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A REMARK ON THE RIEMANN-ROCH FORMULA ON AFFINE SCHEMES ASSOCIATED WITH NOETHERIAN LOCAL RINGS

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Abstract. The aim of this paper is to describe the Riemann-Roch map on affine schemes associated with Noetherian local rings. The Riemann-Roch theorem on singular affine schemes is one of the powerful tools in the commutative ring theory. Our main theorem enables us to calculate the Riemann-Roch maps under some assumption.

1. Main theorem. Let k be a field and $R=\bigoplus_{i\geq 0} R_i$ a graded Noetherian ring which satisfies $R_0=k$ and $R=R_0[R_1]$, i.e., R is a graded k-algebra generated by R_1. We denote by $\mathfrak{m}$ the homogeneous maximal ideal $\bigoplus_{i>0} R_i$.

For an abelian group M, we write $M_{\boldsymbol{Q}}$ for $M\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$, where $\boldsymbol{Z}$ (resp. $\boldsymbol{Q}$) is the ring of integers (resp. the field of rational numbers).

Let $X=\mathrm{Proj}(R)$ be a smooth projective variety over k of dimension d. We denote by $\mathrm{A}_*(X)=\bigoplus_{i=0}^{d}\mathrm{A}_i(X)$ the *Chow group* of X. If we put $\mathrm{CH}^i(X)=\mathrm{A}_{d-i}(X)$ for $i=0,\ldots,d$, then $\mathrm{CH}(X)=\bigoplus_{i=0}^{d}\mathrm{CH}^i(X)$ has the structure of a commutative ring (since X is smooth), and is called the *Chow ring* of X. (We refer the reader to [5] for definitions and basic facts.)

We put $c=c_1(\mathcal{O}_X(1))\cap[X]\in\mathrm{A}_{d-1}(X)_{\boldsymbol{Q}}=\mathrm{CH}^1(X)_{\boldsymbol{Q}}$, where $c_1(\mathcal{O}_X(1))$ denotes the *first Chern class* of the invertible sheaf $\mathcal{O}_X(1)$, i.e., c stands for the Cartier divisor corresponding to the line bundle $\mathcal{O}_X(1)$. Let

$$\pi:\mathrm{CH}(X)_{\boldsymbol{Q}}\to\mathrm{CH}(X)_{\boldsymbol{Q}}/(c) \tag{1.1}$$

be the natural surjective ring homomorphism ((c) is the principal ideal of $\mathrm{CH}(X)_{\boldsymbol{Q}}$ generated by c), and

$$\tau:\mathrm{K}_0(R_{\mathfrak{m}})_{\boldsymbol{Q}}\to\mathrm{A}_*(\mathrm{Spec}\,R_{\mathfrak{m}})_{\boldsymbol{Q}} \tag{1.2}$$

the Riemann-Roch map for the affine scheme $\mathrm{Spec}\,R_{\mathfrak{m}}$ (cf. [5, chap. 18]), where $\mathrm{K}_0(R_{\mathfrak{m}})$ is the *Grothendieck group* of finitely generated $R_{\mathfrak{m}}$-modules and $\mathrm{A}_*(\mathrm{Spec}\,R_{\mathfrak{m}})$ is the Chow group of $\mathrm{Spec}\,R_{\mathfrak{m}}$.

Our main theorem is the following:

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THEOREM (1.3). *In the notation as above, we have an isomorphism of graded modules*

$$\xi : \mathrm{CH}(X)_{\boldsymbol{Q}}/(c) \to \mathrm{A}_*(\operatorname{Spec} R_{\mathrm{m}})_{\boldsymbol{Q}} \tag{1.4}$$

such that $\xi\pi(\mathrm{td}(\Omega_X^\vee)) = \tau([R_{\mathrm{m}}])$ *is satisfied, where* $\mathrm{td}(\Omega_X^\vee)$ *is the Todd class of the locally free sheaf* $\Omega_X^\vee$, *and* $[R_{\mathrm{m}}]$ *is the element in* $\mathrm{K}_0(R_{\mathrm{m}})_{\boldsymbol{Q}}$ *corresponding to* R_{m}. *Here,* $\xi = \bigoplus_{i=0}^{d} \xi_i$ *is an isomorphism of graded modules such that*

$$\begin{aligned} \xi_i : [\mathrm{CH}(X)_{\boldsymbol{Q}}/(c)]_i &\simeq \mathrm{A}_{d+1-i}(\operatorname{Spec} R_{\mathrm{m}})_{\boldsymbol{Q}} \\ (0) &\simeq \mathrm{A}_0(\operatorname{Spec} R_{\mathrm{m}})_{\boldsymbol{Q}} . \end{aligned} \tag{1.5}$$

We have the following corollary immediately from Theorem (1.3).

COROLLARY (1.6). *In the notation as in Theorem* (1.3), *the following are equivalent*:

(1) $\tau([R_{\mathrm{m}}])$ *is contained in* $\mathrm{A}_{d+1}(\operatorname{Spec} R_{\mathrm{m}})_{\boldsymbol{Q}}$.
(2) $\mathrm{td}(\Omega_X^\vee) \equiv 1 \bmod (c)$.

We shall prove Theorem (1.3) in Section 4. The next section is devoted to motivation. By using Theorem (1.3), we show in Section 3 that $\tau([A])$ is not contained in $\mathrm{A}_{\dim A}(\operatorname{Spec} A)_{\boldsymbol{Q}}$ when A is a Gorenstein local ring defined in (3.3). (As far as the author knows, it is the first example of a Gorenstein ring such that $\tau([A]) \notin \mathrm{A}_{\dim A}(\operatorname{Spec} A)_{\boldsymbol{Q}}$.)

2. Motivation. In the present section we shall explain why we would like to prove Theorem (1.3) or what we want to do. The reader can skip this section because the results introduced here are not used after this section.

Let k be a perfect field of characteristic p (>0) and A a complete Noetherian local ring of dimension r with coefficient field k.

Let $f : A \to A$ be the Frobenius map (i.e., $f(x) = x^p$ for any $x \in A$). Then f is a (module-)finite morphism since A is a complete local ring whose coefficient field is perfect. We denote by $f^e : A \to A$ the e-th iteration of the Frobenius map f, i.e., $f^e(x) = x^{p^e}$ for $x \in A$. We write ${}^eA$ for the A-module A whose module structure is given by f^e.

Let $\mathrm{K}_0(A)$ be the Grothendieck group of finitely generated A-modules and we write $\mathrm{K}_0(A)_{\boldsymbol{Q}}$ for $\mathrm{K}_0(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. For a finitely generated A-module M, $[M]$ stands for the element in $\mathrm{K}_0(A)_{\boldsymbol{Q}}$ corresponding to M.

If a ring homomorphism $g : B \to C$ is (module-)finite, every finitely generated C-module naturally has a structure of a finitely generated B-module. Therefore we can define a group homomorphism $g^* : \mathrm{K}_0(C)_{\boldsymbol{Q}} \to \mathrm{K}_0(B)_{\boldsymbol{Q}}$. Since the Frobenius map $f : A \to A$ is finite in our case, we have an endomorphism $f^* : \mathrm{K}_0(A)_{\boldsymbol{Q}} \to \mathrm{K}_0(A)_{\boldsymbol{Q}}$.

We put

$$\mathrm{L}_i\mathrm{K}_0(A)_{\boldsymbol{Q}} = \{ c \in \mathrm{K}_0(A)_{\boldsymbol{Q}} \mid f^*(c) = p^i c \} \tag{2.1}$$

for $i = 0, 1, \ldots, r$, i.e., $\mathrm{L}_i\mathrm{K}_0(A)_{\boldsymbol{Q}}$ is the eigenspace of f^* with eigenvalue p^i. Then we can easily prove

$$\mathrm{K}_0(A)_{\boldsymbol{Q}}=\bigoplus_{i=0}^{r}\mathrm{L}_i\mathrm{K}_0(A)_{\boldsymbol{Q}} \tag{2.2}$$

by induction on the dimension of cycles. Hence we have a unique representation

$$[A]=q_r+q_{r-1}+\cdots+q_0\,,\qquad (q_i\in\mathrm{L}_i\mathrm{K}_0(A)_{\boldsymbol{Q}})\,. \tag{2.3}$$

Such a decomposition of $[A]$ inherits various properties of our ring A.

Our aim in the present paper is to study how to decompose $[A]$ as above.

Since $(f^*)^e(q_i)=p^{ie}q_i$ for $i=0, 1, \ldots, r$ (cf. (2.1)), we have

$$(f^*)^e([A])=[{}^eA]=p^{re}q_r+p^{(r-1)e}q_{r-1}+\cdots+p^eq_1+q_0\,.$$

Therefore, it is immediate to see

$$q_r=\lim_{e\to\infty}\frac{1}{p^{re}}[{}^eA]\qquad\text{in }\mathrm{K}_0(A)_{\boldsymbol{Q}}\,. \tag{2.4}$$

Suppose that A is a regular local ring. Then ${}^1A$ is an A-free module (of rank p^r) by Kunz's theorem [10]. Hence we get $f^*([A])=[{}^1A]=p^r[A]$, and therefore $[A]\in\mathrm{L}_r\mathrm{K}_0(A)_{\boldsymbol{Q}}$. That is to say, $[A]=q_r$ and $q_{r-1}=\cdots=q_0=0$ are satisfied. (When A is a regular local ring, we immediately obtain $\mathrm{K}_0(A)_{\boldsymbol{Q}}=\boldsymbol{Q}\cdot[A]$ since the global dimension of A is finite. Furthermore it is easy to see $\mathrm{K}_0(A)_{\boldsymbol{Q}}=\mathrm{L}_r\mathrm{K}_0(A)_{\boldsymbol{Q}}$ and $\mathrm{L}_i\mathrm{K}_0(A)_{\boldsymbol{Q}}=(0)$ for $i=0, 1, \ldots, r-1$.)

When A is a complete Noetherian local ring of characteristic $p>0$ whose coefficient field is perfect, we obtain a natural decomposition as (2.2) or (2.3) using the Frobenius endomorphism. As we shall see below, however, we can make a decomposition as (2.2) or (2.3) over an arbitrary Noetherian local ring A (under a mild condition) by using localized Chern characters, the singular Riemann-Roch theorem [5] or Adams operations (cf. Gillet-Soulé [6], [7]).

In the rest of this section we merely assume that our ring A is a homomorphic image of a regular local ring unless otherwise specified. (A does not have to include a field.) Put $r=\dim A$.

Letting $\mathrm{A}_*(\operatorname{Spec}A)_{\boldsymbol{Q}}=\bigoplus_{i=0}^{r}\mathrm{A}_i(\operatorname{Spec}A)_{\boldsymbol{Q}}$ be the Chow group of the affine scheme $\operatorname{Spec}A$ with rational coefficients, we can construct a natural isomorphism

$$\tau:\mathrm{K}_0(A)_{\boldsymbol{Q}}\to\mathrm{A}_*(\operatorname{Spec}A)_{\boldsymbol{Q}} \tag{2.5}$$

called the *Riemann-Roch map* (cf. [5]). When A is a complete Noetherian local ring of characteristic $p>0$ with perfect coefficient field, it is easy to see that

$$\tau^{-1}(\mathrm{A}_i(\operatorname{Spec}A)_{\boldsymbol{Q}})=\mathrm{L}_i\mathrm{K}_0(A)_{\boldsymbol{Q}} \tag{2.6}$$

is satisfied for each i. That is to say, in order to study the decomposition as (2.2) or (2.3), we have only to investigate the Riemann-Roch map (2.5), and therefore, we do not have to assume that A includes a field of positive characteristic.

Put

(2.7) $$\tau([A]) = q_r + q_{r-1} + \cdots + q_0 \qquad (q_i \in \mathrm{A}_i(\operatorname{Spec} A)_{\boldsymbol{Q}}).$$

Our aim is to get the decomposition (2.7).

The following are basic facts about the decomposition (2.7). We omit the proof.

REMARK (2.8). Let A be a homomorphic image of a regular local ring.

(1) We have $q_r \neq 0$. (By [5], $q_r = [\operatorname{Spec} A] \in \mathrm{A}_r(\operatorname{Spec} A)_{\boldsymbol{Q}}$ is satisfied, where $[\operatorname{Spec} A]$ is the cycle corresponding to $\operatorname{Spec} A$.)

(2) If A is a complete intersection, then we have $q_{r-1} = \cdots = q_0 = 0$ (cf. [5, Cor. 18.1.2]).

(3) Assume that A is equidimensional. If the dimension of the non-Cohen-Macaulay locus of A is l, then we get

$$\tau([K_A]) \equiv q_r - q_{r-1} + \cdots + (-1)^i q_{r-i} + \cdots \mod(\textstyle\bigoplus_{j=0}^{l} \mathrm{A}_j(\operatorname{Spec} A)_{\boldsymbol{Q}}),$$

where K_A denotes the canonical module of A. In particular, when A is a Cohen-Macaulay ring, we have $\tau([K_A]) = q_r - q_{r-1} + \cdots + (-1)^i q_{r-i} + \cdots$. Therefore, if A is a Gorenstein ring, we get $q_{r-1} = q_{r-3} = q_{r-5} = \cdots = 0$ because A is a Cohen-Macaulay ring with $K_A \simeq A$.

(4) Suppose that A is a normal domain. Then we have a natural isomorphism $\phi\colon \mathrm{A}_{r-1}(\operatorname{Spec} A)_{\boldsymbol{Q}} \to \mathrm{Cl}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ such that $\phi(q_{r-1}) = \mathrm{cl}(K_A)/2$ holds ([12, Lemma 3.5]). Here $\mathrm{cl}(K_A)$ stands for the isomorphism class which the reflexive A-module K_A belongs to. Especially, in this case, $q_{r-1} = 0$ is satisfied if and only if $\mathrm{cl}(K_A)$ is a torsion element of $\mathrm{Cl}(A)$.

It seems to be natural to ask the following question:

QUESTION (2.9). When does $\tau([A]) = q_r$ hold? (When A is a complete Noetherian local ring of characteristic $p > 0$ with perfect coefficient field, the equality $\tau([A]) = q_r$ holds if and only if $[{}^1A] = p^r[A]$ is satisfied in $\mathrm{K}_0(A)_{\boldsymbol{Q}}$ (cf. (2.6)).)

As we saw in (2) of Remark (2.8), $\tau([A]) = q_r$ is satisfied if A is a complete intersection. Furthermore, by (4) of Remark (2.8), we know $q_{r-1} \neq 0$ if A is a normal domain whose canonical class $\mathrm{cl}(K_A)$ is not a torsion element in its divisor class group $\mathrm{Cl}(A)$, and therefore, we can easily make examples of Cohen-Macaulay normal rings with $q_{r-1} \neq 0$. Hence we would like to discuss Question (2.9) in the case where A is a Gorenstein ring. By using the main theorem (Theorem (1.3)), we shall construct an example of a Gorenstein ring with $q_{r-2} \neq 0$ in Section 3 (cf. (3.3)).

In the rest of this section, we state some properties of a local ring with $\tau([A]) = q_r$. We can find many examples of Noetherian local rings which satisfy $\tau([A]) = q_r$ by Corollary (1.6).

REMARK (2.10). (1) As Roberts [16] pointed out, if a Noetherian local ring A satisfies $\tau([A]) = q_r$, then the vanishing theorem holds, i.e., $\sum_i (-1)^i l_A(\mathrm{Tor}_i^A(M, N)) = 0$

is satisfied for finitely generated A-modules M and N such that $\mathrm{pd}_A M < \infty$, $\mathrm{pd}_A N < \infty$, $l_A(M \otimes_A N) < \infty$ and $\dim M + \dim N < \dim A$.

Therefore, if A is a complete intersection, then the vanishing theorem holds (cf. Roberts [16] and Gillet-Soulé [7]) since $\tau([A]) = q_r$ is satisfied by (2) of Remark (2.8). We note that there are many examples of Noetherian local rings such that the vanishing theorem holds but $\tau([A]) \neq q_r$ (see [16], [12]).

(2) Assume that A is a complete Noetherian local ring of characteristic $p > 0$ whose coefficient field is perfect.

Let $d\colon F \to G$ be an A-linear map of free A-modules. By choosing free bases of F and G, we denote by (d_{ij}) the matrix corresponding to the A-linear map d. For a positive integer e, let ${}^e d\colon F \to G$ denote the A-linear map corresponding to the matrix $(d_{ij}^{p^e})$. Let

$$F.: 0 \longrightarrow F_l \xrightarrow{d_l} F_{l-1} \xrightarrow{d_{l-1}} \cdots \xrightarrow{d_1} F_0 \longrightarrow 0$$

be a complex of finitely generated free A-modules such that $H_i(F.)$ has finite length for every i. Choosing bases of $F_0, \dots, F_l$, we define a complex $F.^{[e]}$ by

$$F.^{[e]}: 0 \longrightarrow F_l \xrightarrow{{}^e d_l} F_{l-1} \xrightarrow{{}^e d_{l-1}} \cdots \xrightarrow{{}^e d_1} F_0 \longrightarrow 0 .$$

It is easy to see that $F.^{[e]}$ is certainly a complex and is uniquely determined, i.e., the construction does not depend (up to isomorphism) on the choice of the bases of $F_0, \dots, F_l$. Furthermore $H_i(F.^{[e]})$ has finite length for every i. (As is easily seen, $F.^{[e]}$ coincides with $F. \otimes_A {}^e A$ regarded as an ${}^e A$-free complex.)

In the notation as above, Szpiro [20] conjectured the following:

Conjecture (2.11). Put $r = \dim A$. Then

$$\sum_i l_A(H_i(F.^{[e]})) = p^{re} \sum_i l_A(H_i(F.)) .$$

It is now known that there exists a counterexample to the above conjecture. Indeed, let k be a perfect field of characteristic p (>0) and put

$$B = k[[x_0, x_1, x_2, y_0, y_1]] \Big/ I_2\begin{pmatrix} x_0 & x_1 & y_0 \\ x_1 & x_2 & y_1 \end{pmatrix} .$$

It is easy to see that B is a Cohen-Macaulay normal domain of dimension 3. Modifying the famous example due to Dutta-Hochster-MacLaughlin [4], one can construct a finite B-free complex $G.$ such that $\sum_i l_B(H_i(G.^{[1]})) \neq p^3 \sum_i l_B(H_i(G.))$.

Here assume that A is an r-dimensional Noetherian local ring which satisfies $\tau([A]) = q_r$. Then we obtain $[{}^e A] = p^{re}[A]$ in $\mathrm{K}_0(A)_{\mathbf{Q}}$ for every e. Then, it is easy to see

$$\sum_i (-1)^i l_A(H_i(F_.^{[e]})) = \sum_i (-1)^i l_A(H_i(F_. \otimes_A {}^eA))$$
$$(2.12) \qquad = p^{re} \cdot \sum_i (-1)^i l_A(H_i(F_.)) .$$

As a result, Szpiro's conjecture (2.11) is true if $\tau([A]) = q_r$ is satisfied. Therefore, for example, when A is a complete intersection, that is true (by (2) of Remark (2.8)) as it has been already pointed out by Gillet-Soulé [7, Theorem B].

In the notation as in Conjecture (2.11), we put

$$(2.13) \qquad \boldsymbol{D}_A(F_.) = \lim_{e \to \infty} \frac{1}{p^{re}} \sum_i (-1)^i l_A(H_i(F_.^{[e]}))$$

and call it the *Dutta multiplicity*. It is known that $\boldsymbol{D}_A(F_.)$ is a rational number (cf. Dutta [2]). In general, $\boldsymbol{D}_A(F_.)$ need not coincide with $\sum_i (-1)^i l_A(H_i(F_.))$ (see [4]). By (2.4), we have

$$\boldsymbol{D}_A(F_.) = \sum_i (-1)^i l_A(H_i(F_. \otimes_A q_r))$$

in general. (Therefore, once $\tau([A]) = q_r$ is satisfied, we immediately get $\boldsymbol{D}_A(F_.) = \sum_i (-1)^i l_A(H_i(F_.))$.) Geometrically, the Dutta multiplicity is nothing but the localized Chern character (cf. [5]), i.e., we have

$$\boldsymbol{D}_A(F_.) = \mathrm{ch}^{\mathrm{Spec}(A)}_{\mathrm{Spec}(A/\mathfrak{m})}(F_.) \cap [\mathrm{Spec}(A)]$$

by [17], where $\mathfrak{m}$ is the maximal ideal of A. Calculation of localized Chern characters plays important roles in studying intersection multiplicities.

3. An application. In this section, using Theorem (1.3), we shall make an example of a Gorenstein ring A of dimension r with $q_{r-2} \neq 0$, as we announced before. Here,

$$(3.1) \qquad \tau : \mathrm{K}_0(A)_{\boldsymbol{Q}} \to \mathrm{A}_*(\mathrm{Spec}\, A)_{\boldsymbol{Q}}$$

is the Riemann-Roch map and we put

$$(3.2) \qquad \tau([A]) = q_r + q_{r-1} + q_{r-2} + \cdots + q_0 , \qquad (q_i \in \mathrm{A}_i(\mathrm{Spec}\, A)_{\boldsymbol{Q}}) .$$

As far as the author knows, it is the first counterexample to Question (2.9) when A is a Gorenstein ring. (In the present section we shall prove that q_{r-2} does not coincide with 0 when

$$(3.3) \qquad A = R_{(x_{ij} \mid i=1,\dots,n;\, j=1,\dots,m)} ,$$

where R is the ring defined in (3.5).)

Let k be a field. Let n and m be positive integers such that $n \leq m$. Furthermore, let r and d be the integers satisfying $r = d + 1 = n + m + 1$. We denote by $\boldsymbol{P}^n$ (resp. $\boldsymbol{P}^m$) the projective space over k of dimension n (resp. m), i.e., put $\boldsymbol{P}^n=\operatorname{Proj}(k[x_0,\dots,x_n])$ (resp. $\boldsymbol{P}^m=\operatorname{Proj}(k[x_0,\dots,x_m])$). Setting $X=\boldsymbol{P}^n\times\boldsymbol{P}^m$, let

$$(3.4)\qquad X\hookrightarrow \boldsymbol{P}^{nm+n+m}$$

be the Segre embedding. Obviously X is a smooth projective variety over k of dimension d. Let R be the homogeneous coordinate ring of the embedding (3.4), i.e.,

$$(3.5)\qquad R=k[\{x_{ij}\mid i=0,\dots,n;\,j=0,\dots,m\}]/I_2(x_{ij})$$

where $k[\{x_{ij}\mid i=0,\dots,n;\,j=0,\dots,m\}]$ is the polynomial ring with variables $\{x_{ij}\mid i=0,\dots,n;\,j=0,\dots,m\}$ and $I_2(x_{ij})$ is the ideal generated by all 2 by 2 minors of the $n+1$ by $m+1$ matrix (x_{ij}). Then it is well-known that the graded ring R is a Cohen-Macaulay normal domain of dimension r. Furthermore, it is a Gorenstein ring (resp. complete intersection) if and only if $n=m$ (resp. $n=m=1$) is satisfied (cf. [1]).

The Chow ring of the projective spaces are well-known. Indeed, we have an isomorphism of graded rings

$$\mathrm{CH}(\boldsymbol{P}^n)_{\boldsymbol{Q}}=\boldsymbol{Q}[a]/(a^{n+1})$$

where $a=c_1(\mathcal{O}_{\boldsymbol{P}^n}(1))$ (cf. [5]). We set $\deg a=1$. In the same way, we get an isomorphism

$$\mathrm{CH}(\boldsymbol{P}^m)_{\boldsymbol{Q}}=\boldsymbol{Q}[b]/(b^{m+1}),$$

where $b=c_1(\mathcal{O}_{\boldsymbol{P}^m}(1))$.

In this case, it is known (cf. [5]) that the Chow ring of $X=\boldsymbol{P}^n\times\boldsymbol{P}^m$ is isomorphic to the tensor product of $\mathrm{CH}(\boldsymbol{P}^n)_{\boldsymbol{Q}}$ and $\mathrm{CH}(\boldsymbol{P}^m)_{\boldsymbol{Q}}$. Therefore, we get

$$(3.6)\qquad \mathrm{CH}(\boldsymbol{P}^n\times\boldsymbol{P}^m)_{\boldsymbol{Q}}=\mathrm{CH}(\boldsymbol{P}^n)_{\boldsymbol{Q}}\otimes_{\boldsymbol{Q}}\mathrm{CH}(\boldsymbol{P}^m)_{\boldsymbol{Q}}=\boldsymbol{Q}[a,b]/(a^{n+1},b^{m+1}).$$

Let $p_1: X\to\boldsymbol{P}^n$ and $p_2: X\to\boldsymbol{P}^m$ be the projections. Then the line bundle $\mathcal{O}_X(1)$ with respect to the Segre embedding (3.4) coincides with $p_1^*\mathcal{O}_{\boldsymbol{P}^n}(1)\otimes_{\mathcal{O}_X}p_2^*\mathcal{O}_{\boldsymbol{P}^m}(1)$. Hence we obtain

$$(3.7)\qquad c_1(\mathcal{O}_X(1))=a+b\in\boldsymbol{Q}[a,b]/(a^{n+1},b^{m+1})=\mathrm{CH}(\boldsymbol{P}^n\times\boldsymbol{P}^m)_{\boldsymbol{Q}}.$$

Let A be the local ring of R at the homogeneous maximal ideal $(x_{ij}\mid i=1,\dots,n;\ j=1,\dots,m)$. As we noted before, A is a Cohen-Macaulay local ring of dimension $r=n+m+1$. Then, by Theorem (1.3), we have an isomorphism ξ of graded modules such that $\xi\pi(\mathrm{td}(\Omega_X^\vee))=\tau([A])$, where

$$\mathrm{CH}(X)_{\boldsymbol{Q}}\xrightarrow{\pi}\mathrm{CH}(X)_{\boldsymbol{Q}}/(a+b)\xrightarrow{\xi}\mathrm{A}_*(\operatorname{Spec}A)_{\boldsymbol{Q}}\xleftarrow{\tau}\mathrm{K}_0(A)_{\boldsymbol{Q}}.$$

In order to determine the decomposition (3.2), we shall investigate $\pi(\mathrm{td}(\Omega_X^\vee))\in\mathrm{CH}(X)_{\boldsymbol{Q}}/(a+b)$.

Since n is at most m, we have an isomorphism of graded rings as

$$\text{(3.8)} \qquad \mathrm{CH}(X)_{\boldsymbol{Q}}/(a+b)=\boldsymbol{Q}[a,b]/(a^{n+1},b^{m+1},a+b)=\boldsymbol{Q}[a]/(a^{n+1}).$$

Then we note that the composite map

$$\text{(3.9)} \qquad \mathrm{CH}(X)_{\boldsymbol{Q}}=\boldsymbol{Q}[a,b]/(a^{n+1},b^{m+1}) \xrightarrow{\pi} \mathrm{CH}(X)_{\boldsymbol{Q}}/(a+b)=\boldsymbol{Q}[a]/(a^{n+1})$$

is given by $\pi(a)=a$, $\pi(b)=-a$.

We shall calculate the Todd class of the tangent bundle of X. The following results are immediate consequences of the definition of Todd classes (cf. [5, chap. 3]).

REMARK (3.10). (1) If the sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of locally free sheaves is exact, then $\mathrm{td}(M_2)=\mathrm{td}(M_1)\cdot\mathrm{td}(M_3)$ holds.

(2) Let L be an invertible sheaf on X and put $c=c_1(L)\cap[X]\in\mathrm{CH}^1(X)_{\boldsymbol{Q}}$. Then we have

$$\mathrm{td}(L)=\frac{c}{1-e^{-c}}=1+\frac{1}{2}c+\frac{1}{12}c^2+\cdots.$$

($c/(1-e^{-c})$ is contained in $\mathrm{CH}(X)_{\boldsymbol{Q}}=\bigoplus_{i=0}^{d}\mathrm{CH}^i(X)_{\boldsymbol{Q}}$, since $\mathrm{CH}^i(X)_{\boldsymbol{Q}}=(0)$ for $i\gg 0$. Note that $\mathrm{td}(L)=1\in\mathrm{CH}(X)_{\boldsymbol{Q}}$ if $c=0$).

Since X is the fibre product of $\boldsymbol{P}^n$ and $\boldsymbol{P}^m$, we have

$$\text{(3.11)} \qquad \Omega_X=p_1^*\Omega_{\boldsymbol{P}^n}\oplus p_2^*\Omega_{\boldsymbol{P}^m}.$$

Furthermore, we have the following famous exact sequence of locally free sheaves on $\boldsymbol{P}^n$:

$$\text{(3.12)} \qquad 0\to\Omega_{\boldsymbol{P}^n}\to\mathcal{O}_{\boldsymbol{P}^n}(-1)^{n+1}\to\mathcal{O}_{\boldsymbol{P}^n}\to 0.$$

We refer the reader to [8] for basic facts on algebraic geometry. Take the inverse image by p_1^* of (3.12) and take the $\mathcal{O}_X$-dual. Then we obtain the following exact sequence of locally free $\mathcal{O}_X$-modules:

$$\text{(3.13)} \qquad 0\to\mathcal{O}_X\to p_1^*\mathcal{O}_{\boldsymbol{P}^n}(1)^{n+1}\to p_1^*\Omega_{\boldsymbol{P}^n}^\vee\to 0.$$

In the same way, we get

$$\text{(3.14)} \qquad 0\to\mathcal{O}_X\to p_2^*\mathcal{O}_{\boldsymbol{P}^m}(1)^{m+1}\to p_2^*\Omega_{\boldsymbol{P}^m}^\vee\to 0.$$

Then by (1) in Remark (3.10), (3.11), (3.13) and (3.14), we have

$$\begin{aligned}\text{(3.15)} \qquad \mathrm{td}(\Omega_X^\vee)&=\mathrm{td}(p_1^*\Omega_{\boldsymbol{P}^n}^\vee)\cdot\mathrm{td}(p_2^*\Omega_{\boldsymbol{P}^m}^\vee)=\mathrm{td}(p_1^*\mathcal{O}_{\boldsymbol{P}^n}(1))^{n+1}\cdot\mathrm{td}(p_1^*\mathcal{O}_{\boldsymbol{P}^m}(1))^{m+1}\\ &=\left(\frac{a}{1-e^{-a}}\right)^{n+1}\cdot\left(\frac{b}{1-e^{-b}}\right)^{m+1}\in\boldsymbol{Q}[a,b]/(a^{n+1},b^{m+1})=\mathrm{CH}(X)_{\boldsymbol{Q}}.\end{aligned}$$

Then, by (3.9) and (3.15), we have

$$\pi(\mathrm{td}(\Omega_X^\vee))=\left(\frac{a}{1-e^{-a}}\right)^{n+1}\cdot\left(\frac{-a}{1-e^{a}}\right)^{m+1}$$
$$=\left(\frac{-a^2}{(1-e^{-a})(1-e^{a})}\right)^{n+1}\cdot\left(\frac{-a}{1-e^{a}}\right)^{m-n}. \tag{3.16}$$

It is easy to see that

$$\left(\frac{-a}{1-e^{a}}\right)^{m-n}=\left(\sum_{k\geq 0}\frac{a^k}{(k+1)!}\right)^{-(m-n)}$$
$$=1-\frac{m-n}{2}a+\cdots\in \boldsymbol{Q}[a]/(a^{n+1}), \tag{3.17}$$

$$\left(\frac{-a^2}{(1-e^{-a})(1-e^{a})}\right)^{n+1}=\left(\sum_{k\geq 0}\frac{2a^{2k}}{(2k+2)!}\right)^{-(n+1)}$$
$$=1-\frac{n+1}{12}a^2+\cdots\in \boldsymbol{Q}[a]/(a^{n+1}). \tag{3.18}$$

By the equations (3.16), (3.17) and (3.18), we get

$$\pi(\mathrm{td}(\Omega_X^\vee))=1-\frac{m-n}{2}a+a^2(\cdots).$$

By the isomorphisms (3.8) and (1.5) in Theorem (1.3), we have $\xi(-(m-n)a/2)=q_{r-1}$ (see (3.2)). Here note that, if m is bigger than n, then $-(m-n)a/2\in \mathrm{CH}(X)_{\boldsymbol{Q}}/(a+b)=\boldsymbol{Q}[a]/(a^{n+1})$ is not equal to 0. Therefore $q_{r-1}\neq 0$ if and only if $n\neq m$. (Since A is a Cohen-Macaulay normal domain of dimension r, we have $q_{r-1}=0$ if and only if the canonical class $\mathrm{cl}(K_A)$ is a torsion element in the divisor class group $\mathrm{Cl}(A)$, as we saw in (4) of Remark (2.8). In fact, A is a Gorenstein ring if and only if $n=m$ is satisfied. Furthermore, when $n\neq m$ is satisfied, $\mathrm{cl}(K_A)$ is not a torsion element in $\mathrm{Cl}(A)$. We refer the reader to [1] for the ring-theoretic properties of A.)

Assume $n=m$, i.e., A is a Gorenstein ring. Then we have

$$\pi(\mathrm{td}(\Omega_X^\vee))=1-\frac{n+1}{12}a^2+a^4(\cdots)\in \boldsymbol{Q}[a]/(a^{n+1}).$$

Therefore, if n is at least 2, then $-(n+1)a^2/12\neq 0\in \mathrm{CH}(X)_{\boldsymbol{Q}}/(a+b)$ is satisfied. Hence, when $n=m\geq 2$ is satisfied, we get $q_{r-2}=\xi(-(n+1)a^2/12)\neq 0$.

That is to say, in this case, A is a Gorenstein ring with $q_{r-2}\neq 0$.

In the case $n=m=1$, A is a complete intersection. Therefore, by (2) of Remark (2.8), $\tau([A])$ coincides with q_r, i.e., $\mathrm{td}(\Omega_X^\vee)=1\in \boldsymbol{Q}[a]/(a^2)$ is satisfied.

In the case where a local ring A has an isolated singularity, Theorem (1.3) sometimes enables us to calculate $\tau([A])$. If the singular locus has positive dimension, however, it is very hard to calculate $\tau([A])$.

4. Proof of the main theorem. In the present section we shall prove Theorem (1.3).

To begin with, we shall investigate the relation between $A_*(\operatorname{Spec} R)$ and $A_*(\operatorname{Spec} R_{\mathfrak{m}})$. The following lemma must be well-known, but we shall give a proof because the author does not know an adequate reference.

LEMMA (4.1). *Let $T=\bigoplus_{i\geq 0} T_i$ be a Noetherian graded ring such that T_0 is a field. (T need not coincide with $T_0[T_1]$.) We denote by $\mathfrak{n}$ the homogeneous maximal ideal $\bigoplus_{i>0} T_i$. Then we have an isomorphism of graded modules*

$$\bar{\psi}: A_*(\operatorname{Spec} T) \to A_*(\operatorname{Spec} T_{\mathfrak{n}})$$

such that, for each prime ideal $\mathfrak{p}$ in T,

$$\bar{\psi}([\operatorname{Spec}(T/\mathfrak{p})])=\begin{cases} [\operatorname{Spec}(T_{\mathfrak{n}}/\mathfrak{p}T_{\mathfrak{n}})] & \text{if } \mathfrak{p}\subseteq\mathfrak{n} \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

is satisfied.

PROOF OF LEMMA (4.1). We define $\operatorname{Rat}_i(-)$ and $Z_i(-)$ as in [5, Chap. 1]. For each i, we define $\psi_i: Z_i(\operatorname{Spec} T)\to Z_i(\operatorname{Spec} T_n)$ by

$$\psi_i([\operatorname{Spec}(T/\mathfrak{p})])=\begin{cases} [\operatorname{Spec}(T_{\mathfrak{n}}/\mathfrak{p}T_{\mathfrak{n}})] & \text{if } \mathfrak{p}\subseteq\mathfrak{n}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.3}$$

Note that, if $\mathfrak{p}$ is a prime ideal of T such that $\dim T/\mathfrak{p}T=i$ and $\mathfrak{p}\subseteq\mathfrak{n}$, then we have $\dim T_{\mathfrak{n}}/\mathfrak{p}T_{\mathfrak{n}}=i$. Hence ψ_i is well-defined and is surjective. Let $\mathfrak{q}$ be a prime ideal in T such that $\dim T/\mathfrak{q}=i+1$. For $d\in Q(T/\mathfrak{q})\setminus\{0\}$, we have

$$\psi_i(\operatorname{div}_{\operatorname{Spec}(T/\mathfrak{q})}(d))=\begin{cases} \operatorname{div}_{\operatorname{Spec}(T_{\mathfrak{n}}/\mathfrak{q}T_{\mathfrak{n}})}(d) & \text{if } \mathfrak{q}\subseteq\mathfrak{n} \\ 0 & \text{otherwise}\,, \end{cases}$$

where $Q(T/\mathfrak{q})$ is the field of fractions of $T/\mathfrak{q}$, and $\operatorname{div}_{\operatorname{Spec}(T/\mathfrak{q})}(d)$ (resp. $\operatorname{div}_{\operatorname{Spec}(T_{\mathfrak{n}}/\mathfrak{q}T_{\mathfrak{n}})}(d)$) is an i-cycle defined by

$$\operatorname{div}_{\operatorname{Spec}(T/\mathfrak{q})}(d)=\sum_{\operatorname{ht}\mathfrak{p}/\mathfrak{q}=1}\operatorname{ord}_{\operatorname{Spec}(T/\mathfrak{p})}(d)\cdot[\operatorname{Spec}(T/\mathfrak{p})]$$

$$\left(\text{resp. } \operatorname{div}_{\operatorname{Spec}(T_{\mathfrak{n}}/\mathfrak{q}T_{\mathfrak{n}})}(d)=\sum_{\operatorname{ht}\mathfrak{p}T_{\mathfrak{n}}/\mathfrak{q}T_{\mathfrak{n}}=1}\operatorname{ord}_{\operatorname{Spec}(T_{\mathfrak{n}}/\mathfrak{p}T_{\mathfrak{n}})}(d)\cdot[\operatorname{Spec}(T_{\mathfrak{n}}/\mathfrak{p}T_{\mathfrak{n}})]\right)$$

(cf. [5, Chap. 1]). Therefore

$$\psi_i(\operatorname{Rat}_i(\operatorname{Spec} T))=\operatorname{Rat}_i(\operatorname{Spec} T_{\mathfrak{n}}) \tag{4.4}$$

is satisfied and we get an induced map $\bar{\psi}_i: A_i(\operatorname{Spec} T)\to A_i(\operatorname{Spec} T_{\mathfrak{n}})$ which makes the following diagram commutative for each i:

$$
\begin{array}{ccccccccc}
0 & \to & \mathrm{Rat}_i(\operatorname{Spec} T) & \to & \mathrm{Z}_i(\operatorname{Spec} T) & \to & \mathrm{A}_i(\operatorname{Spec} T) & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \bar{\psi}_i} & & \\
0 & \to & \mathrm{Rat}_i(\operatorname{Spec} T_{\mathfrak{n}}) & \to & \mathrm{Z}_i(\operatorname{Spec} T_{\mathfrak{n}}) & \to & \mathrm{A}_i(\operatorname{Spec} T_{\mathfrak{n}}) & \to & 0 .
\end{array}
\tag{4.5}
$$

Put $\bar{\psi} = \bigoplus_i \bar{\psi}_i$. By definition (cf. (4.3)), it is easily checked that $\bar{\psi}$ satisfies (4.2).

Next we shall prove that $\bar{\psi}_i$ is an isomorphism for each i. Since ψ_i is surjective, so is $\bar{\psi}_i$. Therefore we have only to show that $\bar{\psi}_i$ is injective for each i. By (4.4), we have only to show the following:

CLAIM (4.6). *In the notation as in Lemma* (4.1), *let* $\mathfrak{p}$ *be a prime ideal with* $\dim T/\mathfrak{p} = i$ *such that* $\mathfrak{p} \not\subseteq \mathfrak{n}$. *Then we have* $[\operatorname{Spec}(T/\mathfrak{p})] = 0$ *in* $\mathrm{A}_i(\operatorname{Spec} T)$.

PROOF OF CLAIM (4.6). Let $\mathfrak{p}$ be a prime ideal with $\dim T/\mathfrak{p} = i$ such that $\mathfrak{p} \not\subseteq \mathfrak{n}$. We denote by $\mathfrak{p}^*$ the ideal generated by all homogeneous elements contained in $\mathfrak{p}$. Then it is easy to see that $\mathfrak{p}^*$ is a homogeneous prime ideal with $\dim T/\mathfrak{p}^* = i+1$. Let $f: \operatorname{Spec}(T/\mathfrak{p}^*) \to \operatorname{Spec} T$ be the closed immersion. Then we get an induced homomorphism $f_*: \mathrm{A}_*(\operatorname{Spec}(T/\mathfrak{p}^*)) \to \mathrm{A}_*(\operatorname{Spec} T)$ such that $f_*([\operatorname{Spec}(T/\mathfrak{p})]) = [\operatorname{Spec}(T/\mathfrak{p})]$ (cf. [5, Chap. 1]). Therefore, replacing T by $T/\mathfrak{p}^*$, we may assume that T is an integral domain and $\mathfrak{p}$ is a prime ideal with $\operatorname{ht} \mathfrak{p} = 1$ such that $\mathfrak{p} \not\subseteq \mathfrak{n}$.

Suppose that $\mathfrak{p}$ is a prime ideal in a Noetherian graded domain T such that $\operatorname{ht} \mathfrak{p} = 1$ and $\mathfrak{p} \not\subseteq \mathfrak{n}$. Then note that $\mathfrak{p}^*$ coincides with (0). Let $\tilde{T}$ be the normalization of T. It is well-known that $\tilde{T}$ has the natural structure of a graded ring, i.e., $\tilde{T}$ is a graded ring $\bigoplus_{i \geq 0} \tilde{T}_i$ such that $\tilde{T}_0$ is a field and $T_i \subseteq \tilde{T}_i$ for each i.

Since $T \subseteq \tilde{T}$ is (module-)finite, the associated morphism $g: \operatorname{Spec} \tilde{T} \to \operatorname{Spec} T$ is proper. Hence we have an induced map $g_*: \mathrm{A}_*(\operatorname{Spec} \tilde{T}) \to \mathrm{A}_*(\operatorname{Spec} T)$ (cf. [5, Chap. 1]).

On the other hand, $\tilde{T} \otimes_T T_{\mathfrak{p}}$ is the normalization of $T_{\mathfrak{p}}$. Put

$$M = \{a \mid a \text{ is a non-zero homogeneous element of } T\} . \tag{4.7}$$

Since $\mathfrak{p}^* = (0)$, we have $\mathfrak{p} \cap M = \varnothing$. Therefore $T_{\mathfrak{p}}$ is a local ring of $T[M^{-1}]$. It is easy to see that $T[M^{-1}]$ is isomorphic to a one-dimensional Laurent polynomial ring over a field. In particular, $T_{\mathfrak{p}}$ is normal. Hence we have $T_{\mathfrak{p}} = \tilde{T} \otimes_T T_{\mathfrak{p}}$. Therefore, there is only one prime ideal $\tilde{\mathfrak{p}}$ (in $\tilde{T}$) lying over $\mathfrak{p}$. Furthermore,

$$T_{\mathfrak{p}} = \tilde{T}_{\tilde{\mathfrak{p}}} \tag{4.8}$$

is satisfied.

By the definition of g_* (cf. [5, Chap. 1]), we obtain

$$g_*([\operatorname{Spec}(\tilde{T}/\tilde{\mathfrak{p}})]) = [Q(\tilde{T}/\tilde{\mathfrak{p}}) : Q(T/\mathfrak{p})] \cdot [\operatorname{Spec}(T/\mathfrak{p})] ,$$

where $Q(-)$ is the field of fractions. By (4.8), $[Q(\tilde{T}/\tilde{\mathfrak{p}}) : Q(T/\mathfrak{p})] = 1$ is satisfied. Hence we have $g_*([\operatorname{Spec}(\tilde{T}/\tilde{\mathfrak{p}})]) = [\operatorname{Spec}(T/\mathfrak{p})]$. Therefore we have only to show $[\operatorname{Spec}(\tilde{T}/\tilde{\mathfrak{p}})] = 0$

in $A_*(\operatorname{Spec} \tilde{T})$. Replacing T by $\tilde{T}$, we may assume that T is a normal domain.

Suppose that T is a normal domain and $\mathfrak{p}$ is a prime ideal in T such that $\operatorname{ht}\mathfrak{p}=1$ and $\mathfrak{p}\not\subseteq\mathfrak{n}$. Put $r=\dim T$. By definition, we may identify $A_{r-1}(\operatorname{Spec} T)$ with the divisor class group $\operatorname{Cl}(T)$. Hence it suffices to prove $\mathfrak{p}\simeq T$ as a T-module.

The rest of this proof is due to Kei-ichi Watanabe. The author thanks him for his important suggestions.

Since $T[M^{-1}]$ is a principal ideal domain (cf. (4.7)), there exists an element a in $\mathfrak{p}$ such that $\mathfrak{p}T[M^{-1}]=aT[M^{-1}]$. Take the minimal primary decomposition of aT:

$$aT=\mathfrak{p}\cap\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_r\,. \tag{4.9}$$

If $aT=\mathfrak{p}$, then there is nothing to prove. Suppose $r>0$. $\mathfrak{q}_i$ is a homogeneous primary ideal for each i, since $\mathfrak{p}T[M^{-1}]=aT[M^{-1}]$. Put $\mathfrak{p}_i=\sqrt{\mathfrak{q}_i}$. Since T is normal, there is a positive integer n_i such that $\mathfrak{q}_i$ coincides with the n_i-th symbolic power $\mathfrak{p}_i^{(n_i)}$ of $\mathfrak{p}_i$. Let b be a homogeneous element such that

$$b\in(\mathfrak{p}_1^{(n_1)}\cap\cdots\cap\mathfrak{p}_r^{(n_r)})\setminus(\mathfrak{p}_1^{(n_1+1)}\cup\cdots\cup\mathfrak{p}_r^{(n_r+1)})\,.$$

Take the minimal primary decomposition of bT;

$$bT=\mathfrak{p}_1^{(n_1)}\cap\cdots\cap\mathfrak{p}_r^{(n_r)}\cap\mathfrak{q}_{r+1}\cap\cdots\cap\mathfrak{q}_l\,. \tag{4.10}$$

Then, by (4.9) and (4.10), we have

$$\operatorname{cl}(\mathfrak{p})=-\operatorname{cl}(\mathfrak{q}_1)-\cdots-\operatorname{cl}(\mathfrak{q}_r)=\operatorname{cl}(\mathfrak{q}_{r+1})+\cdots+\operatorname{cl}(\mathfrak{q}_l)\,,$$

where, for a divisorial ideal J, $\operatorname{cl}(J)$ stands for the isomorphism class (in $\operatorname{Cl}(T)$) which J belongs to. Put $I=\mathfrak{q}_{r+1}\cap\cdots\cap\mathfrak{q}_l$. Then I is a homogeneous ideal, since $b, \mathfrak{q}_{r+1},\dots,\mathfrak{q}_l$ are homogeneous. By the construction of I, we have $\operatorname{cl}(\mathfrak{p})=\operatorname{cl}(I)$. Therefore $\mathfrak{p}$ is isomorphic to I as a T-module. Then $\mathfrak{p}T_\mathfrak{n}$ coincides with $T_\mathfrak{n}$ because $\mathfrak{p}\not\subseteq\mathfrak{n}$. Therefore we have $IT_\mathfrak{n}\simeq\mathfrak{p}T_\mathfrak{n}=T_\mathfrak{n}$. Hence we get

$$\dim_{T/\mathfrak{n}}(I/\mathfrak{n}I)=\dim_{T_\mathfrak{n}/\mathfrak{n}T_\mathfrak{n}}(IT_\mathfrak{n}/\mathfrak{n}IT_\mathfrak{n})=\dim_{T_\mathfrak{n}/\mathfrak{n}T_\mathfrak{n}}(T_\mathfrak{n}/\mathfrak{n}T_\mathfrak{n})=1\,. \tag{4.11}$$

Since I is a homogeneous ideal, the equality (4.11) implies that I is a principal ideal. Therefore $\mathfrak{p}$ is isomorphic to T. q.e.d.

Now we start to prove Theorem (1.3). We define k, R, $\mathfrak{m}$, X as in the beginning of this paper.

Proof of Theorem (1.3). Put $R=k[x_0,\dots,x_n]/I$, where $k[x_0,\dots,x_n]$ is the graded polynomial ring with $\deg(x_i)=1$ for $i=0,\dots,n$ and I is a homogeneous ideal. Consider the morphisms

$$X=\operatorname{Proj}(R)\hookrightarrow\operatorname{Proj}(k[x_0,\dots,x_n])=\boldsymbol{P}^n=V_+(x_{n+1})\hookrightarrow\boldsymbol{P}^{n+1}=\operatorname{Proj}(k[x_0,\dots,x_{n+1}])\,.$$

Here $V_+(M)$ stands for the closed subscheme of a projective space defined by a set of homogeneous polynomials M and we denote by $D_+(M)$ the complement of $V_+(M)$.

We put $t=(0,\dots,0,1)\in \boldsymbol{P}^{n+1}$ and $\tilde{X}=V_+(Ik[x_0,\dots,x_{n+1}])\hookrightarrow \boldsymbol{P}^{n+1}$. Since $X=\tilde{X}\cap V_+(x_{n+1})\subseteq \boldsymbol{P}^{n+1}$, we have

$$\tilde{X}\setminus X=\tilde{X}\cap D_+(x_{n+1})\hookrightarrow D_+(x_{n+1})=\operatorname{Spec}\left(k\left[\frac{x_0}{x_{n+1}},\dots,\frac{x_n}{x_{n+1}}\right]\right).$$

Then we have

$$\tilde{X}\setminus X=\operatorname{Spec}\left(k\left[\frac{x_0}{x_{n+1}},\dots,\frac{x_n}{x_{n+1}}\right]\Big/J\right),$$

where $J=(f(x_0/x_{n+1},\dots,x_n/x_{n+1})\mid f\in I)$. Therefore we have an isomorphism $\tilde{X}\setminus X\simeq \operatorname{Spac} R$ of k-schemes. Let $l\colon X\hookrightarrow \tilde{X}\setminus\{t\}$ be the inclusion and $\eta\colon \tilde{X}\setminus\{t\}\to X$ the projection, i.e., $\eta((s_0,\dots,s_{n+1}))=(s_0,\dots,s_n)$:

$$X\xrightarrow{\ l\ }\tilde{X}\setminus\{t\}\xrightarrow{\ \eta\ }X.$$

Then it is easy to see that η coincides with the vector bundle $\operatorname{Spec}_X(\operatorname{Sym}\mathcal{O}_X(-1))\to X$ and l is its zero section. (Here $\operatorname{Spec}_X(\operatorname{Sym}\mathcal{O}_X(-1))$ is the vector bundle on X whose sheaf of sections coincides with $\mathcal{O}_X(1)$.) In particular, η is a smooth morphism of relative dimension 1 whose relative tangent bundle is isomorphic to $\eta^*\mathcal{O}_X(1)$. Let $i\colon X\to\tilde{X}$, $i_t\colon \{t\}\to\tilde{X}$, $j\colon \tilde{X}\setminus\{t\}\to\tilde{X}$, $k\colon \tilde{X}\setminus X\to\tilde{X}$ be immersions. Note that η, j and k (resp. i and i_t) are flat (resp. proper) morphisms. Then we have a diagram

(4.12)
$$\begin{tikzcd}
 & 0 & & \\
A_*(X) \arrow[r, "\eta^*"] & A_*(\tilde{X}\setminus\{t\}) \arrow[u] & & \\
A_*(X) \arrow[r, "i_*"] & A_*(\tilde{X}) \arrow[u, "j^*"] \arrow[r, "k^*"] & A_*(\tilde{X}\setminus X) \arrow[r] \arrow[d, equal] & 0 \\
\boldsymbol{Z}[t] \arrow[r, phantom, "="] & A_*(t) \arrow[u, "(i_t)_*"] & A_*(\operatorname{Spec} R) &
\end{tikzcd}$$

where the vertical and horizontal sequences are exact. Since η is a vector bundle, η^* is the direct sum of isomorphisms $A_i(X)\to A_{i+1}(\tilde{X}\setminus\{t\})$ $(i=0,\dots,d)$ and $A_0(\tilde{X}\setminus\{t\})=(0)$ is satisfied. Taking the homogeneous component of degree 0 of (4.12), we obtain the following diagram:

$$
(4.13)\qquad
\begin{tikzcd}
 & 0 & & \\
A_0(X) \arrow[r, "i_*"] & A_0(\tilde{X}) \arrow[u] \arrow[r, "k^*"] & A_0(\tilde{X} \setminus X) \arrow[r] \arrow[d, equal] & 0\,. \\
\mathbf{Z}[t] \arrow[r, phantom, "="] & A_0(t) \arrow[u, "(i_t)_*"] & A_0(\operatorname{Spec} R) &
\end{tikzcd}
$$

Since the sequences in the diagram (4.13) are exact, $A_0(\operatorname{Spec} R)$ is generated by $k^*(i_t)_*([t])=[\operatorname{Spec}(R/\mathfrak{m})]$. Since $[\operatorname{Spec}(R/\mathfrak{m})]$ is a torsion element of $A_0(\operatorname{Spec} R)$, we have

$$
(4.14)\qquad A_0(\operatorname{Spec} R)_{\mathbf{Q}}=A_0(\operatorname{Spec} R_{\mathfrak{m}})_{\mathbf{Q}}=(0)
$$

by Lemma (4.1).

Let v be a positive integer and take the homogeneous component in degree v of (4.12).

$$
(4.15)\qquad
\begin{tikzcd}
 & 0 & & \\
A_{v-1}(X) \arrow[r, "\eta^*"] & A_v(\tilde{X} \setminus \{t\}) \arrow[u] & & \\
A_v(X) \arrow[r, "i_*"] & A_v(\tilde{X}) \arrow[u, "j^*"] \arrow[r, "k^*"] & A_v(\tilde{X} \setminus X) \arrow[r] \arrow[d, equal] & 0\,. \\
 & 0 \arrow[u] & A_v(\operatorname{Spec} R) &
\end{tikzcd}
$$

Here note that η^* is an isomorphism. By the diagram (4.15), we have an exact sequence

$$
A_v(X) \xrightarrow{(\eta^*)^{-1}j^*i_*} A_{v-1}(X) \xrightarrow{k^*(j^*)^{-1}\eta^*} A_v(\tilde{X} \setminus X) \longrightarrow 0\,.
$$

Since $l_*=j^*i_*$ (cf. [5, Proposition 1.7]), we obtain $(\eta^*)^{-1}j^*i_*=(\eta^*)^{-1}l_*$. Furthermore, by [5, Example 3.3.2], $(\eta^*)^{-1}l_*(\alpha)=c_1(\mathcal{O}_X(1))\cap\alpha$ is satisfied for any $\alpha\in A_v(X)$ since $\eta\colon \tilde{X}\setminus\{t\}\to X$ is the vector bundle whose sheaf of sections coincides with $\mathcal{O}_X(1)$. Therefore, for each $v>0$, we have the following exact sequence:

$$
(4.16)\qquad
\begin{array}{ccccc}
A_v(X) & \longrightarrow & A_{v-1}(X) & \xrightarrow{k^*(j^*)^{-1}\eta^*} & A_v(\tilde{X} \setminus X) \longrightarrow 0\,. \\
\alpha & \longmapsto & c_1(\mathcal{O}_X(1))\cap\alpha & &
\end{array}
$$

Put $c=c_1(\mathcal{O}_X(1))\in A_{d-1}(X)=\mathrm{CH}^1(X)$. Let π be the map (1.1). By the exact sequence (4.16), we have an isomorphism $\xi'_{d-v+1}\colon \mathrm{CH}^{d-v+1}(X)_{\mathbf{Q}}/c\cdot\mathrm{CH}^{d-v}(X)_{\mathbf{Q}}\to A_v(\tilde{X}\setminus X)_{\mathbf{Q}}$ such that

$$k^*(j^*)^{-1}\eta^* = \xi'_{d-v+1}\pi_{d-v+1} \tag{4.17}$$

holds, where π_{d-v+1} is the homogeneous component of π of degree $d-v+1$. On the other hand, by Lemma (4.1), we have an isomorphism

$$\bar{\psi}_i \colon \mathrm{A}_i(\operatorname{Spec} R)_{\boldsymbol{Q}} \to \mathrm{A}_i(\operatorname{Spec} R_{\mathfrak{m}})_{\boldsymbol{Q}}$$

for $i=0, \ldots, r$. Put

$$\xi_i = \bar{\psi}_i \xi'_i \tag{4.18}$$

for $i=0, \ldots, r$. Then ξ_i satisfies (1.5) in Theorem (1.3) (cf. (4.14)).

Put $\xi = \bigoplus_i \xi_i$, $\xi' = \bigoplus_i \xi'_i$ and $\bar{\psi} = \bigoplus_i \bar{\psi}_i$.

Next we shall prove $\xi\pi(\mathrm{td}(\Omega_X^\vee)) = \tau([R_{\mathfrak{m}}])$, where τ is the Riemann-Roch map (cf. (1.2)). By the Riemann-Roch theorem ([5, Chap. 18]), we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{K}_0(X)_{\boldsymbol{Q}} & \xrightarrow{\tau_X} & \mathrm{A}_*(X)_{\boldsymbol{Q}} \\
\downarrow \eta^* & & \downarrow \mathrm{td}(\mathrm{T}_\eta)\eta^* \\
\mathrm{K}_0(\tilde{X}\setminus\{t\})_{\boldsymbol{Q}} & \xrightarrow{\tau_{\tilde{X}\setminus\{t\}}} & \mathrm{A}_*(\tilde{X}\setminus\{t\})_{\boldsymbol{Q}} \\
\uparrow j^* & & \uparrow j^* \\
\mathrm{K}_0(\tilde{X})_{\boldsymbol{Q}} & \xrightarrow{\tau_{\tilde{X}}} & \mathrm{A}_*(\tilde{X})_{\boldsymbol{Q}} \\
\downarrow k^* & & \downarrow k^* \\
K_0(\tilde{X}\setminus X)_{\boldsymbol{Q}} & \xrightarrow{\tau_{\tilde{X}\setminus X}} & \mathrm{A}_*(\tilde{X}\setminus X)_{\boldsymbol{Q}} \\
\| & & \| \\
\mathrm{K}_0(R)_{\boldsymbol{Q}} & \xrightarrow{\tau} & \mathrm{A}_*(\operatorname{Spec} R)_{\boldsymbol{Q}}
\end{array} \tag{4.19}$$

where τ_X, $\tau_{\tilde{X}\setminus\{t\}}$, $\tau_{\tilde{X}}$ and $\tau_{\tilde{X}\setminus X}$ are the Riemann-Roch maps ([5, Chap. 18]). Here note that

$$\begin{aligned}
&[\mathcal{O}_{\tilde{X}\setminus\{t\}}] = \eta^*([\mathcal{O}_X]) = j^*([\mathcal{O}_{\tilde{X}}])\,, \\
&[R] = [\mathcal{O}_{\tilde{X}\setminus X}] = k^*([\mathcal{O}_{\tilde{X}}])\,.
\end{aligned} \tag{4.20}$$

Since $\eta \colon \tilde{X}\setminus\{t\} \to X$ is the vector bundle whose sheaf of sections is $\mathcal{O}_X(1)$, the relative tangent bundle T_η coincides with $\eta^*\mathcal{O}_X(1)$. Then, by (4.19) and (4.20), we have

$$\begin{aligned}
\tau_{\tilde{X}\setminus\{t\}}([\mathcal{O}_{\tilde{X}\setminus\{t\}}]) &= \tau_{\tilde{X}\setminus\{t\}}(\eta^*([\mathcal{O}_X])) = \mathrm{td}(\mathrm{T}_\eta) \cap \eta^*\tau_X([\mathcal{O}_X]) \\
&= \mathrm{td}(\eta^*\mathcal{O}_X(1)) \cap \eta^*\tau_X([\mathcal{O}_X]) = \eta^*(\mathrm{td}(\mathcal{O}_X(1)) \cap \tau_X([\mathcal{O}_X]))
\end{aligned} \tag{4.21}$$

$$\tau_{\tilde{X}\setminus\{t\}}([\mathcal{O}_{\tilde{X}\setminus\{t\}}]) = \tau_{\tilde{X}\setminus\{t\}}(j^*([\mathcal{O}_{\tilde{X}}])) = j^*\tau_{\tilde{X}}([\mathcal{O}_{\tilde{X}}]) \tag{4.22}$$

$$k^*(\tau_{\tilde{X}}([\mathcal{O}_{\tilde{X}}])) = \tau_{\tilde{X}\setminus X}([\mathcal{O}_{\tilde{X}\setminus X}]) = \tau([R])\,. \tag{4.23}$$

By (4.21), (4.22) and (4.23), we get

$$\eta^*([\mathrm{td}(\mathcal{O}_X(1))\cap\tau_X([\mathcal{O}_X])]_{v-1})=[\tau_{\tilde{X}\setminus\{t\}}([\mathcal{O}_{\tilde{X}\setminus\{t\}}])]_v \tag{4.24}$$

$$j^*([\tau_{\tilde{X}}([\mathcal{O}_{\tilde{X}}])]_v)=[\tau_{\tilde{X}\setminus\{t\}}([\mathcal{O}_{\tilde{X}\setminus\{t\}}])]_v \tag{4.25}$$

$$k^*([\tau_{\tilde{X}}([\mathcal{O}_{\tilde{X}}])]_v)=[\tau([R])]_v \tag{4.26}$$

where $[-]_v$ denotes the homogeneous component of degree v. By the equations (4.24), (4.25) and (4.26), we obtain

$$k^*(j^*)^{-1}\eta^*([\mathrm{td}(\mathcal{O}_X(1))\cap\tau_X([\mathcal{O}_X])]_{v-1})=[\tau([R])]_v\,. \tag{4.27}$$

We define rational numbers $d_1, d_2, \ldots$ by

$$\mathrm{td}(\mathcal{O}_X(1))=\frac{c}{1-e^{-c}}=1+d_1c+d_2c^2+d_3c^3+\cdots,$$

where $c=c_1(\mathcal{O}_X(1))$. It is known that $d_1=1/2$, $d_2=1/12$, $d_3=0$, $d_4=-1/720, \ldots$ ([5, Example 3.2.4]). Then we have

$$\begin{aligned}&[\mathrm{td}(\mathcal{O}_X(1))\cap\tau_X([\mathcal{O}_X])]_{v-1}\\ &\quad=[\tau_X([\mathcal{O}_X])]_{v-1}+d_1c\cap[\tau_X([\mathcal{O}_X])]_v+\cdots+d_ic^i\cap[\tau_X([\mathcal{O}_X])]_{v-1+i}+\cdots.\end{aligned} \tag{4.28}$$

By the exactness of (4.16), we get

$$\mathrm{Ker}\left(\mathrm{A}_{v-1}(X)_{\mathbf{Q}}\xrightarrow{k^*(j^*)^{-1}\eta^*}\mathrm{A}_v(\tilde{X}\setminus X)_{\mathbf{Q}}\right)=\{c\cap\alpha\mid\alpha\in\mathrm{A}_v(X)_{\mathbf{Q}}\}\,. \tag{4.29}$$

Then, by (4.27), (4.28) and (4.29),

$$[\tau([R])]_v=k^*(j^*)^{-1}\eta^*([\tau_X([\mathcal{O}_X])]_{v-1}) \tag{4.30}$$

is satisfied. By (4.14), (4.17) and (4.30), we get $\tau([R])=\xi'\pi(\tau_X([\mathcal{O}_X]))$. Since X is smooth over k, $\tau_X([\mathcal{O}_X])=\mathrm{td}(\Omega_X^\vee)$ holds by the definition of the Riemann-Roch map τ_X (cf. [5, Chap. 18]). Therefore we obtain

$$\tau([R])=\xi'\pi(\mathrm{td}(\Omega_X^\vee))\,. \tag{4.31}$$

Let $S=k[x_0, \ldots, x_n]$ be a polynomial ring. Recall that R is a homomorphic image of S, i.e., $R=S/I$ (I is a homogeneous ideal of S). Let

$$\boldsymbol{F}.: 0\longrightarrow F_l\xrightarrow{d_l}F_{l-1}\xrightarrow{d_{l-1}}\cdots\xrightarrow{d_1}F_0\longrightarrow 0$$

be a minimal free S-free resolution of R. Then, by definition,

$$\tau([R])=\mathrm{ch}^{\mathrm{Spec}\,S}_{\mathrm{Spec}\,R}(\boldsymbol{F}.)\cap\mathrm{td}(\Omega_{S/k}^\vee)\cap[\mathrm{Spec}\,S] \tag{4.32}$$

where $\mathrm{ch}^{\mathrm{Spec}\,S}_{\mathrm{Spec}\,R}(\boldsymbol{F}.)$ is the localized Chern character ([5, Chap. 18]). Since the differential module $\Omega_{S/k}$ is a free S-module, we obtain $\mathrm{td}(\Omega_{S/k}^\vee)=1$. On the other hand, by definition, we have

$$\tau([R_{\mathfrak{m}}]) = \mathrm{ch}^{\mathrm{Spec}\,S_{\mathfrak{m}}}_{\mathrm{Spec}\,R_{\mathfrak{m}}}(F. \otimes_S S_{\mathfrak{m}}) \cap [\mathrm{Spec}\,S_{\mathfrak{m}}] \tag{4.33}$$

where $\mathfrak{m} = (x_0, \ldots, x_n)$. By (4.32), (4.33) and the commutativity of the diagram

$$\begin{array}{ccc} A_*(\mathrm{Spec}\,S)_{\boldsymbol{Q}} & \xrightarrow{\mathrm{ch}^{\mathrm{Spec}\,S}_{\mathrm{Spec}\,R}(F.)} & A_*(\mathrm{Spec}\,R)_{\boldsymbol{Q}} \\ \downarrow{\scriptstyle \bar{\psi}} & & \downarrow{\scriptstyle \bar{\psi}} \\ A_*(\mathrm{Spec}\,S_{\mathfrak{m}})_{\boldsymbol{Q}} & \xrightarrow{\mathrm{ch}^{\mathrm{Spec}\,S_{\mathfrak{m}}}_{\mathrm{Spec}\,R_{\mathfrak{m}}}(F. \otimes_S S_{\mathfrak{m}})} & A_*(\mathrm{Spec}\,R_{\mathfrak{m}})_{\boldsymbol{Q}}\,, \end{array}$$

we get $\bar{\psi}\tau([R]) = \tau([R_{\mathfrak{m}}])$, where $\bar{\psi}$'s are maps defined in Lemma (4.1). Then, by (4.18) and (4.31), we obtain

$$\tau([R_{\mathfrak{m}}]) = \bar{\psi}\tau([R]) = \bar{\psi}\xi'\pi(\mathrm{td}(\Omega_X^{\vee})) = \xi\pi(\mathrm{td}(\Omega_X^{\vee}))\,.$$

q.e.d.

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