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A CHARACTERIZATION OF HYPERBOLIC CYLINDERS IN THE DE SITTER SPACE

SEBASTIÁN MONTIEL

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Abstract. We characterize a class of hyperbolic cylinders of the de Sitter spacetime as the only complete non-compact spacelike hypersurfaces with constant lowest mean curvature and having more than one topological end.

1. Introduction. Constant non-zero mean curvature and maximal spacelike hypersurfaces in Lorentzian spaces are objects of a great amount of interest from both physical and mathematical points of view since some works, in the last fifteen years, have revealed that they are convenient initial data for the Cauchy problem of the Einstein equation in general relativity. Also, hypersurfaces with non-zero constant mean curvature become asymptotically null as they approach infinity and because of this they are particularly suitable for studying propagation of gravitational waves (cf. [B], [CB], [CFM], [G], [S]). The ambient spaces mostly considered have been Lorentzian space forms such as Minkowski and de Sitter spaces or some spacetimes which are close, in a geometrical sense, to them.

From a mathematical point of view, the attention was focused upon these hypersurfaces because they exhibit Bernstein type properties. Firstly, Calabi [C] proved that a maximal spacelike entire graph in the Minkowski space $\boldsymbol{R}_1^{n+1}$ with $n \leq 4$ is a linear hyperplane. Later, Cheng and Yau [CY] showed that the same holds for arbitrary n. The case of entire graphs with non-zero constant mean curvature in this same ambient spacetime has a completely different flavour, as was pointed out by Treibergs [T].

Another physically relevant spacetime is the de Sitter space $\boldsymbol{S}_1^{n+1}$ where the role of the linear hyperplanes of the Minkowski space is played by the umbilical hypersurfaces, obtained by intersecting $\boldsymbol{S}_1^{n+1}$ with linear hyperplanes through the origin of the Minkowski space $\boldsymbol{R}_1^{n+2}$ where the de Sitter space can be viewed as a hypersphere. Goddard [G] conjectured that complete space-like hypersurfaces in the de Sitter space with constant mean curvature must be umbilical. In [Mo], the author solved this conjecture in the affirmative provided that the hypersurface is compact by using an easy integral formula (the case $n=2$ had been settled earlier in [Ak] by Akutagawa). This result had been obtained in the maximal case in [CFM]. Also, in this way, an earlier theorem due to Akutagawa [Ak] and also in part to Ramanathan [R] was generalized. This result asserted that, if a complete spacelike hypersurface has mean curvature H

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satisfying $H^2<4(n-1)/n^2$ if $n>2$, or $H^2\leq 1$ if $n=2$, then it is umbilical. The relation between the afore-mentioned theorems in [Mo] and [Ak] can be established as follows: if the mean curvature H of the hypersurface satisfies that inequality, it is immediate, from the Gauss equation, that its Ricci tensor is bounded below by a positive number. So, it is compact from the Bonnet-Myers theorem. Later, Oliker [Ol] has shown that the Bernstein type property is stable relative to perturbation of the data.

This theorem by Akutagawa and the author is the best possible for complete spacelike hypersurfaces in S_1^{n+1} having constant mean curvature H. Indeed, in [Ak], non-umbilical examples with $n=2$ and $H^2>1$ were constructed and in [Mo] we remarked that for $n>2$ and $H^2\geq 4(n-1)/n^2$ there are, besides the corresponding umbilical examples (isometric to spheres if $H^2<1$, to Euclidean spaces if $H^2=1$ and to hyperbolic spaces if $H^2>1$), also non-umbilical hypersurfaces. The example invoked there was nothing but the so-called *hyperbolic cylinder* (cf. [A], [KKN]) which is isometric to the Riemannian product $H^1\times S^{n-1}$ of a hyperbolic line and an n-dimensional sphere of radii $\sinh r$ and $\cosh r$, respectively. The corresponding mean curvature H satisfies $H^2=(\coth r+(n-1)\tanh r)^2/n^2$ which is always greater than or equal to $4(n-1)/n^2$ and this precise value is attained by the hyperbolic cylinder with $\coth^2 r=n-1$.

So, it seems natural to look for complete spacelike hypersurfaces in the de Sitter space S_1^{n+1}, $n>2$, with constant mean curvature exactly equal to this boundary value $2\sqrt{n-1}/n$. We conjectured that they are only the corresponding umbilical spheres and the above hyperbolic cylinder. In this work we will solve affirmatively this problem with a topological restriction on the hypersurface. Concretely, we will prove the following theorem:

A complete spacelike hypersurface in the de Sitter spacetime S_1^{n+1} with constant mean curvature H satisfying $H^2=4(n-1)/n^2$ which is not connected at infinity must be, up to rigid motion, a certain hyperbolic cyclinder.

Here, *connected at infinity* is used in the sense of Freudenthal, that is, the hypersurface is not compact and has exactly one topological end. An end is a connected component in the complement of balls with radii increasing to infinity.

2. Preliminaries. Consider Lorentz-Minkowski space R_1^{n+2} as the real vector space R^{n+2} endowed with the Lorentzian metric $\langle\ ,\ \rangle$ given by

$$\langle u, v\rangle=-u_0v_0+u_1v_1+\cdots+u_{n+1}v_{n+1}$$

for $u, v\in R^{n+2}$. Then, de Sitter space of dimension $n+1$ can be defined as the following hyperquadric of R_1^{n+2}

$$S_1^{n+1}=\{u\in R_1^{n+2}\,|\,|u|^2=1\}\ .$$

In this way, S_1^{n+1} inherits from $\langle\ ,\ \rangle$ a metric which makes it a Lorentzian manifold with constant sectional curvature one.

The standard examples of spacelike hypersurfaces, umbilical ones and hyperbolic cylinders, with constant mean curvature in the de Sitter spacetime can be found in almost any of the references in the bibliography. The umbilical ones are given by

$$M=\{p\in \boldsymbol{S}_1^{n+1} \mid \langle p, a\rangle=\tau\}$$

where $a\in \boldsymbol{R}_1^{n+2}$, $|a|^2=\rho=1, 0, -1$ and $\tau^2>\rho$. The corresponding mean curvature H satisfies $H^2=\tau^2/(\tau^2-\rho)$. As one can see in [Mo], for instance, M is isometric to a hyperbolic space, a Euclidean space or a sphere according as $\rho=1$, $\rho=0$ or $\rho=-1$, respectively. On the other hand, hyperbolic cylinders are the hypersurfaces given by

$$M=\{p\in \boldsymbol{S}_1^{n+1} \mid p_2^2+\cdots+p_{n+1}^2=\cosh^2 r\},$$

where r is a real positive number and $n>2$. One can easily show, or see [Mo] for example, that the mean curvature is

$$H=\frac{1}{n}[\coth r+(n-1)\tanh r].$$

From this we get $H^2\geq 4(n-1)/n^2$ and the equality is attained only if $\coth^2 r=n-1$. These hypersurfaces are of course not umbilical. They have two different constant principal curvatures at each point, one with multiplicity one, and have parallel second fundamental form. This property characterizes them among complete spacelike hypersurfaces in the de Sitter space. Another characterization of these hyperbolic cylinders, in the two-dimensional case, is as the only surfaces with constant mean curvature in $\boldsymbol{S}_1^3$ which are uniformly non-umbilical (see [A] and [M]). Moreover, they are isometric to the Riemannian product $\boldsymbol{H}^1(\sinh r)\times \boldsymbol{S}^{n-1}(\cosh r)$ of a hyperbolic line and an $(n-1)$-dimensional sphere.

Let M be an n-dimensional manifold immersed into $\boldsymbol{S}_1^{n+1}$ as a spacelike hypersurface and represent by σ its second fundamental form

$$\sigma(X, Y)=\bar{\nabla}_X Y-\nabla_X Y,$$

where X, Y are vector fields on M, and $\bar{\nabla}$ and ∇ are the metric connections of $\boldsymbol{S}_1^{n+1}$ and M, respectively. If N is a local unit normal field for our immersion, we have

$$\sigma(u, v)=-\langle Au, v\rangle N$$

where $u, v\in TM$ and A is the Weingarten endomorphism corresponding to N. So, the mean curvature H of the immersion, corresponding to the choice of N, is $H=(1/n)\mathrm{tr}\, A$. If R and S denote the curvature and Ricci tensors of ∇, we have the classical Gauss and Codazzi equations

$$\begin{aligned} &R(u, v)w=\langle v, w\rangle u-\langle u, w\rangle v-\langle Av, w\rangle Au+\langle Au, w\rangle Av\\ (1)\qquad &S(u, v)=(n-1)\langle u, v\rangle-nH\langle Au, v\rangle+\langle A^2u, v\rangle\\ &(\nabla A)(u, v)=(\nabla A)(v, u)\end{aligned}$$

with u, v, w tangent to M. Hence, using Codazzi's equation and Ricci's identities, the rough Laplacian of the endomorphism field A is given by

$$(\Delta A)u=\sum_{i=1}^{n}(\nabla^2 A)(e_i, e_i, u)=\sum_{i=1}^{n}(\nabla^2 A)(e_i, u, e_i)$$
$$=\sum_{i=1}^{n}\{(\nabla^2 A)(u, e_i, e_i)+R(e_i, u)Ae_i-AR(e_i, u)e_i\}$$

for $u\in TM$. *If the mean curvature H of the immersion is supposed to be constant*, then

$$\sum_{i=1}^{n}(\nabla^2 A)(u, e_i, e_i)=n\nabla_u H=0\,.$$

So, using the Gauss equation (1), we obtain

$$(\Delta A)u=nAu-nHu+(\operatorname{tr}A^2)Au-nHA^2u$$

for each u tangent to M. Observing now that

$$\frac{1}{2}\Delta\operatorname{tr}A^2=|\nabla A|^2+\langle A, \Delta A\rangle\,,$$

we arrive at the following Simons type formula which, in several forms, has already been used in [A], [Ak], [CY], [KKN], [N], [R], [T], and whose derivation we have included for completeness,

$$\frac{1}{2}\Delta\operatorname{tr}A^2=|\nabla A|^2+n\operatorname{tr}A^2-n^2H^2-nH\operatorname{tr}A^3+(\operatorname{tr}A^2)^2\,.$$

Instead of the Weingarten endomorphism A, we will use the traceless symmetric tensor $T=A-HI$ which vanishes identically if and only if the immersion is umbilical. Substituting it in the above Simons formula, we have

$$\frac{1}{2}\Delta\operatorname{tr}T^2=|\nabla T|^2+(\operatorname{tr}T^2)^2-nH\operatorname{tr}T^3+n(1-H^2)\operatorname{tr}T^2\,. \tag{2}$$

3. Results and proofs. We will obtain the announced characterization of hyperbolic cylinders in the de Sitter space among complete spacelike hypersurfaces whose mean curvature is constantly the boundary value $H=2\sqrt{n-1}/n$. The tools which we will use are suitable manipulations of the Simons formula (2), the celebrated splitting theorem of Cheeger and Gromoll and the following maximum principle at infinity for complete manifolds due to Omori and Yau:

THEOREM 1. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below and $f\colon M\to\boldsymbol{R}$ a smooth function bounded from below. Then, for each $\varepsilon>0$ there exists a point $p_\varepsilon\in M$ such that*

(i) $|\nabla f|(p_\varepsilon)<\varepsilon$,
(ii) $\Delta f(p_\varepsilon)>-\varepsilon$,
(iii) $\inf f\leq f(p_\varepsilon)<\inf f+\varepsilon$.

In fact, we start by taking into account the fact that the tensor T is diagonalizable and using Lagrange multipliers in order to find extrema of $\operatorname{tr} T^3$, considered as a function of its eigenvalues, subjected to the constraints $\operatorname{tr} T=0$ and $\operatorname{tr} T^2$ fixed. So, we obtain the following inequality

$$|\operatorname{tr} T^3|\leq\frac{n-2}{\sqrt{n(n-1)}}(\operatorname{tr} T^2)^{3/2} \tag{3}$$

which already appeared in [O], for example. The equality holds here only when T has two different eigenvalues, one of them with multiplicity one. We want to use this inequality in the formula (2). For this purpose, suppose that we have chosen the unit normal field N in such a way that the constant mean curvature H is non-negative. Also, we define h to be the non-negative smooth function $\operatorname{tr} T^2$ defined on M. From (2), we get

$$\begin{aligned}\frac{1}{2}\Delta h&\geq|\nabla T|^2+h^2+n(1-H^2)h-\frac{n(n-2)H}{\sqrt{n(n-1)}}h^{3/2}\\&=|\nabla T|^2+h\left(h-\frac{n(n-2)H}{\sqrt{n(n-1)}}h^{1/2}+n(1-H^2)\right).\end{aligned}$$

As a conclusion, the function h satisfies the following second order inequality

$$\Delta h\geq 2|\nabla T|^2+2hP_H(h^{1/2})\geq 2hP_H(h^{1/2}), \tag{4}$$

where P_H is the real polynomial given by

$$P_H(x)=x^2-\frac{n(n-2)H}{\sqrt{n(n-1)}}x+n(1-H^2). \tag{5}$$

Now, we can state our first result, which, in a weaker verison, was also obtained in [KKN]:

PROPOSITION 2. *Let M be a complete spacelike hypersurface immersed into de Sitter space S_1^{n+1} with constant mean curvature $H=2\sqrt{n-1}/n$. Then, either M is umbilical (and so compact) or $n>2$ and the scalar curvature r of M satisfies*

$$\sup_{p\in M} r(p)=(n-2)^2.$$

This supremum is attained if and only if M is the corresponding hyperbolic cylinder, up to rigid motion of S_1^{n+1}.

PROOF. Consider the positive smooth function f on M defined by

$$f=\frac{1}{\sqrt{1+h}},$$

where $h=\operatorname{tr} T^2$, as before. It is immediate to check that

$$|\nabla f|^2=\frac{1}{4}\frac{|\nabla h|^2}{(1+h)^3}$$

and that

$$\Delta f=-\frac{1}{2}\frac{\Delta h}{(1+h)^{3/2}}+\frac{3}{4}\frac{|\nabla h|^2}{(1+h)^{5/2}}.$$

On the other hand, using the expression in (1) concerning the Ricci tensor of M and that $nH=2\sqrt{n-1}$, we have

$$S(v,v)=\langle(A-\sqrt{n-1}I)^2v,v\rangle=|(A-\sqrt{n-1}I)v|^2 \tag{6}$$

for each v tangent to M. Hence, the Ricci curvature of M is non-negative and, in particular, bounded from below. Consequently, we may apply Theorem 1 to the function f. So, it is possible to find in M a sequence of points p_k, $k\in N$, such that

$$\lim_{k\to\infty} f(p_k)=\inf f,\quad \Delta f(p_k)>-\frac{1}{k},\quad |\nabla f|^2(p_k)<\frac{1}{k^2}.$$

Using the formulae for the gradient and the Laplacian of f in terms of the function h which we have found above, we get $\lim_{k\to\infty} h(p_k)=\sup_{p\in M} h(p)$ and

$$-\frac{1}{k}<-\frac{1}{2}\frac{\Delta h}{(1+h)^{3/2}}(p_k)+\frac{3}{k^2}\left(1+h(p_k)\right)^{1/2}.$$

Hence we get

$$\frac{\Delta h}{(1+h)^2}(p_k)<\frac{2}{k}\left(\frac{1}{\sqrt{1+h(p_k)}}+\frac{3}{k}\right).$$

Now, taking into account the inequality (4) which h satisfies since the mean curvature H of M is constant, we obtain

$$\frac{hP_H(h^{1/2})}{(1+h)^2}(p_k)<\frac{1}{k}\left(\frac{1}{\sqrt{1+h(p_k)}}+\frac{3}{k}\right)$$

where, in this case, by putting the concrete value of H in (5), the polynomial P_H is exactly

$$P_H(x)=x^2-\frac{2(n-2)}{\sqrt{n}}x+\frac{(n-2)^2}{n}=\left(x-\frac{n-2}{\sqrt{n}}\right)^2. \tag{7}$$

From this inequality, we firstly have $\sup_{p\in M} h(p)<\infty$ and secondly

$$\text{either}\quad P_H(\sqrt{\sup h})\leq 0 \qquad \text{or}\quad h\equiv 0\,.$$

The latter says that the immersion is umbilical. If the first alternative occurs, we have from (7)

$$0\geq P_H(\sqrt{\sup h})=\left(\sqrt{\sup h}-\frac{n-2}{\sqrt{n}}\right)^2,$$

that is, $\sup h=(n-2)^2/n$. Then $n>2$ because, if $n=2$, then $h\equiv 0$ again. Now, by recalling that $|\sigma|^2=\operatorname{tr} A^2=h+nH^2$, we have $\sup \operatorname{tr} A^2=n$. Computing from (1) the scalar curvature of M in terms of $\operatorname{tr} A^2$ we conclude that $\sup r=(n-2)^2$, as we wanted. If this value is attained at some point of M, then the function h would reach its supremum on M. But, from (4) and (7), we know that this function h is subharmonic. Then h would be constant because of the maximum principle. Again from (4) and the above inequalities we conclude that $\nabla T=0$, that is, the second fundamental form of the immersion is parallel and, as the equality holds in (3), the hypersurface has two different constant principal curvatures $1/\sqrt{n-1}$ with multiplicity $n-1$ and $\sqrt{n-1}$ with multiplicity 1. So, the proof is finished.

Finally, we can state the main result of this paper, which gives us the uniqueness property of hyperbolic cylinders in the de Sitter space which we had stated in the introduction.

THEOREM 3. *Let M be a complete spacelike hypersurface of the de Sitter space S_1^{n+1}, $n>2$, with constant mean curvature $H=2\sqrt{n-1}/n$. If M is not connected at infinity, that is, M has at least two ends, then M is, up to isometry, a hyperbolic cylinder.*

PROOF. Our hypersurface cannot be umbilical because, in that case, it would be compact since $H=2\sqrt{n-1}/n<1$ and it would have no topological ends. Hence, from Proposition 2, we have $\sup_{p\in M} r(p)=(n-2)^2$. But, using the Gauss equation (1), we have $r=(n-2)^2-n+|\sigma|^2$, and so $\sup_{p\in M}|\sigma|^2(p)=n$. It suffices to prove, according to Proposition 2, that this supremum of $|\sigma|^2$ is attained. In order to show this, recall that our hypothesis about the value of the mean curvature H implies (see (6)) that the Ricci curvature of M is non-negative. Then we are in a position to apply the Cheeger-Gromoll splitting theorem [CG] because our manifold has at least two ends and, so, the existence of a geodesic line in M is warranted. Then, the hypersurface M is isometric to a Riemannian product $N\times \boldsymbol{R}$ of an $(n-1)$-dimensional complete manifold N and a Euclidean line. So, at each point of M, there exists a direction where the Ricci curvature vanishes. From the relation (6) between the Ricci tensor and the Weingarten endomorphism, this means that $\lambda_1=\sqrt{n-1}$ is a constant principal curvature of the hypersurface. Moreover, since the sum $|\sigma|^2$ of the squares of all principal curvatures $\lambda_1\geq\cdots\geq\lambda_n$ at each point is always less than or equal to n, we have as a conclusion that all the remaining principal curvatures $\lambda_2 \geq \cdots \geq \lambda_n$ have absolute value less than or equal to one. As we assume $n > 2$, we conclude that the principal curvature $\lambda_1 = \sqrt{n-1}$ has multiplicity exactly one at each point. Again by (6), we see that zero is a single eigenvalue of the Ricci curvature of M. Hence, the Ricci tensor of N is bounded below by the positive number $(\lambda_2 - \sqrt{n-1})^2$ and, so, the Bonnet-Myers theorem says that N must be compact.

On the other hand, if X is a principal vector field on M corresponding to λ_1, which can be taken to be parallel, and $E_1 = X, E_2, \ldots, E_n$ form a local orthonormal reference on M (so, $E_2, \ldots, E_n$ are tangent to N), one can see that

$$X|\sigma|^2 = X \sum_{i=1}^{n} \langle AE_i, AE_i \rangle = 2 \sum_{i=1}^{n} \langle (\nabla_X A)E_i, AE_i \rangle$$

because $\nabla_X E_i = 0$ for each $i = 1, 2, \ldots, n$. Using the Codazzi equation (1), we obtain

$$X|\sigma|^2 = \sum_{i=1}^{2} \langle (\nabla_{E_i} A)X, AE_i \rangle = 0$$

because $AX = \sqrt{n-1}\,X$ and X is a parallel vector field. Then, the function $|\sigma|^2$ does not depend on the Euclidean factor of the splitting of M. Hence, as N is compact, $|\sigma|^2$ attains its maximum on $N \times \{t\}$ for each $t \in \boldsymbol{R}$ and the same holds for $|\sigma|^2$ on M. In this way, we have proved the theorem.

References

[A] R. Aiyama, On complete spacelike surfaces with constant mean curvature in a Lorentzian 3-space form, Tsukuba J. Math. 15 (1991), 235–247.

[Ak] K. Akutagawa, On spacelike hypersurfaces with constant mean curvature in the de Sitter space, Math. Z. 196 (1987), 13–19.

[B] R. Bartnik, Existence of maximal surfaces in asymptotically flat spacetimes, Comm. Math. Phys. 94 (1984), 155–175.

[C] E. Calabi, Examples of Bernstein problems for some nonlinear equations, Proc. Symp. Pure and Appl. Math. 15 (1970), 223–230.

[CG] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Differential Geometry 6 (1971), 119–128.

[CY] S. Y. Cheng and S. T. Yau, Maximal spacelike hypersurfaces in the Lorentz-Minkowski space, Ann. of Math. 104 (1976), 407–419.

[CB] Y. Choquet-Bruhat, Maximal submanifolds and submanifolds of constant extrinsic curvature of a Lorentzian manifold, Ann. Scuola Norm. Sup. Pisa 3 (1976), 361–376.

[CFM] Y. Choquet-Bruhat, A. E. Fischer and J. E. Marsden, Maximal hypersurfaces and positivity of mass, Proc. of the Enrico Fermi Summer School of the Italian Physical Society, J. Ehlers, Ed., North-Holland, 1979.

[G] A. J. Goddard, Some remarks on the existence of spacelike hypersurfaces of constant mean curvature, Math. Proc. Cambridge Phil. Soc. 82 (1977), 489–495.

[KKN] U. H. Ki, H. J. Kim and H. Nakagawa, On spacelike hypersurfaces with constant mean curvature of a Lorentz space form, Tokyo J. Math. 14 (1991), 205–216.

[M] T. K. Milnor, Harmonic maps and classical surface theory in Minkowski 3-space, Trans. Amer. Math. Soc. 280 (1983), 161–185.

[Mo] S. Montiel, An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature, Indiana Univ. Math. J. 37 (1988), 909–917.

[N] S. Nishikawa, On maximal spacelike hypersurfaces in a Lorentzian manifold, Nagoya Math. J. 95 (1984), 117–124.

[O] M. Okumura, Hypersurfaces and a pinching problem on the second fundamental tensor, Amer. J. Math. 96 (1974), 207–213.

[Ol] V. Oliker, A priori estimates of the principal curvatures of spacelike hypersurfaces in de Sitter space with applications to hypersurfaces in hyperbolic space, Amer. J. Math. 114 (1992), 605–626.

[Om] H. Omori, Isometric immersions of Rimannian manifolds, J. Math. Soc. Japan 19 (1967), 205–214.

[R] J. Ramanathan, Complete spacelike hypersurfaces of constant mean curvature in de Sitter space, Indiana Univ. Math. J. 36 (1987), 349–359.

[S] S. Stumbles, Hypersurfaces of constant mean extrinsic curvature, Ann. of Physics 133 (1981), 28–56.

[T] A. Treibergs, Entire spacelike hypersurfaces of constant mean curvature in Minkowski space, Invent. Math. 66 (1982), 39–56.

[Y] S. T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure and Appl. Math. 28 (1975), 201–228.

Departamento de Geometría y Topología
Universidad de Granada
18071 Granada
Spain