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CELLS IN CERTAIN SETS OF MATRICES

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Abstract. We decompose the canonical bases for q-Schur algebras and the modified quantized enveloping algebras of type A into two-sided cells in terms of some combinatorics on certain sets of matrices.

Introduction. Let $\mathscr{A}$ be an associative algebra over a field K and $\mathscr{B}$ a basis for $\mathscr{A}$ as a K-vector space. Then $\mathscr{B}$ is divided into *cells*, by Lusztig [Lu2, 29.4], via the equivalence relations on $\mathscr{B}$ defined as follows:

If the elements $c_{b,b',b''}\in K$ with $b, b', b''\in\mathscr{B}$ denote the structure constants of $\mathscr{A}$ i.e., they satisfy $bb'=\sum_{b''\in\mathscr{B}}c_{b,b',b''}b''$, then we say for $b, b'\in\mathscr{B}$ that $b'\leq_L b$ (resp. $b'\leq_R b$) if there are sequences $b_1=b, b_2, \dots, b_n=b'$ and $\beta_1, \dots, \beta_{n-1}$ in $\mathscr{B}$ such that $c_{\beta_i,b_i,b_{i+1}}\neq 0$ (resp. $c_{b_i,\beta_i,b_{i+1}}\neq 0$) for all $i=1, \dots, n-1$. These are preorders on $\mathscr{B}$. We define $\leq_{LR}$ to be the preorder on $\mathscr{B}$ generated by $\leq_L$ and $\leq_R$. For $x\in\{L, R, LR\}$ and $b, b'\in\mathscr{B}$ we say $b\sim_x b'$ if $b\leq_x b'\leq_x b$. Thus $\sim_L$, $\sim_R$ and $\sim_{LR}$ are equivalence relations on $\mathscr{B}$. The corresponding equivalence classes are called *left*, *right* and *two-sided cells* of $\mathscr{B}$ respectively.

In certain nice circumstances, cells are important in the study of representation theory and can be classified combinatorially. For example, if $\mathscr{A}=\mathscr{H}(W)$ is a Hecke algebra associated with a Coxeter group W and $\mathscr{B}$ is the Kazhdan-Lusztig basis of $\mathscr{H}(W)$, then cells in the sense above are the Kazhdan-Lusztig cells (see [KL]). When W is a Weyl group or an affine Weyl group of type A, K-L cells can be classified in terms of partitions, tableaux, Robinson-Schensted maps, etc. (see, e.g., [Sh]).

In this paper, we shall consider two more examples in the case of type A, namely, $\mathscr{A}$ is a q-Schur algebra $\mathscr{S}_q(n, r)$ or a modified quantized enveloping algebra $\dot{U}$ of type A and $\mathscr{B}$ is the K-L basis of $\mathscr{S}_q(n, r)$ (cf. [Du1]) or the canonical basis of $\dot{U}$ (cf. [Lu2]). These bases are indexed by certain sets of matrices. So the cell decomposition of $\mathscr{B}$ induces a cell decomposition of these matrix sets. We shall give a combinatorial description for those two-sided cells.

In Section 1 we first generalize a result of Greene [G], which associates partitions to finite posets, to a result on a cerain matrix semigroup $M(n)$, namely one associates partitions to the matrices in $M(n)$. Thus, we decompose $M(n)$ into subsets in terms of partitions. Our main result is Theorem 2.1 which shows that these subsets are two-sided

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cells of the K-L bases of q-Schur algebras. We shall prove the theorem in Section 3. In Section 4 we apply the result to modified quantum groups $\dot{U}$ of type A.

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1. A generalization of Greene's result. We first recall Greene's method [G] which associates partitions to finite partially ordered sets.

Let P be a finite partially ordered set. A *chain* in P is a subset of P which is totally ordered by the induced order of P. A *k-chain family* is a subset of P which is a disjoint union of k chains. We denote by $c_k(P)$ the maximum cardinality of k-chain families and let $\lambda_k(P)=c_k(P)-c_{k-1}(P)$ (with the convention that $c_0(P)=0$).

1.1. THEOREM (cf. [G]). *Maintain the notation introduced above. Then*

$$\lambda_1(P)\geq \cdots \geq \lambda_p(P) \tag{1.1}$$

is a partition of $|P|$ *where* p *is the maximum cardinality of the subsets which contain no chain of length* 2. (*These subsets are called antichains according to Greene.*)

Let N be the set of nonnegative integers. For any $n\in N$, let $[1, n]$ denote the interval from 1 to n in N. We order $N^2=N\times N$ by setting $(i,j)\leq(i',j')$ if $i\geq i'$ and $j\leq j'$. Thus, each $[1,n]^2$ is a finite poset with the induced order. Clearly, the partition of n^2 associated to $[1,n]^2$ is $(2n-1, 2n-3, \dots, 3, 1)$.

We now generalize Greene's idea to associate partitions to $n\times n$ matrices with entries in N. Let $M(n)$ be the set of all $n\times n$ matrices with entries in N, and let $\mathfrak{s}: M(n)\to N$ be the map sending a matrix $A=(a_{ij})\in M(n)$ to its entry sum $\mathfrak{s}(A)=\sum_{i,j}a_{ij}$. More generally, if F is a k-chain family of $[1,n]^2$, we define $\mathfrak{s}_F(A)$ to be the sum of the entries a_{ij} with $(i,j)\in F$. We call the map $\mathfrak{s}_F: M(n)\to N$ an F-sum map and $\mathfrak{s}_F(A)$ the F-sum of A. Let $\mathfrak{s}_k(A)$ be the maximum value of F-sums of A for all k-chain families F.

1.2. THEOREM. *Maintain the notation above. For* $A\in M(n)$ *with* $\mathfrak{s}(A)=r$, *we define* $\sigma_i(A)=\mathfrak{s}_i(A)-\mathfrak{s}_{i-1}(A)$ (*with the convention that* $\mathfrak{s}_0(A)=0$) *for all* $i\in[1,n]$. *Then* $\sigma_1(A)\geq\sigma_2(A)\geq\cdots\geq\sigma_n(A)$ *is a partition of* r. *Therefore, we obtain a surjective map*

$$\sigma: A\longmapsto \sigma(A)=(\sigma_1(A), \sigma_2(A), \dots, \sigma_n(A))$$

from $M(n)$ *onto the set* $\Lambda^+(n)$ *of partitions with at most* n *parts.*

PROOF. The idea of the proof is to associate a finite poset, hence a partition by (1.1), to a matrix in $M(n)$.

For $A=(a_{ij})\in M(n)$, let $P=P(A)$, $\leq_A$ be a finite poset satisfying the following conditions:

(i) $P=\dot{\bigcup}_{1\leq i,j\leq n}P_{ij}$ (a disjoint union) with $|P_{ij}|=a_{ij}$. Hence $|P|=\mathfrak{s}(A)$.

(ii) Every P_{ij} is a chain in P.

(iii) For $a \in P_{ij}$, $b \in P_{i'j'}$ with $(i, j) \neq (i', j')$, $a <_A b$ if and only if $(i, j) < (i', j')$.

Note that condition (iii) means that $\leq_A$ induces a partial order on the set $\bar{P} = \{P_{ij} \mid (i, j) \in [1, n]^2\}$ and $\bar{P}$ with the induced order is isomorphic to the poset $[1, n]^2$. Also, one sees easily from (ii) and (iii) that if $(i, j) \leq (i', j')$ then $P_{ij} \cup P_{i'j'}$ is a chain.

Our construction of P implies also the following relationship between the posets $P(A)$, $\leq_A$ and $[1, n]^2$, $\leq$: Let S be a subset of P which is a union of P_{ij} and let

$$I = I(S) = \{(i, j) \mid P_{ij} \subseteq S\} .$$

Then we have $\mathfrak{s}_I(A) = |S|$. Further S is a chain if and only if $I(S)$ is a chain, while S is a k-chain family if and only if $I(S)$ is a k-chain family.

For any subset S of P, we define its *closure* $\bar{S}$ as the minimal subset of P containing S and being a union of some P_{ij}. Clearly, if S is a chain, so is $\bar{S}$. Also, if S is a k-chain family, then so is $\bar{S}$. In particular, if S is a k-chain family in P satisfying $c_k(P) = |S|$, then $S = \bar{S}$. Therefore, in this case, $I(S)$ is a k-chain family in $[1, n]^2$ and $c_k(P) = |S| = \mathfrak{s}_{I(S)}(A) \leq \mathfrak{s}_k(A)$. Conversely, if F is a k-chain family in $[1, n]^2$ with $\mathfrak{s}_k(A) = \mathfrak{s}_F(A)$, then $S_F = \bigcup_{(i,j) \in F} P_{ij}$ is a k-chain family in P. So $\mathfrak{s}_k(A) = |S_F| \leq c_k(P)$. So we have proved that $\mathfrak{s}_k(A) = c_k(P)$. Now, our first assertion follows from (1.1). The proof of the rest is obvious. $\square$

1.3. REMARK. In Section 3 we shall look at the poset $P(A)$ in terms of certain pseudo-matrices. These matrices will be used to define the longest element of the double coset defined by A. A special case for this is given in Example 1.4 below.

We now define an equivalence relation $\sim$ on $M(n)$ by setting, for $A, B \in M(n)$, $A \sim B$ if $\sigma(A) = \sigma(B)$. The corresponding equivalence classes, i.e., the fibres of σ, are called *cells* of $M(n)$. Clearly, for any $r \in N$, the subset $M(n, r) = \mathfrak{s}^{-1}(r)$ of $M(n)$ (consisting of all matrices in $M(n)$ whose entry sum is r) is a union of cells. For a partition $\lambda \in \Lambda^+(n)$, we denote by $M(n)_\lambda = \sigma^{-1}(\lambda)$ the cell corresponding to λ. We shall see that these cells agree with the Kazhdan-Lusztig two-sided cells in the sense of [Du2] as indicated in the following example.

1.4. EXAMPLE. Let $P(n)$ be the set of permutation matrices in $M(n)$. Then, $P(n)$ is a group isomorphic to the symmetric group $\mathfrak{S}_n$ on $[1, n]$. We take this isomorphism from $\mathfrak{S}_n$ to $P(n)$ by sending w to A_w where if w maps j to j' then A_w is the matrix with 1 at the (j', j)-entry for all j, and 0 elsewhere. Since, any entry sum of A_w over a chain in $[1, n]^2$ gives the cardinality of the subset of $[1, n]$ consisting of the column indices of the non-zero entries over the chain and the natural order of this subset is reversed by w, it follows that the number $\mathfrak{s}_k(A_w)$ is the maximal cardinality of a subset of $[1, n]$ which is a disjoint union of k subsets each of which has its natural order reversed by w. Thus, the partition $\sigma(A_w)$ associated to A_w as in 1.2 is the partition associated to w as defined in [Lu1, §6] which is actually based on the general result (1.1) of Greene. Therefore, by [loc. cit., (7.1)], we have the following.

1.5. COROLLARY. *The sets* $M(n)_\lambda \cap P(n)$ ($\lambda \vdash n$, *a partition of* n) *are all distinct two-sided Kazhdan-Lusztig cells of* $\mathfrak{S}_n$.

We shall generalize this result to the matrix semigroup $M(n)$.

2. Kazhdan-Lusztig cells of the matrix semi-group $M(n)$. We now recall from [Du2, 4.2] the definition of Kazhdan-Lusztig cells for $M(n)$. This definition via q-Schur algebras and their K-L bases (cf. [Du1]) is a natural generalization of the original one as given in [KL], and is equivalent to the one given in the introduction.

Fix a positive integer r, and let W denote the symmetric group $\mathfrak{S}_r$ on r letters. Let λ be a composition of r, namely,

$$\lambda = (\lambda_1, \ldots, \lambda_m)$$

with $\lambda_1 + \cdots + \lambda_m = r$ and $\lambda_i \geq 0$, for all i. The set of all compositions of r into n parts is denoted by $\Lambda(n, r)$. Let $\Lambda(n) = \bigcup_{r \geq 0} \Lambda(n, r)$. Note that $\Lambda^+(n)$ is a subset of $\Lambda(n)$.

Each $\lambda \in \Lambda(n)$ corresponds to a standard Young subgroup W_λ generated by basic transpositions. This correspondence is given as follows: We first partition the set $\{1, 2, 3, \ldots, r\}$ into subsets R_i^λ such that the subset R_1^λ contains the first λ_1 entries, R_2^λ the next λ_2 entries and so on. The Young subgroup is then the one generated by those transpositions which interchange numbers in the same subset.

For example, if $\lambda = (2, 1, 3)$, then the corresponding subsets are

$$R_1^\lambda = \{1, 2\}, \quad R_2^\lambda = \{3\}, \quad R_3^\lambda = \{4, 5, 6\}$$

and the Young subgroup is generated by

$$\{(12), (45), (56)\}.$$

It is well-known that we may identify the matrix set $M(n, r)$ with the set of double cosets $D(n, r)$ of W with respect to standard Young subgroups associated with compositions in $\Lambda(n, r)$. The bijection is defined as follow: if $D = W_\lambda w W_\mu$ is a double coset with $\lambda, \mu \in \Lambda(n, r)$, then $A_D = (a_{ij}) \in M(n, r)$ is the matrix satisfying $a_{ij} = |R_i^\lambda \cap w(R_j^\mu)|$ for all i, j. Note that a_{ij} is independent of the selection of the representative w of the double coset (cf. [JK, (1.3.10)]).

Let $\mathcal{S}_q(n, r)$ be the q-Schur algebra of degree (n, r) over the field $A = \mathbf{Q}(t)$ with $q = t^2$. It has the standard and K-L bases indexed by the double cosets in $D(n, r)$ (see, e.g., [Du1, 2.2]). Then, with the previous identification, these bases may be labelled by the matrix set $M(n, r)$. Let

$$\mathbf{B}(n, r) = \{\theta_A \mid A \in M(n, r)\} \tag{2.0}$$

denote the K-L basis of $\mathcal{S}_q(n, r)$ (cf. [Du1, 2.3]). Thus, we may define the K-L cells for $M(n, r)$, hence for $M(n)$, via these K-L bases as done in [KL], or equivalently, as in the introduction.

To do so, we first define preorders $\leq_L$, $\leq_R$, $\leq_{LR}$ on $M(n,r)$. We say $A\leq_L B$ if θ_A appears with nonzero coefficient in the product $\theta_C\theta_B$ for some $C\in M(n,r)$, and $A\leq_R B$ if $A^t\leq_L B^t$ where $(\ )^t$ is the transpose of a matrix. Let $\leq_{LR}$ be the preorder generated by $\leq_L$ and $\leq_R$, and let $\sim_L$, $\sim_R$, $\sim_{LR}$ be the associated equivalence relations on $M(n,r)$. We call the corresponding equivalence classes the *left*, *right* and *two-sided* K-L *cells* of $M(n,r)$ respectively. Thus, the matrix semigroup $M(n)$ are decomposed into these K-L cells.

The following theorem is a generalization of Corollary 1.5.

2.1. THEOREM. *The cells $M(n)_\lambda$ $(\lambda\in\Lambda^+(n))$ defined in* 1.3 *are all distinct two-sided K-L cells.*

The proof is divided into two steps. First, we reduce it in the rest of this section to a problem on symmetric groups. We then solve the symmetric group problem in the next section.

We need some notation first. For each $A=(a_{ij})\in M(n)$, let

$$\mathrm{ro}(A)=\left(\sum_i a_{1i},\ldots,\sum_i a_{ni}\right)\quad\text{and}\quad \mathrm{co}(A)=\left(\sum_i a_{i1},\ldots,\sum_i a_{in}\right).$$

Both ro and co define two maps from $M(n)$ to $\Lambda(n)$. Let w_A denote the longest element in the double coset $W_{\mathrm{ro}(A)}wW_{\mathrm{co}(A)}$ corresponding to A, and w_λ $(\lambda\in\Lambda(n))$ the longest element in the Young subgroup corresponding to λ. We denote by B_w $(w\in W)$ the K-L basis of the generic Hecke algebra associated to W. We shall use the same notation $\leq_L$, $\leq_R$, $\leq_{LR}$ and $\sim_L$, $\sim_R$, $\sim_{LR}$ to denote the preorders and the associated equivalence relations on W as defined in [KL].

We now have the following reduction.

2.2. LEMMA. *For any A, $B\in M(n)$ with $\mathfrak{s}(A)=\mathfrak{s}(B)$, we have*

$$A\sim_{LR}B\iff w_A\sim_{LR}w_B\,.$$

PROOF. Without loss of generality, we may assume $A,B\in M(n,r)$ for some $r\geq 0$.

We first claim that $A\leq_L B$ if and only if $\mathrm{co}(A)=\mathrm{co}(B)$ and $w_A\leq_L w_B$. Indeed, the "only if" part is obvious (see [Du1, 3.4]). Conversely, suppose $\mathrm{co}(A)=\mathrm{co}(B)$ and $w_A\leq_L w_B$. Then, there is a $w\in W$ such that B_{w_A} appears (with nonzero coefficient) in the product $B_wB_{w_B}$. This implies by the positivity property on the structure constants with respect to K-L bases that B_{w_A} appears in the product $B_{w_{\mathrm{ro}(A)}}B_wB_{w_{\mathrm{ro}(B)}}B_{w_B}$. However, $B_{w_{\mathrm{ro}(A)}}B_wB_{w_{\mathrm{ro}(B)}}$ is a linear combination of the elements B_{w_D} $(D\in W_{\mathrm{ro}(A)}\backslash W/W_{\mathrm{ro}(B)})$ where w_D is the longest element in D. So B_{w_A} appears in a product $B_{w_D}B_{w_B}$. It follows from [loc.cit.] and the hypothesis $\mathrm{co}(A)=\mathrm{co}(B)$ that there exists a matrix $C\in M(n,r)$ such that $w_C=w_D$, $\mathrm{ro}(C)=\mathrm{ro}(A)$ and $\mathrm{co}(C)=\mathrm{ro}(B)$ and such that θ_A appears (with nonzero coefficient) in the product $\theta_C\theta_B$. That is, we have $A\leq_L B$, and our claim is proved.

We now turn to the proof of the lemma. The "only if" part of our assertion is obviously true. Conversely, suppose $w_A \sim_{LR} w_B$. By [Lu3, 3.1(k)(l)], there exists $x \in W$ such that $w_A \sim_L x \sim_R w_B$. Thus, by [KL, 2.4], we have $x = w_C$ for some $C \in W_{\mathrm{ro}(B)} \backslash W / W_{\mathrm{co}(A)}$. Now the claim above implies that $A \sim_L C \sim_R B$ and therefore, $A \sim_{LR} B$. □

Note that one may use [Du2, 4.2(d)] to prove the above lemma. However, the previous proof gives a result on left cells.

2.3. COROLLARY. *For any $A, B \in M(n)$ with $\mathfrak{s}(A) = \mathfrak{s}(B)$, we have*

$$A \sim_L B \Longleftrightarrow \mathrm{co}(A) = \mathrm{co}(B) \quad \textit{and} \quad w_A \sim_L w_B .$$

We end this section with an example.

2.4. EXAMPLE. For $n=2$, Theorem 2.1 gives the following easy description for the two-sided K-L cells of the matrix semigroup $M(2)$: Two matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

in $M(2)$ are in the same two-sided cell if and only if $a+b+c+d = a'+b'+c'+d'$ and $\min(a, d) = \min(a', d')$. This can be seen easily from Lemma 2.2 and the fact that for any $A = (a_{ij}) \in M(2)$ we have $\sigma_2(A) = \min(a_{11}, a_{22})$.

3. Proof of Theorem 2.1. The rest part of the proof of Theorem 2.1 is based on an interesting description of the longest element in a double coset defined by a matrix $A \in M(n)$. This result was carried out by the collaboration of Tim Sturge, a vacation student of UNSW from Canterbury University, New Zealand.

Fix a matrix $A = (a_{ij}) \in M(n, r)$. We define numbers

$$b_{ij} = b_{ij}(A) = \sum_{k=1}^{i} \sum_{l=1}^{n} a_{kl} - \sum_{l=1}^{j} a_{il}$$

and decreasing sequences

$$c_{ij} = c_{ij}(A) = (b_{ij} + a_{ij}, \ldots, b_{ij} + 1) .$$

We combine these sequences into a sequence

$$(j_1, j_2, \ldots, j_r) = (c_{n1}, \ldots, c_{11}, c_{n2}, \ldots, c_{12}, \ldots, c_{nn}, \ldots, c_{1n}) .$$

This sequence defines a permutation y_A sending i to j_i. We write

$$y_A = (j_1, j_2, \ldots, j_r) .$$

We note that the c_{ij} can be arranged in a pseudo-matrix (c_{ij}) associated to A. Thus, the numbers $\{1, \ldots, r\}$ in (c_{ij}) are arranged so that they are in the natural order when read from right to left inside the sequence and from right to left along the rows, followed by top to bottom down the successive rows. Then the permutation y_A may be read off from this matrix by reading from left to right inside the sequences and from bottom to top up the columns, followed by left to right along the successive columns.

For example if the matrix (a_{ij}) is

$$\begin{pmatrix} 2 & 0 & 3 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix} \tag{3.1}$$

then the psuedo-matrix associated to A is

$$\begin{pmatrix} (5,4) & \varnothing & (3,2,1) \\ (8) & (7,6) & \varnothing \\ (12,11) & (10) & (9) \end{pmatrix}$$

which gives the permutation

$$y_A = (12, 11, 8, 5, 4, 10, 7, 6, 9, 3, 2, 1)\,.$$

Let l be the length function on the symmetric group W.

3.2. Lemma. *Maintain the notation above. If $A=(a_{ij}) \in M(n, r)$, then y_A is the longest element w_A of the double coset corresponding to A and*

$$l(w_A) = \binom{r}{2} - \sum_{\substack{1 \le i < k \le n \\ 1 \le j < l \le n}} a_{ij} a_{kl}\,.$$

Proof. We shall use the notation given at the beginning of Sections 1 and 2. For simplicity, we denote by N the number given by the summation above. That is,

$$N = \sum_{\substack{1 \le i < k \le n \\ 1 \le j < l \le n}} a_{ij} a_{kl}\,.$$

Note that the subset $\{(i,j),(k,l)\}$, where $1 \le i < k \le n$, $1 \le j < l \le n$, is an antichain and the sum is taken over all antichains of length 2 in $[1, n]^2$.

Let w be an element in the double coset $W_\lambda w_A W_\mu$ corresponding to A. Thus, $\lambda = \mathrm{ro}(A)$ and $\mu = \mathrm{co}(A)$. For any i, j, k, l such that $1 \le i < k \le n$ and $1 \le j < l \le n$, we pick $c \in R_i^\lambda \cap w(R_j^\mu)$ and $d \in R_k^\lambda \cap w(R_l^\mu)$. Clearly, there are N ways of making this selection. Let $a = w^{-1}(c)$, $b = w^{-1}(d)$. Then $i<k$ and $j<l$ imply $w(a)<w(b)$ and $a<b$, respectively. So there are at least N pairs (a, b) satisfying $1 \le a < b \le r$ and $w(a) < w(b)$. In other words, we have

$$N \le |\{a, b) \mid 1 \le a < b \le r,\ w(a) < w(b)\}|\,. \tag{3.3}$$

On the other hand, it is well known that

$$\binom{r}{2} = |\{(a, b) \mid 1 \le a < b \le r\}|$$
$$= |\{(a, b) \mid a < b, w(a) > w(b)\}| + |\{(a, b) \mid a < b, w(a) < w(b)\}|$$
$$\ge l(w) + N .$$

Therefore, we have $l(w) \le \binom{n}{2} - N$, and in particular, $l(w_A) \le \binom{n}{2} - N$. By the uniqueness of the longest element of a double coset, we only need to prove that $y_A \in W_\lambda w_A W_\mu$ and $l(y_A) = \binom{n}{2} - N$.

Let $C_{ij} = C_{ij}(A)$ denote the set of members of the sequence c_{ij}. We note from the definition of y_A that

$$\bigcup_{1 \le l \le n} C_{il} = R_i^\lambda \quad \text{and} \quad \bigcup_{1 \le k \le n} C_{kj} = y_A(R_j^\mu) .$$

Thus, we have $C_{ij} = R_i^\lambda \cap y_A(R_j^\mu)$, and hence $a_{ij} = |R_i^\lambda \cap y_A(R_j^\mu)|$, which proves that $y_A \in W_\lambda w_A W_\mu$ by [JK, 1.3.10]. Note that $y_A^{-1}(C_{ij}(A)) = C_{ji}(A^t)$.

We now prove that y_A has the required length. Consider a, b such that $1 \le a < b \le r$ and $y_A(a) < y_A(b)$. There exist i, j such that $y_A(a) \in C_{ij}$ and k, l such that $y_A(b) \in C_{kl}$. Note that $a \in C_{ji}(A^t)$ and $b \in C_{lk}(A^t)$. As $a < b$, we have either $j < l$ or $j = l$ and $i > k$ while as $y_A(a) < y_A(b)$ we have either $i < k$ or $i = k$ and $j > l$. These together imply $i < k$ and $j < l$. Thus $|\{(a, b) \mid a < b, y_A(a) < y_A(b)\}| \le N$. However $|\{(a, b) \mid a < b, y_A(a) < y_A(b)\}| \ge N$ by (3.3). Therefore, we have $|\{(a, b) \mid a < b, y_A(a) < y_A(b)\}| = N$. □

The shortest element w_A^- of the double coset corresponding to A may be constructed similarly. Given $A = (a_{ij})$ as before, we define numbers

$$b_{ij}^- = b_{ij}^-(A) = \sum_{k=1}^{i} \sum_{l=1}^{n} a_{kl} - \sum_{l=j}^{n} a_{il}$$

and increasing sequences

$$c_{ij}^- = c_{ij}^-(A) = (b_{ij} + 1, \ldots, b_{ij} + a_{ij}) .$$

We combine these sequences into a permutation

$$y_A^- = (c_{11}^-, \ldots, c_{n1}^-, c_{12}^-, \ldots, c_{n2}^-, \ldots, c_{1n}^-, \ldots, c_{nn}^-) .$$

Note that the element y_A^- could be easily read off from the pseudo-matrix (c_{ij}^-) by reading from left to right inside the sequences and down the successive columns.

For example if the matrix (a_{ij}) is given as in (3.1), then the associated pseudo-matrix for the shortest element is

$$\begin{pmatrix} (1,2) & \varnothing & (3,4,5) \\ (6) & (7,8) & \varnothing \\ (9,10) & (11) & (12) \end{pmatrix}.$$

Reading off from this matrix, we obtain

$$y_A^- = (1, 2, 6, 9, 10, 7, 8, 11, 3, 4, 5, 12).$$

Similarly to Lemma 3.2, we have the following.

3.4. LEMMA. *Maintain the notation above. If* $A = (a_{ij}) \in M(n, r)$, *then* y_A^- *is the shortest element* w_A^- *of the double coset corresponding to* A *and*

$$l(w_A^-) = \sum_{\substack{1 \le i < k \le n \\ 1 \le l < j \le n}} a_{ij} a_{kl}.$$

The proof of the lemma is similar to that of Lemma 3.2. We leave this to the reader.

3.5. PROOF OF THEOREM 2.1. By Lemma 3.2, it is easy to see that, given a matrix $A \in M(n)$ and a chain C in $[1, n]^2$, the subset of $[1, r]$ consisting of the numbers whose images under w_A are in the sets $C_{ij}(A)$ where $(i, j) \in C$ has the cardinality $\mathfrak{s}_C(A)$ and the natural order reversed by w_A. This implies that the number $\mathfrak{s}_k(A)$ defined in Section 1 is the maximal cardinality of a subset of $[1, r]$ which is a disjoint union of k subsets each of which has its natural order reversed by w_A (compare the proof of (1.2)). Therefore, by [Lu1, 7.1], we have $\sigma(A) = \sigma(B)$ (cf. Lemma 1.2) if and only if $w_A \sim_{LR} w_B$. Now, our result follows immediately from Lemma 2.2. □

3.6. COROLLARY. *Fro any* A, $B \in M(n)$ *and integer* $l \ge 0$, *we have*

$$A \sim_{LR} B \iff A + lI_n \sim_{LR} B + lI_n.$$

PROOF. Clearly, it suffices to prove the case where $l = 1$.

Let F be a k-chain family satisfying $\mathfrak{s}_k(A) = \mathfrak{s}_F(A)$. We may extend the chains in F to obtain a k-chain family F_1 such that $F \subseteq F_1$ and $|F_1 \cap \Delta| = k$, where Δ is the diagonal $\{(i, i) \mid i \in [1, n]\}$ of $[1, n]^2$. For example, if $k = 1$, we may take F_1 to be a chain of length $2n - 1$ starting with $(n, 1)$ and ending at $(1, n)$. So we may assume $F = F_1$ without loss of generality. Thus we have $\mathfrak{s}_k(A + I_n) \ge \mathfrak{s}_F(A + I_n) = \mathfrak{s}_k(A) + k$.

On the other hand, if F' is a k-chain family satisfying $\mathfrak{s}_k(A + I_n) = \mathfrak{s}_{F'}(A + I_n)$, then $\mathfrak{s}_k(A + I_n) \le \mathfrak{s}_{F'}(A) + k \le \mathfrak{s}_k(A) + k$. Therefore, we have $\mathfrak{s}_k(A + I_n) = \mathfrak{s}_k(A) + k$ for any A. Now the assertion follows immediately from 2.1 and 1.2. □

Let $M(n)_0 = \{A = (a_{ij}) \in M(n) \mid a_{ii} = 0 \text{ for some } i\}$.

3.7. COROLLARY. *The matrix set* $M(n)_0$ *is a union of two-sided K-L cells.*

PROOF. Since

$$M(n)_0=(\bigcup_{r=0}^{n-1} M(n,r))\cup(\bigcup_{r\geq n} M(n,r)\setminus(M(n,r-n)+I_n))\,,$$

where $M(n,r-n)+I_n$ denotes the set consisting of $A+I_n$ ($A\in M(n,r-n)$), and $M(n,r-n)+I_n$, is a union of two-sided K-L cells by Corollary 3.6, our assertion follows immediately. $\square$

4. An application to quantum groups of type A. Let $\boldsymbol{U}$ be the Drinfeld-Jimbo quantized enveloping algebra over the field $\boldsymbol{A}=\boldsymbol{Q}(t)$ associated with the special linear Lie algebra $\mathfrak{sl}_n$ and $\dot{\boldsymbol{U}}$ the modified form of $\boldsymbol{U}$ (see [Lu2, chpt. 23]). It is well-known that, for each $r\geq 0$, there is an algebra homomorphism from $\boldsymbol{U}$ to the q-Schur algebra $\mathscr{S}_q(n,r)$. This induces a homomorphism

$$\alpha_r\colon \dot{\boldsymbol{U}}\longrightarrow \mathscr{S}_q(n,r)$$

with the following property: If $\dot{\boldsymbol{B}}$ denotes the canonical basis of $\dot{\boldsymbol{U}}$, then α_r sends an element in $\dot{\boldsymbol{B}}$ either to a K-L basis element in $\boldsymbol{B}(n,r)$ (2.0) of the q-Schur algebra or to zero (cf. [Gr, 3.5]). We label the elements of $\dot{\boldsymbol{B}}$ by the index set $M(n)_0$:

$$\dot{\boldsymbol{B}}=\{\Theta_A\mid A\in M(n)_0\}$$

such that the labeling is compatible with that of K-L basis of $\mathscr{S}_q(n,r)$, that is, if $\alpha_r(\Theta_B)=\theta_A$ then $A=B$. Such a labelling can also be easily obtained via the canonical basis of the corresponding quantum coordinate algebra (see [Lu2, 29.5] and [Du3, 3.6]).

We now can describe the two-sided cells of $\dot{\boldsymbol{B}}$ in terms of the combinatorics in Section 1.

4.1. THEOREM. *Let $\sim_{LR}$ be the equivalence relation on $\dot{\boldsymbol{B}}$ introduced in the introduction. Then we have $\Theta_A\sim_{LR}\Theta_B$ ($A, B\in M(n)_0$) if and only if $\sigma(A)=\sigma(B)$.*

PROOF. If $\Theta_A\sim_{LR}\Theta_B$, then by [Lu2, 29.4.1-2] Θ_A, $\Theta_B\in\dot{\boldsymbol{B}}[\lambda]$ for some dominant weight λ. (The definition of $\dot{\boldsymbol{B}}[\lambda]$ is in [loc. cit., 29.1.1].) However, the image of $\dot{\boldsymbol{B}}[\lambda]$ under some α_r is a two-sided cell of the K-L basis $\boldsymbol{B}(n,r)$ of $\mathscr{S}_q(n,r)$. This is seen from the definition of $\dot{\boldsymbol{B}}[\lambda]$ and the facts that the basis $\boldsymbol{B}(n,r)$ can also be defined by the canonical bases of those tensor products which define $\dot{\boldsymbol{B}}$ and that the action of $\dot{\boldsymbol{U}}$ on the tensor products factors through α_r. (One can actually define $\boldsymbol{B}(n,r)[\lambda]$ similarly.) Therefore, $\Theta_A\sim_{LR}\Theta_B$ if and only if $\theta_A\sim_{LR}\theta_B$, i.e., $A\sim_{LR}B$ and the result follows from Theorem 2.1. $\square$

We end the paper with an example (compare Example 2.4).

4.2. EXAMPLE. Let $\boldsymbol{U}=\boldsymbol{U}(\mathfrak{sl}_2)$ and E, F, K, K^{-1} the generators. With the notation in [Lu2, 25.3.1] we have

$$\dot{\boldsymbol{B}}=\{E^{(a)}1_{-n}F^{(b)}, F^{(b)}1_nE^{(a)}\mid a, b, n\in\boldsymbol{N}, n\geq a+b\}\,.$$

Note that

$$(*) \qquad E^{(a)}1_{-n}F^{(b)}=F^{(b)}1_nE^{(a)} \qquad \text{for} \quad n=a+b\,.$$

We define

$$\Theta_A=\begin{cases} E^{(u)}1_{-\mathfrak{s}(A)}F^{(v)}\,, & \text{if } y=0\,,\\ F^{(v)}1_{\mathfrak{s}(A)}E^{(u)}\,, & \text{if } x=0\,,\end{cases} \qquad \text{for any} \quad A=\begin{pmatrix} x & u\\ v & y\end{pmatrix}\in M(2)_0\,.$$

The element Θ_A is well-defined by $(*)$ when $x=y=0$. So $\dot{\boldsymbol{B}}=\{\Theta_A \mid A\in M(2)_0\}$. The two-sided cells $\dot{\boldsymbol{B}}[n]$ of $\dot{\boldsymbol{B}}$ corresponds bijectively to the elements n in $\boldsymbol{N}$. By the previous theorem we have

$$\begin{aligned}\dot{\boldsymbol{B}}[n]&=\{\Theta_A \mid \sigma(A)=n\}\\ &=\{E^{(a)}1_{-n}F^{(b)} \mid n\geq a+b\}\cup\{F^{(b)}1_nE^{(a)} \mid n\geq a+b\}\,.\end{aligned}$$

The results agree with those given in [Lu2, 29.4.3] which are obtained in a different way.

References

[Du1] Jie Du, Kazhdan-Lusztig bases and isomorphism theorems for q-Schur algebras, Contemp. Math. 139 (1992), 121–140.

[Du2] Jie Du, q-Schur algebras, asymptotic forms and quantum SL_n, J. Algebra 177 (1995), 385–408.

[Du3] Jie Du, Global IC basis for quantum linear groups, J. Pure Appl. Algebra (to appear).

[G] C. Greene, Some partitions associated with a partially ordered set, J. Combinatorial Theory A 20 (1979), 69–79.

[Gr] I. Grojnowski, The coproduct for quantum GL_n, preprint, IAS.

[JK] G. James and A. Kerber, The representation theory of the symmetric group, Encyclopedia of Math., vol. 16, Addison-Wesley, Mass, 1981.

[KL] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), 165–184.

[Lu1] G. Lusztig, The two-sided cells of the affine Weyl group of type $\tilde{A}_n$, MSRI Publ. (Berkeley, California) 4 (1985), 275–283.

[Lu2] G. Lusztig, Introduction to quantum groups, Birkhäuser, Boston (1993).

[Lu3] G. Lusztig, Leading coefficients of character values of Hecke algebras, Proc. Symp. Pure Math. 47 (1987), 235–262.

[Sh] J. Shi, The Kazhdan-Lusztig cells in certain affine Weyl groups, Lecture Notes in Math. 1179 (1986), Springer.

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