Tôhoku Math. J.
47 (1995), 509–519

GRADIENT BOUNDS AND LIOUVILLE'S TYPE THEOREMS FOR THE POISSON EQUATION ON COMPLETE RIEMANNIAN MANIFOLDS

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(Received May 9, 1994, revised November 18, 1994)

Abstract. We prove a gradient estimate and Liouville type theorems for the solutions of the Poisson equation on a complete manifold whose Ricci curvature is suitably restricted.

1. Introduction and results. Throughout this paper M will denote a complete, connected, non-compact Riemannian manifold of dimension $m \geq 2$. Our main aim is to establish various a priori estimates for the gradient of solutions to the Poisson equation $\Delta u = f(u)$ on M under suitable assumptions on the Ricci curvature (unless otherwise specified, the function f will be assumed to be of class C^1). Our first result is:

THEOREM 1. *Let $F \in C^2(\boldsymbol{R})$ be a function such that*

$$\text{(1)} \qquad \text{(i)} \quad \inf_{\boldsymbol{R}} F = 0 \qquad \text{(ii)} \quad F'(u) = f(u)\,.$$

Let u be a bounded solution of

$$\text{(2)} \qquad \Delta u = f(u) \qquad \text{on} \quad M$$

and assume that $\mathrm{Ricci}(M) \geq 0$. *Then*

$$\text{(3)} \qquad |\nabla u|^2(x) \leq 2F(u(x)) \qquad \text{for all} \quad x \in M\,.$$

COROLLARY 1. *Under the assumptions of Theorem* 1, *suppose that there exists $x_0 \in M$ such that $F(u(x_0)) = 0$. Then u is constant.*

The special case where M is $\boldsymbol{R}^m$ with its Euclidean metric is due to Modica [7]. One of the difficulties to recover Modica's theorem in our non-flat context is to prove that bounded solutions to (2) have bounded gradient. Towards this end, we use a method inspired by the old work of Ahlfors [1]: basically, we obtain estimates by studying the inequality $\Delta G \leq 0$ which holds at any relative maximum of G, where G is a suitable function of u, $|\nabla u|^2$ and r, the distance function from a base point. More generally, our analysis leads us to the following gradient estimate which should prove useful in other contexts:

1991 *Mathematics Subject Classification*. Primary 58G03.

PROPOSITION 1. *Suppose that* Ricci$(M) \geq -A$, *where* A *is a nonnegative constant. Let* u *be a bounded solution of the Poisson equation* (0.2). *Then* $|\nabla u|$ *is bounded on* M.

An important tool in our analysis is the Weitzenböck formula; however, when we apply it to the study of (2) on manifolds whose Ricci curvature is allowed to be negative we must require a strong convexity assumption on F (see (4) below). That is not surprising, at least because we know through work of Serrin [11] that the convexity of F implies that bounded solutions of (2) on $\boldsymbol{R}^m$ are constant. Denoting by $B_a(p)$ the geodesic ball of radius a centered at a point $p \in M$, our results are:

THEOREM 2. *Suppose that* Ricci$(M) \geq -A$, $A \geq 0$. *Let* u *be a solution of* (2) *such that*

$$f'(u) \geq A \qquad on \quad M. \tag{4}$$

We set $N(a) = \mathrm{Sup}\{|u|\}$ *on* $B_a(p)$, *and require that*

$$\liminf_{a \to +\infty} \frac{(N(a))^2(1+aA)}{a^2} = 0. \tag{5}$$

Thus u *is constant.*

THEOREM 3. *Suppose that* Ricci$(M) \geq -A$, $A \geq 0$. *Let* u *be a solution of* (2) *which verifies* (4) *and such that*

$$\text{(i)} \quad |\nabla u| \text{ is bounded on } M \qquad \text{(ii)} \quad \operatorname*{Inf}_M \{|\nabla u|\} = 0. \tag{6}$$

Thus u *is constant.*

REMARKS 1. (i) Because of Proposition 1 above, the assumptions (6) are automatically satisfied if u is bounded.

(ii) Theorem 2 includes as a special case the well-known fact that harmonic functions with sublinear growth on complete manifolds with nonnegative Ricci curvature are constant (see [12]).

Our techniques can also be adapted to estimate the rate of decay of ground states, i.e. positive solutions which tend to zero as the distance function $r(x)$ from a base point increases to $+\infty$ (see [8], [9], for instance). To illustrate this more precisely, we state

PROPOSITION 2. *Suppose that* Ricci$(M) \geq 0$. *Let* u *be a ground state for the equation*

$$\Delta u = u^q - \lambda u \qquad on \quad M \quad (q > 1, \lambda > 0) \tag{7}$$

such that

$$0 < u < \lambda^{1/(q-1)} \qquad on \quad M. \tag{8}$$

Then

$$\liminf_{a\to+\infty} T(a)a < +\infty\,, \tag{9}$$

where $T(a)=\mathrm{Inf}\{u^{(q-1)}\}$ *on* $B_a(p)$.

We mention here that methods related to those of the present paper have been used to derive a priori estimates in other geometric problems: see [3], [4] and [10], for instance.

ACKNOWLEDGEMENTS. This article is a revised version of an unpublished manuscript "Gradient estimates for the Poisson equation on complete Riemannian manifolds" which was written in October 1990, while the two authors were at the university of Warwick. They wish to express here their thanks to this institution for hospitality.

1. Proof of the results.

PROOF OF THEOREM 2, PROPOSITIONS 1 AND 2. Step 1. We denote by $B_a(p)$ the geodesic ball of radius a centered at a point $p\in M$, and by r the distance function from p. On $B_a(p)$ we consider the function

$$G=(a^2-r^2)^2|\nabla u|^2 g(u)\,, \tag{10}$$

where u is a solution of the Poisson equation $\Delta u=f(u)$ and g is a positive differentiable function to be chosen later. If there exists a positive maximum $q\in B_a(p)$ of G, then at q we must have

$$\text{(i)}\quad \nabla(\log G)=0 \quad\text{and}\quad \text{(ii)}\quad \Delta(\log G)\le 0\,. \tag{11}$$

(Note that, using a trick of Calabi (see [2] or [3]), we can assume that r is C^2 in a neighborhood of q). In the following lemma we compute explicitly (11) (ii) and derive two inequalities which will play a key role in the proof of our theorems.

LEMMA 12. *Suppose that* $\mathrm{Ricci}(M)\ge -A$, $A\ge 0$. *If* $q\in B_a(p)$ *is a positive maximum of the function G in* (10), *then the following two inequalities hold at* q:

$$\begin{aligned} 0\ge -&\left\{\frac{2C(1+Aa)}{(a^2-r^2)}+\frac{16a^2}{(a^2-r^2)^2}-2f'(u)+2A-\frac{g'(u)f(u)}{g(u)}\right\}\\ &-\left\{\frac{4a|g'(u)|}{(a^2-r^2)|g(u)|}\right\}|\nabla u|+\left\{\frac{2g(u)g''(u)-3g'^2(u)}{2g^2(u)}\right\}|\nabla u|^2 \end{aligned} \tag{13}$$

and

$$(14)\qquad 0\geq -\left\{\frac{2C(1+Aa)}{(a^2-r^2)}+\frac{24a^2}{(a^2-r^2)^2}-2f'(u)+2A\right\}$$
$$-\left\{\frac{8a|g'(u)|}{(a^2-r^2)|g(u)|}\right\}|\nabla u|+\left\{\frac{8g(u)g''(u)-(16+m)g'^2(u)}{8g^2(u)}\right\}|\nabla u|^2,$$

where $m=\dim M$ and C is a positive constant which depends only on M.

PROOF OF LEMMA 12. From the definition of G and computing we see that (11) (i) is equivalent to

$$(15)\qquad \frac{g'(u)\nabla u}{g(u)}+\frac{\nabla|\nabla u|^2}{|\nabla u|^2}-\frac{2\nabla r^2}{(a^2-r^2)}=0\,.$$

We recall that, for any differentiable function ψ, we have

$$\operatorname{div}[\psi(u)\nabla u]=\psi(u)\Delta u+\psi'(u)|\nabla u|^2\,.$$

Then a simple computation shows that (11) (ii) takes the form

$$(16)\qquad 0\geq -\frac{2\Delta r^2}{(a^2-r^2)}-\frac{2|\nabla r^2|^2}{(a^2-r^2)^2}+\frac{g'(u)\Delta u}{g(u)}+\left\{\frac{g(u)g''(u)-(g')^2(u)}{g^2(u)}\right\}|\nabla u|^2$$
$$+\frac{\Delta|\nabla u|^2}{|\nabla u|^2}-\frac{|\nabla|\nabla u|^2|^2}{|\nabla u|^4}\,.$$

Now, from the Weitzenböck formula

$$(17)\qquad \Delta|\nabla u|^2=2|\operatorname{Hess}(u)|^2+2\operatorname{Ricci}(M)(\nabla u,\nabla u)+2\langle\nabla\Delta u,\nabla u\rangle$$

and the assumption $\operatorname{Ricci}(M)\geq -A$, together with $\Delta u=f(u)$, we deduce

$$(18)\qquad \Delta|\nabla u|^2\geq 2|\operatorname{Hess}(u)|^2-2A|\nabla u|^2+2f'(u)|\nabla u|^2\,.$$

On the other hand, the Schwartz inequality immediately gives

$$(19)\qquad |\nabla|\nabla u|^2|^2\leq 4|\nabla u|^2|\operatorname{Hess}(u)|^2\,.$$

Putting together (18) and (19) we obtain

$$(20)\qquad \frac{\Delta|\nabla u|^2}{|\nabla u|^2}\geq\frac{|\nabla|\nabla u|^2|^2}{2|\nabla u|^4}-2A+2f'(u)\,.$$

Next, we recall that, since $\operatorname{Ricci}(M)\geq -A$,

$$(21)\qquad \Delta r^2\leq C(1+Ar)\,,$$

where C is a positive constant which depends only on M (see [5]). Now we use (20) and (21) in (16), together with $\Delta u=f(u)$ and the Gauss lemma (i.e., $|\nabla r|=1$): that yields

$$0 \geq -\frac{2C(1+Aa)}{(a^2-r^2)} - \frac{8a^2}{(a^2-r^2)^2} + \frac{g'(u)f(u)}{g(u)} + \left\{\frac{g(u)g''(u)-(g')^2(u)}{g^2(u)}\right\}|\nabla u|^2 - \frac{|\nabla|\nabla u|^2|^2}{2|\nabla u|^4} - 2A + 2f'(u)\,. \tag{22}$$

Finally, we observe that (15) implies

$$\frac{|\nabla|\nabla u|^2|^2}{2|\nabla u|^4} \leq \frac{(g')^2(u)}{2g^2(u)}|\nabla u|^2 + \frac{8a^2}{(a^2-r^2)^2} + \left\{\frac{4a|g'(u)|}{(a^2-r^2)|g(u)|}\right\}|\nabla u|\,. \tag{23}$$

Now (13) follows readily by the inequality (23) put into (22).

In order to prove (14), we recall Newton's inequality

$$|\mathrm{Hess}(u)|^2 \geq (1/m)|\Delta u|^2 \qquad (m=\dim M)\,. \tag{24}$$

Next, we use the inequality

$$-2\langle \xi, \eta\rangle \leq \varepsilon|\xi|^2 + (1/\varepsilon)|\eta|^2\,,$$

which holds for any $\varepsilon > 0$, to deduce that

$$\frac{g'(u)\Delta u}{g(u)} \geq -\frac{2|\Delta u|^2}{m|\nabla u|^2} - \frac{m(g')^2(u)}{8g^2(u)}|\nabla u|^2\,. \tag{25}$$

Putting (24) and (25) into (18) we obtain

$$\frac{\Delta|\nabla u|^2}{|\nabla u|^2} \geq -\frac{g'(u)\Delta u}{g(u)} - \frac{m(g')^2(u)}{8g^2(u)}|\nabla u|^2 - 2A + 2f'(u)\,. \tag{26}$$

Now the inequality (14) follows from the argument used in the proof of (13), if we replace (20) by (26). □

In order to apply successfully Lemma 12, the most delicate point is a good choice of the function $g(u)$, as illustrated in the next steps.

Step 2 (End of the proof of Theorem 2). Assume that, for some $p \in M$, $|\nabla u|^2(p) > \varepsilon^2 > 0$ (note that the hypothesis (5) does not depend upon the choice of p). We derive a contradiction. We use (14) with $g(u) = [3N(a) - u]^{-d}$, with $d > 0$ to be determined. Since q is a positive maximum for the function G, we obtain

$$|\nabla u|^2(q) \geq \frac{a^4|\nabla u|^2(p)g(p)}{(a^2-r^2(q))^2 g(q)} > \frac{a^4\varepsilon^2}{2^d(a^2-r^2(q))^2}\,. \tag{27}$$

Next, we substitute the expression of $g(u)$ into (14): also, we divide both sides of (14) by $|\nabla u|^2(q)$ and use (27) together with $f'(u) \geq A$: That leads us to

$$0 \geq -\frac{2^d[2C(1+Aa)+24]}{a^2\varepsilon^2} - \frac{4d2^{d/2}}{aN(a)\varepsilon} + \frac{d[(d+1)-(16+m)d/8]}{16N^2(a)} . \tag{28}$$

Now we choose d so small as to have the last term in (28) greater than zero and let a tend to $+\infty$. Then it is easy to see that (28) contradicts (5), so ending the proof of Theorem 2. □

Step 3 (End of the proof of Proposition 1). Let $N = \mathrm{Sup}\{|u|\}$ on M. We proceed as in Step 2 above, with $g(u) = [3N - u]^{-d}$ and $a = 1$: Then (28) takes the form

$$0 \geq -\frac{2^d(2C(1+A)+24+2A+2R)}{\varepsilon^2} - \frac{4d2^{d/2}}{N\varepsilon} + \frac{d((d+1)-(16+m)d/8)}{16N^2} , \tag{29}$$

where $R = \mathrm{Sup}\{|f'(u)|\}$ on M and d is small as above. We observe the (29) must hold at any point $p \in M$ at which $|\nabla u|^2(p) > \varepsilon^2$; but, if ε^2 is large, then (29) does not hold. This is a contradiction unless $|\nabla u|$ is bounded. □

Step 4 (End of the proof of Proposition 2). Let u be a solution of $\Delta u = f(u)$ on M. Assuming that f is of class C^2, we introduce the following two functions:

$$W(a) = \mathrm{Inf}\{-f''(u)f(u)\} \quad \text{on} \quad B_a(p) \tag{30}$$

$$R(a) = \mathrm{Inf}\left\{\frac{-f''(u)f(u)}{|f'(u)|}\right\} \quad \text{on} \quad B_a(p) . \tag{31}$$

In particular, if $f(u) = [u^q - \lambda u]$ we have

$$W(a) = \mathrm{Inf}\{q(q-1)u^{(q-1)}[-u^{(q-1)} + \lambda]\} \quad \text{on} \quad B_a(p) \tag{32}$$

$$R(a) = \mathrm{Inf}\left\{q(q-1)u^{(q-1)}\frac{[-u^{(q-1)} + \lambda]}{|qu^{(q-1)} - \lambda|}\right\} \quad \text{on} \quad B_a(p) . \tag{33}$$

Now Proposition 2 follows immediately if we take $f(u) = u^q - \lambda u$ in the following more general result:

THEOREM A. *Suppose that* $\mathrm{Ricci}(M) \geq 0$ *and let u be a solution of* $\Delta u = f(u)$ *on M, where f is of class C^2. Let* $W(a)$, $R(a)$ *be as in* (30), (31) *and assume that* $W(a) > 0$ *for all* $a > 0$. *Then either*

$$\liminf_{a \to +\infty} W(a)a^2 < +\infty , \tag{34}$$

or

$$\liminf_{a \to +\infty} R(a)a < +\infty . \tag{35}$$

PROOF OF THEOREM A. We set $g(u) = 1/f^2(u)$ (note that, since W is a positive function, $f(u)$ cannot vanish at any point of M). In particular, it is clear that there exist a point p of M and $\varepsilon>0$ such that $|\nabla u|^2(p)>\varepsilon^2/g(p)$. If $q\in B_a(p)$ is a maximum of the function G in (10), we have

$$|\nabla u|^2(q)>\frac{a^4\varepsilon^2}{g(q)(a^2-r^2(q))^2}\,. \tag{36}$$

Next, we apply (13) with $g(u)=1/f^2(u)$ and $A=0$; we also divide both sides of the inequality (13) by $|\nabla u|^2(q)$ and use (36). That leads us to

$$0\geq-\frac{C+8}{\varepsilon^2}-f'(u)f(u)a^2\left\{1+\frac{4|f'(u)|}{\varepsilon f''(u)f(u)a}\right\}. \tag{37}$$

It is now easy—using the definition of W and R—to conclude that, if both (34) and (35) are false, then we contradict (37), so ending Theorem A and Step 4. □

PROOF OF THEOREMS 1 AND 3. These two theorems are special cases of the following more general result:

THEOREM B. *Suppose that* Ricci$(M)\geq-A$, $A\geq0$. *Let u be a solution of $\Delta u=f(u)$ on M such that* (6) *holds and assume that there exists a function Q such that*

$$\text{(38)}\qquad \text{(i)}\quad Q(u),\ Q'(u)\ \textit{are bounded}\qquad \text{(ii)}\quad \operatorname*{Inf}_M\{Q(u)\}=0\,.$$

$$\text{(iii)}\quad [Q'(u)-2f(u)]Q'(u)\geq0\qquad \text{(iv)}\quad [2f'(u)-2A-Q''(u)]\geq0\,.$$

Then

$$|\nabla u|^2\leq Q(u)\qquad \textit{on}\quad M\,. \tag{39}$$

Indeed, because of Remark 1 (i), Theorem 1 follows immediately by applying Theorem B with $Q(u)=2F(u)$ and $A=0$. Theorem 3 is Theorem B in the special case $Q\equiv0$. Thus we are left with the following:

PROOF OF THEOREM B. We apply the method of [7] to the function

$$P=|\nabla u|^2-Q(u)\,. \tag{40}$$

Although some parts of our analysis reproduce [7], we include the details for the sake of completeness. First we need to establish the following lemma:

LEMMA 41. *Let a, ε, $L>0$, A, $N\geq0$ be fixed constants. Then there exists a function $\eta_{\varepsilon,a}=\eta:[a,+\infty)\to\boldsymbol{R}$ with the following properties*:

(42) (i) *η is of class C^2 on $(a,+\infty)$*;

(ii) $\eta(a)=1$, $\eta>0$, $\eta'<0$, $\lim_{r\to+\infty}\eta(r)=0$;

(iii) $\lim_{\varepsilon\to 0}\eta_{\varepsilon,a}(r)=1$ *for each* $r\geq a$;

(iv) $\dfrac{\eta^2}{\eta'^2}\left\{\dfrac{2\eta'^2}{\eta}-\left(\dfrac{N}{\varepsilon}+\dfrac{m-1}{r}+\sqrt{(m-1)A}\right)\eta'-\eta''\right\}\leq\dfrac{\varepsilon}{L}$ *for* $r\geq a$.

PROOF OF LEMMA 41. We define $g_\varepsilon\colon[0,1]\to\boldsymbol{R}$ by setting

$$g_\varepsilon(t)=\int_t^1\frac{\exp(-\varepsilon/Ls)}{s^2}\,ds$$

and observe that $g'_\varepsilon<0$ so that $g_\varepsilon\colon[0,1]\to[0,g_\varepsilon(0)]$ has an inverse $(g_\varepsilon)^{-1}\colon[0,g_\varepsilon(0)]\to[0,1]$. We define $h_{\varepsilon,a}\colon[a,+\infty)\to\boldsymbol{R}$ by setting

$$h_{\varepsilon,a}(t)=\int_a^t\frac{\exp\{-((N/\varepsilon)+\sqrt{(m-1)A})s\}}{s^{m-1}}\,ds\,.$$

We observe that $h_{\varepsilon,a}$ is increasing, $h_{\varepsilon,a}(a)=0$ and $h_{\varepsilon,a}$ is bounded above by the positive number $A_\varepsilon=\lim h_{\varepsilon,a}(t)$ as t increases to $+\infty$. Renormalizing it to $(g_\varepsilon(0)/A_\varepsilon)h_{\varepsilon,a}$, we set

$$\eta_{\varepsilon,a}(r)=(g_\varepsilon)^{-1}\left(\frac{g_\varepsilon(0)}{A_\varepsilon}h_{\varepsilon,a}(r)\right)\quad\text{on}\quad[a,+\infty)\,.$$

Having defined η in this way, properties (42) (i), (ii) and (iii) are easily verified. As for (42) (iv), we consider the identity

$$\int_{\eta(r)}^1\frac{\exp(-\varepsilon/Ls)}{s^2}\,ds=[g_\varepsilon(0)/A_\varepsilon]\int_a^r\frac{\exp\{-[(N/\varepsilon)+\sqrt{(m-1)A}]s\}}{s^{m-1}}\,ds\,.$$

Differentiating this with respect to r, taking the logarithm of the resulting equation and differentiating the result once more we obtain (42) (iv). □

We are now in a position to prove Theorem B. Let us fix $d>0$: because of (6), there exists $p\in M$ such that

$$|\nabla u|^2(p)<d\,. \tag{43}$$

We define a function $v\colon M/B_a(p)\to\boldsymbol{R}$ by setting $v(x)=\eta(r(x))P(x)$, where $P(x)$ is the function in (40) and $\eta=\eta_{\varepsilon,a}$ is as in Lemma 41. We may assume that $v>0$ somewhere, for otherwise, since $\eta>0$, $P\leq 0$ on $M/B_a(p)$. Because of the assumptions (6) and (38) (i), P is bounded: thus (42) (ii) implies that $v(x)$ tends to 0 as $r(x)$ tends to $+\infty$. First we prove that, for an arbitrary $\varepsilon>0$, we have

$$v(x)\leq\max\left\{\varepsilon,\ \max_{\partial B_a(p)}v(x)\right\}. \tag{44}$$

For this purpose, it is enough to show that $v(\underline{x})\leq\varepsilon$ at any interior maximum point $\underline{x}$, if there is any. At $\underline{x}$ we must have $\nabla v=0$ and $\Delta v\leq 0$, which are equivalent to

$$\nabla P = -\frac{\eta'(r)}{\eta(r)} P\nabla r \quad \text{and} \quad 0 \geq P\eta'(r)\Delta r + P\eta''(r) + \eta(r)\Delta P + 2\eta'(r)\langle \nabla P, \nabla r\rangle$$

respectively. From these we deduce

$$0 \geq P\eta'(r)\Delta r + P\eta''(r) + \eta(r)\Delta P - 2\frac{(\eta')^2(r)}{\eta(r)} P . \tag{45}$$

Now we need to estimate ΔP: we compute it directly using (40) and apply Weitzenböck's formula as in (18) to obtain

$$\begin{aligned} \Delta P &\geq 2|\mathrm{Hess}(u)|^2 - 2A|\nabla u|^2 + 2f'(u)|\nabla u|^2 - f(u)Q'(u) - Q''(u)|\nabla u|^2 \\ &\geq 2|\mathrm{Hess}(u)|^2 - f(u)Q'(u) , \end{aligned} \tag{46}$$

where the last inequality is due to the assumption (38) (iv). Next, we observe that, since $\eta' < 0$,

$$\begin{aligned} |\nabla|\nabla u|^2|^2 = |\nabla P + Q'(u)\nabla u|^2 &= \left| -\frac{\eta'(r)}{\eta(r)} P\nabla r + Q'(u)\nabla u\right|^2 \\ &\geq \frac{(\eta')^2(r)}{\eta^2(r)} P^2 + (Q')^2(u)|\nabla u|^2 + 2\frac{\eta'(r)}{\eta(r)} P|\nabla u||Q'(u)| . \end{aligned}$$

Now, by the Schwartz inequality as in (19),

$$2|\mathrm{Hess}(u)|^2|\nabla u|^2 \geq \frac{1}{2}\frac{(\eta')^2(r)}{\eta^2(r)} P^2 + \frac{1}{2}(Q')^2(u)|\nabla u|^2 + \frac{\eta'(r)}{\eta(r)} P|\nabla u||Q'(u)| . \tag{47}$$

Using (47) into (46) we get

$$\begin{aligned} |\nabla u|^2\Delta P &\geq \frac{1}{2}\frac{(\eta')^2(r)}{\eta^2(r)} P^2 + \frac{\eta'(r)}{\eta(r)} P|\nabla u| + \left\{\frac{1}{2}(Q')^2(u) - f(u)Q'(u)\right\}|\nabla u|^2 \\ &\geq \frac{1}{2}\frac{(\eta')^2(r)}{\eta^2(r)} P^2 + \frac{\eta'(r)}{\eta(r)} P|\nabla u||Q'(u)| , \end{aligned} \tag{48}$$

where the last inequality follows from (38) (iii). Now we put (48) into (45) to get

$$0 \geq |\nabla u|^2 P\left\{\eta'(r)\Delta r + \eta''(r) - 2\frac{(\eta')^2(r)}{\eta(r)}\right\} + \frac{1}{2}\frac{(\eta')^2(r)}{\eta(r)} P^2 + \eta'(r)P|\nabla u||Q'(u)| . \tag{49}$$

If $|\nabla u|^2(\underline{x}) \leq \varepsilon$, then $v \leq P$ and $Q \geq 0$ imply immediately that $v(\underline{x}) \leq \varepsilon$. Thus we may assume $|\nabla u|^2(\underline{x}) > \varepsilon$. Set

$$L = \operatorname*{Sup}_M \{2|\nabla u|^2\} , \qquad N = \operatorname*{Sup}_M \{|\nabla u||Q'(u)|\}$$

and recall that (see [5]), since $\mathrm{Ricci}(M) \geq -A$,

$$\Delta r \le \left[\frac{m-1}{r} + \sqrt{(m-1)A}\right].$$

Then it is easy to deduce from (49) and (42) (iv) that, at $\underline{x}$,

$$v = \eta P \le L \frac{\eta^2}{(\eta')^2} \left\{ \frac{2(\eta')^2}{\eta} - \left[\frac{N}{\varepsilon} + \frac{m-1}{r} + \sqrt{(m-1)A}\right]\eta' - \eta'' \right\} \le \varepsilon ,$$

so proving (44). Now we let ε tend to 0 in (44). Because of (42) (iii) and the fact that $v \le P$, we deduce that on $M/B_a(p)$

$$P \le \max\left\{0, \max_{\partial B_a(p)} P\right\}.$$

Letting a tend to 0 in this last inequality we conclude that $P \le \max\{0, \max P(p)\}$ on M. Finally, we use (40), (43) and (38) (ii) to get $P = |\nabla u|^2 - Q(u) \le |\nabla u|^2(p) < d$ on M. Since $d > 0$ was arbitrary, we conclude that $P \le 0$ on M, as required to end Theorem B. □

PROOF OF COROLLARY 1. This is a routine modification of the argument of [7] and so we omit it. □

REMARK 2. Combining the methods of Theorem 2 with those of Karp [6], it is not difficult to obtain various conditions which imply that M has infinite volume. For instance:

PROPOSITION 3. *Suppose that* Ricci$(M) \ge -A$, $A \ge 0$. *Let u be a solution of the Poisson equation* (2) *such that* (4) *holds and ∇u is not parallel. If there exist constants B, $C > 0$ and $q > 1$ such that*

$$|\nabla u|^q(x) \le [Br^2(x)\log(2 + r(x)) + C] \qquad \text{on} \quad M ,$$

then M has infinite volume.

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