi_n)V(\sigma, \psi_{n+1})_c\} .$$

Then, by the proof of [7, Theorem (4.1)], we have $I(\tilde{\sigma}_c) = \Delta_n$, and hence $I(\sigma_1) \subset I(\tilde{\sigma}_c)$. This implies $\rho(\mathscr{H}_{n+1})\rho(\xi_n)W(\sigma_1, \psi_{n+1}) = V(\tilde{\sigma}, \psi_{n+1})_c$. Therefore, $\tilde{\sigma}_c$ is realized as a subquotient of σ_1. This is a contradiction. ■

PROPOSITION 4. *Let σ be an irreducible generic representation of G_n. If σ_1 is irreducible, then $\tilde{\sigma}$ is an admissible representation of Whittaker type and σ_1 is a unique irreducible subrepresentation of $\tilde{\sigma}$.*

PROOF. We note that any irreducible generic representation of G_{n+1} has a nonzero vector fixed by $\rho(\xi_n)$. If V' is an irreducible submodule of $V(\sigma, \psi_{n+1})$, then we have $\rho(\xi_n)W(\sigma_1, \psi_{n+1}) \supset \rho(\xi_n)V' \neq 0$, and hence $W(\sigma_1, \psi_{n+1}) = V'$. We prove that the dimension of $\mathrm{Hom}_{G_{n+1}}(V(\sigma, \psi_{n+1}), W(\psi_{n+1}))$ equals 1. Let L_0 be the natural injection of $V(\sigma, \psi_{n+1})$ to $W(\psi_{n+1})$ and $L \in \mathrm{Hom}_{G_{n+1}}(V(\sigma, \psi_{n+1}), W(\psi_{n+1}))$ an arbitrary nonzero element. If $\mathrm{Ker}\, L$ is nonzero, then $W(\sigma_1, \psi_{n+1}) \subset \mathrm{Ker}\, L$ and $V(\sigma, \psi_{n+1})/W(\sigma_1, \psi_{n+1})$ contains a nonzero generic irreducible subquotient. Therefore, $V(\sigma, \psi_{n+1})/W(\sigma_1, \psi_{n+1})$ has a nonzero vector fixed by $\rho(\xi_n)$. This contradicts $\rho(\xi_n)(V(\sigma, \psi_{n+1})/W(\sigma_1, \psi_{n+1})) = 0$. Thus L must be injective. Then there exists a constant c such that $L|_{W(\sigma_1, \psi_{n+1})} = cL_0|_{W(\sigma_1, \psi_{n+1})}$. Since $W(\sigma_1, \psi_{n+1}) \subset \mathrm{Ker}(L - cL_0)$, we obtain $L - cL_0 = 0$. ■

It is expected that $\tilde{\sigma} = \sigma_1$ for any irreducible generic representation σ. In fact, this is the case if σ is an irreducible generic spherical representation. Namely, we have the following:

PROPOSITION 5. *Let $z = (z_1, \dots, z_n) \in (\boldsymbol{C}^\times)^n$ and $(z, 1) = (z_1, \dots, z_n, 1) \in (\boldsymbol{C}^\times)^{n+1}$. If the spherical representation π_z is irreducible, then $\tilde{\pi}_z = \pi_{(z,1)}$.*

PROOF. Since $W_z(\psi_n^{-1}) \cong W_z(\psi_n)$, we may substitute ψ_n^{-1} by ψ_n in the definitions of W_z and π_z. Let $f^0 \in \mathcal{S}(M_{n,n+1}(F))$ be the characteristic function of $M_{n,n+1}(\mathcal{O})$. Then, from (2.4) and the fact that $\omega(\xi_n)\mathcal{S}(M_{n,n+1}(F)) = \omega(\mathcal{H}_{n+1})f^0$ (cf. [5, Theorem 10.2]), it follows that $V(\pi_z, \psi_{n+1})$ is generated by $V_{(W_z, f^0)}$. By calculation similar to that in the proof of Lemma 3, we obtain $V_{(W_z, f^0)} = W_{(z,1)}$. Therefore, $V(\pi_z, \psi_{n+1})$ coincides with $W_{(z,1)}(\psi_{n+1})$. ■

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