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PERIODIC SOLUTIONS OF DISSIPATIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. We consider periodic, infinite delay differential equations. We investigate dissipativeness for these equations. Massat proved that dissipative, periodic, infinite delay equations have a periodic solution. For our purpose we need a weaker dissipativeness, so we prove Massat's theorem from this weak dissipativeness in an elementary way. Then we extend a theorem of Pliss giving a necessary and sufficient condition for this weak dissipativeness. We also present a theorem using Liapunov functionals to show the weak dissipativeness and hence the existence of a periodic solution.

1. Introduction. Let $f: \boldsymbol{R}\times\boldsymbol{R}^d\to\boldsymbol{R}^d$ be continuous and locally Lipschitz in x with $f(t+T, x)=f(t, x)$ for all (t, x) and some $T>0$. We say that the ordinary differential equation

$$x'=f(t, x) \tag{1}$$

is dissipative, if all solutions become bounded by a fixed constant at some time and remain bounded from that time on. Pliss [9, Theorem 2.1] showed that the ordinary differential equation is dissipative if and only if there is an $r>0$ such that for each (t_0, x_0) there is a $\tau>t_0$ with $|x(\tau, t_0, x_0)|<r$. The author [7] generalized this result for finite delay differential equations stating that dissipativeness is equivalent to every solution becoming bounded by a fixed constant for an interval of length $2h$, where h is the retardation. The author also gave an elementary proof for a result of Hale and Lopes [3], who proved that dissipativeness implies the existence of a periodic solution for finite delay equations. The following Lyapunov-type theorem, which can also be found in [7], proves the existence of a periodic solution through dissipativeness.

THEOREM A. *Suppose there are a functional $V: \boldsymbol{R}\times\mathscr{C}\to\boldsymbol{R}$ and constants $a, b, M, U>0$ such that*

(i) $0\le V(t, \phi)$,

(ii) $V'(t, x_t)\le M$ *and*

(iii) $V'(t, x_t)\le -a|x'(t)|-b$ *for* $|x(t)|\ge U$.

Then the solutions of the finite delay differential equation are dissipative.

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Stronger versions of the conditions asked in this theorem are used in the literature to prove uniform boundedness and uniform ultimate boundedness (see e.g. Theorem 4.2.11 in [1]). There naturally arises the question if these theorems can be generalized for infinite delay. By looking at Theorem A nothing seems to keep this theorem from being applicable to the infinite delay case. Also the theorem of Hale and Lopes can be stated for infinite delay, Massat [8] proved using an axiomatic approach that dissipativeness is enough to have a periodic solution for infinite delay differential equations. Hino and Murakami [4] also used axiomatic setup of the phase space to investigate infinite delay equations. Kato [6] summarized many boundedness-type properties and their connections, which are related to dissipativeness. For a good summary of the recent results concerning dissipativeness see Hale's book [2].

In this paper we consider an equation not satisfying some of the axioms used in the above mentioned papers. We will see that this paper is parallel to [7] although the properties used in this paper are different from the obvious generalizations of that paper. We prove the existence of a periodic solution from a property called weak dissipativity, which is weaker than dissipativity. With this weak dissipativity we prove a generalization of Pliss' Theorem and then use this result to get an exact counterpart of Theorem A.

2. Main results. We now introduce a functional differential equation with infinite delay. Let $(\mathscr{C}, \|\cdot\|_g)$ be the Banach space of continuous functions $\phi\colon(-\infty,0]\to\mathbf{R}^d$ with the so-called g-norm defined by

$$\|\phi\|_g := \sup_{s\le 0}\frac{|\phi(s)|}{g(s)},$$

where $g\colon(-\infty,0]\to[1,\infty)$ is a continuous, decreasing function with $g(0)=1$ and $g(s)\to\infty$ as $s\to-\infty$. If we talk about boundedness or compactness in the following we always mean it in the g-norm. Denote $x_t(s)=x(t+s)$ for $s\le 0$ and let $F\colon \mathbf{R}\times\mathscr{C}\to\mathbf{R}^d$ be continuous and locally Lipschitz in ϕ in the g-norm with $F(t+T,\phi)=F(t,\phi)$. Then

$$x'=F(t,x_t) \tag{2}$$

is a system of functional differential equations and for each $(t_0,\phi)\in\mathbf{R}\times\mathscr{C}$ there is a unique solution $x(t,t_0,\phi)$ which depends continuously on the initial data. The local Lipschitz condition together with the periodicity of F clearly implies that F takes bounded sets of ϕ into bounded sets. Let us denote by $L(M)$ the bound for F when ϕ is bounded by M and assume that L is a strictly increasing function of M.

We asked g to be decreasing in order to simplify our proofs. If one has a not decreasing $g(s)$, one can replace it by $\inf_{u\le s} g(u)$, which is decreasing, smaller than $g(s)$, and hence the conditions on F can be more easily satisfied.

We now show how to use the dissipativeness of the solutions in proving the existence of a periodic solution. We need a few quite technical lemmas, which we will use in the following.

LEMMA 1. *The set*

$$\mathcal{S}(R) := \{\psi \in \mathscr{C} : \text{there is an } S>0 \text{ such that } |\psi(s)| \le \sqrt{g(s)} \text{ for all } s \le -S,\ |\psi(u)-\psi(v)| \le L(\sqrt{g(\min\{u, v\})})\,|u-v| \text{ for } u, v \le -S \text{ and } |\psi(s)| \le R,\ |\psi(u)-\psi(v)| \le L(R)\,|u-v| \text{ for } s, u, v \in [-S, 0]\}$$

is compact in the g-norm.

PROOF. Let $\psi_n \in \mathcal{S}(R)$ be an arbitrary sequence and let S_n denote the constant used for ψ_n. We have two cases:

Case 1. If S_n is bounded above, then take a subsequence, say S_n again, such that $S_n \to S$. Using Ascoli's Theorem, ψ_n has a subsequence, ψ_n again, which converges in the supremum norm on the interval $[-S, 0]$. Now take an interval $[-Q, -S]$. On this interval ψ_n is bounded by $\sqrt{g(-Q)}$ and satisfies a Lipschitz condition with $L(\sqrt{g(-Q)})$ and hence we can apply Ascoli's theorem to prove that ψ_n has a subsequence, which converges in the supremum norm on the interval $[-Q, -S]$. Using this result for $-Q := -S-m$ $(m \to \infty)$ and applying the diagonal method we can find a subsequence of ψ_n, say ψ_n again, which converges to a function ψ uniformly on any finite interval. Now we estimate the g-norm:

$$\|\psi_n - \psi\|_g = \max\left\{ \sup_{s \le -Q} \frac{|\psi_n(s)-\psi(s)|}{g(s)},\ \sup_{s\in[-Q,0]} \frac{|\psi_n - \psi|}{g(s)} \right\}$$

$$\le \max\left\{ \sup_{s \le -Q} \frac{2\sqrt{g(s)}}{g(s)},\ \sup_{s\in[-Q,0]} \frac{|\psi_n - \psi|}{g(s)} \right\}.$$

We can make the first argument small by taking Q large enough and using that $g(s) \to \infty$ as $s \to -\infty$. Also, the second term is small for large enough n, since ψ_n converges uniformly to ψ on the interval $[-Q, 0]$. Hence, ψ_n converges to ψ in the g-norm.

Case 2. If S_n has a subsequence, say S_n again, so that $S_n \to \infty$, then using the usual diagonal method and Ascoli's Theorem we can find a subsequence of ψ_n (again ψ_n) so that it converges to a function ψ uniformly in the supremum norm on any finite interval, where ψ is bounded by R. We now estimate the g-norm:

$$\|\psi^n - \psi\|_g = \max\left\{ \sup_{s \le -S_n} \frac{|\psi^n(s)-\psi(s)|}{g(s)},\ \sup_{s\in[-S_n,0]} \frac{|\psi^n(s) - \psi(s)|}{g(s)} \right\}$$

$$\le \max\left\{ \sup_{s \le -S_n} \frac{\sqrt{g(s)}+R}{g(s)},\ \sup_{s\in[-S_n,0]} \frac{|\psi^n(s) - \psi(s)|}{g(s)} \right\}.$$

Since $g(s) \to \infty$ as $s \to -\infty$ and since $S_n \to \infty$ as $n \to \infty$, the first term tends to 0. Next,

$$\sup_{s\in[-S_n,0]}\frac{|\psi^n(s)-\psi(s)|}{g(s)}\le\max\left\{\sup_{s\in[-S_n,-P]}\frac{2R}{g(s)},\ \sup_{s\in[-P,0]}\frac{|\psi^n(s)-\psi(s)|}{g(s)}\right\}$$

is also small if we choose P independently of n but large enough to make the first term small, and then if n is large enough, the second term will also become small. This proves that ψ^n converges to ψ in the g-norm.

Note that $\|\phi\|_g\le R$ for all $\phi\in\mathcal{S}(R)$, if $R\ge 1$.

LEMMA 2. *Let $R, r>1$, Q constants be given and consider any function x: $(-\infty,\infty)\to\mathbf{R}^d$ with $x_Q\in\mathcal{S}(R)$ and $|x(s)|<r$ for $s\ge Q$. Then we can find an $H\ge Q$ independent of x so that $\|x_t\|_g<r$ for all $t\ge H$.*

PROOF. We estimate $\|x_t\|_g$ in three parts. As $x_Q\in\mathcal{S}(R)$ there is an $S>0$ such that $|x_Q(s)|\le\sqrt{g(s)}$ for $s\le -S$. Then

$$\|x_t\|_g=\max\left\{\sup_{s\le Q-S}\frac{|x(s)|}{g(s-t)},\ \sup_{s\in[Q-S,Q]}\frac{|x(s)|}{g(s-t)},\ \sup_{s\in[Q,t]}\frac{|x(s)|}{g(s-t)}\right\}$$

$$\le\max\left\{\sup_{s\le -S}\frac{\sqrt{g(s)}}{g(s-t+Q)},\ \sup_{s\in[Q-S-t,Q-t]}\frac{R}{g(s)},\ \sup_{s\in[Q,t]}\frac{|x(s)|}{g(s-t)}\right\}$$

$$\le\max\left\{\sup_{s\le -S}\frac{1}{\sqrt{g(s)}},\ \sup_{s\le Q-H}\frac{R}{g(s)},\ \sup_{s\in[Q,t]}\frac{|x(s)|}{g(s-t)}\right\}.$$

The first term is less than $1/\sqrt{g(s)}\le 1<r$ by our assumptions. Since we can have $g(s)>R/r$ for $s\le Q-H$ for large H, we can also make the second expression smaller than r. The third argument is clearly smaller than r from $|x(s)|<r$. Hence we proved that for any large enough t we have $\|x_t\|_g<r$ independent of the x chosen from the given set.

DEFINITION. Equation (2) is *weakly dissipative* (at $t=0$), if there is an $r>0$ such that

$$\limsup_{t\to\infty}|x(t,0,\phi)|<r$$

for all $\phi\in\mathcal{S}(R)$, where $R>0$ is arbitrary. For technical reasons we always assume in the following that $r>1$.

LEMMA 3. *If (2) is weakly dissipative with r, and we start a solution from $\phi\in\mathcal{S}(R)$ (for some $R>0$) then there is an $S>0$ such that $x_t(\cdot,0,\phi)\in\mathcal{S}(r)$ for all $t\ge S$.*

PROOF. Let $x(s):=x(s,0,\phi)$. Using the dissipativity we can find $Q>0$ and $M>R$ such that $|x(s)|<M$ for $s\in[0,Q]$ and $|x(s)|<r$ for $s\ge Q$. Observe that $x_Q\in\mathcal{S}(M)$, and hence from Lemma 2 we find an $H>Q$ such that $\|x_s\|_g<r$ for $s\ge H$. Now let $\bar{H}>0$ be large enough, so that $g(s)>M^2$ for $s\le -\bar{H}$, and let $S:=H+\bar{H}$. Then x_t is in $\mathcal{S}(r)$ by construction for all $t\ge S$. Note that we proved that x_t is in some sense in the inside of

$\mathscr{S}(r)$. This means that if we start a solution from a function $\psi \in \mathscr{S}(R)$ close to ϕ and fix a $t \geq S$, then using the continuous dependence of the solution on the initial data we also have $x_t(\cdot, 0, \psi) \in \mathscr{S}(r)$. We will use this remark in the following.

DEFINITION. Equation (2) is *weakly uniformly bounded* (at $t=0$), if for every $R>0$ there is a $B>0$ such that $|x(t, 0, \phi)|<B$ for all $\phi \in \mathscr{S}(R)$ and $t \geq 0$.

DEFINITION. Equation (2) is *weakly uniformly dissipative* (at $t=0$), if there is an $r>0$ such that for every $R>0$ there is a $P>0$ so that $|x(t, 0, \phi)|<r$ if $\phi \in \mathscr{S}(R)$ and $t \geq P$.

THEOREM 4. *If* (2) *is weakly dissipative, then it is weakly uniformly bounded.*

PROOF. Let $r>0$ be the number in the definition of weak dissipativeness. Suppose that the statement of the theorem does not hold. Then we find $R>r$ and sequences $\phi_n \in \mathscr{S}(R)$ and $t_n \geq 0$ such that $|x(t_n, 0, \phi_n)| \to \infty$. We assume that n is so large that $x_{t_n}(\cdot, 0, \phi_n)$ is not in $\mathscr{S}(R)$, because $\mathscr{S}(R)$ is bounded and $|x(t_n, 0, \phi_n)| \to \infty$. Since $\phi_n = x_0(\cdot, 0, \phi_n) \in \mathscr{S}(R)$, we can define τ_n so that $\psi_n := x_{\tau_n}(\cdot, 0, \phi_n) \in \mathscr{S}(R)$, but $x_t(\cdot, 0, \phi_n) \notin \mathscr{S}(R)$ for $t \in (\tau_n, t_n]$. Using a translation argument we find a $\bar{\tau}_n \in [0, T]$ such that $x(t, \bar{\tau}_n, \psi_n) = x(t+(\tau_n - \bar{\tau}_n), \tau_n, \psi_n) = x(t+(\tau_n - \bar{\tau}_n), 0, \phi_n)$. Since $\mathscr{S}(R)$ and $[0, T]$ are compact, there are subsequences, say ψ_n and $\bar{\tau}_n$ again, such that $\psi_n \to \psi \in \mathscr{S}(R)$ and $\bar{\tau}_n \to \bar{\tau} \in [0, T]$. Using Lemma 3 for this ψ we find a $t \geq \bar{\tau}$ such that $x_t(\cdot, \bar{\tau}, \psi) \in \mathscr{S}(r)$. Let $M := \sup_{x \in [\bar{\tau}, t]} |x(s, \bar{\tau}, \psi)|$. Now take any n large enough to have $|x(t_n, 0, \phi_n)| = |x(t_n - (\tau_n - \bar{\tau}_n), \bar{\tau}_n, \psi_n)| \geq M+1$, $|x(s, \bar{\tau}_n, \psi_n)| < M+1$ for $s \in [\bar{\tau}_n, t]$ and $x_t(\cdot, \bar{\tau}_n, \psi_n) \in \mathscr{S}(r) \subset \mathscr{S}(R)$ using the remark in Lemma 3. Then we must have $t_n - (\tau_n - \bar{\tau}_n) > t$, but this is a contradiction to the choice of τ_n, because we must have $x_t(\cdot, \bar{\tau}_n, \psi_n) \notin \mathscr{S}(R)$ for $t \in (\bar{\tau}_n, t_n - (\tau_n - \bar{\tau}_n)]$. This contradiction shows the required weak uniform boundedness.

THEOREM 5. *If* (2) *is weakly dissipative, then it is weakly uniformly dissipative.*

PROOF. Let $r>0$ be the number in the definition of the dissipativeness and $\bar{r}>0$ be the number of weak uniform boundedness for $\mathscr{S}(r)$ from the previous theorem. We claim that (2) is weakly uniformly dissipative with $\bar{r}$. Suppose for contradiction that there is an $R>0$ and sequences $\phi_n \in \mathscr{S}(R)$ and $t_n \to \infty$ such that $|x(t_n, 0, \phi_n)| \geq \bar{r}$. As $\mathscr{S}(R)$ is compact, there is a subsequence of ϕ_n, say ϕ_n again, such that $\phi_n \to \phi \in \mathscr{S}(R)$. For this ϕ we find a $m \geq 0$ such that $x_{mT}(\cdot, 0, \phi) \in \mathscr{S}(r)$ using Lemma 3. Now take n large enough to have $x_{mT}(\cdot, 0, \phi_n) \in \mathscr{S}(r)$ and $t_n > mT$. Using the weak uniform boundedness we find that $|x(s, 0, \phi_n)| = |x(s, mT, x_{mT}(\cdot, 0, \phi_n))| = |x(s-mT, 0, x_{mT}(\cdot, 0, \phi_n))| < \bar{r}$ for all $s \geq mT$, which is a contradiction to $t_n > mT$ and $|x(t_n, 0, \phi_n)| \geq \bar{r}$. This contradiction shows the required weak uniform dissipativeness.

THEOREM 6. *If* (2) *is weakly dissipative with* $r>0$ (*and hence weakly uniformly dissipative with* $\bar{r}>0$ *from the previous theorem*), *then for all* $R>0$ *there is a* $P>0$ *such that* $x_t(\cdot, 0, \phi) \in \mathscr{S}(\bar{r})$ *for all* $\phi \in \mathscr{S}(R)$ *and* $t \geq P$.

PROOF. This theorem states a stronger version of the weak uniform dissipativeness we will use in the following theorem. From Theorem 4 we have a constant $M > \max\{\bar{r}, R\}$ such that $|x(t, 0, \phi)| \le M$ for all $\phi \in \mathscr{S}(R)$ and $t \ge 0$. We also have a $\bar{P}$ such that $|x(t, 0, \phi)| < \bar{r}$ for all $\phi \in \mathscr{S}(R)$ and $t \ge \bar{P}$. Using Lemma 2 (since $x_{\bar{P}}(\cdot, 0, \phi) \in \mathscr{S}(M)$ for all $\phi \in \mathscr{S}(R)$) take $H > \bar{P}$ large enough to have $\|x_t(\cdot, 0, \phi)\|_g < \bar{r}$ for all $\phi \in \mathscr{S}(R)$ and $t \ge H$. Taking $\bar{H} > 0$ so large that $g(s) \ge M^2$ for $s \le -\bar{H}$ holds and defining $P := H + \bar{H}$ we have $x_t(\cdot, 0, \phi) \in \mathscr{S}(\bar{r})$ for all $\phi \in \mathscr{S}(R)$ and $t \ge P$, which was to be proven.

THEOREM 7. *If* (2) *is weakly dissipative, then it has a T-periodic solution.*

PROOF. From Theorems 4 and 5 we know that (2) is weakly uniformly bounded and weakly uniformly dissipative with $\bar{r} > 0$. Since the proof of this theorem is very similar to the usual proof of the existence of a T-periodic solution assuming uniform boundedness and uniform ultimate boundedness (see [1, Theorem 4.2.2]), we will give only a sketch of the proof. Let $S_0 := \mathscr{S}(\bar{r})$ and define $P: \mathscr{C} \to \mathscr{C}$ by $P\phi := x_T(\cdot, 0, \phi)$. From the weak uniform boundedness we find $B_1 > \bar{r}$ such that $|x(t, 0, \phi)| < B_1$ for $t \ge 0$ and $\phi \in S_0$. Let $S_1 := \mathscr{S}(B_1)$. Once again using the weak uniform boundedness we define $B_2 > B_1$ such that if $S_2 := \mathscr{S}(B_2)$, then $P^n(S_1) \subset S_2$ for all $n \ge 0$. Also, from Theorem 6 we find an $m > 0$ such that $P^n(S_1) \subset S_0$ for $n \ge m$. Now all the conditions of Horn's fixed-point theorem (see [5] or [1, Section 3.4]) are satisfied, and hence there is a fixed point of P, which is (of course) a T-periodic solution of (2). The proof is complete.

Now we generalize a theorem of Pliss for this infinite delay case.

THEOREM 8. *Equation* (2) *is weakly dissipative if and only if there exists an* $r > 0$ *such that for all* $\phi \in \mathscr{S}(R)$ $(R > 0)$ *there is a* $\tau > 0$ *such that* $x_\tau(\cdot, 0, \phi) \in \mathscr{S}(r)$.

PROOF. The implication follows from Lemma 3. To prove the opposite direction, suppose for contradiction that equation (2) is not weakly dissipative, i.e. there is a sequence $\phi_n \in \mathscr{S}(R)$ such that $\limsup_{t \to \infty} |x(t, 0, \phi_n)| > r_n$, where $r_n \to \infty$. By our assumption, take $s_n > 0$ so that $x_{s_n}(\cdot, 0, \phi_n) \in \mathscr{S}(r)$. Let $t_n > s_n$ be any number with $|x(t_n, 0, \phi_n)| > r_n$ and assume that n is large enough to have $r_n > r$. Let $\tau_n \in [s_n, t_n)$ be a number with $\psi_n := x_{\tau_n}(\cdot, 0, \phi_n) \in \mathscr{S}(r)$ and $x_t(\cdot, 0, \phi_n) \notin \mathscr{S}(r)$ for $t \in (\tau_n, t_n]$. The proof from here on is the same as that of Theorem 4; we use the translation argument, take convergent subsequences of $\bar{\tau}_n$ and ψ_n and get a contradiction. This proves the weak dissipativity of equation (2).

THEOREM 9. *Suppose there are a functional* $V: \mathbf{R} \times \mathscr{C} \to \mathbf{R}$ *and constants* $a, b, M, U > 0$ *such that*

(i) $0 \le V(t, \phi)$,

(ii) $V'(t, x_t) \le M$ *and*

(iii) $V'(t, x_t) \le -a|x'(t)| - b$ *for* $|x(t)| \ge U$.

Assume also, that $g(s) \ge c^2 s^4$ *for all* $s \le 0$ *and some* $c > 0$. *Then the solutions of* (2) *are weakly dissipative.*

PROOF. By our previous theorem we need only to prove that there is an $r>0$ such that for every $\phi\in\mathscr{S}(R)$ there is a $\tau\geq 0$ with $x_\tau(\cdot, 0, \phi)\in\mathscr{S}(r)$. Fix $R>0$ and $\phi\in\mathscr{S}(R)$, and let $x(t):=x(t, 0, \phi)$, $V(t):=V(t, x_t)$ and $t_1:=0$. Let $S>0$ be the number for ϕ in the definition of $\mathscr{S}(R)$, and let $t_0:=-S$. Define L to be so large that if $Q\geq U+L>1$ then $(Q-U)a-M(\sqrt{Q/c}+\sqrt[4]{Q/c^2})>d_1>0$, where d_1 is a constant. Clearly we have such an $L>0$. Define $r:=U+L$ and $M_1:=R$. We will do an induction. Suppose, that $t_0, \ldots, t_n$ and $M_1, \ldots, M_n$ are defined so that $|x_{t_n}(s)|\leq\sqrt{g(s)}$ and $|x_{t_n}(u)-x_{t_n}(v)|\leq L(\sqrt{g(\min\{u, v\})})|u-v|$ for $s, u, v\leq -(t_n-t_{n-1})$ and $|x_{t_n}(s)|\leq M_n$ and $|x_{t_n}(u)-x_{t_n}(v)|\leq L(M_n)|u-v|$ for $s, u, v\in[-(t_n-t_{n-1}), 0]$. This inductional assumption clearly holds for $n=1$. Suppose, that it is true for some $n>0$. Then we define $t_{n+1}:=t_n+\sqrt{M_n/c}+\sqrt[4]{M_n/c^2}$. We have two cases:

Case I: If x is bounded by r on the interval $[t_n, t_{n+1}]$ then by construction we have $\|x_t\|_g\leq r$ for $t\in[t_n+\sqrt[4]{M_n/c^2}, t_{n+1}]$, and hence x satisfies a Lipschitz condition with $L(r)$ in that interval. Also, for $s, u, v\leq -\sqrt{M_n/c}$ we have $|x_{t_{n+1}}(s)|\leq\sqrt{g(s)}$ and $|x_{t_{n+1}}(u)-x_{t_{n+1}}(v)|\leq L(\sqrt{g(\min\{u, v\})})|u-v|$. Therefore $x_{t_{n+1}}\in\mathscr{S}(r)$ by construction, and the proof is finished (the induction is terminated).

Case II: Let $M_{n+1}>r$ be the maximum of x on the interval $[t_n, t_{n+1}]$. First, we estimate the decrease in V:

Case 1: If $|x(t)|\geq U$ for all $t\in[t_n, t_{n+1}]$ then using (iii) we have a $d_2>0$ such that

$$V(t_{n+1})-V(t_n)\leq -b(t_{n+1}-t_n)\leq -d_2\leq -d_2-(M_{n+1}-M_n)a$$

if $M_{n+1}\leq M_n$ and

$$V(t_{n+1})-V(t_n)\leq -b(t_{n+1}-t_n)-(M_{n+1}-M_n)a\leq -d_2-(M_{n+1}-M_n)a$$

if $M_{n+1}>M_n$.

Case 2: If there is a $t\in[t_n, t_{n+1}]$ with $|x(t)|\leq U$, then

$$\begin{aligned}V(t_{n+1})-V(t_n)&\leq M(t_{n+1}-t_n)-(M_{n+1}-U)a\\&=M(\sqrt{M_n/c}+\sqrt[4]{M_n/c^2})-(M_n-U)a-(M_{n+1}-M_n)a\\&\leq -d_1-(M_{n+1}-M_n)a\end{aligned}$$

by the definition of L.

In any case we have $V(t_{n+1})-V(t_n)\leq -d-(M_{n+1}-M_n)a$ for some $d>0$. To prove the inductional assumption for t_{n+1} we need to consider two cases again.

Case A: If $M_{n+1}<M_n$ then we redefine t_n to be $t_{n+1}-\sqrt{M_n/c}$. Now, using this new definition of t_n we have $|x_{t_{n+1}}(s)|\leq\sqrt{g(s)}$ and $|x_{t_{n+1}}(u)-x_{t_{n+1}}(v)|\leq L(\sqrt{g(\min\{u, v\})})|u-v|$ for $s, u, v\leq -(t_{n+1}-t_n)$. We also have $\|x_t\|_g\leq M_{n+1}$ for $t\in[t_n, t_{n+1}]$ and hence x satisfies a Lipschitz condition with $L(M_{n+1})$ on that interval.

Case B: If $M_{n+1}\geq M_n$ then we leave t_n as it is, and because $g(s)\geq M_n^2$ for $s\leq t_{n+1}-t_n$ we have $|x_{t_n}(s)|\leq\sqrt{g(s)}$ and $|x_{t_n}(u)-x_{t_n}(v)|\leq L(\sqrt{g(\min\{u, v\})})|u-v|$ for $s, u, v\leq -(t_n-t_{n-1})$. Obviously, x satisfies a Lipschitz condition with $L(M_{n+1})$ on the interval $[t_n, t_{n+1}]$.

This finishes our induction step.

If we ever go in Case I during this induction, then the proof is finished. If we always get Case II, then we have

$$V(t_n)-V(t_1)\leq -(n-1)d-(M_{n+1}-M_1)a\leq -(n-1)d-(r-R)a\,,$$

which is a contradiction for large n. This proves that Case I must happen at least once and the proof is complete.

Note that we can replace b by a function $b\colon \boldsymbol{R}\to\boldsymbol{R}$ integrable on any finite interval with $\int_0^\infty b(s)ds=\infty$ and we do not have to change much in the proof. In this case we argue that we cannot have Case 2 of Case II infinitely many times, and hence there is an $N>0$ such that Case 1 holds for $n\geq N$ and so $V(t_n)-V(t_1)\leq\int_{t_1}^{t_N}|b(s)|ds-\int_{t_N}^{t_n}b(s)ds-(r-R)a$, a contradiction for large n.

In order to make the computations in the proof easier we took a stronger condition in Theorem 9, than it is really necessary. With more careful investigations one could prove that if $g(s)/|s|\to\infty$ as $s\to-\infty$ then the statement of the theorem still holds. For this we must start the proofs from the beginning of the paper by modifying the definition of $\mathscr{S}(R)$ to let the function get closer to g for $s\leq -S$. Then we prove everything the same way as we did modifying the necessary parts of the proofs.

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