Tôhoku Math. J.
47 (1995), 499–508

CONSTANT MEAN CURVATURE HYPERSURFACES IN NONCOMPACT SYMMETRIC SPACES

NANY LEE

(Received May 6, 1994, revised November 10, 1994)

Abstract. Here, we compute the mean curvature of the geodesic sphere at any point in some symmetric spaces and determine the lower bound of the mean curvature of a closed hypersurface of constant mean curvature in it. With the Hessian Comparison Theorem, we also show that there is a lower bound for the mean curvature of any closed hypersurface of constant mean curvature in a manifold with a pole satisfying a curvature condition.

1. Introduction. In this article, we study closed hypersurfaces of constant mean curvature in noncompact symmetric spaces or, more generally, the product of such spaces with a Euclidean space. These closed hypersurfaces of constant mean curvature are called soap bubbles in [HH89] and we refer the readers to this paper as well as [Kap90], [Kap91] and the references there for a discussion of the historical as well as mathematical background of these hypersurfaces. Our main theorem in this direction is the determination of a lower bound of the mean curvature of these hypersurfaces in terms of $\Lambda(M)$, defined as follows. Let M be such a space and let p be any point in M. For $v \in T_pM$, define a symmetric linear map $K_v\colon T_pM \to T_pM$ by

$$K_v(X) = R(X, v)v\,, \qquad \text{for} \quad X \in T_pM\,.$$

We let

$$\Lambda(M) = \max\left\{\sum_{i=1}^{n} c_1(v) : v \in T_p(M) \text{ and } \|v\| = 1\right\}$$

where $\{c_1(v)^2, \dots, c_n(v)^2\}$ are all the eigenvalues of K_v. Throughout this paper, we assume that all the c_i's are nonnegative without loss of generality. This lower bound should be compared with an earlier result in the same direction in [Hsi92]. While Hsiang's result is in terms of roots, we shall show that the bound we obtain here is at least as big as that of [Hsi92]; whether or not they are equal is unclear at this point.

With the Hessian Comparison Theorem, we also prove that there is a lower bound for the mean curvature of any closed hypersurface of constant mean curvature in a manifold with a pole when its radial curvature is $\leq -c^2$ for some nonzero constant c.

1991 *Mathematics Subject Classification*. Primary 53A10; Secondary 53C35.

Key words and phrases: symmetric space, constant mean curvature, manifold with a pole.

Partially supported by Global Analysis Research Center, Seoul National University, Korea.

ACKNOWLEDGMENT. The author would like to express her deepest gratitude to her advisor Professor Hung-Hsi Wu for his fruitful discussions. Also special thanks are due to Professor Wu-yi Hsiang for his valuable advice.

2. Constant mean curvature hypersurfaces in noncompact symmetric spaces.

2.1. Preliminary. We begin with some definitions.

DEFINITION 2.1. Let M be an n-dimensional Riemannian manifold and p be a point in M. If $\exp_p$ is a diffeomorphism in a neighborhood V of the origin in T_pM, and $S=\{X\in T_pM: \|X\|=r\}$ is contained in V, then $\exp_p S$ is called the *geodesic sphere* of radius r around p and is denoted by $S_p(r)$.

DEFINITION 2.2. Let $\langle\ ,\ \rangle$ be a Riemannian metric on M and ∇ be the Levi-Civita connection of M. Let N be a hypersurface in M, let $x\in N$, and let $(T_xN)^\perp$ denote the orthogonal complement of T_xN in T_xM. Choose a unit vector v in $(T_xN)^\perp$. Then the symmetric operator $S_x: T_xN\to T_xN$ given by

$$\langle S_x(X), Y\rangle=\langle\nabla_X Y, v\rangle \qquad \text{for any} \quad X, Y\in T_xN$$

is called the *second fundamental form* of N at x with respect to v. The *mean curvature* of N at x is the trace of S_x, denoted by $h(N)(x)$. In case N has a constant mean curvature, we will omit x.

By convention, we will always choose a unit vector v in $(T_xN)^\perp$ so that the mean curvature is positive.

Now we will prove a useful lemma.

LEMMA 2.1. *Let M be an n-dimensional Riemannian manifold and fix $p\in M$. Suppose $S_p(r)$ is the geodesic sphere of radius r around p for some r. For $x\in S_p(r)$, let γ be the normal geodesic joining p and x and $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis for $T_x(S_p(r))$. Consider the Jacobi fields $\{W_1, \dots, W_{n-1}\}$ such that $W_i(0)=0$ and $W_i(r)=e_i$ for $i=1, \dots, n-1$. Then the mean curvature of $S_p(r)$ at x is equal to $\sum_{i=1}^{n-1}\langle\dot W_i(r), e_i\rangle$.*

PROOF. Let ρ be the distance function relative to p and let $v=-\operatorname{grad}\rho$. Then v is the inward unit normal vector field to $S_p(r)$ and S_x denotes the second fundamental form of $S_p(r)$ at x with respect to v. Let $\{e_1, \dots, e_{n-1}\}$ be any orthonormal vectors in $T_x(S_p(r))$. Now

$$h(S_p(r))(x)=\operatorname{Tr} S_x=\sum_{i=1}^{n-1}\langle S_x(e_i), e_i\rangle=\sum_{i=1}^{n-1}\langle\nabla_{e_i}e_i, v\rangle=-\sum_{i=1}^{n-1}\langle\nabla_{e_i}v, e_i\rangle .$$

For fixed i, consider the variation of γ:

$$\Gamma: [0, r]\times[-c, c]\to M$$

such that

- $\Gamma_r(s)=\Gamma(r, s)\in S_p(r)$
- $\Gamma(t, 0)=\gamma(t)$ and for fixed s, $\Gamma(t, s)$ is a normal geodesic joining p to $\Gamma(r, s)$
- $(\partial\Gamma_r/\partial s)(0)=e_i$.

Let T and V be the tangent vector fields on $[0, r]\times[-c, c]$ corresponding to its first and second variables. We will identify the vectors with their images under Γ. Note that $-T$ is equal to v at $\Gamma_r(s)$ and $W_i(t)$ is equal to the restriction of V to $\Gamma(t, 0)=\gamma(t)$. Now we have

$$-\langle\nabla_{e_i}v, e_i\rangle=-\langle\nabla_V(-T), V\rangle(x)=\langle\nabla_T V, V\rangle(x)=\langle\nabla_T W_i(r), e_i\rangle=\langle\dot{W}_i(r), e_i\rangle .$$

□

2.2. Theorems. Let p be any point in M. For $v\in T_pM$, define a linear map $K_v\colon T_pM\to T_pM$ by

$$K_v(X)=R(X, v)v , \qquad \text{for} \quad X\in T_pM .$$

Note that K_v is symmetric and all the eigenvalues are real. Furthermore, they are all nonnegative if M has nonpositive sectional curvature.

Proposition 2.2. *Let M be an n-dimensional symmetric space of noncompact type or the product of such a space with a Euclidean space and $p\in M$. Let v be any unit vector in T_pM. If $\{c_1^2, \ldots, c_t^2\}$ are all the nonzero eigenvalues of K_v, then the mean curvature of $S_p(r)$ at $\exp_p rv$ is $\sum_{i=1}^t c_i\coth c_ir+(n-t-1)/r$ which is greater than $\sum_{i=1}^t c_i$ for any $r>0$.*

Proof. Let $\gamma(t)=\exp_p tv$ and $x=\exp_p rv$. We will use Lemma 2.1 and so we need to find the Jacobi fields $\{W_1, \ldots, W_{n-1}\}$ along γ such that $W_i(0)=0$ for $i=1, \ldots, n-1$ and $\{W_1(r), \ldots, W_{n-1}(r)\}$ are orthonormal in $T_xS_p(r)$.

Now choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM, consisting of eigenvectors of K_v, that is,

$$K_v(e_i)=c_i^2e_i , \qquad \text{for} \quad i=1, \ldots, n$$

and extend the e_i's to vector fields $\{E_1, \ldots, E_n\}$ along γ by parallel transport. For $\dot\gamma(t)$, $0\le t\le r$, define a linear map $K_{\dot\gamma(t)}\colon T_{\gamma(t)}M\to T_{\gamma(t)}M$ by

$$K_{\dot\gamma(t)}(X)=R(X, \dot\gamma(t))\dot\gamma(t) , \qquad \text{for} \quad X\in T_{\gamma(t)}M .$$

Consider $K_{\dot\gamma(t)}(E_i(t))$, for all $0\le t\le r$. We have

$$\nabla_{\dot\gamma(t)}K_{\dot\gamma(t)}(E_i(t))=\nabla_{\dot\gamma(t)}(R(E_i(t), \dot\gamma(t))\dot\gamma(t))=0 ,$$

since $\nabla R=0$ and $\nabla_{\dot\gamma(t)}\dot\gamma(t)=\nabla_{\dot\gamma(t)}E_i(t)=0$. This implies that $K_{\dot\gamma(t)}(E_i(t))$ is a parallel transport along $\gamma(t)$ of $K_{\dot\gamma(r)}(E_i(r))=K_v(e_i)=c_i^2e_i$. By the uniqueness of parallel transport,

$$K_{\dot\gamma(t)}(E_i(t))=c_i^2E_i(t) .$$

Note that c_i does not depend on t. For simplicity, we may assume $E_n=\dot\gamma(t)$.

Now we are ready to construct the Jacobi fields W_i along γ such that $W_i(0)=0$

and $W_i(r)=E_i(r)$, for $1\leq i\leq n-1$.

If $W_i(t)=\sum_{j=0}^{n}\alpha_i^j(t)E_j(t)$, the coefficients $\alpha_i^j(t)$'s should satisfy the Jacobi equation and the initial conditions: for all $1\leq i\leq n-1$ and $1\leq j\leq n$.

$$\ddot{\alpha}_i^j(t)=\sum_{k=1}^{n-1}\langle R(\dot{\gamma}(t),E_j(t))\dot{\gamma}(t),E_k(t)\rangle\alpha_i^k(t)=-\sum_{k=1}^{n-1}\langle K_{\dot{\gamma}(t)}(E_j(t)),E_k(t)\rangle\alpha_i^k(t)$$
$$=-\sum_{k=1}^{n-1}\langle c_j^2E_j(t),E_k(t)\rangle\alpha_i^k(t)=-c_j^2\alpha_i^j(t)\,,$$

and

$$\alpha_i^j(0)=0\,,\quad\text{and}\quad\alpha_i^j(r)=\delta_i^j\,.$$

Therefore we have

$$\alpha_i^j(t)=0\qquad\text{if}\quad i\neq j$$
$$\alpha_i^i(t)=\begin{cases}t/r & \text{if}\quad c_i=0\\ \sinh c_it/\sinh c_ir & \text{if}\quad c_i>0\,,\end{cases}$$

i.e.,

$$W_i(t)=\begin{cases}(t/r)E_i(t) & \text{if}\quad c_i=0\\ (\sinh c_it/\sinh c_ir)E_i(t) & \text{if}\quad c_i>0\,.\end{cases}$$

Furthermore,

$$\dot{W}_i(t)=\begin{cases}(1/r)E_i(t) & \text{if}\quad c_i=0\\ c_i(\cosh c_it/\sinh c_ir)E_i(t) & \text{if}\quad c_i>0\,,\end{cases}$$

and so

$$\langle\dot{W}_i(r),W_i(r)\rangle=\begin{cases}1/r & \text{if}\quad c_i=0\\ c_i\coth c_ir & \text{if}\quad c_i>0\,,\end{cases}$$

which is monotone decreasing to c_i as r tends to ∞. Recall that by assumption, $c_i\neq 0$ if and only if $1\leq i\leq t$.

By Lemma 2.1, we have

$$h(S_p(r))(x)=\sum_{i=1}^{n-1}\langle\dot{W}_i(r),W_i(r)\rangle=\sum_{i=1}^{n-1}\dot{\alpha}_i^i(r)$$
$$=\sum_{i=1}^{t}c_i\coth c_ir+(n-t-1)/r>\sum_{i=1}^{t}c_i\,.$$

□

Now we can prove the main theorem.

THEOREM 2.3. *Let M be an n-dimensional symmetric space of noncompact type or the product of such a space with a Euclidean space and $p \in M$. Let*

$$\Lambda(M) = \max\left\{ \sum_{i=1}^{n} c_i(v) : v \in T_p(M) \text{ and } \|v\| = 1 \right\}$$

where $\{c_1(v)^2, \dots, c_n(v)^2\}$ are all the eigenvalues of K_v. Then the mean curvature of any closed hypersurface of constant mean curvature in M is greater than $\Lambda(M)$.

PROOF. First we choose $v_0 \in T_pM$ such that $\|v_0\| = 1$ and $\sum_{i=1}^{n} c_i(v_0) = \Lambda(M)$. Then for any $\varepsilon > 0$, there exists a neighborhood N of v_0 in a unit ball in T_pM such that

$$\sum_{i=1}^{n} c_i(v) \geq \Lambda(M) - \varepsilon \qquad \text{for} \quad v \in N.$$

Let Σ be a closed hypersurface of constant mean curvature h in M and O be its center of gravity. We move Σ by transvection which maps O to p and then push it by the transvection T_t along $\exp_p tv_0$ until $T_{t_0}(\Sigma)$ is contained in $\{\exp_p sv : v \in N \text{ and } s \geq 0\}$ for some t_0. Now choose r such that $T_{t_0}(\Sigma)$ is inside $S_p(r)$ and touches it, say, at $\exp_p s'v'$ for some $s' > 0$ and some $v' \in N$. Clearly, the mean curvature $h(\Sigma)$ of Σ must be greater than or equal to the mean curvature $h(S_p(r))(\exp_p s'v')$ of $S_p(r)$ at $\exp_p s'v'$.

On the other hand, by Proposition 2.2, we have

$$h(S_p(r))(\exp_p s'v') > \sum_{i=1}^{n} c_i(v'),$$

which is $\geq \Lambda(M) - \varepsilon$ since $v' \in N$. Therefore we get $h(\Sigma) > \lambda(M)$ since ε is arbitrary. □

REMARK. If λM has a metric multiplied by a constant λ, then

$$\Lambda(\lambda M) = \frac{1}{\lambda} \Lambda(M).$$

REMARK. This result should be compared with a similar one in [Hsi92], where the lower bound of the mean curvature is expressed in terms of $b(M)$ to be defined below. We shall prove that our lower bound is at least as big as $b(M)$; whether or not they are equal is unclear at the moment except for an M with rank ≤ 2.

REMARK. Let $G = I_0(M)$ be the connected component of the isometry group $I(M)$ that contains the identity and $K = G_p = \{g \in G : g(p) = p\}$ and $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. And let θ_p be the involution of $\mathfrak{g}$. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ as the decomposition of $\mathfrak{g}$ into eigenspaces of $\theta_p : \mathfrak{g} \to \mathfrak{g}$. Then the map $p : G \to M$ given by $p(g) = g(p)$ induces the isomorphism $dp : \mathfrak{m} \to T_pM$. Here we have useful facts:

Fact 1. Fix a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{m}$ and let α be a real linear function on $\mathfrak{a}$. Then α is the restriction of a root of $\mathfrak{g}$ if and only if there exists a vector $X \neq 0$ in $\mathfrak{m}$ such that

$$(\operatorname{ad} H)^2 X = \alpha(H)^2 X \qquad \text{for all} \quad H \in \mathfrak{a} .$$

(See [Hel78, ch. VII (2)].)

Fact 2. The curvature tensor of M at p is given by

$$R(X, Y)Z = \operatorname{ad}[X, Y](Z) = [[X, Y], Z]$$

for all X, Y, $Z \in \mathfrak{m}$.

(See [Hel78, ch. IV (4)].)

Note that $K_v(X) = R(X, v)v = [[X, v], v] = [v, [v, X]] = (\operatorname{ad} v)^2 X$. With the identification $dp: \mathfrak{m} \to T_pM$, the above two facts imply that

$$\sum_{i=1}^{n} c_i(v) = \sum_{\alpha \in \Delta(M)} |\alpha(v)| \qquad \text{for} \quad v \in \mathfrak{a} \text{ and } \|v\| = 1 ,$$

where $\Delta(M)$ is the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Therefore, letting

$$b(M) = \max\left\{ \sum_{\alpha \in \Delta(M)} |\alpha(v)| : v \in \mathfrak{a} \text{ and } \|v\| = 1 \right\} ,$$

we have $\Lambda(M) \geq b(M)$.

Remark. This $b(M)$ is not dependent on the choice of the maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$. Indeed, let $\mathfrak{a}'$ be the other maximal abelian subspace in $\mathfrak{m}$. We choose some $k \in K$ such that $\operatorname{Ad}_k \mathfrak{a}' = \mathfrak{a}$. By the above fact, if λ is the restricted root of $\mathfrak{g}$ with respect to $\mathfrak{a}$, then the linear function λ' on $\mathfrak{a}'$ defined by

$$\lambda'(\operatorname{Ad}_k H) = \lambda(H)$$

is also the restricted root of $\mathfrak{g}$ with respect to $\mathfrak{a}'$. From this, we have

$$\sum_{\alpha \in \Delta(M)} |\alpha(v)| = \sum_{\alpha' \in \Delta'(M)} |\alpha'(\operatorname{Ad}_k v)| ,$$

where $\Delta(M)$ and $\Delta'(M)$ are the restricted root systems of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and $\mathfrak{a}'$, respectively. Therefore, if the rank of M is ≤ 2, then

$$\Lambda(M) = b(M) ,$$

since for any nonzero $v \in \mathfrak{m}$, there is a maximal abelian subalgebra containing v.

Before going further, we recall a definition: M is said to be a *manifold of s-positive* (resp. *s-negative*) *curvature* if s is a smallest integer such that for each $p \in M$ and for any $(s+1)$ orthonormal vectors $\{e_0, e_1, \ldots, e_s\}$ in M_p, we have $\sum_{i=1}^{s} K(e_0, e_i) > 0$ (resp. < 0), where $K(e_0, e_i)$ denotes the sectional curvature of the plane spanned by e_0 and e_i. This s is determined for each irreducible symmetric space in [Lee93].

For convenience, we introduce a function $\kappa_s: M \to R$ given by letting $\kappa_s(p)$ to be the maximum of all $\sum_{i=1}^{s} K(e_0, e_i)$ for any $(s+1)$ orthonormal vectors $\{e_0, e_1, \ldots, e_s\}$

in T_pM. And let $\kappa_s(M) = \max_{p\in M} \kappa_s(p)$.

THEOREM 2.4. *Let M be an n-dimensional symmetric space of noncompact type or the product of such a space with a Euclidean space with an s-negative curvature. If $\kappa_s(M) \le -\varepsilon^2$, then the mean curvature of any closed hypersurface of constant mean curvature in M is greater than $(n-s)\varepsilon$.*

PROOF. By the definition of s-negative curvature, there exist orthonormal vectors $\{e_1, \dots, e_s\}$ in T_pM for some $p \in M$ such that $\sum_{i=2}^s K(e_1, e_i) = 0$. Letting $v = e_1$, consider the eigenvalues $\{c_1^2(v) \le c_2^2(v) \le \cdots \le c_n^2(v)\}$ of K_v. Then $c_1(v) = \cdots = c_s(v) = 0$ since $K_v(e_i) = 0$ for all $1 \le i \le s$. Furthermore, by hypothesis, $c_j(v) \ge \varepsilon$ for all $s+1 \le j \le n$. Thus $\sum_{i=1}^n c_i(v) \ge (n-s)\varepsilon$. Theorem 2.3 implies the conclusion. □

THEOREM 2.5. *Let M_i be an n_i-dimensional symmetric space of noncompact type or the product of such a space with a Euclidean space and have an s_i-negative curvature and $\kappa_s(M_i) \le -\varepsilon_i^2$. Then*

$$\Lambda(M_1 \times M_2) \ge \min\{(n_i - s_j)\varepsilon_k : i \ne j,\, i, j, k = 1, 2\}\,.$$

PROOF. It is clear that $M_1 \times M_2$ has s-negative curvature, where $s = \max\{s_1 + n_2, n_1 + s_2\}$ and $\kappa_s(M_1 \times M_2) \le \max\{-\varepsilon_1^2, -\varepsilon_2^2\}$. Then we apply Theorem 2.4. □

3. Constant mean curvature hypersurfaces in manifolds with a pole.

3.1. Preliminary. We begin with some definitions.

DEFINITION 3.1. Let M be an n-dimensional Riemannian manifold. A point p in M is called a *pole* of M if $\exp : M_p \to M$ is a diffeomorphism, and an ordered pair (M, p) a *manifold with a pole*.

DEFINITION 3.2. Given an (M, p), the *radial vector field* is the unit vector field v defined on $M - \{p\}$ such that for all $x \in M - \{p\}$, $v(x)$ is the unit vector tangent to the unique geodesic joining p and x and pointing away from p. And a plane π in M_x is called a *radial plane* if π contains $v(x)$ and the restriction of the sectional curvature function to all the radial planes is called the *radial curvature*.

DEFINITION 3.3. Let ∇ be the Levi-Civita connection and f be a C^2 function on M. We define the *Hessian* of f as the second covariant differential $D^2 f$ of f, i.e.,

$$D^2 f(X, Y) = X(Yf) - (\nabla_X Y)f$$

for all vector fields X, Y on M. Note that $D^2 f$ is a symmetric tensor field of type (0, 2).

Now we have a useful result, whose proof can be found in [GW79, pp. 19–24].

THEOREM 3.1 [Hessian Comparison Theorem]. *Let (M, p) and (N, q) be manifolds with a pole such that* $\dim M \le \dim N$. *Let $\gamma_1 : [0, r] \to M$ and $\gamma_2 : [0, r] \to N$ be normal geodesics with $\gamma_1(0) = p$ and $\gamma_2(0) = q$. ρ_M and ρ_N denote the distance functions on M and*

N relative to p and q, and v_M and v_N the radial vector fields of M and N, respectively. Suppose each radial curvature at $\gamma_2(t)$ is not less than every radial curvature at $\gamma_1(t)$ for all $t\in[0,r]$. Then for all $X\in M_{\gamma_1(r)}$ and for all $Y\in M_{\gamma_2(r)}$ such that $\|X\|=\|Y\|$ and $\langle X, v_M(\gamma_1(r))\rangle=\langle Y, v_N(\gamma_2(r))\rangle$,

$$D^2\rho_M(\gamma_1(r))(X, X)\geq D^2\rho_N(\gamma_2(r))(Y, Y).$$

3.2. Theorem. We will use the Hessian Comparison Theorem to prove the main theorem and so we need the following proposition.

PROPOSITION 3.2. *Let H be an n-dimensional hyperbolic space with a constant curvature $-c^2$ for some positive c. Let p be any point in H and ρ_H be the distance function from p. Then for any $x=\exp_p rv$ with $v\in T_pH$, $\|v\|=1$ and $r>0$,*

$$D^2\rho_H(X, X)=c\coth cr \qquad \text{for any} \quad X\in T_x(S_p(r)),\ \|X\|=1.$$

PROOF. First we will prove that

$$D^2\rho_H(X, X)=\langle\nabla_T W(r), X\rangle,$$

where W is a Jacobi field along $\gamma(t)=\exp_p tv$ such that $W(0)=0$ and $W(r)=X$.

Fix an $X\in T_x(S_p(r))$ with $\|X\|=1$. Consider the normal geodesic $\zeta(s)$ such that $\dot\zeta(0)=X$. Let $\gamma_s\colon[0,r]\to M$ be a normal geodesic joining p to $\zeta(s)$. Then clearly $\gamma_0=\gamma$. Let W be the transversal vector field of γ_s along γ. Note that

- the transversal vector field W of γ_s is a Jacobi field along γ
- $W(0)=0$ and $W(r)=X$
- $W\perp\dot\gamma$.

Let $L(s)$ denote the length of γ_s and $T=\dot\gamma$. Now

$$D^2\rho(X, X)=X\dot\zeta(\rho)-(\nabla_X\dot\zeta)(\rho)$$

$$=\dot\zeta\dot\zeta(\rho)(0)-(\nabla_{\dot\zeta}\dot\zeta)(\rho)(0)=\ddot L(0) \tag{3.1}$$

$$=\langle\nabla_W W, T\rangle]_0^r+\int_0^r\{\langle\nabla_T W, \nabla_T W\rangle-\langle R(W, T)T, W\rangle-(T\langle W, T\rangle)^2\}\,dt \tag{3.2}$$

$$=\int_0^r\{\langle\nabla_T W, \nabla_T W\rangle-\langle R(W, T)T, W\rangle\}\,dt$$

$$=\langle\nabla_T W, W\rangle]_0^r=\langle\nabla_T W(r), W(r)\rangle=\langle\nabla_T W(r), X\rangle. \tag{3.3}$$

We used the fact that ζ is a geodesic and $W\perp T$ in (3.1) and (3.2). And in (3.3), we needed the following by the Jacobi equation for W and integration by parts: if $\{E_i\}_{i=1}^n$ are orthonormal parallel vector fields along γ and let $W=\sum_{i=1}^n\alpha_i E_i$, then

$$\ddot\alpha_i=\sum_{j=1}^n\langle R(T, E_i)T, E_j\rangle\alpha_j,$$

also so

$$\langle \ddot{W}, W\rangle = \sum_{i=1}^{n} \ddot{\alpha}_i\alpha_i = \sum_{i,j=1}^{n} \langle R(T, E_i)T, E_j\rangle \alpha_j\alpha_i$$

$$= \left\langle R\left(T, \sum_{i=1}^{n} \alpha_i E_i\right)T, \sum_{j=1}^{n} \alpha_j E_j\right\rangle = \langle R(T, W)T, W\rangle = \langle R(W, T)T, W\rangle ,$$

i.e.,

$$\nabla_T\langle \dot{W}, W\rangle = \langle \dot{W}, \dot{W}\rangle + \langle \ddot{W}, W\rangle = \langle \dot{W}, \dot{W}\rangle + \langle R(W, T)T, W\rangle .$$

On the other hand, we have

$$W(t) = \frac{\sinh ct}{\sinh cr} E(t)$$

as in Proposition 2.2 of Section 2 where $E(t)$ is a parallel transport of X along γ. So

$$\nabla_T W(r) = c \coth crE(r) = c \coth crX$$

and

$$D^2\rho_H(X, X) = \langle \nabla_T W(r), X\rangle = \langle c \coth crX, X\rangle = c \coth cr .$$

□

Let us prove the main theorem.

THEOREM 3.3. *Let (M, p) be an n-dimensional manifold with a pole. If its radial curvature is $\leq -c^2$ for some positive constant c, then the mean curvature of any closed hypersurface of constant mean curvature in M is greater than $(n-1)c$.*

PROOF. Fix $x = \exp_p rv$ for any $v \in T_pM$, $\|v\| = 1$ and for any $r > 0$. First we will show that the mean curvature $h(S_p(r))(x)$ of $S_p(r)$ at x is greater than $(n-1)c$.

Let ρ be the distance function relative to p and let $v = -\operatorname{grad}\rho$. Then v is the inward unit normal vector field to $S_p(r)$ and S_x denotes the second fundamental form of $S_p(r)$ at x with respect to v. Choose an orthonormal basis $\{e_1, \dots, e_{(n-1)}\}$ for $T_x(S_p(r))$. Then we have

$$h(S_p(r))(x) = \operatorname{Tr} S_x = \sum_{i=1}^{n-1} \langle S_x(e_i), e_i\rangle = \sum_{i=1}^{n-1} \langle \nabla_{e_i} e_i, v\rangle = -\sum_{i=1}^{n-1} \langle \nabla_{e_i} e_i, \operatorname{grad}\rho\rangle$$

$$= -\sum_{i=1}^{n-1} (\nabla_{e_i} e_i)(\rho) = \sum_{i=1}^{n-1} e_i e_i(\rho) - (\nabla_{e_i} e_i)(\rho) = \sum_{i=1}^{n-1} D^2\rho(e_i, e_i) .$$

If H is the n-dimensional hyperbolic space with a constant curvature $-c^2$ and ρ_H be the distance function from some point $q \in H$, then for $X \in T_yS_q(r)$ with $\|X\| = 1$ and $\rho_H(y) = r$,

$$D^2\rho(e_i, e_i) \geq D^2\rho_H(X, X), \quad \text{(by the Hessian Comparison Theorem)},$$

and for such X,

$$D^2\rho_H(X, X) \geq c \coth cr > c \quad \text{(by Proposition 3.2)}.$$

Therefore, from the above three formulas, we obtain

$$h(S_p(r))(x) > (n-1)c.$$

Note that v and r are arbitrary.

Now let N be a closed hypersurface of constant mean curvature in M. Then we can choose v and r such that N is inside $S_p(r)$ and touches it at $x = \exp_p rv$. Clearly this implies the conclusion. □

We close this section with a remark.

REMARK. Let M be a symmetric space of noncompact type. Then for any $p \in M$, (M, p) is a manifold with a pole. But except for an M with rank $M = 1$, there does not exist a positive c such that the radial curvature of (M, p) is $\leq -c^2$. Thus Theorem 2.3 of Section 2 and Theorem 3.3 above overlap only in the case of symmetric spaces of rank one.

REFERENCES

[GW79] R. GREENE AND H. WU, Function theory on manifolds which possess a pole, Lecture Notes in Math., vol. 699, Springer-Verlag, Berlin, New York, 1979.

[Hel78] S. HELGASON, Differential geometry, Lie groups, and symmetric spaces, Academic Press, New York, San Francisco, London, 1978.

[HH89] W. T. HSIANG AND W. Y. HSIANG, On the uniqueness of isoperimetric solutions and imbedded soap bubbles in noncompact symmetric spaces, I, Invent. Math. 98 (1989), 39–58.

[Hsi92] W. Y. HSIANG, On soap bubbles and isoperimetric regions in noncompact symmetric spaces, I, Tôhoku Math. J. (2) 44 (1992), 151–175.

[Kap90] N. KAPOULEAS, Complete constant mean curvature surfaces in Euclidean three-spaces, Ann. of Math. (2) 131 (1990), 239–330.

[Kap91] N. KAPOULEAS, Constant mean curvature surfaces, J. Differential Geom. 33 (1991), 683–715.

[Lee93] NANY LEE, Determination of the partial positivity of the curvature in symmetric spaces, to appear in Annali di Matematica pura ed applicata.

GLOBAL ANALYSIS RESEARCH CENTER
DEPARTMENT OF MATHEMATICS
SEOUL NATIONAL UNIVERSITY
SEOUL 151–742
KOREA

E-mail address: nany@math.snu.ac.kr