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POLARIZED SURFACES OF LOW DEGREES WITH RESPECT TO THE DELTA-GENUS

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Abstract. We classify such polarized surfaces that a certain equality holds between the self-intersection number and the delta-genus and that the complete linear system has finite base locus and defines a non-birational map. The surface obtained by the blowing up at a point of such a surface turns out to be a double cover of a desingularization of a surface with delta-genus zero. We classify these surfaces according to the shape of the inverse image of the image of the exceptional curve. Six of the classes consist of fiber spaces over the projective line and the other class consists of irrational ruled surfaces. Conversely, we show the existence of polarized surfaces in each of the seven classes.

1. Introduction. Let (M, L) be a polarized manifold, i.e., a pair of an n-dimensional complete algebraic manifold M and an ample Cartier divisor L on it. First we recall some definitions and necessary results. The integers $\chi_j(M, L)$ $(j=0, \dots, n)$ are the coefficients of the Hilbert polynomial

$$\chi(M, tL)=\sum\chi_j(M, L)\frac{t^{[j]}}{j!},$$

where $t^{[0]}=1$ and $t^{[j]}=t(t+1)\cdots(t+j-1)$ $(j>0)$. The *sectional genus* of (M, L) is defined as

$$g(M, L):=1-\chi_{n-1}(M, L).$$

By the Riemann-Roch theorem we get $2g(M, L)-2=L^{n-1}\cdot((n-1)L+K_M)$, where K_M is a canonical divisor of M. We define the *Δ-genus* as

$$\Delta(M, L):=n+L^n-h^0(M, L).$$

A prime divisor R_{n-1} in the linear system $|L|$ is called a *rung* of (M, L). We have $g(M, L)=g(R_{n-1}, L_{|R_{n-1}})$. If the restriction map

$$r_{n-1}: H^0(M, \mathcal{O}(L))\to H^0(R_{n-1}, \mathcal{O}(L_{|R_{n-1}}))$$

is surjective, then R_{n-1} is said to be *regular*. A rung R_{n-1} is regular if and only if $\Delta(M, L)=\Delta(R_{n-1}, L_{|R_{n-1}})$. We denote by $\mathrm{Bs}|L|$ the base locus of the linear system $|L|$.

As to the existence of a regular rung, Fujita proved the following:

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THEOREM 1 (Fujita [2]). *Let (M, L) be a polarized manifold. When* $\dim \mathrm{Bs}\,|L| \leq 0$ *and* $g(M, L) \geq \Delta(M, L)$, *the following are satisfied*:

(i) *If* $L^n \geq 2\Delta(M, L) - 1$, *then* (M, L) *has a nonsingular regular rung.*

(ii) *If* $L^n \geq 2\Delta(M, L)$, *then* $\mathrm{Bs}\,|L| = \varnothing$.

(iii) *If* $L^n \geq 2\Delta(M, L) + 1$, *then* $g(M, L) = \Delta(M, L)$ *and* L *is very ample.*

We are interested in polarized manifolds satisfying the equality in the assumptions of the above theorem. From now on we assume $g(M, L) \geq \Delta(M, L)$ and $\dim \mathrm{Bs}\,|L| \leq 0$.

So far there have been the following results under these assumptions.

(1) Classification in the case $L^n = 2\Delta(M, L)$ (Fujita [3]).

(1′) Classification and study of deformations in the case $n=2$, $L^2 = 2\Delta(M, L)$ and $L = K_M$ (Horikawa [9]).

(2) Classification in the case $L^n = 2\Delta(M, L) - 1$ and $\Delta(M, L) \leq 2$ (Fujita [4], [5]).

(2′) Classification and study of deformations in the case $n=2$, $L^n = 2\Delta(M, L) - 1$ and $L = K_M$ (Horikawa [10]).

(2″) Classification in the case $n=2$, $L^n = 2\Delta(M, L) - 1$, $\Delta(M, L) = 3$ and $\deg \Phi_L = 2$ (cf. [12]).

Here we are interested in classifying the other polarized surfaces satisfying $L^2 = 2\Delta(M, L) - 1$. For surfaces $\deg \Phi_L$ is one or two. In this paper we classify those in the case $\deg \Phi_L = 2$ using a method similar to that in [5].

In this case, the base locus of $|L|$ is a point p, and by the blowing up at p, we obtain a surface $\tilde{M}$, where E is the exceptional curve, and a degree two morphism $\Phi_{\tilde{L}} : \tilde{M} \to W_0 := \Phi_{\tilde{L}}(\tilde{M}) \subset \boldsymbol{P}(H^0(\tilde{M}, \mathcal{O}(\tilde{L})))$, where the Δ-genus of the pair of W_0 and a hyperplane section H on it is zero. Moreover, we lift it to a morphism f_1 from $\tilde{M}$ to a Hirzebruch surfaces Σ. We carry out the classification by dividing the surfaces into cases by the type of a divisor $f_1^* f_1(E) \subset \tilde{M}$. We lift f_1 to a finite degree two morphism from $\tilde{M}$ or $\hat{M}$ to a surface obtained from Σ by the blowing up at a few points, where $\hat{M}$ is a surface obtained from $\tilde{M}$ by the blowing up at a point. We describe the branch locus of the double covering. We then show the existence of polarized surfaces for each of these types.

2. Generalities. In the rest of this paper we assume that $g(M, L) \geq \Delta(M, L)$ and $\dim \mathrm{Bs}\,|L| \leq 0$. First we obtain the following:

PROPOSITION 1. $\mathrm{Bs}\,|L|$ *is empty or consists of one point* p. *In the latter case, any* n *general members of* $|L|$ *intersect one another transversely at* p.

PROOF. Since $L^n \geq 2\Delta(M, L) - 1$, the pair (M, L) has a nonsingular regular rung R_{n-1} by Theorem 1. A pair $(R_{n-1}, L_{|R_{n-1}})$ satisfies $(L_{|R_{n-1}})^{n-1} \geq 2\Delta(R_{n-1}, L_{|R_{n-1}}) - 1$. Thus $(R_{n-1}, L_{|R_{n-1}})$ has a nonsingular regular rung R_{n-2}. Hence we have a sequence of rungs $M \supset R_{n-1} \supset \cdots \supset R_2 \supset R_1$. By the definition of a regular rung, it is sufficient to show that $\mathrm{Bs}\,|L_{|R_1}| = \{p\}$. We set L_{R_1} to be a divisor on the curve R_1 which satisfies

$|L_{|R_1}|=|L_{R_1}|+p$. We have $\deg(L_{|R_1})=\deg L_{R_1}+1$. Hence we obtain $\Delta(R_1, L_{R_1})=\Delta(R_1, L_{|R_1})-1$, since $\Delta(M, L)=\Delta(R_1, L_{|R_1})$ and $\deg L_{R_1}=\deg(L_{|R_1})=L^n-1$. Consequently, we have $\deg L_{R_1}=2\Delta(R_1, L_{R_1})$. Moreover, we have $g(R_1, L_{R_1})\geq\Delta(R_1, L_{R_1})$ by the Riemann-Roch theorem applied to the algebraic curve R_1. It follows that $\mathrm{Bs}|L_{|R_1}|=\varnothing$ by Theorem 1, (ii). Hence we have $\mathrm{Bs}|L_{|R_1}|=\{p\}$, and the coefficient for p of $L'_{|R_1}-p$ is equal to zero for any general member L' of $|L|$. Therefore any n general members of $|L|$ intersect one another at p with the local intersection number one.

q.e.d.

If $\mathrm{Bs}|L|=\varnothing$, then we have a morphism

$$\Phi_L : \tilde{M}\to W_0 := \Phi_L(\tilde{M})\subset \boldsymbol{P}(H^0(M, \mathcal{O}(L))) .$$

Since $L^n=2\Delta(M, L)-1$, we obtain $\Delta(M, L)\geq 2$ and $\deg\Phi_L\cdot\deg W_0=2h^0(M, \mathcal{O}(L))-2n+1$. Since $\Delta(W_0, H)\geq 0$, we have $\deg W_0\geq h^0(M, \mathcal{O}(L))-n$. Hence $2h^0(M, \mathcal{O}(L))-2n+1\geq(h^0(M, \mathcal{O}(L))-n)\deg\Phi_L$. By the ampleness of L, we have $h^0(M, \mathcal{O}(L))-n\geq 1$. Consequently, we see that $\deg\Phi_L$ is one or three by the oddness of L^n. When $\deg\Phi_L=3$, we have $\Delta(M, L)=2$. This case is classified in [5]. When $\deg\Phi_L=1$, we have $\Delta(M, L)\geq 3$ and $\Delta(M, L)=\Delta(W_0, H)$.

If $\mathrm{Bs}|L|\neq\varnothing$, then $\mathrm{Bs}|L|$ consists of a point p. We now eliminate the base point in $\mathrm{Bs}|L|$ of the rational map $\Phi_L : M\to W_0 := \Phi_L(M)\subset\boldsymbol{P}(H^0(M, \mathcal{O}(L)))$. Let $\pi := \tilde{M}\to M$ be the blowing up at p, and denote by E the exceptional curve over p. We denote by $\tilde{L}$ the proper transform of a general member of $|L|$. n general members of $|L|$ intersect one another at p transversely by Proposition 1. Thus we have $\pi^*L=\tilde{L}+E$, and n general members of $|\tilde{L}|$ do not intersect one another on E. Hence $|\tilde{L}|$ has no base point. Therefore the rational map

$$\Phi_{\tilde{L}} := \tilde{M}\to W_0 := \Phi_{\tilde{L}}(\tilde{M})\subset\boldsymbol{P}(H^0(\tilde{M}, \mathcal{O}(\tilde{L})))$$

is a morphism such that $\Phi_{\tilde{L}}=\Phi_L\circ\pi$. We see that $\deg\Phi_L$ is one or two as in the $\mathrm{Bs}|L|=\varnothing$ case. Moreover, we have $\Delta(W_0, H)=\Delta(M, L)-1$ if $\deg\Phi_{\tilde{L}}=1$ while $\Delta(W_0, H)=0$ if $\deg\Phi_{\tilde{L}}=2$.

We set $\Gamma_0 := \Phi_{\tilde{L}}(E)$. The pull-back of Γ_0 by $\Phi_{\tilde{L}}$ can be written as $\Phi_{\tilde{L}}^*\Gamma_0=\varepsilon E+E^*+D_0$, where ε is the multiplicity of E in $\Phi_{\tilde{L}}^*\Gamma_0$ and E^* is the sum of the components which are not contracted by $\Phi_{\tilde{L}}$, while D_0 is the sum of the components which are contracted by $\Phi_{\tilde{L}}$. We refer the reader to [5, Lemma 1.5] for the proof of the following:

PROPOSITION 2. *Let x be a point of W_0 such that $X=\Phi_{\tilde{L}}^{-1}(x)$ is of positive dimension. Then X is an irreducible reduced curve with $E\cdot X=1$ and $x\in\Gamma_0$. Moreover, $X\subset E^*$ or $X\subset D_0$.*

From now on we assume the following:

ASSUMPTION. $n=2$ and $\deg\Phi_{\tilde{L}}=2$.

Under the above assumption $\Delta(W_0, H)=0$ holds. Hence W_0 is one of the following [1], [11]:

(I) W_0 is the Hirzebruch surface Σ_d of degree d, and $H=T+((r-1+d)/2)F$ where $r-d-3$ is an even nonnegative integer, T is the minimal section and F is a fiber.

(II) $W_0 \subset \boldsymbol{P}^r$ is the cone over a nonsingular rational curve of degree $r-1$ in $\boldsymbol{P}^{r-1}$, and H is a hyperplane section of W_0.

(III) $W_0=\boldsymbol{P}^2$, and H is a hyperplane.

(IV) W_0 is $\boldsymbol{P}^2$ embedded into $\boldsymbol{P}^5$ by $\mathcal{O}(2)$ and H is a hyperplane section of W_0.

In Case (III) we have $\Delta(M, L)=2$, and this case is classified in [5]. Moreover, Case (IV) is impossible because Γ_0 is a line but W_0 of Case (IV) has no line.

We consider the Case (I). We set $W_1 := W_0$ and W_0 is Σ_d for a d, and H is linearly equivalent to $T+((r-1+d)/2)F$. We set $f_1 := \Phi_{\tilde{L}}$. Since $\tilde{L} \cdot E=1$, we see that $\tilde{M}$ satisfies one of the following:

(I-i) $E \cdot f_1^* T=1$ and $E \cdot f_1^* F=0$.

(I-ii) $E \cdot f_1^* T=0$ and $((r-1+d)/2) E \cdot f_1^* F=1$.

(I-iii) $E \cdot f_1^* T<0$.

In the case (I-ii), we have $\Delta(M, L)=3$, and this case was already classified in [12]. In the case (I-i), consider the natural morphism $\tilde{M} \to W_1 \to \boldsymbol{P}^1$, and set $\Gamma := f_1(E) \subset W_1$. We have $\Gamma=\Gamma_0$ and Γ is a fiber of $W_1 \to \boldsymbol{P}^1$. The proof of the following theorem is similar to that of [12, Theorem 2]

THEOREM 2. *In the case* (I-i), *$f_1^*\Gamma$ is of one of the following types*:

(a) $f_1^*\Gamma=2E+X_1+X_2$.

(b) $f_1^*\Gamma=2E+2X$.

(c) $f_1^*\Gamma=E+E^*+X$, *and* $E \cdot E^*=0$.

(d) $f_1^*\Gamma=E+E^*$, *and* $E \cdot E^*=1$.

Here X_i and X are irreducible reduced curves which are contracted by f_1, and E^* is an irreducible reduced curve birational to Γ.

In the case (I-iii), we have $E \cdot f_1^* T<0$ and f_1 is generically two-to-one, hence we have $-d=T^2<0$, and $T=f_1(E)$. Moreover, since $\tilde{L} \cdot E=1$, we have $r=d+3$.

THEOREM 3. *In the case* (I-iii), *f_1^*T and d are of one of the following types*:

(e) $f_1^*T=2E+X$, *and* $d=1$.

(f) $f_1^*T=2E$, *and* $d=2$.

(g) $f_1^*T=E+E^*$, $E \cdot E^*=0$, *and* $d=1$.

Here X is an irreducible reduced curve contracted by f_1, and E^* is an irreducible reduced curve birational to T. Moreover, we obtain $\Delta(M, L)=4$ for the cases (e) *and* (g), *while $\Delta(M, L)=5$ for the case* (f).

PROOF. Let D (resp. E^*) be the sum of the irreducible reduced curves contracted (resp. not contracted) by f_1. Since $2=f_1^*T \cdot f_1^*F=\varepsilon E \cdot f_1^*F+E^* \cdot f_1^*F$, we obtain $\varepsilon=2$ or $\varepsilon=1$. When $\varepsilon=2$, we have $E^*=0$, since $E^* \cdot f^*F=0$. Since $-d=E \cdot f_1^*T$, we have $d=1$

or 2. If $d=1$, then we have $r=d+3=4$ and so $\Delta(M, L)=4$. Moreover, D is an irreducible reduced curve X by $1=D\cdot E$. If $d=2$, then we have $r=d+3=5$ and so $\Delta(M, L)=5$. Moreover, we get $D=0$ by $0=D\cdot E$. When $\varepsilon=1$, we see that E^* is an irreducible reduced curve since $E^*\cdot f^*F=1$. Since $-d=E\cdot f_1^*T$, we have $d=1$. Thus we have $r=d+3=4$ and so $\Delta(M, L)=4$. Moreover, we get $D=0$ and $E\cdot E^*=0$ because of $0=E\cdot E^*+D\cdot E$.
q.e.d.

We consider the Case (II). We obtain a desingularization of $\Phi_{(r-1)F+T}: \Sigma_{r-1}\to W_0$ by the method in [8, p. 46] and [9, Lemma 1.5]. We can lift $\Phi_{\tilde{L}}: \tilde{M}\to W_0$ to $f_1: \tilde{M}\to W_1=\Sigma_{r-1}$ and E is contained in a fiber of $\tilde{M}\to W_1\to \boldsymbol{P}^1$. We have $\Gamma:=f_1(E)=\Phi^*_{(r-1)F+T}\Gamma_0-T$.

THEOREM 4. *In the Case* (II), $f_1^*\Gamma$ *is of one of the following types*:
(a) $f_1^*\Gamma=2E+X_1+X_2$.
(b) $f_1^*\Gamma=2E+2X$.
(c) $f_1^*\Gamma=E+E^*+X$, *and* $E\cdot E^*=0$.
(d) $f_1^*\Gamma=E+E^*$, *and* $E\cdot E^*=1$.
Here X_i *and* X *are irreducible reduced curves contracted by* f_1, *while* E^* *is an irreducible reduced curve birational to* Γ.

The proof of the above theorem is similar to that of [12, Theorem 2].

3. Classification in Case (I), (a). From this section on, we use the same notation for a divisor and its total transform, when there is no fear of confusion. In this section, we assume that $f_1^*\Gamma$ is in Case (I), (a), i.e., $f_1^*\Gamma=2E+X_1+X_2$. Then the morphism $f_1: \tilde{M}\to W_1$ is not finite. Hence we lift it to a finite morphism from $\tilde{M}$ to W, where W is obtained from W_1 by the blowing up at two points.

We first study the inverse image of $x_i:=f_1(X_i)$ by f_1

LEMMA 1. *The inverse image of* x_i *by* f_1 *is a divisor.*

PROOF. The curves X_1 and X_2 are contracted to distinct points by f_1. Thus we have $X_1\cap X_2=\varnothing$, and we get $X_1\cdot X_2=0$. Therefore we have $0=f_1^*\Gamma\cdot X_i=(2E+X_1+X_2)=2+X_i^2$, and hence $X_i^2=-2$. Let S' and S'' be general members of the linear system which consists of those divisors of the linear system $|T+((r-1+d)/2)F|$ of W_1 which contains x_i. S' and S'' intersect each other transversely at x_i. By $S'\cdot\Gamma=S''\cdot\Gamma=1$ and the generality of S' and S'', the other intersections of S' and S'' are outside Γ. By Proposition 2 the morphism $f_{|\tilde{M}\setminus f_1^{-1}\Gamma}: \tilde{M}\setminus f_1^{-1}\Gamma\to W_1\setminus\Gamma$ is a finite double covering. Let $f_1^*S'=: C'+\mu_1X_i$ and $f_1^*S''=: C''+\mu_2X_i$. Hence $\tilde{L}^2-2\le C'\cdot C''=(S'-\mu_1X_i)\cdot(S''-\mu_2X_i)=S'\cdot S''-2\mu_1\mu_2$. Thus we have $\mu_1=\mu_2=1$. Therefore C' and C'' are elements of $|\tilde{L}-X_i|$, and have no intersection of Γ, since $C'\cdot C''=\tilde{L}^2-2$. Thus $\mathrm{Bs}|\tilde{L}-X_i|$ is empty.
q.e.d.

Let $\sigma: W\to W_1$ be the blowing up at $x_1=f_1(X_1)$ and $x_2=f_1(X_2)$, and let Z_1 and

Z_2 be the exceptional curves over x_1 and x_2, respectively. Let $\hat{M} := \tilde{M}$. We denote by $\tilde{\Gamma}$ the proper transform of Γ. The inverse image of x_i by f_1 is X_i because of the above lemma. Hence by the universality of the blowing up, there exists a morphism $f: \hat{M} \to W$ such that $f = \sigma \circ f_1$ and $f^* Z_i = X_i$. Then f is a finite double covering. Since $\mathrm{Pic}(W) = \boldsymbol{Z}T \oplus \boldsymbol{Z}F \oplus \boldsymbol{Z}Z_1 \oplus \boldsymbol{Z}Z_2$, the branch locus is linearly equivalent to $2A = 2\alpha T + 2\beta F - 2\gamma_1 Z_1 - 2\gamma_2 Z_2$ for a unique quadruple $(\alpha, \beta, \gamma_1, \gamma_2)$ of integers.

THEOREM 5. *In Case* (I), (a) *let* $\sigma: W \to W_1$ *be the blowing up at the two points* $x_1 = f_1(X_1)$ *and* $x_2 = f_1(X_2)$ *with the exceptional curves* Z_1 *and* Z_2 *over* x_1 *and* x_2, *respectively. Let* $\tilde{T}$ *be the proper transform of* T. *Then* $\hat{M}$ $(=\tilde{M})$ *is a finite double covering of* W. *The branch locus* B *is linearly equivalent to* $2\alpha T + 2\beta F - 2\gamma_1 Z_1 - 2\gamma_2 Z_2$. *The integers* α, β, γ_1 *and* γ_2 *satisfy* $\alpha = \gamma_1 + \gamma_2 - 1$, $2\gamma_i - 1 \geq 0$ *and* $(\alpha - 2)(r - d - 1) + 2(\beta - d - 2) \geq 0$. *Moreover, we have the following*:

(1) *When* $d > 0$ *and* $x_1, x_2 \notin T$, *we have* $2\beta \geq 2\alpha d + 1$.

(2) *When* $d > 0$, $x_2 \in T$, *and* $\tilde{T}$ *is a component of* B, *we have* $2\beta - 1 = (2\alpha - 1)d + (2\gamma_2 - 2)$.

(3) *When* $d > 0$, $x_2 \in T$ *and* $\tilde{T}$ *is not a component of* B, *we have* $2\beta - 1 \geq 2\alpha d + (2\gamma_2 - 1)$.

(4) *When* $d = 0$, *we have* $\beta - \gamma_i \geq 0$.

Conversely, for each quadruple $(\alpha, \beta, \gamma_1, \gamma_2)$ *satisfying these conditions, there exists a polarized surface* (M, L) *giving rise to the quadruple.*

The first half of Theorem 5 is proved as follows: Let $A = \alpha T + \beta F - \gamma_1 Z_1 - \gamma_2 Z_2$. Then clearly $B \in |2A|$. Since $f^*\tilde{\Gamma} = f^*\Gamma - f^*Z_1 - f^*Z_2 = 2E$, the curve $\tilde{\Gamma}$ is a component of B. Then we have $B = B' + \tilde{\Gamma}$, where B' is a nonsingular curve. Since the branch locus B is nonsingular, we have $B \cap \tilde{\Gamma} = \emptyset$. Hence we have $0 = B' \cdot \tilde{\Gamma} = (2A - \tilde{\Gamma}) \cdot \tilde{\Gamma} = 2(\alpha - \gamma_1 - \gamma_2 + 1)$, and obtain $\alpha = \gamma_1 + \gamma_2 - 1$. Z_i is not a component of B' because $Z_i \cdot \tilde{\Gamma} = 1$. Thus we obtain $0 \leq B' \cdot Z_i = 2\gamma_i - 1$. Since $g(M, L) \geq \Delta(M, L)$, we have $(\alpha - 2)(r - d - 1) + 2(\beta - d - 2) \geq 0$. Suppose $d > 0$ and $x_1, x_2 \notin T$. T is not a component of B' because $T \cdot \tilde{\Gamma} = 1$. Thus $0 \leq B' \cdot T = (2\alpha T + (2\beta - 1)F - (2\gamma_1 - 1)Z_1 - (2\gamma_2 - 1)Z_2) \cdot T = -2\alpha d + 2\beta - 1$. Therefore we obtain $2\beta \geq 2\alpha d + 1$. Suppose $d > 0$, $x_2 \in T$ and that $\tilde{T}$ is a component of B. Then we have $B' = B'' + \tilde{T}$, where B'' is a nonsingular curve. Then $0 = B'' \cdot \tilde{T} = ((2\alpha - 1)T + (2\beta - 1)F - (2\gamma_1 - 1)Z_1 - (2\gamma_2 - 2)Z_2) \cdot (T - Z_2) = -(2\alpha - 1)d + (2\beta - 1) - (2\gamma_2 - 2)$. Thus we have $(2\beta - 1) = (2\alpha - 1)d + (2\gamma_2 - 2)$. Moreover, d is odd. Suppose $d > 0$ and $x_2 \in T$ and that $\tilde{T}$ is not a component of B'. Thus $0 \leq B' \cdot \tilde{T} = (2\alpha T + (2\beta - 1)F - (2\gamma_1 - 1)Z_1 - (2\gamma_2 - 1)Z_2) \cdot (T - Z_2) = -2\alpha d + (2\beta - 1) - (2\gamma_2 - 1)$. Therefore $2\alpha d + (2\gamma_2 - 1) \leq 2\beta - 1$. Suppose $d = 0$. If $\tilde{T}_i \in |T - Z_i|$ is a component of B', then $B' \cdot \tilde{T}_i = \tilde{T}_i^2 = -1$. If $\tilde{T}_i$ is not a component of B', then $0 \leq B' \cdot \tilde{T}_i$. In either case we have $-1 \leq B' \cdot \tilde{T}_i = 2(\beta - \gamma_i)$. Thus we have $\beta - \gamma_i \geq 0$. Moreover, $\tilde{T}_i$ is not a component of B'.

We now consider the existence of such polarized surfaces. Suppose $d > 0$ and $x_1, x_2 \notin T$. Because $B' \sim (2\beta - 2\alpha d - 1)F + (2\gamma_1 - 1)(T + dF - Z_1) + (2\gamma_2 - 1)(T + dF - Z_2)$ and $2\beta \geq 2\alpha d + 1$ and $2\gamma_i - 1 \geq 0$, it suffices to prove the following:

PROPOSITION 3. $\mathrm{Bs}\,|T+dF-Z_i|$ *is empty.*

PROOF. We see that $|T+dF-Z_1|\ni(d-1)F+T+\tilde{\Gamma}+Z_2$ and $(T+dF-Z_1)\cdot T=(T+dF-Z_1)\cdot\tilde{\Gamma}=(T+dF-Z_1)\cdot Z_2=0$. Hence it suffices to prove that T, $\tilde{\Gamma}$ and Z_2 are not fixed parts of $|T+dF-Z_1|$. By the exact sequence

$$0\to\mathcal{O}_W(T+dF-Z_1)\to\mathcal{O}_W(T+dF)\to\mathcal{O}_{Z_1}(T+dF)\to 0$$

we have $h^0(T+dF-Z_1)\geq h^0(T+dF)-1=d+1$. By the exact sequence

$$0\to\mathcal{O}_W(T+(d-1)F)\to\mathcal{O}_W(T+(d-1)F+\tilde{\Gamma})\to\mathcal{O}_{\tilde{\Gamma}}(T+(d-1)F+\tilde{\Gamma})\to 0$$

we have $h^0(T+(d-1)F+\tilde{\Gamma})=h^0(T+(d-1)F)=d$. Thus Z_2 is not a fixed component. Similarly $\tilde{\Gamma}$ is not a fixed component. By the exact sequence

we have $h^0((d-1)F+\tilde{\Gamma}+Z_2)=h^0((d-1)F+Z_2)$. Similarly we have $h^0((d-1)F+Z_2)=h^0((d-1)F)=d$. Hence we have $h^0((d-1)F+\tilde{\Gamma}+Z_2)=d$. Thus T is not a fixed component. q.e.d.

Suppose $d>0$, $x_2\in T$ and that $\tilde{T}$ is a component of B. Because $B=\tilde{\Gamma}+\tilde{T}+B''$, where $B''\sim(2\alpha-1)T+(2\beta-1)F-(2\gamma_1-1)Z_1-(2\gamma_2-2)Z_2$, it suffices to prove that $|B''|$ is base point free. We have $B''\sim(2\gamma_1-1)(T+dF-Z_1)+(2\gamma_2-2)(T+(d+1)F-Z_2)$. Hence it suffices to prove that $\mathrm{Bs}\,|T+dF-Z_1|$ and $\mathrm{Bs}\,|T+(d+1)F-Z_2|$ are empty.

PROPOSITION 4. $\mathrm{Bs}\,|T+(d+1)F-Z_2|$ *is empty.*

PROOF. We see that $\tilde{T}+|T+(d+1)F|$ and $\tilde{\Gamma}+Z_1+|T+dF|$ are subsets of $|T+(d+1)F-Z_2|$. Thus the base points are on $\tilde{T}\cap(\tilde{\Gamma}\cup Z_1)$. Since $\tilde{T}\cdot\tilde{\Gamma}=\tilde{T}\cdot Z_1=0$, we see that $\tilde{T}\cap\tilde{\Gamma}=\tilde{T}\cap Z_1=\varnothing$. q.e.d.

PROPOSITION 5. $\mathrm{Bs}\,|T+dF-Z_1|$ *is empty.*

PROOF. We see that $\tilde{T}+\tilde{\Gamma}+2Z_2\in|T+dF-Z_1|$ and $(T+dF-Z_1)\cdot\tilde{T}=(T+dF-Z_1)\cdot\tilde{\Gamma}=(T+dF-Z_1)\cdot Z_2=0$. Hence it suffices to prove that $\tilde{T}$, $\tilde{\Gamma}$ and Z_2 are not fixed parts of $|T+dF-Z_1|$. By the exact sequence

$$0\to\mathcal{O}_W(T+dF-Z_1)\to\mathcal{O}_W(T+dF)\to\mathcal{O}_{Z_1}(T+dF)\to 0$$

we have $h^0(T+dF-Z_1)\geq h^0(T+dF)-1=d+1$. By the exact sequences

$$0\to\mathcal{O}_W(dF-Z_1)\to\mathcal{O}_W(dF-Z_1+Z_2)\to\mathcal{O}_{Z_2}(dF-Z_1+Z_2)\to 0$$

$$0\to\mathcal{O}_W((d-1)F+Z_2)\to\mathcal{O}_W(dF-Z_1)\to\mathcal{O}_{\tilde{\Gamma}}(dF-Z_1)\to 0$$

and

$$0\to\mathcal{O}_W((d-1)F)\to\mathcal{O}_W((d-1)F+Z_2)\to\mathcal{O}_{Z_2}((d-1)F+Z_2)\to 0$$

we have $h^0(dF-Z_1+Z_2)=d$. Hence $\tilde{T}$ is not a fixed part. As to Z_2 and $\tilde{\Gamma}$, the proof is similar to Proposition 5. q.e.d.

Supposed $d>0$ and $x_2 \in T$ and that $\tilde{T}$ is not a component of B. Because $B=\tilde{\Gamma}+B'$, where $B' \sim (2\beta-1-2\alpha d-(2\gamma_2-1))F+(2\gamma_1-1)(T+dF-Z_1)+(2\gamma_2-1)(T+(d+1)F-Z_2)$, it suffices to prove that $|B'|$ is base point free. The proof is similar to the above case.

Suppose $d=0$. We see that $B' \sim 2(\beta-\gamma_1)F+(2\gamma_1-1)(T+F-Z_1)+(2\gamma_2-1)(T-Z_2)$.

PROPOSITION 6. $\mathrm{Bs}\,|T+F-Z_1|$ *is empty.*

PROOF. Since $(T+F-Z_1)\cdot\tilde{T}_1=(T+F-Z_1)\cdot(T-Z_1)=0$, it suffices to prove that $\tilde{T}$ is not a fixed part of $|T+F-Z_1|$. By the exact sequence

$$0 \to \mathcal{O}_W(T+F-Z_1) \to \mathcal{O}_W(T+F) \to \mathcal{O}_{Z_1} \to 0$$

we see that $h^0(T+F-Z_1) \geq h^0(T+F)-1=3$. Because $(T+F-Z_1)-\tilde{T} \sim F$, we have $h^0(F)=2$. q.e.d.

Hence we have $\mathrm{Bs}\,|B'| \subset \tilde{T}_2$. Moreover, $\mathrm{Bs}\,|B'| \subset \tilde{T}_1$ similarly. Since $\tilde{T}_1 \cap \tilde{T}_2=\varnothing$, we conclude that $\mathrm{Bs}\,|B'|$ is empty.

We prove the existence of the ample divisor L by a method similar to [12, Proposition 4]. The invariants of M are as follows:

$$K_M^2=2(\alpha-2)(-d\alpha+2\beta-4)-2(\gamma_1-1)^2-2(\gamma_2-1)^2+1\,,$$

$$p_g(M)=\frac{1}{4}(2(\alpha-2)(-d\alpha+2\beta-4)-2(\gamma_1-1)^2-2(\gamma_2-1)^2+1)$$

$$+\frac{1}{2}(-d+1)\alpha+\beta-\frac{11}{4}\,,$$

$$q(M)=0\,.$$

4. Classification in Case (I), (b). In this section, we assume that $f_1^*\Gamma$ is in Case (I), (b). Then the morphism $f_1:\tilde{M}\to W_1$ is not finite. Hence we lift it to a finite morphism from $\hat{M}$ to W.

We first study the inverse image.

PROPOSITION 7. *The inverse image of x by f_1 consists of a divisor and an isolated point.*

PROOF. Since $0=f_1^*F\cdot X=(2E+2X)\cdot X$, we have $X^2=-1$. Let S' and S'' be general members of $|T+((r-1+d)/2)F|$ on W_1 containing x. By $S'\cdot\Gamma=S''\cdot\Gamma=1$ and the generality of S' and S'' the other intersections of S' and S'' are outside Γ. By Proposition 2 the morphism $f_{|\tilde{M}\setminus f_1^{-1}\Gamma}:\tilde{M}\setminus f_1^{-1}\Gamma\to W_1\setminus\Gamma$ is a finite double covering. Let $f_1^*S'=:C'+\mu_1X$ and $f_1^*S''=:C''+\mu_2X$. Since $E\cdot f_1^*S'=E\cdot f_1^*S''=1$, we have $\mu_1=\mu_2=1$ and $C'\cap E=C''\cap E=\varnothing$. Thus $0=X\cdot f_1^*S'=X\cdot C'-1$, so that $X\cap C'=y'$ is a single point. Similarly we see that $X \cap C'' =: y''$ is a single point. Therefore C' and C'' are elements of $|\tilde{L} - X|$, and have intersection of Γ since $C' \cdot C'' = \tilde{L}^2 - 1$. q.e.d.

Thus the inverse image of x by f_1 has an isolated part. Let y be the base point of $|\tilde{L} - X|$. Denote by $\rho : \hat{M} \to \tilde{M}$ the blowing up at y with the exceptional curve Y over y. Let $\tilde{X}$ be the proper transform of X. By Proposition 7, the inverse image of x by $\rho \circ f_1$ is the divisor $\tilde{X} + 2Y$. Let $\sigma_1 : W_2 \to W_1$ be the blowing up at x, and let Z_1 be the exceptional curve over x and $\tilde{\Gamma}$ the proper transform of Γ, respectively. By the universality of the blowing up, there exists a morphism $f_2 : \hat{M} \to W_2$ satisfying $f_1 \circ \rho = \sigma_1 \circ f_2$ and $f_2^* Z_1 = \tilde{X} + 2Y$.

PROPOSITION 8. *The image of $\tilde{X}$ by f_2 is equal to the intersection z of Z_1 with $\tilde{\Gamma}$, and the morphism $f_{2|Y} : Y \to Z_1$ is an isomorphism.*

PROOR. Let q be a point on $Z_1 \cong \boldsymbol{P}^1$. Since $\deg(f_{2|Y})^* q = \deg(f_{2|Y})^*(-Z_1)_{|Z_1} = -f_2^* Z_1 \cdot Y = 1$, we see that $f_{2|Y} : Y \to Z_1 \cong \boldsymbol{P}^1$ is an isomorphism. On the other hand, since $\deg(f_{2|\tilde{X}})^* q = \deg(f_{2|\tilde{X}})^*(-Z_1)_{|Z_1} = -f_2^* Z_1 \cdot \tilde{X} = 0$, the image of $\tilde{x}$ by $f_{2|\tilde{X}} : \tilde{X} \to Z_1 \cong \boldsymbol{P}^1$ is a point. Moreover, since $f_2^* \tilde{\Gamma} = f_2^* \Gamma - f_2^* Z_1 = 2E + \tilde{X}$, we have $f_2(\tilde{X}) \in \tilde{\Gamma}$. Hence $f_2(\tilde{X}) \in \tilde{\Gamma} \cap Z_1$. q.e.d.

Consequently, the morphism f_2 is not finite. Hence we carry out the same operation again. Let $\sigma_2 : W \to W_2$ be the blowing up at z, and let Z_2 be the exceptional curve over z. We denote by $\hat{\Gamma}$ and $\tilde{Z}_1$ the proper transform of $\tilde{\Gamma}$ and Z_1, respectively. By Proposition 8, the inverse image of z is $\tilde{X}$. Thus by the universality of the blowing up, there exists a morphism $f : \hat{M} \to W$ such that $f_2 = \sigma_2 \circ f$ and $f^* Z_2 = \tilde{X}$. Then f is a finite double covering. Since $\mathrm{Pic}(W) = \boldsymbol{Z}T \oplus \boldsymbol{Z}F \oplus \boldsymbol{Z}Z_1 \oplus \boldsymbol{Z}Z_2$, the branch locus is linearly equivalent to $2A = 2\alpha T + 2\beta F - 2\gamma_1 Z_1 - 2\gamma_2 Z_2$ for a unique quadruple $(\alpha, \beta, \gamma_1, \gamma_2)$ of integers.

THEOREM 6. *In Case* (I), (b) *let $\rho : \hat{M} \to \tilde{M}$ be the blowing up at the base point y of the linear system $|\tilde{L} - X|$. Let $\sigma_1 : W_2 \to W_1$ be the blowing up at $x = f_1(X)$, and let Z_1 be the exceptional curve over x, and $\tilde{\Gamma}$ the proper transform of Γ, respectively. The intersection of $\tilde{\Gamma}$ and Z_1 is a point z. Let $\sigma_2 : W \to W_2$ be the blowing up at z, and let Z_2 be the exceptional curve over z. Then $\hat{M}$ is a finite double covering over W. The branch locus B is linearly equivalent to $2\alpha T + 2\beta F - 2\gamma_1 Z_1 - 2\gamma_2 Z_2$. The integers α, β, γ_1 and γ_2 satisfy $\alpha = \gamma_1 + \gamma_2 - 1$, $\gamma_1 = \gamma_2 - 1$, $\gamma_2 - 1 \geq 0$ and $(\alpha - 2)(r - d - 1) + 2(\beta - d - 2) \geq 0$. Moreover, we have the following*:

(1) *When $d > 0$, $x \notin T$, we have $2\alpha d + 1 \leq 2\beta$.*

(2) *When $d > 0$ and $x \in T$, we have $2(\alpha d + \gamma_1) \leq 2\beta - 1$.*

(3) *When $d = 0$, we have $2\beta - \alpha - 1 \geq 0$.*

Conversely, for each quadruple $(\alpha, \beta, \gamma_1, \gamma_2)$ satisfying these conditions, there exists a polarized surface (M, L) giving rise to the quadruple.

The first half of Theorem 6 is proved as follows: Let $A = \alpha T + \beta F - \gamma_1 Z_1 - \gamma_2 Z_2$.

Then clearly $B \in |2A|$. Since $f^*\hat{\Gamma} = f^*\Gamma - f^*Z_1 - f^*Z_2 = 2E$, the curve $\hat{\Gamma}$ is a component of B. Since $f^*Z_1 - f^*Z_2 = 2Y$, the curve $\tilde{Z}_1$ is a component of B. Thus we have $B = B' + \hat{\Gamma} + \tilde{Z}_1$, where B' is a nonsingular curve. Since the branch locus B is nonsingular, we have $B' \cap \hat{\Gamma} = B' \cap \tilde{Z}_1 = \varnothing$. Hence we have $\alpha = \gamma_1 + \gamma_2 - 1$ and $\gamma_1 = \gamma_2 - 1$. Because $Z_2 \cdot \tilde{Z}_1 = 1$, we see that Z_2 is not a component of B'. Thus we have $0 \leq \gamma_2 - 1$. Since $g(M, L) \geq \Delta(M, L)$, we have $(\alpha - 2)(r - d - 1) + 2(\beta - d - 2) \geq 0$. Suppose $d > 0$ and $x \notin T$. Because $T \cdot \hat{\Gamma} = 1$, we see that T is not a component of B'. Thus $0 \leq B' \cdot T = -2\alpha d + (2\beta - 1)$. Suppose $d > 0$ and $x \in T$. Since $\tilde{T} \cdot \tilde{Z}_1 = 1$, we see that $\tilde{T}$ is not a component of B'. Thus we have $0 \leq B' \cdot \tilde{T} = -2\alpha d - 2\gamma_1 + (2\beta - 1)$. Suppose $d = 0$. Since $\tilde{Z}_1 \cdot \tilde{T} = 1$ for $\tilde{T} \in |T - Z_1|$, we see that $\tilde{T}$ is not a component. Thus we have $0 \leq \tilde{T} \cdot B' = 2\beta - 2\gamma_1 - 1$.

We now consider the existence of such polarized surfaces. Suppose $d > 0$ and $x \notin T$. Because $B' \sim (2\beta - 1 - 2\alpha d)F + \alpha(2T + 2dF - Z_1 - Z_2)$, $2\beta - 1 - 2\alpha d \geq 0$ and $\alpha \geq 0$, it suffices to prove that $|2T + 2dF - Z_1 - Z_2|$ has no base point. Since $(T + (2d-1)F) + T + \hat{\Gamma} \in |2T + 2dF - Z_1 - Z_2|$ and $(2T + 2dF - Z_1 - Z_2) \cdot T = (2T + 2dF - Z_1 - Z_2) \cdot \hat{\Gamma} = 0$, it suffices to prove that T and $\hat{\Gamma}$ are not fixed parts of $|2T + 2dF - Z_1 - Z_2|$. By the exact sequence

$$0 \to \mathcal{O}_W(2T + 2dF - Z_1) \to \mathcal{O}_W(2T + 2dF) \to \mathcal{O}_{Z_1}(2T + 2dF) \to 0$$

we see that $h^0(2T + 2dF - Z_1) \geq h^0(2T + 2dF) - 1$. By the exact sequence

$$0 \to \mathcal{O}_W(2T + 2dF - Z_1 - Z_2) \to \mathcal{O}_W(2T + 2dF - Z_1) \to \mathcal{O}_{Z_2}(2T + 2dF - Z_1) \to 0$$

we see that $h^0(2T + 2dF - Z_1 - Z_2) \geq h^0(2T + 2dF - Z_1) - 1$. Thus $h^0(2T + 2dF - Z_1 - Z_2) \geq h^0(2T + 2dF) - 2 = 3d + 1$. Moreover, by the exact sequence

$$0 \to \mathcal{O}_W(T + (2d-1)F) \to \mathcal{O}_W(\hat{\Gamma} + T + (2d-1)F) \to \mathcal{O}_{\hat{\Gamma}}(\hat{\Gamma} + T + (2d-1)F) \to 0$$

we have $h^0(T + (2d-1)F) = 3d$. Consequently, T is not a fixed component of $|2T + 2dF - Z_1 - Z_2|$. Similarly we see that $\hat{\Gamma}$ is not a fixed component.

Supposed $d > 0$ and $x \in T$. Because $B' \sim (2\beta - 1 - \alpha(2d+1))F + \alpha(2T + (2d+1)F - Z_1 - Z_2)$, it suffices to prove that $|2T + (2d+1)F - Z_1 - Z_2|$ is base point free. $2\tilde{T} + \tilde{Z}_1 + |(2d+1)F|$ and $\hat{\Gamma} + |2(T + dF)|$ are sublinear systems of $|2T + (2d+1)F - Z_1 - Z_2|$. Since $\hat{\Gamma} \cap (\tilde{T} \cup \tilde{Z}_1)$ is empty, $|2T + (2d+2)F - Z_1 - Z_2|$ has no base point.

Suppose $d = 0$. We see that $B' \sim (2\beta - \alpha - 1)F + \alpha(2T + F - Z_1 - Z_2)$. Thus it suffices to prove that $|2T + F - Z_1 - Z_2|$ is base point free. We see that $2\tilde{T} + \tilde{Z}_1 + F \in |2T + F - Z_1 - Z_2|$. Since $(2T + F - Z_1 - Z_2) \cdot \tilde{T} = (2T + F - Z_1 - Z_2) \cdot \tilde{Z}_1 = 0$, it suffices to prove that $\tilde{T}$ and $\tilde{Z}_1$ are not fixed components of $|2T + F - Z_1 - Z_2|$. By the exact sequence

$$0 \to \mathcal{O}_W(2T + F - Z_1) \to \mathcal{O}_W(2T + F) \to \mathcal{O}_{Z_1}(2T + F) \to 0$$

we have $h^0(2T + F - Z_1) \geq h^0(2T + F) - 1$. By the exact sequence

$$0 \to \mathcal{O}_W(2T + F - Z_1 - Z_2) \to \mathcal{O}_W(2T + F - Z_1) \to \mathcal{O}_{Z_2}(2T + F - Z_1) \to 0$$

we see that $h^0(2T + F - Z_1 - Z_2) \geq h^0(2T + F - Z_1) - 1$. Thus we have $h^0(2T + F - Z_1 -$

$Z_2) \geq h^0(2T+F)-2=4$. On the other hand by the exact sequence

$$0 \to \mathcal{O}_W(T+Z_1+Z_2) \to \mathcal{O}_W(T+F-Z_2) \to \mathcal{O}_{\tilde{\Gamma}}(T+F-Z_2) \to 0$$

we see that $h^0(T+Z_1+Z_2)=h^0(T+F-Z_2)$. By the exact sequence

$$0 \to \mathcal{O}_W(T+Z_1) \to \mathcal{O}_W(T+Z_1+Z_2) \to \mathcal{O}_{Z_2}(T+Z_1+Z_2) \to 0$$

we see that $h^0(T+Z_1)=h^0(T+Z_1+Z_2)$. By the exact sequence

$$0 \to \mathcal{O}_W(T) \to \mathcal{O}_W(T+Z_1) \to \mathcal{O}_{Z_1}(T+Z_1) \to 0$$

we see that $h^0(T)=h^0(T+Z_1)$. Thus we have $h^0(T+F-Z_2)=h^0(T)=2$. Consequently, $\tilde{T}$ is not a fixed component of $|2T+F-Z_1-Z_2|$.

Moreover, by the exact sequence

$$0 \to \mathcal{O}_W(T+F-Z_1) \to \mathcal{O}_W(2T+F-2Z_1) \to \mathcal{O}_{\tilde{T}}(2T+F-2Z_1) \to 0$$

we see that $h^0(T+F-Z_1)=h^0(2T+F-2Z_1)$. By the exact sequence

$$0 \to \mathcal{O}_W(T) \to \mathcal{O}_W(T+F-Z_1) \to \mathcal{O}_{\tilde{\Gamma}} \to 0$$

we see that $3=h^0(T)+1 \geq h^0(T+F-Z_1)$. Thus we have $3 \geq h^0(2T+F-2Z_1)$. Consequently, $\tilde{Z}_1$ is not a fixed part of $|2T+F-Z_1-Z_2|$. Hence $\mathrm{Bs}\,|B'|$ is empty.

We prove the existence of an ample divisor L by a method similar to [12, Proposition 7]. The invariants are as follows:

$$\begin{aligned}
K_M^2 &= 2(\alpha-2)(-d\alpha+2\beta-4)-2(\gamma_1-1)^2-2(\gamma_2-1)^2+2\,, \\
p_g(M) &= \frac{1}{4}(2(\alpha-2)(-d\alpha+2\beta-4)-2(\gamma_1-1)^2-2(\gamma_2-1)^2+2) \\
&\quad +\frac{1}{2}(-d+1)\alpha+\beta-\frac{12}{4}\,, \\
q(M) &= 0\,.
\end{aligned}$$

5. Classification in Case (I), (c). In this section, we assume that $f_1^*\Gamma$ is in Case (I), (c), i.e., $f_1^*\Gamma=E+E^*+X$ and $E \cdot E^*=0$. We divide surfaces of this type into two subtypes.

PROPOSITION 9. *There are the following two subtypes*:
(c-1) $X \cdot E^*=1$,
(c-2) $X \cdot E^*=0$.

PROOF. Let S be a general member of $|T+((r-1+d)/2)F|$ on W_1 containing x. Let $f_1^*S=:C+\mu X$. Since $E \cdot f_1^*S=1$, we have $\mu=1$ and $C \cap E=\varnothing$. We have $X \cdot E^*+C \cdot E^*=1$ because $\tilde{L} \cdot E^*=1$. Since X is not a component of E^*, we have $X \cdot E^* \geq 0$. On the other hand, since $|\tilde{L}-X|$ has no fixed component, we have $C \cdot E^* \geq 0$. q.e.d.

First we treat the subtype (c-1). The morphism $f_1 : \tilde{M} \to W_1$ is not finite. Hence we lift it to a finite morphism. Let $\sigma : W \to W_1$ be the blowing up at $x := f(X)$, and let Z_1 be the exceptional curve over x and $\tilde{\Gamma}$ the proper transform of Γ, respectively. There exists a double covering $f : \tilde{M} \to W_1$ such that $f_1 = \sigma \circ f$ and $f^*Z = X$ by the universality of the blowing up. Moreover, f is a finite double covering. Since $\mathrm{Pic}(W) = \boldsymbol{Z}T \oplus \boldsymbol{Z}F \oplus \boldsymbol{Z}Z$, the branch locus is linearly equivalent to $2A = 2\alpha T + 2\beta F - 2\gamma Z$ for a unique triple (α, β, γ) of integers.

THEOREM 7. *In the Case* (I), (c-1) *let* $\sigma : W \to W_1$ *be the blowing up at the point* $x = f(X)$ *with the exceptional curve* Z *over* x. *Let* $\tilde{T}$ *be the proper transform of* T. *Then* $\tilde{M}$ *is a finite double covering of* W. *The branch locus* B *is linearly equivalent to* $2\alpha T + 2\beta F - 2\gamma Z$. *The integers* α, β *and* γ *satisfy* $\alpha = \gamma$, $\gamma \geq 0$ *and* $(\alpha - 2)(r - d - 1) + 2(\beta - d - 2) \geq 0$. *Moreover, we have the following*:

(1) *When* $d > 0$ *and* $x \notin T$, *we have* $0 \leq -2\alpha d + 2\beta$.

(2) *When* $d > 0$, $x \in T$ *and that* $\tilde{T}$ *is a component of* B, *we have* $0 = -(2\alpha - 1)(d + 1) + 2\beta$ *and* $2\gamma - 1 \geq 0$.

(3) *When* $d > 0$, $x \in T$ *and that* $\tilde{T}$ *is not a component of* B, *we have* $0 \leq -2\alpha d + 2\beta - 2\alpha$.

(4) *When* $d = 0$, *we have* $0 \leq \beta - \gamma$.

Conversely, for each triple (α, β, γ) *satisfying these conditions, there exists a polarized surface* (M, L) *giving rise to the triple.*

To prove the first half of Theorem 7 let $A = \alpha T + \beta F - \gamma Z$. Since $f^*\tilde{\Gamma} = E + E^*$ and $E \cdot E^* = 0$, f is not branched along $\tilde{\Gamma}$. Thus we have $B \cap \tilde{\Gamma} = \emptyset$. Hence $0 = B \cdot \tilde{\Gamma} = 2\alpha - 2\gamma$. Since $\tilde{\Gamma} \cdot Z = 1$, we see that Z is not a component of B. Thus we have $0 \leq B \cdot Z = 2\gamma$. Since $g(M, L) \geq \Delta(M, L)$, we have $(\alpha - 2)(r - d - 1) + 2(\beta - d - 2) \geq 0$. Suppose $d > 0$ and $x \notin T$. Since $T \cdot \tilde{\Gamma} = 1$, we see that T is not a component of B. Thus we have $0 \leq B \cdot T = -2\alpha d + 2\beta$. Suppose $d > 0$, $x \in T$ and that $\tilde{T}$ is a component of B. Then we have $B = B' + \tilde{T}$, where B' is a nonsingular curve. We have $0 = B' \cdot \tilde{T} = -(2\alpha - 1)(d + 1) + 2\beta$. Moreover, we see that d is odd. Since $Z \cdot \tilde{T} = 1$, we see that Z is not a component of B'. Thus we have $0 \leq B' \cdot Z = 2\gamma - 1$. Suppose $d > 0$, $x \in T$ and that $\tilde{T}$ is not a component of B. We see that $0 \leq B \cdot \tilde{T} = -2\alpha(d + 1) + 2\beta$. Suppose $d = 0$. We see that $-1 \leq B \cdot \tilde{T} = 2\beta - 2\gamma$.

We consider the existence of such polarized surfaces. Suppose $d > 0$ and $x \notin T$. Because $B = (2\beta - 2\alpha d)F + 2\alpha(T + dF - Z)$, it suffices to prove that $|T + dF - Z|$ is base point free.

PROPOSITION 10. $\mathrm{Bs}\,|T + dF - Z|$ *is empty.*

PROOF. Since $T + (d - 1)F + \tilde{\Gamma} \in |T + dF - Z|$, it suffices to prove that T and $\tilde{\Gamma}$ are not fixed components. By the exact sequence

$$0 \to \mathcal{O}_W(T + dF - Z) \to \mathcal{O}_W(T + dF) \to \mathcal{O}_Z(T + dF) \to 0$$

we have $h^0(T + dF - Z) \geq h^0(T + dF) - 1 = d + 1$. On the other hand, by the exact sequence

$$0 \to \mathcal{O}_W((d-1)F) \to \mathcal{O}_W(T+(d-1)F) \to \mathcal{O}_T(T+(d-1)F) \to 0$$

we have $h^0((d-1)F)=h^0(T+(d-1)F)=d$. Similarly we have $h^0((d-1)F)=h^0(T+(d-1)F)=d$. Consequently, T and $\tilde{\Gamma}$ are not fixed components. q.e.d.

Suppose $d>0$, $x\in T$ and that $\tilde{T}$ is a component of B. Because $B'\sim(2\alpha-1)(T+(d+1)F-Z)$, it suffices to prove that $|T+(d+1)F-Z|$ is base point free. We see that $\tilde{T}+|(d+1)F|\subset|T+(d+1)F-Z|$ and $\tilde{\Gamma}+|T+dF|\subset|T+(d+1)-Z|$. Since $|(d+1)F|$ and $|T+dF|$ are base point free and $\tilde{T}\cdot\tilde{\Gamma}=0$, we see that $|T+(d+1)F-Z|$ is base point free. Suppose $d>0$, $x\in T$ and that $\tilde{T}$ is not a component of B. Because $B\sim(2\beta-2\alpha(d+1))F+2\alpha(T+(d+1)F-Z)$, it suffices to prove that $|T+(d+1)F-Z|$ is base point free. The proof is similar to the above situation. Suppose $d=0$. Because $B\sim(2\beta-2\alpha)F+2\alpha(T+F-Z)$, it suffice to prove that $\mathrm{Bs}\,|T+F-Z|=\varnothing$. We see that $\tilde{T}+F$ and $T+\tilde{\Gamma}$ belong to $|T+F-Z|$. Thus the base points are on $\tilde{T}\cap\tilde{\Gamma}$, but $\tilde{T}\cdot\tilde{\Gamma}=0$, a contradiction.

We prove the existence of an ample divisor L by a method similar to [12, Proposition 4]. The invariants are as follows:

$$K_M^2=2(\alpha-2)(-d\alpha+2\beta-4)-2(\gamma-1)^2+1\,,$$

$$p_g(M)=\frac{1}{4}(2(\alpha-2)(-d\alpha+2\beta-4)-2(\gamma-1)^2+1)+\frac{1}{2}(-d+1)\alpha+\beta-\frac{11}{4}\,,$$

$$q(M)=0\,.$$

Let us now treat the subtype (c-2). We lift $f_1:\tilde{M}\to W_1$ to a finite double covering $f:\hat{M}\to W$. We first study the inverse image.

PROPOSITION 11. *The inverse image of x by f_1 is a divisor and an isolated part on E^*.*

PROOF. Since $0=f^*F\cdot X=(E+E^*+X)\cdot X$, we have $X^2=-1$. Let S' and S'' be general members of $|T+((r-1+d)/2)F|$ on W_1 containing x. By $S'\cdot\Gamma=S''\cdot\Gamma=1$ and the generality of S' and S'' the other intersections of S' and S'' are outside of Γ. By Proposition 2 the morphism $f_{|\tilde{M}\setminus f_1^{-1}\Gamma}:\tilde{M}\setminus f_1^{-1}\Gamma\to W_1\setminus\Gamma$ is a finite double covering. Let $f_1^*S'=:C'+\mu_1X$ and $f_1^*S''=:C''+\mu_2X$. Since $E\cdot f_1^*S'=E\cdot f_1^*S''=1$, we have $\mu_1=\mu_2=1$ and $C'\cap E=C''\cap E=\varnothing$. Thus C' and C'' intersect each other on X or E^*. The intersection of C' and C'' is a base point y of $|\tilde{L}-X|$.

Suppose that the base point y of $|\tilde{L}-X|$ lies on X. Let $\sigma:\hat{M}\to\tilde{M}$ be the blowing up at y. By the above observation, the fixed part of $|\rho^*\tilde{L}-X|$ is Y. Moreover, the variable part $|\rho^*\tilde{L}-X-Y|$ has no base point. Thus the inverse image of x by $f_0\circ\rho$ is the divisor $X+Y$. Let $\sigma:W\to W_1$ be the blowing up at x, and let Z be the exceptional curve over x and $\tilde{\Gamma}$ the proper transform of Γ, respectively. By the universality of the blowing up, there exists a morphism $f:\hat{M}\to W$ such that $f_1\circ\rho=\sigma\circ f$ and $f^*Z=X+Y$. Moreover, $f^*\tilde{\Gamma}=f^*\Gamma-f^*Z=E+E^*-Y$, a contradiction to the fact that $f^*\tilde{\Gamma}$ is an effective divisor. q.e.d.

Let $\rho: \hat{M} \to \tilde{M}$ be the blowing up at the base point y of $|\tilde{L}-X|$, and let Y be the exceptional curve over y. We denote by $\tilde{E}^*$ the proper transform of E^*. Let $\sigma: W \to W_1$ be the blowing up at x, and let Z be the exceptional curve over z. We denote by $\tilde{\Gamma}$ the proper transform of Γ. By the argument similar to that for Case (I), (b), we get a morphism $f: \hat{M} \to W$ such that $f_1 \circ \rho = \sigma \circ f$ and $f^*Z = X + Y$. Let q be a point of $Z \cong \boldsymbol{P}^1$. We have $\deg(f_{|X})^*q = \deg(f_{|X})^*(-Z)_{|Z} = -f^*Z \cdot X = 1$. Thus $f_{|X}: X \to Z$ is surjective. Similarly, $f_{|Y}: Y \to Z$ is surjective. Thus f is a finite double covering. Since $\mathrm{Pic}(W) = \boldsymbol{Z}T \oplus \boldsymbol{Z}F \oplus \boldsymbol{Z}Z$, the branch locus is linearly equivalent to $2A = 2\alpha T + 2\beta F - 2\gamma Z$ for a unique triple (α, β, γ) of integers.

THEOREM 8. *In Case* (I), (c-2) *let* $\rho: \hat{M} \to \tilde{M}$ *be the blowing up at the base point* y *of the linear system* $|\tilde{L}-X|$. *Let* $\sigma: W \to W_1$ *be the blowing up at* $x := f_1(X)$, *and let* Z *be the exceptional curve over* x, *and* $\tilde{\Gamma}$ *the proper transform of* Γ. *Then* $\hat{M}$ *is a finite covering over* W. *The branch locus* B *is linearly equivalent to* $2\alpha T + 2\beta F - 2\gamma Z$. *The integers* α, β *and* γ *satisfy* $\alpha = \gamma$, $\gamma = 0$, $\beta \geq 0$ *and* $\beta \geq 2d - r + 3$.

Conversely, for each triple (α, β, γ) *satisfying these conditions, there exists a polarized surface* (M, L) *giving rise to the triple.*

The first half of Theorem 8 is proved as follows: Let $A = \alpha T + \beta F - \gamma Z$. Then clearly $B \in |2A|$. Since $f^*\Gamma = E + E^*$ and $E \cdot E^* = 0$, f is not branched along $\tilde{\Gamma}$. Thus we see that $0 = B \cdot \tilde{\Gamma} = 2\alpha - 2\gamma$. Since $f^*Z = X + Y$ and $X \cdot Y = 0$, we see that f is not branched along Z. Thus we have $0 = B \cdot Z = 2\gamma$. Supposed $d > 0$ and $x \notin T$. Since $T \cdot \tilde{\Gamma} = 1$, we see that T is not a component of B. Thus we have $0 \leq B \cdot T = 2\beta$. Since $g(M, L) \geq \Delta(M, L)$, we have $\beta \geq 2d - r + 3$. Suppose $d > 0$ and $x \in T$. Since $\tilde{T} \cdot Z = 1$, we see that $\tilde{T}$ is not a component of B. Thus we have $0 \leq B \cdot \tilde{T} = 2\beta$. Suppose $d = 0$. For $\tilde{T} \in |T - Z|$ we have $\tilde{T} \cdot Z = 1$. Thus $\tilde{T}$ is not a component of B. Hence we have $0 \leq B \cdot \tilde{T} = 2\beta$.

We prove the existence of an ample divisor L by using [12, Lemma 6]. The invariants are as follows:

$$K_M^2 = 8(2-\beta), \quad p_g(M) = 0, \quad q(M) = \beta - 1 .$$

6. Classification in Case (I), (d). In this section, we assume that $f_1^*\Gamma$ is in Case (1), (d), i.e., $f_1^*\Gamma = E + E^*$, and $E \cdot E^* = 1$. The morphism $f := f_1: \tilde{M} \to W_1 =: W$ is a finite covering. Since $\mathrm{Pic}(W) = \boldsymbol{Z}T \oplus \boldsymbol{Z}F$, the branch locus is linearly equivalent to $2A = 2\alpha T + 2\beta F$ for a unique pair (α, β) of integers.

THEOREM 9. *In Case* (I), (d) *the surface* $\tilde{M}$ *is a finite double covering over* W. *The branch locus* B *is linearly equivalent to* $2\alpha T + 2\beta F$. *The integers* α *and* β *satisfy* $\alpha = 1$, $\beta \geq d$ *and* $2\beta \geq r + d + 3$.

Conversely, for each pair (α, β) *satisfying these conditions, there exists a polarized surface* (M, L) *giving rise to the pair.*

The first half of Theorem 9 is proved as follows: Let $A=\alpha T+\beta F$. Then clearly $B\in|2A|$. Since $f_1\Gamma=E+E^*$ and $E\cdot E^*=1$, we see that B and Γ intersect each other at $f_1(E\cap E^*)$ with multiplicity 2. Thus we have $B\cdot\Gamma=2\alpha=2$, and $0\le B\cdot T=-2\alpha d+2\beta$. Since $g(M,L)\ge\Delta(M,L)$, we have $2\beta\ge r+d+3$.

The existence of such a surface is checked as follows: A general member B of $|2A|$ is nonsingular and irreducible. By $B\cdot F=2$, we see that $(\Phi_F)_{|B}: B\to \boldsymbol{P}^1$ is a finite double covering. Since B is irreducible, $(\Phi_F)_{|B}$ has branch points. There exists a finite double covering f: $\tilde{M}\to W$ such that $f^*\Gamma$ and $E\cdot E^*=1$.

We prove the existence of an ample divisor L by using [12, Lemma 6]. The invariants are as follows:

$$K_M^2=-12\beta-6d+24\,,\quad p_g(M)=0\,,\quad q(M)=0\,.$$

7. Classification in Case (I), (e). In this section, we assume that f^*T is in Case (1), (e), i.e., $f_1^*T=2E+X$ and $d=1$. Then the morphism $f_1: \tilde{M}\to W_1$ is not finite. Hence we lift it to a finite morphism. Let $\sigma: W\to W_1$ be the blowing up at $x:=f(X)$, and let Z be the exceptional curve over x and $\tilde{\Gamma}$ the proper transform of Γ, respectively. There exists a double covering f: $\tilde{M}\to W_1$ such that $f_1=\sigma\circ f$ and $f^*Z=X$ by the universality of the blowing up. Moreover, f is a finite double covering. Since $\mathrm{Pic}(W)=\boldsymbol{Z}T\oplus\boldsymbol{Z}F\oplus\boldsymbol{Z}Z$, the branch locus is linearly equivalent to $2A=2\alpha T+2\beta F-2\gamma Z$ for a unique triple (α,β,γ).

THEOREM 10. *In Case* (I), (e) *let* $\sigma: W\to W_1$ *be the blowing up at the point* $x=f(X)$ *with the exceptional curve* Z *over* x. *Let* $\tilde{T}$ *be the proper transform of* T. *Then* $\tilde{M}$ *is a finite double covering of* W. *The branch locus* B *is linearly equivalent to* $2\alpha T+2\beta F-2\gamma Z$. *The integers* α, β *and* γ *satisfy* $2\beta=(2\alpha-1)+(2\gamma-1)$, $2\alpha-1\ge 0$, $2\gamma-1\ge 0$, $(2\alpha-1)\ge(2\gamma-1)$ *and* $(\alpha-2)(r-2)+2(\beta-3)\ge 0$.

Conversely, for each triple (α,β,γ) *satisfying these conditions, there exists a polarized surface* (M,L) *giving rise to the triple.*

The first half of Theorem 10 is proved as follows: Let $A=\alpha T+\beta F-\gamma Z$. Then clearly $B\in|2A|$. Since $f^*\tilde{T}=2E$, the curve $\tilde{T}$ is a component of B. Then we have $B=B'+\tilde{T}$, where B' is a nonsingular curve. Since the branch locus B is nonsingular, we have $0=B'\cdot\tilde{T}=2\beta-(2\alpha-1)-(2\gamma-1)$. Since $\tilde{T}\cdot F=1$, we see that F is not a component and $0\le B'\cdot F=2\alpha-1$. Since $Z\cdot\tilde{T}=1$, we have $2\gamma-1\ge 0$. Since $g(M,L)\ge\Delta(M,L)$, we have $(\alpha-2)(r-2)+2(\beta-3)\ge 0$. Suppose $\tilde{\Gamma}$ is not a component of B'. Then $0\le B'\cdot\tilde{\Gamma}=(2\alpha-1)-(2\gamma-1)$. Suppose $\tilde{\Gamma}$ is a component of B'. Then we have $B\sim\tilde{T}+\tilde{\Gamma}+B''$. We see that $0=B''\cdot\tilde{T}=(2\alpha-1)-(2\gamma-2)$, a contradiction.

We consider the existence of such polarized surfaces. We see that $B'\sim 2(\alpha-\gamma)(T+F)+(2\gamma-1)(T+2F-Z)$. Thus it suffices to prove $\mathrm{Bs}|T+2F-Z|=\varnothing$. We have $\tilde{T}+|2F|\subset|T+2F-Z|$ and $\tilde{\Gamma}+|T+F|\subset|T+2F-Z|$. Since $\tilde{T}\cdot\tilde{\Gamma}=0$, we see that $|T+2F-Z|$ is base point free.

We prove the existence of an ample divisor L by useing [12, Lemma 6]. The invariants are as follows:

$$K_M^2=2(\alpha-2)(-\alpha+2\beta-4)-2(\gamma-1)^2+1\ ,$$

$$p_g(M)=\frac{1}{4}(2(\alpha-2)(-\alpha+2\beta-4)-2(\gamma-1)^2+1)+\beta-\frac{11}{4}\ ,$$

$$q(M)=0\ .$$

8. Classification in Case (I), (f). In this section, we assume that f_1^*T is in Case (I), (f), i.e., $f_1^*T=2E$ and $d=2$. Then the morphism $f_1:\tilde{M}\to W_1$ is a finite covering. We set $f:=f_1$. Since $\mathrm{Pic}(W)=\boldsymbol{Z}T\oplus\boldsymbol{Z}F$, the branch locus is linearly equivalent to $2A=2\alpha T+2\beta F$ for a unique pair (α,β) of integers.

THEOREM 11. *In Case* (I), (f) *the surface $\tilde{M}$ is a finite double covering over W. The branch locus B is linearly equivalent to* $2\alpha T+2\beta F$. *The integers α and β satisfy* $\beta=2\alpha-1$, $2\alpha-1\geq 0$ *and* $(\alpha-2)(r-3)+2(\beta-4)\geq 0$.

Conversely, for each pair (α,β) *satisfying these conditions, there exists a polarized surface* (M,L) *giving rise to the pair.*

The first half of Theorem 11 is proved as follows: Let $A=\alpha T+\beta F$. Then clearly $B\in|2A|$. Since $f^*T=2E$, the curve T is a component of B. Then we have $B=T+B'$, where B' is a nonsingular curve. Thus $0=B'\cdot T=-2(2\alpha-1)+2\beta$. Since $T\cdot F=1$, we see that F is not a component of B'. Thus we have $0\leq B'\cdot F=2\alpha$. Since $g(M,L)\geq\Delta(M,L)$, we have $(\alpha-2)(r-3)+2(\beta-4)\geq 0$.

We consider the existence of such polarized surfaces. We see that $B'\sim(2\alpha-1)(T+2F)$. Since $\mathrm{Bs}|T+2F|=\emptyset$, we have $\mathrm{Bs}|(2\alpha-1)(T+2F)|=\emptyset$.

We prove the existence of an ample divisor L by using [12, Lemma 6]. The invariants are as follows:

$$K_M^2=4(\alpha-2)(\alpha-4)+1\ ,\quad p_g(M)=\alpha^2-3\alpha+2\ ,\quad q(M)=0\ .$$

9. Classification in Case (I), (g). In this section, we assume that f^*T is in Case (I), (g), i.e., $f_1^*T=E+E^*$, $E\cdot E^*=0$ and $d=1$. Then the morphism $f_1:\tilde{M}\to W_1$ is a finite covering. We set $f:=f_1$. Since $\mathrm{Pic}(W)=\boldsymbol{Z}T\oplus\boldsymbol{Z}F$, the branch locus is linearly equivalent to $2A=2\alpha T+2\beta F$ for a unique pair (α,β) of integers.

THEOREM 12. *In Case* (I), (g) *the surface $\tilde{M}$ is a finite double covering over W. The branch locus B is linearly equivalent to* $2\alpha T+2\beta F$. *The integers α and β satisfy* $\alpha=\beta$, $\alpha\geq 0$ *and* $r\alpha-2(r+1)\geq 0$.

Conversely, for each pair (α,β) *satisfying these conditions, there exists a polarized surface* (M,L) *giving rise to the pair.*

The first half of Theorem 12 is proved as follows: Let $A=\alpha T+\beta F$. Then clearly

$B \in |2A|$. Since $f^*T = E + E^*$ and $E \cdot E^* = 0$, we see that f is not branched along T. Thus $0 = B \cdot T = -2\alpha + 2\beta$. Since $T \cdot F = 1$, we see that F is not a component of B. Thus we have $0 \leq T \cdot F = 2\alpha$. Since $g(M, L) \geq \Delta(M, L)$, we have $r\alpha - 2(r+1) \geq 0$.

Conversely, since $B \sim 2\alpha(T+F)$, there exist such polarized surfaces.

The invariants are as follows:

$$K_M^2 = 2(\alpha-2)(-\alpha+2\beta-4)+1\,,$$

$$p_g(M) = \frac{1}{4}(2(\alpha-2)(-\alpha+2\beta-4)+1)\beta + \frac{1}{2}\alpha - \frac{13}{4}\,,$$

$$q(M) = 0\,.$$

10. Classification in Case (II). In this section, we assume that we are in Case (II). This case is similar to the Case (I) where the points $x = f_1(X)$ and $x_i = f_1(X_i)$ are not on T by Proposition 2. Thus we have the following results:

Case (II), (a): Let $\sigma: W \to W_1$ be the blowing up at the two points $x_1 = f_1(X_1)$ and $x_2 = f_1(X_2)$ with the exceptional curves Z_1 and Z_2 over x_1 and x_2, respectively. Let $\tilde{T}$ be the proper transform of T. Then $\tilde{M}$ is a finite double covering of W. The branch locus B is linearly equivalent to $2\alpha T + 2\beta F - 2\gamma_1 Z_1 - 2\gamma_2 Z_2$. The integers α, β, γ_1 and γ_2 satisfy $\alpha = \gamma_1 + \gamma_2 - 1$, $2\gamma_i - 1 \geq 0$, $2\beta \geq 2\alpha(r-1)+1$ and $2(\beta - r - 1) \geq 0$.

Conversely, for each quadruple $(\alpha, \beta, \gamma_1, \gamma_2)$ satisfying these conditions, there exist a polarized surface (M, L) giving rise to the quadruple.

The invariants are as follows:

$$K_M^2 = 2(\alpha-2)(-(r-1)\alpha+2\beta-4) - 2(\gamma_1-1)^2 - 2(\gamma_2)^2 + 1\,,$$

$$p_g(M) = \frac{1}{4}(2(\alpha-2)(-(r-1)\alpha+2\beta-4) - 2(\gamma_1-1)^2 - 2(\gamma_2-1)^2 + 1)$$
$$+ \frac{1}{2}(-r+2)\alpha + \beta - \frac{11}{4}\,,$$

$$q(M) = 0\,.$$

Case (II), (b): Let $\rho: \hat{M} \to \tilde{M}$ be the blowing up at the base point y of the linear system $|\tilde{L} - X|$. Let $\sigma_1: W_2 \to W_1$ be the blowing up at $x = f_1(X)$, and let Z_1 be the exceptional curve over x, and $\tilde{\Gamma}$ the proper transform of Γ, respectively. The intersection of $\tilde{\Gamma}$ with Z_1 is a point z. Let $\sigma_2: W \to W_2$ be the blowing up at z, and let Z_2 be the exceptional curve over z. Then $\hat{M}$ is a finite double covering over W. The branch locus B is linearly equivalent to $2\alpha T + 2\beta F - 2\gamma_1 Z_1 - 2\gamma_2 Z_2$. The integers α, β, γ_1 and γ_2 satisfy $\alpha = \gamma_1 + \gamma_2 - 1$, $\gamma_1 = \gamma_2 - 1$, $\gamma_2 - 1 \geq 0$, $2\beta \geq 2\alpha(r-1)+1$ and $\beta - r - 1 \geq 0$.

Conversely, for each quadruple $(\alpha, \beta, \gamma_1, \gamma_2)$ satisfying these conditions, there exists a polarized surface (M, L) giving rise to the quadruple.

The invariants are as follows:

$$K_M^2=2(\alpha-2)(-(r-1)\alpha+2\beta-4)-2(\gamma_1-1)^2-(\gamma_2-1)^2+1\,,$$

$$p_g(M)=\frac{1}{4}(2(\alpha-2)(-(r-1)\alpha+2\beta-4)-2(\gamma_1-1)^2-2(\gamma_2-1)^2+2)$$
$$+\frac{1}{2}(-r+2)\alpha+\beta-\frac{12}{4}\,,$$

$$q(M)=0\,.$$

Case (II), (c): We have the following two subtypes.

(c-1) $X\cdot E^*=1$,

(c-2) $X\cdot E^*=0$.

Subtype (c-1): Let $\sigma: W\to W_1$ be the blowing up at the point $x=f(X)$ with the exceptional curve Z over x. Let $\tilde{T}$ be the proper transform of T. Then $\tilde{M}$ is a finite double covering of W. The branch locus B is linearly equivalent to $2\alpha T+2\beta F-2\gamma Z$. The integers α, β and γ satisfy $\alpha=\gamma$, $\gamma\geq 0$, $\beta\geq\alpha(r-1)$ and $\beta-r-1\geq 0$.

Conversely, for each triple (α, β, γ) satisfying these conditions, there exists a polarized surface (M, L) giving rise to the triple.

The invariants are as follows:

$$K_M^2=2(\alpha-2)(-(r-1)\alpha+2\beta-4)-2(\gamma-1)^2+1\,,$$

$$p_g(M)=\frac{1}{4}(2(\alpha-2)(-(r-1)\alpha+2\beta-4)-2(\gamma-1)^2+1)+\frac{1}{2}(-r+2)\alpha+\beta-\frac{11}{4}\,,$$

$$q(M)=0\,.$$

Subtype (c-2): Let $\rho: \hat{M}\to\tilde{M}$ be the blowing up at the base point y of the linear system $|\tilde{L}-X|$. Let $\sigma: W\to W_1$ be the blowing up at $x:=f_1(X)$, and let Z be the exceptional curve over x, and $\tilde{\Gamma}$ the proper transform of Γ. Then $\hat{M}$ is a finite covering over W. The branch locus B is linearly equivalent to $2\alpha T+2\beta F-2\gamma Z$. The integers α, β and γ satisfy $\alpha=\gamma$, $\gamma=0$, $\beta\geq 0$ and $\beta\geq r+2$.

Conversely, for each triplet (α, β, γ) satisfying these conditions, there exists a polarized surface (M, L) giving rise to the triple.

The invariants are as follows:

$$K_M^2=8(2-\beta)\,,\quad p_g(M)=0\,,\quad q(M)=\beta-1\,.$$

Case (II), (d): The surface $\tilde{M}$ is a finite double covering over W. The brahch locus B is linearly equivalent to $2\alpha T+2\beta F$. The integer α and β satisfy $\alpha=1$ and $\beta\geq r-1$.

The invariants are as follows:

$$K_M^2=-12\beta-6(r-1)+24\,,\quad p_g(M)=0\,,\quad q(M)=0\,.$$

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