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BOHMAN-KOROVKIN-WULBERT OPERATORS FROM A FUNCTION SPACE INTO A COMMUTATIVE C^*-ALGEBRA FOR SPECIAL TEST FUNCTIONS

Dedicated to Professor Satoru Igari on his sixtieth birthday

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Abstract. We completely determine a class of Bohman-Korovkin-Wulbert operators from a function space on a compact Hausdorff space into the Banach space of continuous complex-valued functions on another space with respect to the special test functions.

1. Introduction and results. Let X and Y be normed spaces and $B(X, Y)$ the set of all bounded linear operators from X into Y. For a subset S of X and a subset B of $B(X, Y)$, an operator T in $B(X, Y)$ is said to be a Bohman-Korovkin-Wulbert operator (BKW-operator, for shortly) for S and B if every net $\{T_\lambda\}$ in B such that $\lim_\lambda \|T_\lambda\| = \|T\|$ and $\lim_\lambda \|T_\lambda(s) - T(s)\| = 0$ for all $s \in S$ converges strongly to T (cf. [6]). We will omit Y (resp. B) when $X = Y$ (resp. $B = B(X, Y)$). Bohman [1] showed that the identity operator $\mathrm{id}_{C([0,1])}$ on $C([0, 1])$ is a BKW-operator for $\{\mathbf{1}, x, x^2\}$ and special interpolation operators on $C([0, 1])$. Korovkin [2] showed that $\mathrm{id}_{C([0,1])}$ is also a BKW-operator for $\{\mathbf{1}, x, x^2\}$ and positive operators on $C([0, 1])$. Moreover, Wulbert [8] showed that $\mathrm{id}_{C([0,1])}$ is a BKW-operator for $\{\mathbf{1}, x, x^2\}$. "BKW" is an abbreviation for Bohman, Korovkin and Wulbert. Micchelli [4] posed (as suggested in Lorentz [5]) a problem of describing all positive BKW-operators on $C(\Omega)$ for suitable test functions on Ω and positive operators on $C(\Omega)$. However, we are interested in describing all BKW-operators from a function space into another space for suitable test functions.

In [7], we completely described all BKW-operators (resp. all norm one unital BKW-operators) from a function space on the unit interval $[0, 1]$ into the Banach space of continuous complex-valued functions on a compact Hausdorff space for the special test functions $\{\mathbf{1}, x\}$ (resp. $\{\mathbf{1}, x, x^2\}$). Here we consider BKW-operators for general function spaces and obtain generalizations of the results in [7].

Throughout this paper, let Ω and Φ be compact Hausdorff spaces and let h be a nonconstant real-valued function in $C(\Omega)$ and X a function space on Ω such that $\mathrm{span}\{\mathbf{1}, h\} \subsetneqq X$, where "span" denotes the linear span. Set

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$$m=\min_{\omega\in\Omega} h(\omega)\,,$$

$$M=\max_{\omega\in\Omega} h(\omega)\,,$$

$$\Omega_m=\{\omega\in\Omega: h(\omega)=m\}\,,$$

$$\Omega_M=\{\omega\in\Omega: h(\omega)=M\} \quad \text{and}$$

$$\Omega_h=\{\omega\in\Omega: \sharp\{h^{-1}(h(\omega))\}=1\}\,.$$

In this notation, we completely describe all BKW-operators from X into $C(\Phi)$ for the test functions $\{\mathbf{1}, h\}$ as follows:

THEOREM 1. (i) *If Ω_m consists of a single point ω_0 and Ω_M consists of a single point ω_1, then every* BKW-*operator T from X into $C(\Phi)$ for the test functions $\{\mathbf{1}, h\}$ is of the form $T(f)=f(\omega_0)u+f(\omega_1)v$ for every $f\in X$, where u and v are functions in $C(\Phi)$ satisfying the following two conditions*:

(1) $|u(\varphi)|+|v(\varphi)|=\|T\|$ *for all* $\varphi\in\Phi$.

(2) *If* $u(\varphi)\neq 0$ *and* $v(\varphi)\neq 0$, *then* $|u(\varphi)+v(\varphi)|\neq\|T\|$.

In this case, the functions u and v are given by $u=T(\mathbf{1}-\tilde{h})$ and $v=(T\tilde{h})$, where $\tilde{h}=(M-m)^{-1}(h-m\mathbf{1})$. In particular, every norm one unital BKW-*operator T from X into $C(\Phi)$ for $\{\mathbf{1}, h\}$ is of the form $T(f)=f(\omega_0)\chi+f(\omega_1)(\mathbf{1}-\chi)$ for every $f\in X$, where χ is the characteristic function on a closed and open subset of Φ.*

(ii) *If Ω_m consists of a single point ω_0 and Ω_M possesses more than two points, then every* BKW-*operator T from X into $C(\Phi)$ for the test functions $\{\mathbf{1}, h\}$ is of the form $T(f)=f(\omega_0)u$ for every $f\in X$, where u is a functions in $C(\Phi)$ such that $|u(\varphi)|=\|T\|$ for all $\varphi\in\Phi$. In particular, every norm one unital* BKW-*operator T from X into $C(\Phi)$ for $\{\mathbf{1}, h\}$ is of the form $T(f)=f(\omega_0)\mathbf{1}$ for every $f\in X$.*

(iii) *If Ω_M consists of a single point ω_1 and Ω_m possesses more than two points, then every* BKW-*operator T from X into $C(\Phi)$ for the test functions $\{\mathbf{1}, h\}$ is of the form $T(f)=f(\omega_1)v$ for every $f\in X$, where v is a function in $C(\Phi)$ such that $|v(\varphi)|=\|T\|$ for all $\varphi\in\Phi$. In particular, every norm one unital* BKW-*operator T from X into $C(\Phi)$ for $\{\mathbf{1}, h\}$ is of the form $T(f)=f(\omega_1)\mathbf{1}$ for every $f\in X$.*

(iv) *If both Ω_m and Ω_M possess more than two points, then the only zero operator from X into $C(\Phi)$ is a* BKW-*operator for the test functions $\{\mathbf{1}, h\}$.*

Furthermore, we completely describe all norm one unital BKW-operators from X into $C(\Phi)$ for the test functions $\{\mathbf{1}, h, h^2\}$ as follows:

THEOREM 2. *Suppose that $\{1, h, h^2, h^3\}\subset X$, $X^*_\ddagger=\{\mu\in X^*: \|\mu\|=\mu(\mathbf{1})\}$, where X^* denotes the space dual to X and that $h(\Omega)=[m, M]$.*

(i) *If Ω_m consists of a single point ω_0 and Ω_M consists of a single point ω_1, then every norm one unital* BKW-*operator T from X into $C(\Phi)$ for the test functions $\{\mathbf{1}, h, h^2\}$*

is of the form

$$(Tf)(\varphi)=\begin{cases} f(\xi(\varphi)), & \text{if } \varphi\in\Phi\setminus G \\ \dfrac{f(\omega_0)\{M-h(\xi(\varphi))\}+f(\omega_1)\{h(\xi(\varphi))-m\}}{M-m}, & \text{if } \varphi\in G \end{cases}$$

for every $f\in X$, *where* ξ *is a map from* Φ *into* Ω *and* G *is an open subset of* Φ *such that* $m<h(\xi(\varphi))<M$ *for all* $\varphi\in G$, *that* $\xi(\varphi)=\omega_0$ *or* ω_1 *for all* $\varphi\in\partial G$, *that* $\xi(\Phi\setminus G)\subset\Omega_h$, *that* $h\circ\xi$ *is continuous on* Φ *and that* $\xi\,|\,(\Phi\setminus G)$ *is continuous on* $\Phi\setminus G$. *Here* ∂G *denotes the topological boundary of* G *in* Ω.

(ii) *If either* Ω_m *or* Ω_M *possesses more than two points, then every norm one unital* BKW-*operator* T *from* X *into* $C(\Phi)$ *for the test functions* $\{\mathbf{1}, h, h^2\}$ *is of the form*

$$(Tf)(\varphi)=f(\xi(\varphi))$$

for every $\varphi\in\Phi$ *and* $f\in X$, *where* ξ *is a continuous map from* Φ *into* Ω_h.

The following are examples of h in Theorems 1 and 2 when $\Omega=[0, 1]$:

(i) $p>0$, $h(w)=w^p$ $(0\le w\le 1)$.

(ii) $0<\alpha<1$, $h(w)=\begin{cases} \dfrac{1}{\alpha}w, & \text{if } 0\le w\le\alpha \\ 0, & \text{if } \alpha<w\le 1. \end{cases}$

(iii) $0<\alpha<1$, $h(w)=\begin{cases} 1-\dfrac{1}{\alpha}w, & \text{if } 0\le w\le\alpha \\ 0, & \text{if } \alpha<w\le 1. \end{cases}$

(iv) $0<\alpha<\alpha'<\beta'<\beta<1$, $h(w)=\begin{cases} 0, & \text{if } 0\le w\le\alpha \\ \dfrac{w-\alpha}{\alpha'-\alpha}, & \text{if } \alpha<w\le\alpha' \\ 1, & \text{if } \alpha'<w\le\beta' \\ \dfrac{w-\beta}{\beta'-\beta}, & \text{if } \beta'<w\le\beta \\ 0, & \text{if } \beta<w\le 1. \end{cases}$

2. Lemmas. For $S\subset X$ and $F\subset X^*$, we set

$$U_S(F)=\{\mu\in F:\mu=\nu \text{ if } \nu\in F \text{ and } \mu\,|\,S=\nu\,|\,S\}\,.$$

The set $U_S(F)$ is called the uniqueness set of F for S, and plays an essential role in the Korovkin type approximation theory. Let $X_\rho^*=\{\mu\in X^*:\|\mu\|\le\rho\}$ for $\rho>0$. The following lemma, which is basic in our argument, is an immediate consequence of

[7, Theorem 1.4]:

LEMMA 1. *Let $S\subset X$ and $T\in B(X, C(\Phi))$. Then T is a* BKW-*operator for S if and only if $T^*(\delta_\varphi)\in U_S(X^*_{\|T\|})$ for each $\varphi\in\Phi$, where T^* is the adjoint operator of T and δ_φ is the evaluation at $\varphi\in\Phi$.*

Let $\boldsymbol{C}$ be the set of all complex numbers. Then we have the following:

LEMMA 2. (i) *If Ω_m consists of a single point ω_0 and Ω_M consists of a single point ω_1, then*

$$U_{\{\mathbf{1},h\}}(X^*_1)=\{a\delta_{\omega_0}|X+b\delta_{\omega_1}|X: a, b\in \boldsymbol{C}, |a|+|b|=1 \text{ and } |a+b|\neq 1 \text{ (if } a\neq 0,\ b\neq 0)\}.$$

(ii) *If Ω_m consists of a single point ω_0 and Ω_M possesses more than two points, then $U_{\{\mathbf{1},h\}}(X^*_1)=\{a\delta_{\omega_0}|X: |a|=1\}$.*

(iii) *If Ω_M consists of a single point ω_1 and Ω_m possesses more than two points, then $U_{\{\mathbf{1},h\}}(X^*_1)=\{a\delta_{\omega_1}|X: |a|=1\}$.*

(iv) *If both Ω_m and Ω_M possess more than two points, the $U_{\{\mathbf{1},h\}}(X^*_1)$ is empty.*

PROOF. Set $\tilde{h}=(M-m)^{-1}(h-m\mathbf{1})$. Then $\mathrm{span}\{\mathbf{1}, h\}=\mathrm{span}\{\mathbf{1}, \tilde{h}\}$ and hence $U_{\{\mathbf{1},h\}}(X^*_1)=U_{\{\mathbf{1},\tilde{h}\}}(X^*_1)$. Therefore, we may assume without loss of generality that $m=0$ and $M=1$. Let $\mu\in U_{\{\mathbf{1},h\}}(X^*_1)$. Put $a=\mu(\mathbf{1}-h)$ and $b=\mu(h)$. Then $|a|\leq 1$ and $|b|\leq 1$. For any $\alpha, \beta\in\boldsymbol{C}$, we have

$$|\alpha a+\beta b|=|\mu(\alpha(\mathbf{1}-h)+\beta h)|\leq\|\mu\|\,\|\alpha(\mathbf{1}-h)+\beta h\|_\infty\leq\max_{0\leq t\leq 1}|\alpha(1-t)+\beta t|.$$

In particular, for $\alpha=\bar{a}/|a|$ and $\beta=\bar{b}/|b|$, we have $|a|+|b|\leq\max_{0\leq t\leq 1}\{|\alpha|(1-t)+|\beta|t\}=1$. Now choose $\xi_0\in\Omega_m$ and $\xi_1\in\Omega_M$ arbitrarily and set $v=a\delta_{\xi_0}|X+b\delta_{\xi_1}|X$, hence $\|v\|\leq|a|+|b|\leq 1$. Also $v(\mathbf{1})=a+b=\mu(\mathbf{1})$ and $v(h)=b=\mu(h)$. Then we have $\mu=a\delta_{\xi_0}|X+b\delta_{\xi_1}|X$, because $\mu\in U_{\{\mathbf{1},h\}}(X^*_1)$. Moreover by [7, Lemma 2.1] we have $\|\mu\|=1$, so that $1\leq|a|+|b|$ and hence $|a|+|b|=1$.

If Ω_m consists of a single point ω_0 and Ω_M possesses two points ω_1 and ω_2, then by the above argument, we have $\mu=a\delta_{\omega_0}|X+b\delta_{\omega_1}|X$ and $\mu=a\delta_{\omega_0}|X+b\delta_{\omega_2}|X$. Hence $b\{x(\omega_1)-x(\omega_2)\}=0$ for all $x\in X$. This implies $b=0$, since X separates the points of Ω. Accordingly $\mu=a\delta_{\omega_0}|X$ and $|a|=1$. Also, if both Ω_m and Ω_M possess more than two points, then $a=b=0$, a contradiction. Hence $U_{\{\mathbf{1},h\}}(X^*_1)$ must be empty and so (iv) has been shown.

Suppose that Ω_m consists of a single point ω_0 and Ω_M consists of a single point ω_1. In this case, if $a\neq 0$ and $b\neq 0$, then $|a+b|\neq 1$. Indeed, if $|a+b|=1$, then we can find $t>0$ such that $b=ta$. Also choose a function $g\in X\setminus\mathrm{span}\{\mathbf{1}, h\}$ and put

$$f=g-g(\omega_0)\mathbf{1}+\{g(\omega_0)-g(\omega_1)\}h.$$

Then $f\in X$ and $f\neq 0$, hence there exists $\omega_2\in\Omega$ such that $f(\omega_2)\neq 0$. Note that $\omega_2\neq\omega_0, \omega_1$, so $0<h(\omega_2)<1$ by hypothesis. Set $s=h(\omega_2)$. Then $(s-t+st)/s<1$, hence we can take a positive number ρ such that $\max\{0, (s-t+st)/s\}<\rho<1$. Set

$$\alpha=\rho a, \quad \beta=\frac{(1-\rho)a}{1-s}, \quad \gamma=\frac{(1-s)b-s(1-\rho)a}{1-s}$$

and

$$\mu_1=\alpha\delta_{\omega_0}\,|\,X+\beta\delta_{\omega_2}\,|\,X+\gamma\delta_{\omega_1}\,|\,X\,.$$

Then we can easily see that $\mu_1(\mathbf{1})=\mu(\mathbf{1})$ and $\mu_1(h)=\mu(h)$. Also we have

$$\begin{aligned}|\alpha|+|\beta|+|\gamma|&=\rho|a|+\frac{(1-\rho)|a|}{1-s}+\frac{|(1-s)t-s(1-\rho)||a|}{1-s}\\&=|a|\left\{\rho+\frac{1-\rho}{1-s}+\frac{(1-s)t-s(1-\rho)}{1-s}\right\}\quad\left(\text{since }\frac{s-t+st}{s}<\rho\right)\\&=|\alpha|(1+t)=|a|+|b|=1\,,\end{aligned}$$

hence $\|\mu_1\|\leq 1$. However $\mu_1(f)=\beta f(\omega_2)\neq 0$ and $\mu(f)=af(\omega_0)+bf(\omega_1)=0$, so $\mu_1\neq\mu$, a contradiction to $\mu\in U_{\{\mathbf{1},h\}}(X_1^*)$.

Conversely, it is easy to see that $\{a\delta_{\omega_0}\,|\,X:|a|=1\}\subset U_{\{\mathbf{1},h\}}(X_1^*)$ when $\Omega_m=\{\omega_0\}$, so (ii) has been shown in view of the above argument. Since $U_{\{\mathbf{1},-h\}}=U_{\{\mathbf{1},h\}}$, (iii) follows immediately from (ii). To show (i), assume that $\Omega_m=\{\omega_0\}$ and $\Omega_M=\{\omega_1\}$, and let $a, b\in \mathbf{C}$ be such that $|a|+|b|=1$ and $|a+b|\neq 1$ if $a\neq 0$, $b\neq 0$. Then we need to show that $a\delta_{\omega_0}\,|\,X+b\delta_{\omega_1}\,|\,X\in U_{\{\mathbf{1},h\}}(X_1^*)$. To do so, let $\mu\in X_1^*$ be such that $\mu(\mathbf{1})=a+b$ and $\mu(h)=b$. By the Hahn-Banach extension theorem, we can find a Radon measure $\tilde{\mu}$ on Ω such that $\tilde{\mu}\,|\,X=\mu$ and $\|\tilde{\mu}\|=\|\mu\|$. Let $\tilde{\mu}=u|\tilde{\mu}|$ be the polar decomposition of $\tilde{\mu}$, i.e.,

$$\int_\Omega f(\omega)d\tilde{\mu}(\omega)=\int_\Omega f(\omega)u(\omega)d|\tilde{\mu}|(\omega)$$

for all $f\in L^1(\Omega, |\tilde{\mu}|)$, where $|\tilde{\mu}|$ is the total variation of $\tilde{\mu}$ and u is a measurable function on Ω with $|u(\omega)|=1$ for all $\omega\in\Omega$ (see [3, Corollary 19.38]). Then we have the following inequality:

$$\begin{aligned}1&=|a|+|b|=|\mu(\mathbf{1}-h)|+|\mu(h)|\\&=\left|\int_\Omega(1-h(\omega))u(\omega)d|\tilde{\mu}|(\omega)\right|+\left|\int_\Omega h(\omega)u(\omega)d|\tilde{\mu}|(\omega)\right|\\&\leq\int_\Omega(1-h(\omega))d|\tilde{\mu}|(\omega)+\int_\Omega h(\omega)d|\tilde{\mu}|(\omega)=\int_\Omega d|\tilde{\mu}|=\|\tilde{\mu}\|=\|\mu\|\leq 1\,.\end{aligned}$$

If $a\neq 0$ and $b\neq 0$, then by [7, Lemma 2.2] we have $\{1-h(\omega)\}u(\omega)=e^{i\alpha}\{1-h(\omega)\}$ ($|\tilde{\mu}|$-a.e.) and $h(\omega)u(\omega)=e^{i\beta}h(\omega)$ ($|\tilde{\mu}|$-a.e.), where $\alpha=\mathrm{Arg}(a)$ and $\beta=\mathrm{Arg}(b)$. Hence we have $1=|(1-h(\omega))e^{i\alpha}+h(\omega)e^{i\beta}|$ ($|\tilde{\mu}|$-a.e.). Since $|a+b|\neq 1$ and hence $\alpha\neq\beta$ (mod 2π), it follows that $|\tilde{\mu}|(\Omega\setminus\{\omega_0,\omega_1\})=0$, i.e., $\mathrm{supp}(|\tilde{\mu}|)\subset\{\omega_0,\omega_1\}$ by the above equation. If $a=0$, then the above inequality implies that $\int_\Omega\{1-h(\omega)\}d|\tilde{\mu}|(\omega)=0$ and hence $\mathrm{supp}(|\tilde{\mu}|)=\{\omega_1\}$. If $b=0$, then the same inequality implies that $\int_\Omega h(\omega)d|\tilde{\mu}|(\omega)=0$ and hence $\mathrm{supp}(|\tilde{\mu}|)=\{\omega_0\}$. Then $|\tilde{\mu}|$ can be expressed as $|\tilde{\mu}|=c\delta_{\omega_0}+d\delta_{\omega_1}$ for some complex numbers c and d. Therefore $\tilde{\mu}=cu(\omega_0)\delta_{\omega_0}+du(\omega_1)\delta_{\omega_1}$. Hence we can easily see that $\mu=a\delta_{\omega_0}|X+b\delta_{\omega_1}|X$. We thus obtain $a\delta_{\omega_0}|X+b\delta_{\omega_1}|X\in U_{\{1,h\}}(X_1^*)$. q.e.d.

LEMMA 3. *Assume that* $\{1,h,h^2,h^3\}\subset X$, $X_+^*=\{\mu\in X^*:\|\mu\|=\mu(1)\}$ *and* $h(\Omega)=[m,M]$.

(i) *If Ω_m consists of a single point ω_0 and Ω_M consists of a single point ω_1, then* $U_{\{1,h,h^2\}}(X_1^*)\cap X_+^*=\{\delta_\omega|X:\omega\in\Omega_h\}\cup\{(1-a)\delta_{\omega_0}|X+a\delta_{\omega_1}|X:0<a<1\}$.

(ii) *If either Ω_m or Ω_M possesses more than two points, then* $U_{\{1,h,h^2\}}(X_1^*)\cap X_+^*=\{\delta_\omega|X:\omega\in\Omega_h\}$.

PROOF. Set $\tilde{h}=(M-m)^{-1}(h-m1)$. Then $\mathrm{span}\{1,h,h^2\}=\mathrm{span}\{1,\tilde{h},\tilde{h}^2\}$ and hence $U_{\{1,h,h^2\}}(X_1^*)=U_{\{1,\tilde{h},\tilde{h}^2\}}(X_1^*)$. Therefore we may assume without loss of generality that $m=0$ and $M=1$. Let $0\le a\le 1$ and $\omega\in\Omega_h$. Then $\delta_\omega|X$ is in X_+^*. To show that $\delta_\omega|X\in U_{\{1,h,h^2\}}(X_1^*)$, let $\nu\in X_1^*$ be such that $\nu(h^k)=\delta_\omega(h^k)$ for $k=0,1,2$. Then $1=\nu(1)\le\|\nu\|\le 1$. Choose a Radon measure $\tilde{\nu}$ on Ω such that $\tilde{\nu}|X=\nu$ and $\|\tilde{\nu}\|=\|\nu\|$. Then $\|\tilde{\nu}\|=\tilde{\nu}(1)=1$, so $\tilde{\nu}$ is positive and also we have

$$\tilde{\nu}((h-h(\omega)1)^2)=\nu(h^2)-2h(\omega)\nu(h)+h(\omega)^2\nu(1)=h(\omega)^2-2h(\omega)^2+h(\omega)^2=0\,.$$

Hence, the support of $\tilde{\nu}$ consists of the single point ω, since $(h(\xi)-h(\omega))^2>0$ for all $\xi\in\Omega\setminus\{\omega\}$. This immediately implies that $\tilde{\nu}=\delta_\omega$, so $\nu=\delta_\omega|X$ and hence $\delta_\omega|X\in U_{\{1,h,h^2\}}(X_1^*)$. Suppose next that $\Omega_m=\{\omega_0\}$ and $\Omega_M=\{\omega_1\}$, hence $(1-a)\delta_{\omega_0}|X+a\delta_{\omega_1}|X$ is in X_+^*. To show that $(1-a)\delta_{\omega_0}|X+a\delta_{\omega_1}|X\in U_{\{1,h,h^2\}}(X_1^*)$, let $\nu\in X_1^*$ be such that $\nu(h^k)=((1-a)\delta_{\omega_0}+a\delta_{\omega_1})(h^k)$ for $k=0,1,2$. Then $1=\nu(1)\le\|\nu\|\le 1$. Choose a Radon measure $\tilde{\nu}$ on Ω such that $\tilde{\nu}|X=\nu$ and $\|\tilde{\nu}\|=\|\nu\|$. Then $\tilde{\nu}$ is positive and $\tilde{\nu}(h-h^2)=\nu(h-h^2)=a-a=0$. Hence, the support of $\tilde{\nu}$ is contained in $\{\omega_0,\omega_1\}$, since $h(\xi)-h(\xi)^2>0$ for all $\xi\in\Omega\setminus\{\omega_0,\omega_1\}$. This immediately implies that $\nu=(1-a)\delta_{\omega_0}|X+a\delta_{\omega_1}|X$, and hence $(1-a)\delta_{\omega_0}|X+a\delta_{\omega_1}|X\in U_{\{1,h,h^2\}}(X_1^*)$.

Conversely, let $\mu\in U_{\{1,h,h^2\}}(X_1^*)\cap X_+^*$. By [7, Lemma 2. 1], $\|\mu\|=1$, and so $\mu(1)=1$. Choose a positive Radon mesure $\tilde{\mu}$ on Ω such that $\tilde{\mu}|X=\mu$ and $\|\tilde{\mu}\|=\|\mu\|$. Put $\alpha=\mu(h)$ and $\beta=\mu(h^2)$. Then we have $0\le\alpha,\beta\le 1$, $\beta\le\alpha$ and $\alpha^2\le\beta$ by Schwarz's inequality. If $0<\beta=\alpha<1$, then $\mu=(1-\alpha)\delta_{\tilde{\omega}_0}|X+\alpha\delta_{\tilde{\omega}_1}|X$ for every $\tilde{\omega}_0\in\Omega_m$ and $\tilde{\omega}_1\in\Omega_M$ because $\mu\in U_{\{1,h,h^2\}}(X_1^*)\cap X_+^*$. Therefore since X separates the points of Ω, we have $\mu=(1-\alpha)\delta_{\omega_0}|X+\alpha\delta_{\omega_1}|X$ only when $\Omega_m=\{\omega_0\}$ and $\Omega_M=\{\omega_1\}$. If also $\alpha^2=\beta$, then $\tilde{\mu}((h-\alpha 1)^2)=\beta-2\alpha^2+\alpha^2=0$. But since $\tilde{\mu}\ne 0$, there must be $\omega\in\Omega$ such that $h(\omega)=\alpha$. Then $\mu(h^k)=\delta_\omega(h^k)$ for $k=0,1,2$ and hence $\mu=\delta_\omega|X$ because $\mu\in U_{\{1,h,h^2\}}(X_1^*)$. In this case, the point ω must be in Ω_h. Actually, if ξ is a point of Ω such that $h(\xi)=h(\omega)$, then $\mu=\delta_\xi|X$ by the above argument and hence $\xi=\omega$ because X separates the points of Ω. We finally show that the case $0<\alpha^2<\beta<\alpha<1$ does not occur. Suppose the contrary.

Let $\omega_0 \in \Omega_m$ and $\omega_1 \in \Omega_M$ be fixed arbitrarily. For each $0<\lambda<1$, we can take a point $\omega_\lambda \in \Omega$ such that $h(\omega_\lambda)=\lambda$ because $h(\Omega)=[0, 1]$. Set

$$\mu_\lambda = a(\lambda)\delta_{\omega_0}|X + b(\lambda)\delta_{\omega_\lambda}|X + c(\lambda)\delta_{\omega_1}|X,$$

where $a(\lambda)=\{\lambda-(1+\lambda)\alpha+\beta\}/\lambda$, $b(\lambda)=(\alpha-\beta)/\lambda(1-\lambda)$ and $c(\lambda)=(\beta-\lambda\alpha)/(1-\lambda)$. Then we have $\mu_\lambda(\mathbf{1})=1=\mu(\mathbf{1})$, $\mu_\lambda(h)=\alpha=\mu(h)$ and $\mu_\lambda(h^2)=\beta=\mu(h^2)$. Note that $0<(\alpha-\beta)/(1-\alpha)<\beta/\alpha<1$ and so take real numbers s and t such that $(\alpha-\beta)/(1-\alpha)<s<t<\beta/\alpha$. Then we see that $a(s)>0$, $a(t)>0$, $b(s)>0$, $b(t)>0$, $c(s)>0$ and $c(t)>0$, so that $\|\mu_s\|=\|\mu_t\|=1$, and hence $\mu_s=\mu=\mu_t$ because $\mu \in U_{\{\mathbf{1},h,h^2\}}(X_1^*)$. Therefore we have

$$\begin{aligned} 0 &= (\mu_s-\mu_t)(h-h^3) = b(s)\{h(\omega_s)-h(\omega_s)^3\} - b(t)\{h(\omega_t)-h(\omega_t)^3\} \\ &= b(s)(s-s^3)-b(t)(t-t^3)=(\alpha-\beta)(s-t)\neq 0, \end{aligned}$$

a contradiction. q.e.d.

The proof of the following fundamental result is straightforward, and left to the reader.

LEMMA 4. *Let Ψ be a topological space, G an open subset of Ψ. Let f and g be two maps from Ψ to another topological space such that $f(x)=g(x)$ for each $x \in \partial G$. If f is continuous on $\Psi \setminus G$ and g is continuous on Ψ, then k defined on Ψ by*

$$k(x)=\begin{cases} f(x), & \text{if } x \in \Psi \setminus G \\ g(x), & \text{if } x \in G \end{cases}$$

is continuous on Ψ.

3. The proofs of the main theorems.

PROOF OF THEOREM 1. (i) Let T be a bounded linear operator from X into $C(\Phi)$. Without loss of generality, we may assume that the norm of T is one. By Lemma 1, T is a BKW-operator from X into $C(\Phi)$ for the test functions $\{\mathbf{1}, h\}$ if and only if $T^*(\delta_\varphi) \in U_{\{\mathbf{1},h\}}(X_1^*)$ for all $\varphi \in \Phi$. Also by Lemma 2-(i), $T^*(\delta_\varphi) \in U_{\{\mathbf{1},h\}}(X_1^*)$ for all $\varphi \in \Phi$ if and only if for each $\varphi \in \Phi$, there exists a pair of complex numbers $(u(\varphi), v(\varphi))$ such that $T^*(\delta_\varphi)=u(\varphi)\delta_{\omega_0}|X+v(\varphi)\delta_{\omega_1}|X$, $|u(\varphi)|+|v(\varphi)|=1$ and $|u(\varphi)+v(\varphi)|\neq 1$ when $u(\varphi)\neq 0$ and $v(\varphi)\neq 0$. Note that $T^*(\delta_\varphi)=u(\varphi)\delta_{\omega_0}|X+v(\varphi)\delta_{\omega_1}|X$ means that $(Tf)(\varphi)=f(\omega_0)u(\varphi)+f(\omega_1)v(\varphi)$ for all $f \in X$. We thus obtain that $T(f)=f(\omega_0)u+f(\omega_1)v$ for all $f \in X$. Moreover, this equality easily implies that $u=T(\mathbf{1}-\tilde{h})$ and $v=T(\tilde{h})$, where $\tilde{h}=(M-m)^{-1}(h-m\mathbf{1})$, and so u and v are in $C(\Phi)$. In particular, if T is unital, then we have

$$1=(T\mathbf{1})(\varphi)=u(\varphi)+v(\varphi)$$

for all $\varphi \in \Phi$ and hence $\Phi=\Phi_u \cup \Phi_v$ and $\Phi_u \cap \Phi_v=\varnothing$, where $\Phi_u=\{\varphi \in \Phi : u(\varphi)\neq 0\}$ and $\Phi_v=\{\varphi \in \Phi : v(\varphi)\neq 0\}$. Hence u and v equal the characteristic functions on Φ_u and Φ_v, respectively. Of course, $u+v=\mathbf{1}$, so that by putting $\chi=u$, we obtain that the desired equality and (i).

The same argument implies (ii) and (iv). Since T is a BKW-operator for $\{\mathbf{1}, h\}$ if and only if it is a BKW-operator for $\{\mathbf{1}, -h\}$, (iii) immediately follows from (ii).

q.e.d.

Proof of Theorem 2. We may assume without loss of generality that $m=0$ and $M=1$.

(i) Suppose that Ω_m consists of a single point ω_0 and Ω_M consists of a single point ω_1. Let T be a norm one unital BKW-operator from X into $C(\Phi)$ for the test functions $\{\mathbf{1}, h, h^2\}$. If $\varphi\in\Phi$, then $T^*(\delta_\varphi)\in U_{\{\mathbf{1},h,h^2\}}(X_1^*)$ by Lemma 1, and so $\|T^*(\delta_\varphi)\|=1$ by [7, Lemma 2.1]. Note also that $(T^*\delta_\varphi)(1)=\mathbf{1}(\varphi)=1$. Therefore $T^*(\delta_\varphi)\in X_+^*$ for all $\varphi\in\Phi$. Hence by Lemma 3-(i), we have $\Phi=F_T\cup G_T$, where F_T is the set of all $\varphi\in\Phi$ such that $T^*(\delta_\varphi)\in\{\delta_\omega|X:\omega\in\Omega_h\}$ and G_T is the set of all $\varphi\in\Phi$ such that $T^*(\delta_\varphi)\in\{(1-a)\delta_{\omega_0}|X+a\delta_{\omega_1}|X:0<a<1\}$. Note that $F_T\cap G_T=\varnothing$ and hence F_T equals the set of all $\varphi\in\Phi$ such that $T^*(\delta_\varphi)\in\{\delta_\varphi|X:\omega\in\Omega\}$. Therefore since the map: $\varphi\to T^*(\delta_\varphi)$ is weak*-continuous on Φ and the set $\{\delta_\omega|X:\omega\in\Omega\}$ is weak*-closed in X^*, F_T must be closed and so G_T is open. Now let $\varphi\in\Phi$. If $\varphi\in F_T$, then we can find a unique point $\omega\in\Omega_h$ such that $T^*(\delta_\varphi)=\delta_\omega|X$. Set $\omega=\xi(\varphi)$. Then we have

$$(Tf)(\varphi)=f(\xi(\varphi))$$

for each $f\in X$. If $\varphi\in G_T$, then there is a unique number $0<a<1$ such that $T^*(\delta_\varphi)=(1-a)\delta_{\omega_0}|X+a\delta_{\omega_1}|X$. Moreover, there is a point $\omega\in\Omega$ such that $a=h(\omega)$ because $h(\Omega)=[0, 1]$. Set $\omega=\xi(\varphi)$. Then we have

$$(Tf)(\varphi)=f(\omega_0)\{1-h(\xi(\varphi))\}+f(\omega_1)h(\xi(\varphi))$$

for each $f\in X$. Of course, ξ is a map from Φ into Ω such that $\xi(\Phi\setminus G_T)\subset\Omega_h$ and $0<h(\xi(\varphi))<1$ for each $\varphi\in G_T$. Also since $h(\xi(\varphi))=(Th)(\varphi)$ for each $\varphi\in\Phi$, we see that $h\circ\xi$ is continuous on Φ. To see that $\xi|F_T$ is continuous on F_T, let $\varphi\in F_T$ and let $\{\varphi_\lambda\}$ be a net of F_T which converges to φ. Consider any subnet $\{\xi(\varphi_{\lambda'})\}$ of the net $\{\xi(\varphi_\lambda)\}$. Then there exists a convergent subnet $\{\xi(\varphi_{\lambda''})\}$ of $\{\xi(\varphi_{\lambda'})\}$. Let ω be the limit point of $\{\xi(\varphi_{\lambda''})\}$. Then we have

$$h(\omega)=\lim_{\lambda''}h(\xi(\varphi_{\lambda''}))=\lim_{\lambda''}(Th)(\varphi_{\lambda''})=(Th)(\varphi)=h(\xi(\varphi)),$$

and so $\omega=\xi(\varphi)$ because $\xi(\varphi)\in\Omega_h$. This observation implies that $\lim_\lambda\xi(\varphi_\lambda)=\xi(\varphi)$ and hence $\xi|F_T$ is continuous on F_T. We next see that $\xi(\varphi)=\omega_0$ or ω_1 for each $\varphi\in\partial G_T$. To do so, let $\varphi\in\partial G_T$. Then $\xi(\varphi)\in\Omega_h$ and $T^*(\delta_\varphi)=\delta_{\xi(\varphi)}|X$. Also since φ is in the closure of G_T, we can take a net $\{\varphi_\lambda\}$ of G_T which converges to φ. Then for each λ, we have $T^*(\delta_{\varphi_\lambda})=\{1-h(\xi(\varphi_\lambda))\}\delta_{\omega_0}|X+h(\xi(\varphi_\lambda))\delta_{\omega_1}|X$ and hence

$$f(\xi(\varphi)) = (Tf)(\varphi) = \lim_{\lambda} (Tf)(\varphi_\lambda)$$

$$= \lim_{\lambda} f(\omega_0)\{1 - h(\xi(\varphi_\lambda))\} + \lim_{\lambda} f(\omega_1) h(\xi(\varphi_\lambda))$$

$$= f(\omega_0)\{1 - h(\xi(\varphi))\} + f(\omega_1) h(\xi(\varphi))$$

for all $f \in X$. In particular, by putting $f = h^2$, we have $h(\xi(\varphi))^2 = h(\xi(\varphi))$ and so $h(\xi(\varphi)) = 0$ or 1, hence $\xi(\varphi) = \omega_0$ or ω_1 since Ω_m consists of a single point ω_0 and Ω_M consists of a single point ω_1.

Conversely, let ξ be a map from Φ into Ω and G is an open subset of Φ such that $0 < h(\xi(\varphi)) < 1$ for all $\varphi \in G$, that $\xi(\varphi) = \omega_0$ or ω_1 for all $\varphi \in \partial G$, that $\xi(\Phi \setminus G) \subset \Omega_h$, that $h \circ \xi$ is continuous on Ω and that $\xi \,|\, (\Phi \setminus G)$ is continuous on $\Phi \setminus G$. For each $f \in X$, put

$$(T_\xi f)(\varphi) = \begin{cases} f(\xi(\varphi)), & \text{if } \varphi \in \Phi \setminus G \\ f(\omega_0)\{1 - h(\xi(\varphi))\} + f(\omega_1) h(\xi(\varphi)), & \text{if } \varphi \in G. \end{cases}$$

Since $\xi(\varphi) = \omega_0$ or ω_1 for all $\varphi \in \partial G$, it follows that

$$f(\xi(\varphi)) = f(\omega_0)\{1 - h(\xi(\varphi))\} + f(\omega_1) h(\xi(\varphi))$$

for all $\varphi \in \partial G$. Then for each $f \in X$, $T_\xi(f)$ is a complex-valued continuous function on Φ by Lemma 4. Moreover, we can easily see that T_ξ is a norm one unital linear operator from X into $C(\Phi)$. Note also that

$$T_\xi^*(\delta_\varphi) \in \{\delta_\omega \,|\, X : \omega \in \Omega_h\} \cup \{(1-a)\delta_{\omega_0} \,|\, X + a\delta_{\omega_1} \,|\, X : 0 < a < 1\}$$

for all $\varphi \in \Phi$. Then T_ξ is a BKW-operator for the test functions $\{\mathbf{1}, h, h^2\}$ by Lemmas 1 and 3-(i).

(ii) Suppose that either Ω_m or Ω_M possesses more than two points. Let T be a norm one unital BKW-operator from X into $C(\Phi)$ for the test functions $\{\mathbf{1}, h, h^2\}$. If $\varphi \in \Phi$, then $T^*(\delta_\varphi) \in U_{\{\mathbf{1},h,h^2\}}(X_1^*) \cap X_+^*$ as observed in the proof of (i). Hence we can find a unique point $\omega \in \Omega_h$ such that $T^*(\delta_\varphi) = \delta_\omega \,|\, X$ by Lemmas 3-(ii). Set $\omega = \xi(\varphi)$. Then we have

$$(Tf)(\varphi) = f(\xi(\varphi))$$

for each $f \in X$. Of course, ξ is a map from Φ into Ω_h and we see that ξ is continuous on Φ by the same method used in the proof of (i).

Conversely, let ξ be a continuous map from Φ into Ω_h. Set $(T_\xi f)(\varphi) = f(\xi(\varphi))$ for each $f \in X$ and $\varphi \in \Phi$. Then we can easily see that T_ξ is a norm one unital linear operator from X into $C(\Phi)$ such that $T_\xi^*(\delta_\varphi) \in \{\delta_\omega \,|\, X : \omega \in \Omega_h\}$ for all $\varphi \in \Phi$. Then T_ξ is a KBW-operator for the test functions $\{\mathbf{1}, h, h^2\}$ by Lemmas 1 and 3-(ii). q.e.d.

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