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ON CONTRACTIBLE CURVES IN THE COMPLEX AFFINE PLANE

RAJENDRA VASANT GURJAR AND MASAYOSHI MIYANISHI

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Abstract. Concerning the topologically contractible curves embedded in the affine plane defined over the complex numbers we shall present new conceptual proofs to the theorem of Abhyankar-Moh and the theorem of Lin-Zaidenberg which are based on the structure theorems of non-complete algebraic surfaces.

1. Introduction. In this paper we give new proofs of the following two results from affine algebraic geometry.

THEOREM 1. *Let $C \subset \boldsymbol{C}^2$ be a closed embedding of the affine line $\boldsymbol{A}^1$. Then there is an algebraic automorphism of $\boldsymbol{C}^2$ which maps C onto the line $\{X=0\}$, where X, Y are some affine coordinates on $\boldsymbol{C}^2$.*

THEOREM 2. *Let $C \subset \boldsymbol{C}^2$ be an irreducible algebraic curve which is topologically contractible. Then there exist affine coordinates X, Y on $\boldsymbol{C}^2$ such that in terms of these coordinates C is defined by the equation $\{X^m = Y^n\}$, where* $\gcd(m,n)=1$.

Theorem 1 was first proved by Abhyankar and Moh [1] and independently by Suzuki [21]. Later on, several proofs of this result were found chronologically by Miyanishi [14], Rudolph [20], Richman [19] and Kang [8].

Most of these proofs involve either somewhat heavy calculations or detailed analysis of the singularity at infinity for the curve C. The proof of Rudolph is short and based on some basic results from knot theory (hence somewhat inaccessible to algebraic geometers).

Theorem 2 was first proved by Lin and Zaidenberg [13]. Their proof involves deep results from Teichmüller theory.

In view of the importance of these results for affine algebraic geometry, it is useful to have different ways of looking at these results. Our proofs of these results use essentially the same ideas from the theory of non-complete algebraic surfaces. An inequality of Miyaoka-Yau type proved by R. Kobayashi, S. Nakamura and F. Sakai plays a crucial role in the proofs. The results from the theory of non-complete algebraic varieties that we use have become by now well-known (except possibly for the inequality of Miyaoka-Yau type). Once the basic results about non-complete algebraic surfaces are assumed, our proofs become rather short and, we believe, more conceptual. In

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addition to these results, we only need some elementary facts about the singular fibres of $\boldsymbol{P}^1$-fibrations on smooth surfaces (cf. Lemmas 7, 8, 9 in this paper).

2. Preliminaries. All algebraic varieties considered in this paper are defined over the field of complex numbers $\boldsymbol{C}$.

An affine space of dimension n is denoted by $\boldsymbol{C}^n$. We denote $\boldsymbol{C}^1$ by $\boldsymbol{A}^1$.

A smooth projective, rational curve with self-intersection number $-n$ on a smooth algebraic surface is called a $(-n)$-curve.

A morphism φ from a smooth projective surface Y to a smooth curve B is called a $\boldsymbol{P}^1$-fibration if a general fibre of φ is isomorphic to $\boldsymbol{P}^1$. Similarly, an $\boldsymbol{A}^1$-fibration and a $\boldsymbol{C}^*$-fibration is defined, where $\boldsymbol{C}^*$ denotes $\boldsymbol{P}^1-\{\text{two points}\}$. A $\boldsymbol{C}^*$-fibration is said to be *untwisted* if it is a Zariski-locally trivial fibration on a non-empty Zariski open subset of the base. Otherwise, it is said to be *twisted.*

For any smooth irreducible algebraic variety X, let $\bar{\kappa}(X)$ denote the logarithmic Kodaira dimension of X as defined by Iitaka in [6]. We will implicitly use the following easy results about $\bar{\kappa}$.

(1) A smooth irreducible, affine curve B has $\bar{\kappa}(B)=-\infty$ if and only if it is isomorphic to $\boldsymbol{A}^1$, and $\bar{\kappa}(B)=0$ if and only if B is isomorphic to $\boldsymbol{C}^*$.

(2) Let $f\colon V\to W$ be a dominant morphism with V and W smooth varieties of same dimension. Then $\bar{\kappa}(V)\geq\bar{\kappa}(W)$. If further f is a proper birational morphism, then the equality holds. If V is a Zariski open set of W, this implies $\bar{\kappa}(V)\geq\bar{\kappa}(W)$.

For any topological space T, $e(T)$ denotes its Euler-Poincaré characteristic.

In what follows, by a surface we mean an algebraic surface and by a curve we mean an algebraic curve.

Let W be a smooth quasi-projective surface and $\overline{W}$ a smooth projective compactification of W such that the divisor $\Delta:=\overline{W}-W$ has simple normal crossings. We say that $(\overline{W},\Delta)$ is a *minimal normal* compactification of W if any (-1)-curve in Δ meets at least 3 other irreducible components of Δ.

Let W be a smooth surface with a morphism $g\colon W\to B$, where B is a smooth curve. For any scheme-theoretic fibre G of g, the greatest common divisor of the multiplicities of the irreducible components of G is called the *multiplicity* of G. If the multiplicity of G is greater than 1, then we call G a *multiple fibre*.

We now collect together the results from the theory of non-complete algebraic surfaces that will be used frequently in our proofs.

LEMMA 1 (cf. [9]). *Let X be a smooth irreducible surface with a morphism $\varphi\colon X\to B$ onto a smooth curve B such that a general fibre F of φ is irreducible. Then $\bar{\kappa}(X)\geq\bar{\kappa}(F)+\bar{\kappa}(B)$.*

LEMMA 2 (cf. [10]). *Let X be a smooth irreducible surface with $\bar{\kappa}(X)=1$. Then X contains a Zariski open subset U with a morphism $\varphi\colon U\to B$, where B is a smooth curve and the general fibre of φ is isomorphic to either $\boldsymbol{C}^*$ or an elliptic curve.*

The next result was proved by Miyanishi-Sugie and Fujita (cf. [16, Chap. I, §3]).

LEMMA 3. *Let X be a smooth affine surface with $\bar{\kappa}(X) = -\infty$. Then X admits an A^1-fibration.*

R. Kobayashi, S. Nakamura and F. Sakai have proved an inequality between logarithmic Chern numbers of a smooth algebraic surface analogous to the well-known Miyaoka-Yau inequality. We need the following consequence of the inequality (cf. [11], [12], [17]).

LEMMA 4. *Let X be a smooth affine surface with $e(X) \leq 0$. Then $\bar{\kappa}(X) \leq 1$.*

This result can be regarded as a generalization of Castelnuovo's result that the Euler-Poincaré characteristic of a surface of general type is strictly positive.

LEMMA 5. *Let V be a smooth affine surface and let $\varphi: V \to B$ be a morphism onto a smooth curve B. Then*

$$e(V) = e(B)e(F) + \sum (e(F_i) - e(F)),$$

where F is a general fiber of φ and the summation is over all the singular fibers of φ. Further, $e(F_i) \geq e(F)$ for all i and the equality occurs if and only if either $F \cong \boldsymbol{C}$ or $F \cong \boldsymbol{C}^*$ and $(F_i)_{\mathrm{red}} \cong F$ (cf. [21, §9] *and* [22, *Lemma* 3.2]).

The next result follows from R. H. Fox's solution of Fenchel's conjecture (cf. [2], [3]).

LEMMA 6. *Let $r \geq 3$ and let $p_1, p_2, \dots, p_r$ be distinct points in $\boldsymbol{P}^1$ and $m_1, m_2, \dots, m_r$ arbitrary integers ≥ 2. Then there exists a finite Galois covering $\tau: B \to \boldsymbol{P}^1$ such that the ramification index at a point over p_i is m_i for $i = 1, 2, \dots, r$. If $r = 2$ and $\gcd(m_1, m_2) = d > 1$, then we can construct a finite cyclic covering $\tau: B \to \boldsymbol{P}^1$ with the ramification index d at p_1, p_2.*

We will often use the following elementary result about singular fibres of a $\boldsymbol{P}^1$-fibration on a smooth projective surface (cf. [16, Chap. I, 4.4.1]).

LEMMA 7. *Let $g: Y \to B$ be a $\boldsymbol{P}^1$-fibration on a smooth projective surface and let F be a singular fibre of g, i.e. F is not isomorphic to $\boldsymbol{P}^1$. Then the following assertions are true.*

(i) *The reduced curve F_{red} is a divisor with simple normal crossings and each irreducible component of F_{red} is isomorphic to $\boldsymbol{P}^1$. Further, the dual graph of F is a tree.*

(ii) *If a (-1)-curve E occurs with multiplicity 1 in the scheme-theoretic fibre F, then F contains another (-1)-curve.*

REMARK. From (i) it follows that any (-1)-curve in a singular fibre of g meets at most two other irreducible components of the fibre. From (ii) it follows that in obtaining a relatively minimal $\boldsymbol{P}^1$-fibration starting from Y, we can successively contract (-1)-curves other than E so that the image of E becomes a full fibre in a relatively minimal fibration.

We now recall some easy properties of twisted and untwisted $\boldsymbol{C}^*$-fibrations. Let $\varphi: V \to B$ be a $\boldsymbol{C}^*$-fibration on a smooth quasi-projective surface V. Then there exists a projective embedding $V \subset \bar{V}$ such that φ extends to a $\boldsymbol{P}^1$-fibration $\Phi: \bar{V} \to \bar{B}$. If φ is twisted, then $D := \bar{V} - V$ has a unique irreducible component which dominates $\bar{B}$. We call this irreducible component the *horizontal component* of D. If φ is untwisted, then D has exactly two irreducible components dominating $\bar{B}$. They are cross-sections of Φ and also called the *horizontal components* of D.

The next result describes the nature of a singular fibre of a $\boldsymbol{P}^1$-fibration arising from a $\boldsymbol{C}^*$-fibration on an affine surface. For the elementary proofs, we refer the reader to a paper of Fujita (cf. [4, Lemmas 7.6 and 7.7]).

LEMMA 8. *Let $f: S \to B$ be an untwisted $\boldsymbol{C}^*$-fibration on a smooth affine surface S. Let $S \subset Y$ be a smooth compactification such that $D := Y - S$ has simple normal crossings and Y admits a $\boldsymbol{P}^1$-fibration $\tilde{f}$ extending f. Denote by D_1, D_2 the horizontal components of D. Let F be a singular fibre of $\tilde{f}$. Assume that no irreducible component of D contained in F is a (-1)-curve. Then we have the following assertions.*

(1) *Suppose that $F \cap S$ is irreducible and occurs with multiplicity >1 in F. If further $D \cap F$ is disconnected, then the dual graph of F is linear and the closure of $F \cap S$ is the unique (-1)-curve in F.*

(2) *Suppose that F is irreducible and $F \cap S \neq \emptyset$. If $D \cap F$ is connected, then F is the closure of $F \cap S$ and $D_1 \cap D_2 \cap F \neq \emptyset$.*

REMARK. Lemma 8 is proved by a repeated use of Lemma 7 and the fact that S is affine (and hence D is connected).

The next result follows easily from the observation that the irreducible components at infinity in a smooth compactification X for $\boldsymbol{C}^2$ generate Pic(X) freely (since the Picard group of $\boldsymbol{C}^2$ is trivial) (cf. [5, Proof of Lemma 3.2]).

LEMMA 9. *Let $\varphi: \boldsymbol{C}^2 \to B$ be a $\boldsymbol{C}^*$-fibration. Then we have:*

(1) *The fibration is untwisted, and B is isomorphic to $\boldsymbol{P}^1$ or $\boldsymbol{A}^1$.*

(2) *If $B \cong \boldsymbol{P}^1$, then every fibre of φ is irreducible and there is exactly one fibre isomorphic to $\boldsymbol{A}^1$, if taken with reduced structure.*

(3) *If $B \cong \boldsymbol{A}^1$ then exactly one fibre is reducible and it contains two irreducible components, say C_1, C_2. Further, either $C_1 \cong C_2 \cong \boldsymbol{A}^1$ and C_1, C_2 intersect each other transversally in a single point, or $C_1 \cong \boldsymbol{A}^1$, $C_2 \cong \boldsymbol{C}^*$ and they are disjoint. All other fibres of φ are isomorphic to $\boldsymbol{C}^*$ if taken with reduced structure.*

The next result is the main point in the proof of the famous topological characterization of $\boldsymbol{C}^2$ due to C. P. Ramanujam (cf. [13]).

LEMMA 10. *Let (X, D) be a minimal normal compactification of $\boldsymbol{C}^2$. Then the dual graph of D is linear.*

3. Proof of Theorem 1. In this section we prove Theorem 1. So, let $C \subset \boldsymbol{C}^2$ be a closed embedding of the curve C isomorphic to $\boldsymbol{A}^1$. Denote the complement $\boldsymbol{C}^2 - C$ by V. Then $e(V)=0$. By Lemma 4, $\bar{\kappa}(V) \leq 1$. The next result is the main step in the proof of Theorem 1.

PROPOSITION. $\bar{\kappa}(V) = -\infty$.

PROOF. Suppose that $\bar{\kappa}(V)=0$ or 1.

First consider the case $\bar{\kappa}(V)=0$. We choose a regular function f on $\boldsymbol{C}^2$ such that the divisor of f is C. Consider the morphism $\varphi\colon \boldsymbol{C}^2 \to \boldsymbol{A}^1$ given by f. Since f is a prime element in the coordinate ring of $\boldsymbol{C}^2$, a general fibre of φ is irreducible. By Lemma 1 applied to the map $\varphi|_V\colon V \to \boldsymbol{C}^*$. we get

$$0 = \bar{\kappa}(V) \geq \bar{\kappa}(F) + \bar{\kappa}(\boldsymbol{C}^*) = \bar{\kappa}(F)\,,$$

where F is a general fibre of φ. Suppose $\bar{\kappa}(F) = -\infty$. Then $F \cong \boldsymbol{A}^1$ and φ is an $\boldsymbol{A}^1$-fibration. Any $\boldsymbol{A}^1$-fibration on a smooth quasi-projective surface is trivial on a Zariski open subset of the base curve. Hence V contains a Zariski open subset which is isomorphic to a cylinder-like open set $B \times \boldsymbol{A}^1$. It follows that $\bar{\kappa}(V) = -\infty$ (cf. [16, Chap. I, §2.2]).

Hence assume that $\bar{\kappa}(F)=0$ and F is isomorphic to $\boldsymbol{C}^*$. Namely, φ is a $\boldsymbol{C}^*$-fibration and C is a full fibre of φ. But this contradicts Lemma 9 because the fibre containing C must be reducible as $C \cong \boldsymbol{A}^1$.

Assume now that $\bar{\kappa}(V)=1$. By Lemma 2, V contains a Zariski open subset U with a $\boldsymbol{C}^*$-fibration φ'. We need to consider three cases.

Case 1. φ' does not extend to a morphism on $\boldsymbol{C}^2$.

Then the closures of the fibres of φ' have a unique common point, say p, in $\boldsymbol{C}^2$. This point cannot lie in V. For, otherwise V will contain a family of affine rational curves with one-place at infinity and passing through p. By blowing up successively at p and its infinitely near points, we resolve the base locus and get a smooth surface $\tilde{V}$ which admits a morphism to a curve with general fibre isomorphic to $\boldsymbol{A}^1$. But then $\bar{\kappa}(\tilde{V}) = -\infty$ as remarked above. Since $\tilde{V}$ is obtained from V by a sequence of blowing-ups, $\bar{\kappa}(\tilde{V}) = \bar{\kappa}(V)$, contradicting the assumption that $\bar{\kappa}(V)=1$. Hence $p \in C$. This gives a $\boldsymbol{C}^*$-fibration $\varphi\colon \boldsymbol{C}^2 - \{p\} \to B$ such that $\varphi' = \varphi|_V$.

We claim that $B \cong \boldsymbol{P}^1$. To see this we observe that since the closure of every fibre of φ passes through p, every fibre of φ intersects the boundary of a small neighborhood of p. Hence this boundary, which is a compact set, maps onto B and hence B is compact.

Clearly, the Picard group of $\boldsymbol{C}^2 - \{p\}$ is trivial and any non-constant regular function on $\boldsymbol{C}^2 - \{p\}$ vanishes at some point. Hence the proof of Lemma 9 applies and shows that all the fibres of φ are irreducible. This also implies that φ' is a $\boldsymbol{C}^*$-fibration on

V. Note that clearly $\boldsymbol{C}^2-\{p\}$ is simply-connected. Suppose φ has three or more multiple fibres $m_1F_1, \ldots, m_sF_s$ lying over the points $p_1, \ldots, p_s$ of $\boldsymbol{P}^1$. Using Lemma 6 we can construct a finite Galois covering $\tau: B \to \boldsymbol{P}^1$ such that the ramification index at a point over p_i is m_i. Then the normalization of the fibre product $(\boldsymbol{C}^2-\{p\}) \times_{\boldsymbol{P}^1} B$, which we denote by $\{(\boldsymbol{C}^2-\{p\}) \times_{\boldsymbol{P}^1} B\}^-$, is a finite unramified covering of $\boldsymbol{C}^2-\{p\}$, which is impossible. Hence there are at most two multiple fibres for φ. If φ' has at most one multiple fibre, then V contains a Zariski open subset U isomorphic to $\boldsymbol{C}^* \times \boldsymbol{C}^*$ and hence $\bar{\kappa}(V) \leq \bar{\kappa}(U)=0$. This contradicts the assumption that $\bar{\kappa}(V)=1$. The same argument shows that $C-\{p\}$ is a reduced fibre of φ, for otherwise φ' has only one multiple fibre. Hence φ' has exactly two multiple fibres. Let m_1F_1, m_2F_2 be the two multiple fibres. Then again by Lemma 6, $\gcd(m_1, m_2)=1$.

Let X be a smooth projective compactification of $\boldsymbol{C}^2-\{p\}$ such that the divisor D at infinity has simple normal crossings and there is a $\boldsymbol{P}^1$-fibration $\Phi: X \to \boldsymbol{P}^1$ extending φ. The fibration φ' is an untwisted fibration with one horizontal irreducible component of D lying over p and the other coming from the compactification divisor for $\boldsymbol{C}^2$. We now apply Lemma 7. Let $\bar{C}$ be the closure of C in X. Since $\bar{C}$ occurs with multiplicity one in the fibre of Φ which contains it, using Lemma 7 repeatedly we can assume that X has the following properties:

(1) D has two horizontal components D_1, D_2 such that D_1 lies over p and D_2 is an irreducible component at infinity for $\boldsymbol{C}^2$.

(2) The closure $\bar{C}$ of C in X is a full fibre of Φ.

(3) $D_1 \cap D_2 \cap \bar{C} = \varnothing$.

(4) The fibre $\tilde{F}_i$ of Φ containing F_i is a linear chain of smooth rational curves, the closure $\bar{F}_i$ of F_i is the unique (-1)-curve in $\tilde{F}_i$ and D_1, D_2 intersect the two end irreducible components of $\tilde{F}_i$ for $i=1, 2$ (cf. [4, Lemmas 7.6, 7.7]).

(5) $\tilde{F}_1$, $\tilde{F}_2$ are the only singular fibres of Φ.

Indeed, as $\bar{C}$ has multiplicity one in the fibre, by applying Lemma 7 repeatedly we can make $\bar{C}$ a full fibre of Φ contracting (-1)-curves other than $\bar{C}$ in the fibre, which also lie outside $\boldsymbol{C}^2-\{p\}$. In this process, the components D_1, D_2 do not meet on $\bar{C}$. Hence the properties (1), (2), (3) hold. By a similar argument, (5) holds. For the proof of (4), we use Lemma 8. For this, let E be any (-1)-curve contained in $D \cap \tilde{F}_i$. By the remark after Lemma 7, E meets at most two other irreducible components of $\tilde{F}_i$, and if it meets two such irreducible components then it occurs with multiplicity ≥ 2 in $\tilde{F}_i$. In the latter case E cannot meet either of D_1, D_2 as these are cross-sections. Hence we can contract E and get a smaller compactification of $\boldsymbol{C}^2-\{p\}$ which satisfies all the properties of X. By this process we reach a situation where $D \cap \tilde{F}_i$ does not contain any (-1)-curve. The hypothesis of Lemma 8 is therefore satisfied, and we may apply it.

Let Δ_1, Δ_2 be the connected components of D containing D_1, D_2 respectively, where Δ_1 contracts to p smoothly. Since no irreducible component of Δ_1 other than D_1 is a (-1)-curve, we see easily that D_1 is a (-1)-curve and Δ_1 is a linear chain. As D_1 is a cross-section and $m_i > 1$, $\tilde{F}_i$ contains at least one irreducible component of Δ_1 for

$i=1$, 2 which intersects D_1. Clearly, X contains a Zariski open subset, say W, which is obtained from $\boldsymbol{C}^2$ by a finite succession of blowing-ups at the point p such that Δ_1 is the complete exceptional divisor. By the above observation, Δ_1 contains at least three irreducible components and it is easy to see that after successive contractions of (-1)-curves from Δ_1 and its images, the image of C' becomes singular at the point p, where C' is the proper transform of C in W. This contradicts the assumption that C is a smooth curve. Hence the case 1 cannot occur.

Next we consider the case where φ' extends to a morphism $\varphi: \boldsymbol{C}^2 \to B$. Then we have two separate cases depending on whether or not C is contained in a fiber of φ. If C is contained in a fiber, φ is a $\boldsymbol{C}^*$-fibration, and if C is not contained in a fiber, φ is an $\boldsymbol{A}^1$-fibration and C is a cross-section.

Case 2. φ' extends to a $\boldsymbol{C}^*$-fibration $\varphi: \boldsymbol{C}^2 \to B$.

Then φ is untwisted and B is necessarily isomorphic to $\boldsymbol{A}^1$ or $\boldsymbol{P}^1$ by Lemma 9. Clearly, $\varphi(C)$ is a point, say p_0. The proof in this case is somewhat similar to the case 1 above. If $B \cong \boldsymbol{P}^1$, let $\Phi: X \to B$ be a $\boldsymbol{P}^1$-fibration on a suitable compactification of $\boldsymbol{C}^2$. Then as in the proof of the case 1, the closure $\bar{C}$ of C is a reduced fibre of Φ, and φ has exactly two multiple fibres $m_1 F_1$, $m_2 F_2$ of relatively prime multiplicities. By minimizing the compactification X of $\boldsymbol{C}^2$ by contracting (-1) curves contained in D and the fibres of Φ, we may assme that the corresponding fibres of Φ are linear chains of nonsingular rational curves, where $D := X - \boldsymbol{C}^2$. The horizontal components D_1, D_2 of D meet each other at a point in $\bar{C}$ (cf. [4, Lemma 7.7(1)]). Let $\tilde{F}_i$ be the fibre of Φ containing F_i for $i=1, 2$. Since $m_i > 1$, $\tilde{F}_i$ contains at least two extra irreducible components which meet the cross-sections D_1, D_2 and are contained in D. Then D_1 and D_2 are branch points for the dual graph of D, where a branch point of the dual graph signifies a vertex from which sprout three or more edges. In this case (X, D) is a minimal normal compactification of $\boldsymbol{C}^2$, contradicting Lemma 10.

Next consider the case where $B \cong \boldsymbol{A}^1$. By Lemma 9, the fibre of φ containing C is of the form $C \cup C'$, where $C' \cong \boldsymbol{A}^1$ or $\boldsymbol{C}^*$. Consider also a suitable compactification X of $\boldsymbol{C}^2$ with a $\boldsymbol{P}^1$-fibration $\Phi: X \to \boldsymbol{P}^1$. The fibre over the point $p_\infty := \boldsymbol{P}^1 - B$ is contained in $D := X - \boldsymbol{C}^2$. As in the case 1, there are exactly two multiple fibres for the morphism $\varphi': V \to \boldsymbol{A}^1$. Hence the morphism $\varphi|_{V-C'}: V - C' = \boldsymbol{C}^2 - (C \cup C') \to \boldsymbol{C}^*$ has a multiple fibre, say mF_1 with $m \geq 2$. Let $\rho: \Delta \to \boldsymbol{P}^1$ be a cyclic covering of degree m totally ramified over $\varphi(F_1)$ and p_∞. Then the normalization Y of the fibre product $(\Delta - \rho^{-1}(p_\infty)) \times_{\boldsymbol{A}^1} \boldsymbol{C}^2$ is an unramified covering of degree m of $\boldsymbol{C}^2$, a contradiction.

Case 3. φ' extends to an $\boldsymbol{A}^1$-fibration $\varphi: \boldsymbol{C}^2 \to B$.

By Lemma 9, $B \cong \boldsymbol{P}^1$ or $\boldsymbol{A}^1$. As $e(\boldsymbol{C}^2) = 1$, using Lemma 5 it is easy to see that $B \cong \boldsymbol{A}^1$. Also, all the fibres of φ are irreducible by the count of the Picard number. They are all reduced, for otherwise we get a finite unramified covering of $\boldsymbol{C}^2$ by the same argument as in the latter case of the case 2 above. But then every fibre of φ is isomorphic to $\boldsymbol{A}^1$ and φ is a trivial $\boldsymbol{A}^1$-bundle by [7]. Hence V is isomorphic to $\boldsymbol{A}^1 \times \boldsymbol{C}^*$, which has $\bar{\kappa} = -\infty$, contradicting the assumption that $\bar{\kappa}(V) = 1$.

This completes the proof of the proposition.

The rest of the proof is quite well-known. We briefly sketch it for the sake of completeness. By Lemma 3, there is an $\boldsymbol{A}^1$-fibration φ' on V. The fibres of this morphism are closed in $\boldsymbol{C}^2$, for otherwise $\boldsymbol{C}^2$ contains complete curves. Hence this morphism extends to an $\boldsymbol{A}^1$-fibration $\varphi: \boldsymbol{C}^2 \to \boldsymbol{B}$ on $\boldsymbol{C}^2$ with the base curve $B \cong \boldsymbol{A}^1$. All the fibres of this extended morphism φ are reduced and irreducible. As in the case 3 of the proposition, φ is a trivial $\boldsymbol{A}^1$-bundle and C is a fibre of φ. So there exists coordinates X, Y on $\boldsymbol{C}^2$ such that C is defined by $X = 0$. This completes the proof of Theorem 1.

4. The proof of Theorem 2. Let C be a contractible irreducible curve in $\boldsymbol{C}^2$ and denote by V the complement $\boldsymbol{C}^2 - C$. Then $e(V) = 0$. By Lemma 4, $\bar{\kappa}(V) \leq 1$. We consider three cases as in the proof of Theorem 1.

Case 1. $\bar{\kappa}(V) = -\infty$.

Then by Lemma 3, there is an $\boldsymbol{A}^1$-fibration $\varphi: V \to \boldsymbol{B}$. Since $\boldsymbol{C}^2$ is affine, it does not contain any complete curves. Hence φ extends to an $\boldsymbol{A}^1$-fibration $\phi: \boldsymbol{C}^2 \to \boldsymbol{B}'$, where B' contains $\boldsymbol{B}$ as a Zariski open subset. Next, ϕ extends to a $\boldsymbol{P}^1$-fibration on a smooth compactification of $\boldsymbol{C}^2$. Clearly, C is contained in a fibre of ϕ. By Lemma 7, any fibre of a $\boldsymbol{P}^1$-fibration on a smooth projective surface is a union of $\boldsymbol{P}^1$'s, hence C is smooth. Therefore $C \cong \boldsymbol{A}^1$ and the result is already proved in Theorem 1.

Case 2. $\bar{\kappa}(V) = 0$.

Let f be a prime element in the coordinate ring of $\boldsymbol{C}^2$ such that C is defined by $f = 0$. By Lemma 1,

$$\bar{\kappa}(V) \geq \bar{\kappa}(F) + \bar{\kappa}(\boldsymbol{C}^*),$$

where F is a general fibre of the morphism given by f. Hence $\bar{\kappa}(F) \leq 0$. But $\bar{\kappa}(F) \neq -\infty$ as in the beginning of the proof of Proposition in §3, for otherwise, $\bar{\kappa}(V) = -\infty$. Hence $\bar{\kappa}(F) = 0$, so that $F \cong \boldsymbol{C}^*$. By Lemma 7 as before, any irreducible component of a $\boldsymbol{C}^*$-fibration on an affine surface is smooth. Therefore C is smooth and we are again through.

Case 3. $\bar{\kappa}(V) = 1$.

Since V is affine, by Lemma 2 and by the same argument as in the beginning of the proof of the case 1 of Proposition in §3, there is a $\boldsymbol{C}^*$-fibration $\varphi: V \to \boldsymbol{B}$. If this morphism extends to a $\boldsymbol{C}^*$-fibration on $\boldsymbol{C}^2$ then C is mapped to a point and C is smooth as above. If φ extends to an $\boldsymbol{A}^1$-fibration on $\boldsymbol{C}^2$, then C is a cross-section and hence smooth and the result is proved by Theorem 1.

Now assume that C is not smooth and that the rational mapping $\varphi: \boldsymbol{C}^2 \to \boldsymbol{P}^1$ given by φ is not defined at a point p on C.

CLAIM 1. *The point p is the only singular point of C, and C is analytically irreducible at p.*

PROOF. Resolve the indeterminacies of the rational mapping $\varphi: \boldsymbol{C}^2 \to \boldsymbol{P}^1$ by a sequence of blowing-ups $\sigma: W \to \boldsymbol{C}^2$ at the point p and its infinitely near points, where W is a smooth surface. Then the rational mapping φ gives rise to an $\boldsymbol{A}^1$-fibration $\rho: W \to \boldsymbol{P}^1$ such that the last exceptional curve of σ is a cross-section of ρ and the proper transform C' of C is contained in a fibre of ρ. The $\boldsymbol{A}^1$-fibration ρ is extended to a $\boldsymbol{P}^1$-fibration $\tilde{\rho}: \tilde{W} \to \boldsymbol{P}^1$ on a smooth compactification $\tilde{W}$ of W. By Lemma 7, every singular fibre of $\tilde{\rho}$ is a tree of nonsingular rational curves. Hence C' is smooth. This implies that the point p is the unique singular point of C. Contractibility of C implies that C is homeomorphic to $\boldsymbol{A}^1$. Hence C is analytically irreducible at p.

The closure of any fibre of $\varphi: V \to B$ passes through the point p. As in the proof of the case 1 of Proposition in §3, the restriction of ρ gives an untwisted $\boldsymbol{C}^*$-fibration $\phi: \boldsymbol{C}^2 - \{p\} \to \boldsymbol{P}^1$.

CLAIM 2. (1) *Every fibre of ϕ is isomorphic to $\boldsymbol{C}^*$ if taken with reduced structure.*

(2) *There are exactly two multiple fibres $m_1 F_1$, $m_2 F_2$ and ϕ is a trivial $\boldsymbol{C}^*$-fibration outside the union of F_1 and F_2.*

(3) *Let $\bar{F}_i$ be the closure of F_i in $\boldsymbol{C}^2$ for $i = 1, 2$. Then $\bar{F}_1$ and $\bar{F}_2$ are isomorphic to $\boldsymbol{A}^1$. Hence $C - \{p\}$ is a reduced fibre of ϕ.*

(4) *The curves $\bar{F}_1$ and $\bar{F}_2$ meet each other transversally at the point p.*

PROOF. (1) Let F be a general fibre of ϕ. Then $F \cong \boldsymbol{C}^*$ and $\boldsymbol{C}^2 - \{p\} - F$ is an affine surface with $e(\boldsymbol{C}^2 - \{p\} - F) = 0$. Applying Lemma 5 to the $\boldsymbol{C}^*$-fibration $\phi|_{\boldsymbol{C}^2 - \{p\} - F}: \boldsymbol{C}^2 - \{p\} - F \to \boldsymbol{A}^1$ we see that every fibre of ϕ other than F is isomorphic to $\boldsymbol{C}^*$ if taken with reduced structure.

(2) If ϕ has three or more multiple fibres, then by Lemma 6 we can construct a suitable ramified covering $\Delta \to \boldsymbol{P}^1$ such that the normalized fibre product $\{(\boldsymbol{C}^2 - \{p\}) \times_{\boldsymbol{P}^1} \Delta\}^-$ is an unramified covering of $\boldsymbol{C}^2 - \{p\}$. But this is a contradiction because $\boldsymbol{C}^2 - \{p\}$ is simply connected. Hence ϕ has at most two multiple fibres and if there are two multiple fibres then their multiplicities are relatively prime. Suppose ϕ has at most one multiple fibre. Then $\bar{\kappa}(\boldsymbol{C}^2 - C) \leq \bar{\kappa}(\boldsymbol{C}^* \times \boldsymbol{C}^*) = 0$, which is a contradiction. Hence ϕ has exactly two multiple fibres $m_1 F_1$, $m_2 F_2$ and ϕ is a trivial $\boldsymbol{C}^*$-fibration outside the union of these two multiple fibres.

(3) Since ϕ is untwisted, $\boldsymbol{C}^2 - \bar{F}_i$ contains $\boldsymbol{C}^* \times \boldsymbol{C}^*$ as a Zariski open subset for $i = 1, 2$. Hence $\bar{\kappa}(\boldsymbol{C}^2 - \bar{F}_i) \leq 0$. Note that $\bar{F}_i$ is a contractible curve because F_i is isomorphic to $\boldsymbol{C}^*$ and $\bar{F}_i$ is the union of F_i and the point p. Then, by the proofs of the cases 1 and 2 above, we see that $\bar{F}_i$ is isomorphic to $\boldsymbol{A}^1$, hence $\bar{F}_i$ is smooth. Since C is a singular

curve, C is distinct from $\bar{F}_1$ and $\bar{F}_2$. So, $C-\{p\}$ is a reduced fibre of ϕ.

(4) Note that $\boldsymbol{C}^2-(\bar{F}_1 \cup \bar{F}_2)$ is a trivial $\boldsymbol{C}^*$-bundle over $\boldsymbol{C}^*$, hence isomorphic to $\boldsymbol{C}^* \times \boldsymbol{C}^*$. Hence $\bar{\kappa}(\boldsymbol{C}^2-\bar{F}_1 \cup \bar{F}_2)=0$. Then the assertion follows from Lemma 11 below.

LEMMA. 11. *Let C_1 and C_2 be irreducible curves on $\boldsymbol{C}^2$ such that C_1 and C_2 are isomorphic to $\boldsymbol{A}^1$ and that C_1 and C_2 meet in one point p with local intersection multiplicity n. Let $V=\boldsymbol{C}^2-(C_1 \cup C_2)$. Then $\bar{\kappa}(V)=0$ or 1, and $\bar{\kappa}(V)=0$ if and only if $n=1$.*

PROOF. Since $e(V)=0$, $\bar{\kappa}(V)\leq 1$ by Lemma 4. If $\bar{\kappa}(V)=-\infty$ then there is an $\boldsymbol{A}^1$-fibration ρ: $V \rightarrow B$, which extends to an $\boldsymbol{A}^1$-fibration $\tilde{\rho}$ on $\boldsymbol{C}^2$ so that $C_1 \cup C_2$ is contained in a fibre. This is impossible because any fibre of $\tilde{\rho}$ is isomorphic to $\boldsymbol{A}^1$ by Theorem 1. Hence $\bar{\kappa}(V)=0$ or 1.

Suppose $\bar{\kappa}(V)=0$. Let $f=0$ be a defining equation of C_1+C_2 in $\boldsymbol{C}^2$ and consider the morphism $f: \boldsymbol{C}^2 \rightarrow \boldsymbol{A}^1$ defined by f. Let F be a general fibre of $f|_V$: $V \rightarrow \boldsymbol{C}^*$. Since $\bar{\kappa}(V) \neq -\infty$, we must have $\bar{\kappa}(F) \neq -\infty$. By Lemma 1 applied to $f|_V$, we know that $f|_V$ is a $\boldsymbol{C}^*$-fibration. Then Lemma 9 implies that $n=1$.

Conversely, suppose $n=1$. Then we may choose affine coordinates X_1, X_2 on $\boldsymbol{C}^2$ so that C_i is defined by $X_i=0$ (cf. [15, Theorem 3.2, p. 40]). Hence $V \cong \boldsymbol{C}^* \times \boldsymbol{C}^*$ and $\bar{\kappa}(V)=0$.

CLAIM 3. *There exist affine coordinates X_1, X_2 of $\boldsymbol{C}^2$ such that C is defined by $X_1^{m_1}-X_2^{m_2}=0$.*

PROOF. As in the proof of Lemma 11 above, we choose affine coordinates X_1, X_2 on $\boldsymbol{C}^2$ so that $\bar{F}_i$ is defined by $X_i=0$ for $i=1$, 2. The fibration ϕ: $\boldsymbol{C}^2-\{p\} \rightarrow \boldsymbol{P}^1$ defines a pencil, say Λ, on $\boldsymbol{C}^2$ of which $m_1\bar{F}_1$ and $m_2\bar{F}_2$ are members. Then we can choose an inhomogeneous coordinate t on $\boldsymbol{P}^1$ such that $t=X_1^{m_1}/X_2^{m_2}$. Since the given curve C is a member of the pencil Λ, we may assume that C is defined by $t=1$. So, C is defined by $X_1^{m_1}-X_2^{m_2}=0$.

This completes the proof of the Lin-Zaidenberg theorem.

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SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI-BHABHA ROAD
BOMBAY 400005
INDIA

E-mail address: gurjar@tifrvax.tifr.res.in

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
OSAKA UNIVERSITY
TOYONAKA, OSAKA 560
JAPAN

E-mail address: miyanisi@math.sci.osaka-u.ac.jp