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MAGNETIC FLOWS OF ANOSOV TYPE

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Abstract. We regard a closed 2-form on a Riemannian manifold as a magnetic field and define a magnetic flow which is a perturbation of a geodesic flow. A sufficient condition is given for a magnetic flow to become an Anosov flow.

Introduction. The geodesic flow on the unit tangent bundle of a compact Riemannian manifold with negative sectional curvature is one of typical examples of Anosov flows. A geodesic curve on a Riemannian manifold may be considered as a trajectory of a particle subject only to forces of constraint. As a perturbation of a geodesic curve, we consider a trajectory of a charged particle under the Lorentz force generated by a magnetic field. The flow defined in terms of the trajectories will be called a *magnetic flow*.

If a magnetic field is weak enough, it follows from the structural stability of Anosov flows that the associated magnetic flow on a compact Riemannian manifold with negative sectional curvature is an Anosov flow. Concrete examples of magnetic flows of Anosov type are investigated in [1], [11], [12]. In this paper, we give a sufficient condition for a magnetic flow to become an Anosov flow. The main theorem is stated as follows:

THEOREM 1. *Let (M, g) be a compact Riemannian manifold with negative sectional curvature, and let $\kappa_{\max}(M)$ be the maximum of the sectional curvature of M. Given a magnetic field B (a closed 2-form) on M, we let $\Omega: TM \to TM$ be the operator defined by $g_p(u, \Omega(v)) = B_p(u, v)$ $(u, v \in T_pM, p \in M)$. If*

$$\max_{u,w \in S_1M} \{rg(u, (\nabla\Omega)(w; w)) + g(\Omega(w), \Omega(w))\} < -r^2\kappa_{\max}(M),$$

then the magnetic flow $\varphi_t: S_rM \to S_rM$ associated with B is of Anosov type.

1. Lorentz forces on Riemannian manifolds.

A magnetic field in $\boldsymbol{R}^3$ is a vector field $B = (b_1, b_2, b_3)$ satisfying the equation

$$\nabla \cdot B = \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3} = 0.$$

The Lorentz force generated by the magnetic field B on a moving unit charged particle

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in $\boldsymbol{R}^3$ is given by

$$F=v\times B=\begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

where v is the velocity vector. Therefore, we obtain the Newtonian equation of the particle

$$\dot{v}=F=v\times B\,.$$

We should note that the matrix determined by B is skew-symmetric and that F is perpendicular to v. Since we have used the vector product $v\times B$, the above discussion depends on the choice of the orientation of $\boldsymbol{R}^3$. In changing the orientation of $\boldsymbol{R}^3$, we need to change B into $-B$ in order that the definition of the Lorentz force is independent of the orientation of $\boldsymbol{R}^3$. To eliminate this dependency, we usually identify B with a 2-form

$$B=b_1dx_2\wedge dx_3+b_2dx_3\wedge dx_1+b_3dx_1\wedge dx_2\,.$$

Then, the equation $\nabla\cdot B=0$ turns out to be equivalent to

$$dB=\left(\frac{\partial b_1}{\partial x_1}+\frac{\partial b_2}{\partial x_2}+\frac{\partial b_3}{\partial x_3}\right)dx_1\wedge dx_2\wedge dx_3=0\,,$$

where d denotes the exterior differentiation.

In the case of a Riemannian manifold (M, g) of dimension n, we consider a closed 2-form on M as a magnetic field on M and will define the Lorentz force on M as follows. First, we define an operator $\Omega\colon TM\to TM$ by

$$g_p(u, \Omega(v))=B_p(u, v)\,,$$

where $u, v\in T_pM$ and $p\in M$. From the definition, it is obvious that Ω is skew-symmetric. Now, we define the Lorentz force on M as

$$F=\Omega(v)\,,$$

where $v\in TM$ is the velocity vector of a moving unit-charged particle on M. It is easy to see that F is perpendicular to v. We define the Newtonian equation of the particle on M by

$$\frac{D}{dt}\dot{c}=\Omega(\dot{c})\,, \tag{1}$$

where D/dt is the covariant derivative along the curve c and $\dot{c}$ is the velocity vector field. In particular, if $B=0$, the equation (1) reduces to the equation of geodesic

$$\frac{D}{dt}\dot{c}=0\,.$$

When B has a globally defined vector potential A, that is to say, when there exists a 1-form A satisfying the equation $B=dA$, the equation (1) is obtained as the Euler-Lagrange equation associated with the action integral

$$E_A(c)=\int_c L_A=\int_\alpha^\beta \left\{\frac{1}{2}g(\dot{c},\dot{c})+A(\dot{c})\right\}dt\,,$$

where $c:[\alpha,\beta]\to M$ is an arbitrary smooth curve on M. Indeed, if c_s $(-\varepsilon<s<\varepsilon)$ is a one-parameter variation of smooth curves with $c_0=c$, $c_s(\alpha)\equiv c(\alpha)$, $c_s(\beta)\equiv c(\beta)$, then the first variation formula of E_A is given by

$$\frac{d}{ds}E_A(c_s)\big|_{s=0}=-\int_\alpha^\beta g\left(W,\frac{D}{dt}\dot{c}-\Omega(\dot{c})\right)dt\,,$$

where ∇ is the Levi-Civita connection of (M,g) and $W=(\partial/\partial s)c_s|_{s=0}$. See [11] for detailed computation. Therefore, we see that the Euler-Lagrange equation for the Lagrangian L_A is the equation (1). However, it is important that the equation (1) is well-defined without a globally defined vector potential.

We shall require a condition on Ω which is equivalent to $dB=0$.

LEMMA 1.1. *The condition $dB=0$ is equivalent to*

$$g((\nabla\Omega)(X;Y),Z)+g((\nabla\Omega)(Y;Z),X)+g((\nabla\Omega)(Z;X),Y)=0$$

for every triple of vector fields X, Y and Z on M, where $(\nabla\Omega)(X;Y)$ denotes $(\nabla_Y\Omega)(X)$.

This is a consequence of the well-known identity

$$dB(X,Y,Z)=g((\nabla\Omega)(X;Y),Z)+g((\nabla\Omega)(Y;Z),X)+g((\nabla\Omega)(Z;X),Y)\,.$$

REMARK. We should note that the condition $dB=0$ is not used essentially in defining the equation (1). In other words, we can define the equation (1) for a general 2-form. However, we will see that the condition $dB=0$ plays an important role in the dynamics under B on Riemannian manifolds.

2. Jacobi fields under magnetic fields. In Section 1, we mentioned the first variation formula of the action integral E_A when there exists a globally defined vector potential A of B. We will derive the second variation formula to find out a suitable concept of a Jacobi field for the functional E_A.

Let c be a solution curve of the equation (1), and let $c_{(s_1,s_2)}$ $(-\varepsilon<s_1,s_2<\varepsilon)$ be a 2-parameter variation of smooth curves with $c_{(0,0)}=c$, $c_{(s_1,s_2)}(\alpha)\equiv c(\alpha)$, $c_{(s_1,s_2)}(\beta)\equiv c(\beta)$. Then, we shall compute

$$\frac{\partial^2}{\partial s_1\partial s_2}E_A(c_{(s_1,s_2)})\big|_{s_1=s_2=0}\,.$$

First, we find

$$\frac{\partial}{\partial s_2}E_A(c_{(s_1,s_2)})\big|_{s_2=0}=-\int_\alpha^\beta g\left(W_{s_1},\frac{D}{\partial t}\frac{\partial}{\partial t}c_{(s_1,0)}-\Omega\left(\frac{\partial}{\partial t}c_{(s_1,0)}\right)\right)dt\,,$$

where $W_{s_1}=(\partial/\partial s_2)c_{(s_1,s_2)}\big|_{s_2=0}$. Next,

$$\begin{aligned}\frac{\partial^2}{\partial s_1\partial s_2}E_A(c_{(s_1,s_2)})\big|_{s_1=s_2=0}&=-\int_\alpha^\beta g\left(\frac{D}{\partial s_1}W_{s_1}\big|_{s_1=0},\frac{D}{dt}\dot c-\Omega(\dot c)\right)dt\\&\quad-\int_\alpha^\beta g\left(W_{s_1},\frac{D}{\partial s_1}\frac{D}{\partial t}\frac{\partial}{\partial t}c_{(s_1,0)}-\frac{D}{\partial s_1}\left\{\Omega\left(\frac{\partial}{\partial t}c_{(s_1,0)}\right)\right\}\right)dt\bigg|_{s_1=0}\\&=-\int_\alpha^\beta g\left(W_2,\frac{D}{\partial s_1}\frac{D}{\partial t}\frac{\partial}{\partial t}c_{(s_1,0)}-\frac{D}{\partial s_1}\left\{\Omega\left(\frac{\partial}{\partial t}c_{(s_1,0)}\right)\right\}\bigg|_{s_1=0}\right)dt\,,\end{aligned}$$

where $W_2=W_{s_1}\big|_{s_1=0}=(\partial/\partial s_2)c_{(s_1,s_2)}\big|_{s_1=s_2=0}$. By standard computation, we get

$$\frac{D}{\partial s_1}\frac{D}{\partial t}\frac{\partial}{\partial t}c_{(s_1,0)}\bigg|_{s_1=0}=\frac{D^2}{dt^2}W_1+R(\dot c,W_1)\dot c\,,$$

where R is the curvature tensor and $W_1=(\partial/\partial s_1)c_{(s_1,s_2)}\big|_{s_1=s_2=0}$. Therefore, we have only to compute

$$\begin{aligned}\frac{D}{\partial s_1}\left\{\Omega\left(\frac{\partial}{\partial t}c_{(s_1,0)}\right)\right\}\bigg|_{s_1=0}&=\left(\frac{D}{\partial s_1}\Omega\right)\left(\frac{\partial}{\partial t}c_{(s_1,0)}\right)+\Omega\left(\frac{D}{\partial s_1}\frac{\partial}{\partial t}c_{(s_1,0)}\right)\bigg|_{s_1=0}\\&=\left(\frac{D}{\partial s_1}\Omega\right)\left(\frac{\partial}{\partial t}c_{(s_1,0)}\right)+\Omega\left(\frac{D}{\partial t}\frac{\partial}{\partial s_1}c_{(s_1,0)}\right)\bigg|_{s_1=0}\\&=(\nabla\Omega)(\dot c;W_1)+\Omega\left(\frac{D}{dt}W_1\right).\end{aligned}$$

Therefore, the second variation formula of E_A at c is

$$\begin{aligned}&\frac{\partial^2}{\partial s_1\partial s_2}E_A(c_{(s_1,s_2)})\bigg|_{s_1=s_2=0}\\&=-\int_\alpha^\beta g\left(W_2,\frac{D^2}{dt^2}W_1+R(\dot c,W_1)\dot c-(\nabla\Omega)(\dot c;W_1)-\Omega\left(\frac{D}{dt}W_1\right)\right)dt\,.\end{aligned}$$

We should note that the right-hand side of the second variation formula depends only on Ω. Namely, without a globally defined vector potential of B, the right-hand side of the above formula is meaningful. Therefore, we may define a Jacobi field under B along a solution curve of the equation (1) in a way similar to that in the definition of a Jacobi field along a geodesic.

DEFINITION 2.1. Let c be a solution curve of the equation (1). The Jacobi equation under B along c is defined by

$$\frac{D^2}{dt^2}J+R(\dot{c},J)\dot{c}-(\nabla\Omega)(\dot{c};J)-\Omega\left(\frac{D}{dt}J\right)=0\,. \tag{2}$$

A solution of the Jacobi equation is called *a Jacobi field under B*.

It is easy to see that $\dot{c}$ is a Jacobi field under B along c. Let c_s $(-\varepsilon<s<\varepsilon)$ be a one-parameter variation of c, not necessarily keeping the end points fixed, such that $c_0=c$ and c_s is a solution curve of the equation (1) in fixing s. That is to say,

$$\frac{D}{\partial t}\frac{\partial}{\partial t}c_s-\Omega\left(\frac{\partial}{\partial t}c_s\right)=0\,.$$

Then, the variation vector field

$$W(t)=\frac{\partial}{\partial s}c_s(t)\bigg|_{s=0}$$

is a Jacobi field under B along c.

3. Decomposition of Jacobi fields under magnetic fields. Let c be a solution curve of the equation (1). In this section, we will show that the Jacobi equation (2) is decomposed into the equations of the tangential component and normal components of c.

LEMMA 3.1. *Let X, Y and Z be smooth vector fields on M. Then*

$$(\nabla B)(X,Y;Z)=g(X,(\nabla\Omega)(Y;Z))$$

where $(\nabla B)(X,Y;Z)$ denotes $(\nabla_Z B)(X,Y)$.

LEMMA 3.2. *Let X, Y and Z be smooth vector fields on M. Then*

$$g(X,(\nabla\Omega)(Y;Z))=-g(Y,(\nabla\Omega)(X;Z))\,.$$

PROOF. Since B is a 2-form, we have

$$\begin{aligned}(\nabla B)(X,Y;Z)&=Z\{B(X,Y)\}-B(\nabla_Z X,Y)-B(X,\nabla_Z Y)\\&=-Z\{B(Y,X)\}+B(Y,\nabla_Z X)+B(\nabla_Z Y,X)\\&=-(\nabla B)(Y,X;Z)\,.\end{aligned}$$

We are done by Lemma 3.1. q.e.d.

LEMMA 3.3. *Let J be a Jacobi field under B along c. Then, $g((D/dt)J,\dot{c})$ is constant.*

PROOF. By Lemma 3.2, we have

$$\frac{d}{dt}\left\{g\left(\frac{D}{dt}J,\dot{c}\right)\right\}=g\left(\frac{D^2}{dt^2}J,\dot{c}\right)+g\left(\frac{D}{dt}J,\frac{D}{dt}\dot{c}\right)=g\left(\frac{D^2}{dt^2}J-\Omega\left(\frac{D}{dt}J\right),\dot{c}\right)$$

$$=g(-R(\dot{c}, J)\dot{c}+(\nabla\Omega)(\dot{c}; J), \dot{c})=g(\dot{c}, (\nabla\Omega)(\dot{c}; J))=0$$

q.e.d.

Let $v_1=\dot{c}(0)/r$ and $r=\{g(\dot{c}(0), \dot{c}(0))\}^{1/2}$, and let us choose $v_2, \dots, v_n \in T_{c(0)}M$ so that $\{v_1, v_2, \dots, v_n\}$ is an orthonormal basis in $T_{c(0)}M$. We define a vector field V_i $(i=1, \dots, n)$ along c as a solution of the differential equation

$$\frac{D}{dt}V_i-\Omega(V_i)=0\,, \quad V_i(0)=v_i\,.$$

In particular, $V_1=\dot{c}/r$.

LEMMA 3.4. *$V_1, \dots, V_n$ are orthonormal vector fields along c. In particular, $g(\dot{c}, \dot{c})\equiv r^2$.*

PROOF. By the definition of $(V_1, \dots, V_n)$, we have

$$\begin{aligned}\frac{d}{dt}\{g(V_i, V_j)\}&=g\left(\frac{D}{dt}V_i, V_j\right)+g\left(V_i, \frac{D}{dt}V_j\right)=g(\Omega(V_i), V_j)+g(V_i, \Omega(V_j))\\&=g(\Omega(V_i), V_j)-g(\Omega(V_i), V_j)=0\,.\end{aligned}$$

q.e.d.

Let J be a Jacobi field under B along c. Let J be expressed as $J=\sum_{i=1}^n f_iV_i$ where each f_i is a smooth function along c. Then,

$$\frac{D}{dt}J=\sum_{i=1}^n \dot{f}_iV_i+\sum_{i=1}^n f_i\Omega(V_i)\,,$$

$$\frac{D^2}{dt^2}J=\sum_{i=1}^n \ddot{f}_iV_i+2\sum_{i=1}^n \dot{f}_i\Omega(V_i)+\sum_{i=1}^n f_i\Omega^2(V_i)+\sum_{i=1}^n f_i(\nabla\Omega)(V_i; \dot{c})\,.$$

Therefore, we obtain

$$\begin{aligned}&\frac{D^2}{dt^2}J+R(\dot{c}, J)\dot{c}-(\nabla\Omega)(\dot{c}; J)-\Omega\left(\frac{D}{dt}J\right)\\&=\sum_{i=1}^n \ddot{f}_iV_i+\sum_{i=1}^n \dot{f}_i\Omega(V_i)+\sum_{i=1}^n f_i\{R(\dot{c}, V_i)\dot{c}+(\nabla\Omega)(V_i; \dot{c})-(\nabla\Omega)(\dot{c}; V_i)\}\,.\end{aligned}$$

DEFINITION 3.5. Let $v\in TM$. A linear endomorphism $\hat{R}_v$ of $T_{\pi(v)}M$ is defined by

$$\hat{R}_v(w)=R(v, w)v+(\nabla\Omega)(w; v)-(\nabla\Omega)(v; w)$$

where $\pi: TM\to M$ is the canonical projection.

The equation (2) is written as the differential equation of the components $f=(f_1, \dots, f_n)$:

$$\ddot{f}+\Omega_{\dot{c}}\dot{f}+\hat{R}_{\dot{c}}f=0 . \tag{3}$$

Since $g(V_1, (\nabla\Omega)(\dot{c}; V_j))=0$,

$$\begin{aligned}\hat{R}^1_{\dot{c},j}&=g(V_1, (\nabla\Omega)(V_j; \dot{c}))\\ &=\frac{d}{dt}\{g(V_1, \Omega(V_j))\}-g(\Omega(V_1), \Omega(V_j))-g(V_1, \Omega^2(V_j))=\dot{\Omega}^1_{\dot{c},j}\end{aligned}$$

for $j=1, \dots, n$. The first column of the equation (3) is written as

$$\ddot{f}_1+\sum_{j=2}^{n}\Omega^1_{\dot{c},j}\dot{f}_j+\sum_{j=2}^{n}\dot{\Omega}^1_{\dot{c},j}f_j=0 ,$$

where we should note that $\Omega^1_{\dot{c},1}=0$. Integrating the above equation, we have

$$\dot{f}_1+\sum_{j=2}^{n}\Omega^1_{\dot{c},j}f_j=g\left(\frac{D}{dt}J, V_1\right)\equiv\frac{C}{r} ,$$

where we set $g((D/dt)J, \dot{c})\equiv C\in \boldsymbol{R}$. Therefore, we obtain

$$\dot{f}_1=\frac{1}{r}g(\Omega(\dot{c}), J)+\frac{C}{r} . \tag{4}$$

DEFINITION 3.6. Let $v\in TM\setminus(0)$. A linear endomorphism $\tilde{R}_v$ of $T_{\pi(v)}M$ is defined by

$$\begin{aligned}\tilde{R}_v(w)&=\hat{R}_v(w)+\frac{1}{g(v, v)}g(\Omega(v), w)\Omega(v)\\ &=R(v, w)v+(\nabla\Omega)(w; v)-(\nabla\Omega)(v; w)+\frac{1}{g(v, v)}g(\Omega(v), w)\Omega(v) .\end{aligned}$$

Let pr_v be the projection map onto the normal subspace of v in $T_{\pi(v)}M$. Let $\Omega_{v,\perp}$ and $\tilde{R}_{v,\perp}$ denote $\mathrm{pr}_v\,\Omega\,\mathrm{pr}_v$ and $\mathrm{pr}_v\,\tilde{R}_v\,\mathrm{pr}_v$, respectively. Substituting the equation (4) for the equation (3), we obtain

$$\ddot{f}_\perp+\Omega_{\dot{c},\perp}\dot{f}_\perp+\tilde{R}_{\dot{c},\perp}f_\perp+\frac{C}{r^2}\Omega(\dot{c})=0 , \tag{5}$$

where we should note that $\hat{R}^i_{\dot{c},1}=0$ for $i=1,\dots,n$. Therefore, the equation (2) is decomposed into the equations of tangential and normal components of c.

For example, let (M, g) be an orientable surface. Then, an arbitrary closed 2-form on M is expressed as $b\,\mathrm{vol}_M$ where $b\in C^\infty(M)$ and vol_M is the canonical volume form determined by g. For $v\in TM\setminus(0)$, let $v_\perp$ denote a unique element in $T_{\pi(v)}M$ such that $\pi(v)=\pi(v_\perp)$, $g(v_\perp, v)=0$ and $\mathrm{vol}_M(v_\perp, v)=1$. In this case, the equation (3) is

$$\ddot{f}+\begin{pmatrix} 0 & -b(c) \\ b(c) & 0 \end{pmatrix}\dot{f}+\begin{pmatrix} 0 & -db(\dot{c}) \\ 0 & r^2R(c)-r^2db(\dot{c}_\perp) \end{pmatrix}f=0\,.$$

The equations (4) and (5) are

$$\begin{cases} \dot{f}_1=b(c)f_2+\dfrac{C}{r} \\ \ddot{f}_2+\{r^2R(c)-r^2db(\dot{c}_\perp)+b(c)^2\}f_2+\dfrac{C}{r}b(c)=0\,. \end{cases}$$

In particular, if $R(p)\equiv\kappa\in\boldsymbol{R}$ and $b(p)\equiv b\in\boldsymbol{R}$ on M, then the above equation is

$$\begin{cases} \dot{f}_1=bf_2+\dfrac{C}{r} \\ \ddot{f}_2+(r^2\kappa+b^2)f_2+\dfrac{C}{r}b=0\,. \end{cases}$$

REMARK. In the computation in this section, we did not use the assumption that B is closed.

4. Matrix differential equations. In this section, we shall study the real $m\times m$ matrix differential equation on $\boldsymbol{R}$

$$\ddot{Y}(t)+A(t)Y(t)=0\,, \tag{6}$$

where the derivative is taken componentwise and $A(t)$ is smooth and symmetric on $\boldsymbol{R}$. First, let $Y_0(t)$ be a solution of the equation (6) with $Y_0(0)=0$ and $\dot{Y}_0(0)=I_m$. The following lemma is easily shown in the same way as in the proof of the comparison theorem of Jacobi fields along geodesics.

LEMMA 4.1. *Suppose that there exists some $a<0$ with $A(t)\le aI_m$ for all $t\in\boldsymbol{R}$. Then, we have*

$$\|Y_0(t)x\|\ge\frac{1}{\sqrt{-a}}|\sinh\sqrt{-a}t|$$

for all unit vectors $x\in\boldsymbol{R}^m$ and all $t\in\boldsymbol{R}$. Therefore, $\det Y_0(t)\neq 0$ *for all $t\neq 0$.*

Next, we describe a useful method of Green [6]. Suppose that $\det Y_0(t)\neq 0$ for all $t\neq 0$. Let $\tau\neq 0$. Then $Y_\tau(t)$ is defined as a solution of the equation (6) with $Y_\tau(\tau)=0$ and $\dot{Y}_\tau(\tau)=-(Y_0^\dagger)^{-1}(\tau)$ where the dagger denotes the transpose operation.

LEMMA 4.2. *Let $\tau\neq 0$. Then, we have*:
1. *$Y_\tau(t)$ is a unique solution of the equation* (6) *with $Y_\tau(0)=I_m$ and $Y_\tau(\tau)=0$.*
2. $\det Y_\tau(t)\neq 0$ *if $t\neq\tau$.*

3. *Both* $\lim_{\tau\to+\infty}\dot{Y}_\tau(0)$ *and* $\lim_{\tau\to-\infty}\dot{Y}_\tau(0)$ *exist.*

For the sake of simplicity, we make it a rule that ∞ denotes one of $+\infty$ and $-\infty$ as the case may be. We may define $Y_\infty(t)$ as a solution of the equation (6) with $Y_\infty(0)=I_m$ and $\dot{Y}_\infty(0)=\lim_{\tau\to\infty}\dot{Y}_\tau(0)$.

LEMMA 4.3. *For all* $t\in\boldsymbol{R}$, *we have*
1. $Y_\infty(t)=\lim_{\tau\to\infty}Y_\tau(t)$,
2. $\det Y_\infty(t)\neq 0$.

Let us set $U_\infty(t)=\dot{Y}_\infty(t)Y_\infty^{-1}(t)$. It is easy to see that $U_\infty(t)$ is a symmetric solution of the Riccati matrix differential equation

$$\dot{U}_\infty(t)+U_\infty^2(t)+A(t)=0\,.$$

The construction of $U_\infty(t)$ is independent of the position of $t=0$ in the following sense: Let $Y(t;v,\tau)$ be a unique solution of the equation (6) with $Y(v;v,\tau)=I_m$ and $Y(\tau;v,\tau)=0$. Then, $Y(t;v)=\lim_{\tau\to\infty}Y(t;v,\tau)$ exists and we have the identity $\dot{Y}(t;v)Y^{-1}(t;v)=U_\infty(t)$.

LEMMA 4.4. *Suppose that there exists some* $\tilde{a}<0$ *with* $A(t)\geq\tilde{a}I_m$ *for all* $t\in\boldsymbol{R}$. *Then*

$$|(U_\infty(t)x,x)|\leq\sqrt{-\tilde{a}}$$

for all unit vectors $x\in\boldsymbol{R}^m$ *and all* $t\in\boldsymbol{R}$.

LEMMA 4.5. *Suppose that there exists some* $a<0$ *with* $A(t)\leq aI_m$ *for all* $t\in\boldsymbol{R}$. *Then*

$$(U_{+\infty}(t)x,x)\leq-\sqrt{-a}\,,\quad (U_{-\infty}(t)x,x)\geq\sqrt{-a}$$

for all unit vectors $x\in\boldsymbol{R}^m$ *and all* $t\in\boldsymbol{R}$.

COROLLARY 4.6. *Suppose that there exists some* $a<0$ *with* $A(t)\leq aI_m$ *for all* $t\in\boldsymbol{R}$. *Then*
1. $\|Y_{+\infty}(t)x\|\leq\exp(-\sqrt{-a}\,t)$, $\|Y_{-\infty}(t)x\|\geq\exp(\sqrt{-a}\,t)$, $(t\geq 0)$,
2. $\|Y_{+\infty}(t)x\|\geq\exp(-\sqrt{-a}\,t)$, $\|Y_{-\infty}(t)x\|\leq\exp(\sqrt{-a}\,t)$, $(t\leq 0)$,

for all unit vectors $x\in\boldsymbol{R}^m$.

5. Magnetic flows on Riemannian manifolds. Let (M,g) be a complete Riemannian manifold. Then, every solution curve of the equation (1) extends to a global solution curve. The magnetic flow associated with B on M is defined as follows:

DEFINITION 5.1. *The magnetic flow associated with* B on M is a flow $\varphi_t\colon TM\to TM$ defined by

$$\varphi_t(v)=\dot{c}_v(t)\,,$$

where c_v is a solution curve of the equation (1) with $\dot{c}_v(0)=v\in TM$. $\varphi_t(v)$ is the velocity vector of c_v at time t.

LEMMA 5.2. *The magnetic flow φ_t leaves the tangent sphere bundle $S_rM=\{v\in TM; g(v, v)=r^2\}$ invariant for all $t\in R$.*

First, we shall state the difference between the geodesic flow and a magnetic flow. Let γ_v be a geodesic with $\dot\gamma_v(0)=v\in TM$, and let ϕ_t: $TM\to TM$ be the geodesic flow $\phi_t(v)=\dot\gamma_v(t)$. Given $\lambda>0$, we obtain the identity

$$\lambda\phi_{\lambda t}\left(\frac{v}{\lambda}\right)=\phi_t(v)\,.$$

This identity is owing to the fact that if $\gamma(t)$ is a geodesic, then $\gamma(\lambda t)$ also is a geodesic. However, this identity no longer holds for a magnetic flow. Indeed, setting $c_v^\lambda(t)\equiv c_{v/\lambda}(\lambda t)$,

$$\dot c_v^\lambda(0)=\lambda\dot c_{v/\lambda}(0)=\lambda\frac{v}{\lambda}=v\,,$$

$$\frac{D}{dt}\dot c_v^\lambda=\lambda^2\frac{D}{ds}\dot c_{v/\lambda}=\lambda^2\Omega(\dot c_{v/\lambda})=\lambda\Omega(\dot c_v^\lambda)\,,$$

where $s=\lambda t$. Namely, c_v^λ is not a solution curve of the equation (1) but a solution curve of the equation

$$\frac{D}{dt}\dot c=\lambda\Omega(\dot c)\,, \tag{7}$$

which is the Newtonian equation of a moving-charged particle under λB. Therefore, we obtain the identity

$$\lambda\varphi_{\lambda t}\left(\frac{v}{\lambda}\right)=\varphi_t^\lambda(v)\,.$$

Next, we shall define a connection map K: $T(TM)\to TM$ such that K: $T_v(TM)\to T_{\pi(v)}M$ is linear for all $v\in TM$. Given a vector $\xi\in T_v(TM)$, let Z_ξ: $(-\varepsilon, \varepsilon)\to TM$ be a smooth curve with the initial condition ξ. Then, we define

$$K(\xi)=\frac{D}{dt}Z_\xi\Big|_{t=0}\in T_{\pi(v)}M$$

where D/dt is the covariant derivative along $\sigma_\xi=\pi(Z_\xi)$. $d\pi(\xi)$ denotes $(d/dt)\sigma_\xi|_{t=0}$ by the definition of $d\pi$: $T(TM)\to TM$. It is obvious that $d\pi(\xi)$ and $K(\xi)$ depend only on ξ. The kernels of $d\pi$ and K are called the vertical and horizontal subspaces of $T_v(TM)$, respectively. $T_v(TM)$ is the direct sum of the horizontal and vertical subspaces. Therefore, we may identify $T_v(TM)$ with $T_{\pi(v)}M\oplus T_{\pi(v)}M$ by the correspondence

$$T_v(TM)\ni\xi\leftrightarrow(d\pi(\xi), K(\xi))\in T_{\pi(v)}M\oplus T_{\pi(v)}M\,.$$

Let $J(c_v)$ denote the $2n$-dimensional vector space of Jacobi fields under B along c_v, and let J_ξ be a unique element in $J(c_v)$ with $J_\xi(0)=d\pi(\xi)$ and $(D/dt)J_\xi(0)=K(\xi)$. In a way similar to that the case of the geodesic flow, the following lemma is proved.

LEMMA 5.3. *Let $v\in TM$. Then we have*:

1. *A map $T_v(TM)\ni\xi\to J_\xi\in J(c_v)$ is a linear isomorphism of $T_v(TM)$ onto $J(c_v)$.*
2. *$J_\xi(t)=d\pi(d\varphi_t(\xi))$ and $(D/dt)J_\xi(t)=K(d\varphi_t(\xi))$ for all $t\in\boldsymbol{R}$.*
3. *$\xi\in T_v(TM)$ lies in $T_v(S_rM)$ for $v\in S_rM$ if and only if*

$$g(K(d\varphi_t(\xi)),\ \varphi_t(v))=g\left(\frac{D}{dt}J_\xi(t),\ \dot c_v(t)\right)\equiv 0$$

for all $t\in\boldsymbol{R}$.

We shall define a metric on TM with respect to which the horizontal and vertical subspaces of $T_v(TM)$ are orthogonal. Given $\xi,\eta\in T_v(TM)$, we define the metric $\tilde g$ by

$$\tilde g_v(\xi,\eta)=g_{\pi(v)}(d\pi(\xi),\ d\pi(\eta))+g_{\pi(v)}(K(\xi),\ K(\eta))\ .$$

By Lemma 5.3, it follows that for all $t\in\boldsymbol{R}$ and all $\xi\in T_v(TM)$,

$$\tilde g_{\varphi_t(v)}(d\varphi_t(\xi),\ d\varphi_t(\xi))=g_{c_v}(J_\xi(t),\ J_\xi(t))+g_{c_v}\left(\frac{D}{dt}J_\xi(t),\ \frac{D}{dt}J_\xi(t)\right).$$

6. Stable and unstable subspaces. From now on, we will study the magnetic flow φ_t restricted to S_rM.

Let c_v be a solution curve of the equation (1) with $\dot c_v(0)=v$ for all $v\in S_rM$. In view of Lemma 5.3, it is useful to study a Jacobi field J under B along c_v such that $g((D/dt)J,\ \dot c_v)\equiv 0$. Let J be expressed as $J=\sum_{i=1}^n f_iV_i$, where $V_1,\dots,V_n$ are orthonormal vector fields along c_v defined in Section 3. From the equations (4) and (5), we find

$$\dot f_1=\frac{1}{r}g(\Omega(\dot c_v),\ J)=-\sum_{j=2}^n\Omega^1_{\dot c_v,j}f_j,\quad \ddot f_\perp+\Omega_{\dot c_v,\perp}\dot f_\perp+\tilde R_{\dot c_v,\perp}f_\perp=0\ ,$$

since $C=g((D/dt)J,\ \dot c_v)\equiv 0$. We shall study the real $(n-1)\times(n-1)$-matrix differential equation along c_v

$$\ddot X+\Omega_{\dot c_v,\perp}\dot X+\tilde R_{\dot c_v,\perp}X=0\ . \tag{8}$$

Let X be a solution of the equation (8), and let us set Y as

$$Y=\exp(\theta_{v,\perp})X\ ,\quad \theta_{v,\perp}(t)=\frac{1}{2}\int_0^t\Omega_{\dot c_v,\perp}ds\ .$$

Substituting this for the equation (8), we have

$$0=\ddot{X}+\Omega_{\dot{c}_v,\perp}\dot{X}+\tilde{R}_{\dot{c}_v,\perp}X$$
$$=\exp(-\theta_{v,\perp})\left\{\ddot{Y}+\exp(\theta_{v,\perp})\left(\tilde{R}_{\dot{c}_v,\perp}-\frac{1}{2}(\nabla_{\dot{c}_v}\Omega)_{\dot{c}_v,\perp}+\frac{1}{4}\Omega^{\dagger}_{\dot{c}_v,\perp}\Omega_{\dot{c}_v,\perp}\right)\exp(-\theta_{v,\perp})Y\right\}.$$

LEMMA 6.1. *Let $v\in TM\setminus(0)$. Then*

$$\Omega^{\dagger}_{v,\perp}\Omega_{v,\perp}(w)=(\Omega^{\dagger}\Omega)_{v,\perp}(w)-\frac{1}{g(v,v)}g(\Omega(v),\mathrm{pr}_v(w))\Omega(v).$$

PROOF. It is enough to prove the identity for w perpendicular to v. First,

$$\Omega_{v,\perp}(w)=\mathrm{pr}_v\,\Omega(w)=\Omega(w)-\frac{1}{g(v,v)}g(v,\Omega(w))v.$$

Since Ω is skew-symmetric, we obtain

$$\begin{aligned}\Omega^{\dagger}_{v,\perp}\Omega_{v,\perp}(w)&=\mathrm{pr}_v\,\Omega^{\dagger}(\Omega_{v,\perp}(w))\\&=(\Omega^{\dagger}\Omega)_{v,\perp}(w)-\frac{1}{g(v,v)}g(v,\Omega(w))\Omega^{\dagger}(v)\\&=(\Omega^{\dagger}\Omega)_{v,\perp}(w)-\frac{1}{g(v,v)}g(\Omega(v),w)\Omega(v).\end{aligned}$$

q.e.d.

DEFINITION 6.2. Let $v\in TM\setminus(0)$. A linear endomorphism $\tilde{K}_v$ of $T_{\pi(v)}M$ is defined by

$$\tilde{K}_v(w)=\tilde{R}_v(w)-\frac{1}{2}(\nabla\Omega)(w;v)+\frac{1}{4}\Omega^{\dagger}\Omega-\frac{1}{4g(v,v)}g(\Omega(v),w)\Omega(v)$$
$$=R(v,w)v+\frac{1}{2}(\nabla\Omega)(w;v)-(\nabla\Omega)(v;w)+\frac{1}{4}\Omega^{\dagger}\Omega(w)+\frac{3}{4g(v,v)}g(\Omega(v),w)\Omega(v).$$

The following result is important.

LEMMA 6.3. *$\tilde{K}_v$ is a symmetric matrix in $T_{\pi(v)}M$ for all $v\in TM\setminus(0)$ if and only if $dB=0$.*

PROOF. Suppose that $dB=0$.

$$g\left(u,\frac{1}{2}(\nabla\Omega)(w;v)-(\nabla\Omega)(v;w)\right)=\frac{1}{2}g(u,(\nabla\Omega)(w;v))+g(v,(\nabla\Omega)(u;w))$$
$$=-\frac{1}{2}g(u,(\nabla\Omega)(w;v))-g(w,(\nabla\Omega)(v;u))$$

$$=g\left(w, \frac{1}{2}(\nabla\Omega)(u; v)-(\nabla\Omega)(v; u)\right),$$

where we have used Lemma 1.1 in the second equality. This implies that $\tilde{K}_v$ is symmetric. It is easy to prove the converse. q.e.d.

Thus, Y is a solution of the real $(n-1)\times(n-1)$-matrix differential equation along c_v

$$\ddot{Y}+\exp(\theta_{v,\perp})\tilde{K}_{\dot{c}_v,\perp}\exp(-\theta_{v,\perp})Y=0, \tag{9}$$

where $\tilde{K}_{\dot{c}_v,\perp}$ denotes $\mathrm{pr}_{\dot{c}_v}\tilde{K}_{\dot{c}_v}\mathrm{pr}_{\dot{c}_v}$. Conversely, if Y is a solution of the equation (9), then $X=\exp(-\theta_{v,\perp})Y$ is a solution of the equation (8). Therefore, we have only to study the equation (9).

Let (M, g) be compact from now on. We define

$$\tilde{K}_{\max,\perp}(M, r)=\max_{v\in S_rM}\max_{w\in T_{\pi(v)}M, w\perp v}\frac{g(\tilde{K}_{v,\perp}(w), w)}{g(w, w)}.$$

If $\tilde{K}_{\max,\perp}(M, r)<0$, that is to say, if $\tilde{K}_{v,\perp}$ is negative definite for all $v\in S_rM$, then one may apply the results obtained in Section 4 to the equation (9) along c_v. Let $\mathscr{Y}_{v,\tau}$, $\mathscr{Y}_{v,\infty}$ and $\mathscr{U}_{v,\infty}$ be the matrices along c_v which correspond to Y_τ, Y_∞ and U_∞ in Section 4, respectively. First, by Lemma 4.2, the following lemma is obtained.

Lemma 6.4. *Suppose that $\tilde{K}_{v,\perp}$ is negative definite for all $v\in S_rM$. Let $\tau\neq 0$. Then for all $v\in S_rM$ and all $\xi\in T_v(S_rM)$, there exists a unique vector $\xi_\tau\in T_v(S_rM)$ such that $\mathrm{pr}_v(d\pi(\xi_\tau))=\mathrm{pr}_v(d\pi(\xi))$ and $d\pi(d\varphi_\tau(\xi_\tau))=0$.*

Proof. Let $J_\xi=\sum_{i=1}^n f_{\xi,i}V_i$. Then let us set $\boldsymbol{f}_\tau$ as

$$f_{\tau,1}=-\int_\tau^t\left(\sum_{j=2}^n \Omega^1_{\dot{c}_v,j}f_{\tau,j}\right)ds, \quad \boldsymbol{f}_{\tau,\perp}=\exp(-\theta_{v,\perp})\mathscr{Y}_{v,\tau}\boldsymbol{f}_{\xi,\perp}(0).$$

ξ_τ is uniquely determined as the element of $T_v(S_rM)$ which corresponds to $\sum_{i=1}^n f_{\tau,i}V_i\in J(c_v)$. q.e.d.

By Lemma 4.3 and Corollary 4.6, the following lemma is proved.

Lemma 6.5. *Suppose that $\tilde{K}_{v,\perp}$ is negative definite for all $v\in S_rM$. Then for all $v\in S_rM$ and all $\xi\in T_v(S_rM)$, there exists a unique vector $\xi_\infty\in T_v(S_rM)$ such that*

$$\xi_\infty=\lim_{\tau\to\infty}\xi_\tau.$$

Proof. Let $J_\xi=\sum_{i=1}^n f_{\xi,i}V_i$. Then let us set $\boldsymbol{f}_\infty$ as

$$f_{\infty,1}=-\int_\infty^t\left(\sum_{j=2}^n \Omega^1_{\dot{c}_v,j}f_{\infty,j}\right)ds, \quad \boldsymbol{f}_{\infty,\perp}=\exp(-\theta_{v,\perp})\mathscr{Y}_{v,\infty}\boldsymbol{f}_{\xi,\perp}(0).$$

By Corollary 4.6, it is shown that $f_{\infty,1}$ is well-defined. Indeed, as $\tau\to+\infty$,

$$\left|\int_0^{+\infty}\left(\sum_{j=2}^{n}\Omega^1_{\dot{c}_v,j}f_{\infty,j}\right)ds\right|\le\int_0^{+\infty}\left(\sum_{j=2}^{n}|\Omega^1_{\dot{c}_v,j}|\right)\exp(-\{-\tilde{K}_{\max,\perp}(M,r)\}^{1/2}s)\|f_{\xi,\perp}(0)\|ds$$
$$\le\frac{n-1}{\{-\tilde{K}_{\max,\perp}(M,r)\}^{1/2}}\max_{w\in S_1M}\{g(\Omega(w),\Omega(w))\}^{1/2}\|f_{\xi,\perp}(0)\|<+\infty\,.$$

ξ_∞ is uniquely determined as the element of $T_v(S_rM)$ which corresponds to $\sum_{i=1}^{n}f_{\infty,i}V_i\in J(c_v)$. q.e.d.

DEFINITION 6.6. Let $\tilde{K}_{v,\perp}$ be negative definite for all $v\in S_rM$. Then

$$E^s(v)\equiv\{\xi\in T_v(S_rM)\,;\,\xi_{+\infty}=\xi\}\,,$$
$$E^u(v)\equiv\{\xi\in T_v(S_rM)\,;\,\xi_{-\infty}=\xi\}\,.$$

$E^s(v)$ and $E^u(v)$ are respectively called *the stable and unstable subspaces* determined by v.

For example, let $\xi\in E^s(v)$. Let $J_\xi=\sum_{i=1}^{n}f_{\xi,i}V_i$. Then there exists some $x\in\boldsymbol{R}^{n-1}$ such that

$$f_{\xi,1}=\int_t^{+\infty}\left(\sum_{j=2}^{n}\Omega^1_{\dot{c}_v,j}f_{\xi,j}\right)ds\,,\quad f_{\xi,\perp}=\exp(-\theta_{v,\perp})\mathscr{Y}_{v,+\infty}x\,.$$

From this, we have the following lemma.

LEMMA 6.7. *Suppose that $\tilde{K}_{v,\perp}$ is negative definite for all $v\in S_rM$. Then for all $v\in S_rM$,*

1. $\dim E^s(v)=\dim E^u(v)=n-1$,
2. $E^s(v)\cap E^u(v)=\{0\}$,
3. $E^s(v)\oplus E^u(v)\not\ni\xi_v$,

where $\xi_v\equiv(d/dt)\varphi_t(v)|_{t=0}$ and $E^0(v)\equiv\{\xi\in T_v(S_rM)\,;\,\xi=\alpha\xi_v,\,\alpha\in\boldsymbol{R}\}$. Therefore,

$$T_v(S_rM)=E^0(v)\oplus E^s(v)\oplus E^u(v)\,.$$

LEMMA 6.8. *If $\tilde{K}_{v,\perp}$ is negative definite for all $v\in S_rM$, then (M,g) is a Riemannian manifold with negative sectional curvature.*

PROOF. Let $w\in T_{\pi(v)}M$ such that $w\perp v$. Then,

$$g(R(v,w)v,w)<-g(v,(\nabla\Omega)(w\,;w))-\frac{1}{4}g(\Omega(w),\Omega(w))-\frac{3}{4g(v,v)}g(\Omega(v),w)^2\,.$$

If $g(v,(\nabla\Omega)(w\,;w))<0$, then $g(-v,(\nabla\Omega)(w\,;w))>0$. Therefore, $g(R(v,w)v,w)<0$.

q.e.d.

7. Magnetic flows of Anosov type. In this section, we will give a sufficient condition for the magnetic flow $\varphi_t\colon S_rM\to S_rM$ associated with B to become an Anosov flow.

First, we recall the definition of Anosov flows.

DEFINITION 7.1. Let ψ_t be a complete C^∞-flow on a compact Riemannian manifold $(N\langle , \rangle)$ of dimension $n \geq 3$. The flow is said to be *of Anosov type* if the following conditions are satisfied:

1. The vector field V defined by the flow never vanishes on N.
2. For all $p \in N$, the tangent space T_pN splits into a direct sum as follows:
$$T_pN = E^0(p) \oplus E^s(p) \oplus E^u(p) ,$$
where $E^0(p)$ is generated by $V(p)$, and there exist positive constants α, β, γ such that
(a) for any $\xi \in E^s(p)$
$$\|d\psi_t(\xi)\|_p \leq \alpha \|\xi\|_p \exp(-\gamma t) \qquad \text{for} \quad t \geq 0 ,$$
$$\|d\psi_t(\xi)\|_p \geq \beta \|\xi\|_p \exp(-\gamma t) \qquad \text{for} \quad t \leq 0 ,$$
(b) for any $\xi \in E^u(p)$
$$\|d\psi_t(\xi)\|_p \leq \alpha \|\xi\|_p \exp(\gamma t) \qquad \text{for} \quad t \leq 0 ,$$
$$\|d\psi_t(\xi)\|_p \geq \beta \|\xi\|_p \exp(\gamma t) \qquad \text{for} \quad t \geq 0 .$$
3. ψ_t leaves $E^0 \equiv \bigcup_{p \in N} E^0(p)$, $E^s \equiv \bigcup_{p \in N} E^s(p)$ and $E^u \equiv \bigcup_{p \in N} E^n(p)$ invariant respectively for all $t \in \boldsymbol{R}$.
4. E^0, E^s and E^u are C^0-subbundles in TN.

REMARK. The third and fourth conditions of Definition 7.1 are proved by the first and second conditions. See [2], [10] for details. Therefore, we have only to show that a given flow satisfies the first and second conditions of Definition 7.1 in order to prove that the flow is of Anosov type.

Now, we state the main result.

THEOREM 7.2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$. If $\tilde{K}_{v,\perp}$ is negative definite for all $v \in S_rM$, then the magnetic flow $\varphi_t: S_rM \to S_rM$ associated with B is of Anosov type.*

It is obvious that the magnetic flow $\varphi_t: S_rM \to S_rM$ satisfies the first condition. Under the assumption that $\tilde{K}_{v,\perp}$ is negative definite for all $v \in S_rM$, we shall prove that the second condition is satisfied. Let $\gamma(M, r)$ denote $\{-\tilde{K}_{\max,\perp}(M, r)\}^{1/2}$ for the sake of simplicity.

LEMMA 7.3. *There exists some $\alpha_1(M, r) > 0$ such that*

1. *for all $\xi \in E^s(v)$*
$$g(J_\xi(t), J_\xi(t)) \leq \alpha_1(M, r) \exp(-2\gamma(M, r)t) g(d\pi(\xi), d\pi(\xi)) \qquad (t \geq 0) ,$$

2. *for all* $\xi \in E^u(v)$

$$g(J_\xi(t), J_\xi(t)) \le \alpha_1(M, r)\exp(2\gamma(M, r)t) g(d\pi(\xi), d\pi(\xi)) \qquad (t \le 0).$$

PROOF. Let $\xi \in E^s(v)$, and let $J_\xi = \sum_{i=1}^n f_{\xi,i} V_i$. From Corollary 4.6, we have

$$\|f_{\xi,\perp}(t)\| \le \exp(-\gamma(M, r)t)\|f_{\xi,\perp}(0)\| ,$$

$$|f_{\xi,1}(t)| \le \frac{(n-1)}{\gamma(M, r)} \max_{w \in S_1 M} \{g(\Omega(w), \Omega(w))\}^{1/2} \exp(-\gamma(M, r)t)\|f_{\xi,\perp}(0)\| ,$$

for $t \ge 0$. Let us set

$$\alpha_1(M, r) = 1 + \frac{(n-1)^2}{\gamma(M, r)^2} \max_{w \in S_1 M} g(\Omega(w), \Omega(w)) > 1 .$$

Then,

$$\begin{aligned} g(J_\xi(t), J_\xi(t)) &\le \alpha_1(M, r)\exp(-2\gamma(M, r)t)\|f_{\xi,\perp}(0)\|^2 \\ &\le \alpha_1(M, r)\exp(-2\gamma(M, r)t) g(d\pi(\xi), d\pi(\xi)) \end{aligned}$$

which implies the first inequality. The second inequality is proved in the same way.

q.e.d.

We define

$$\tilde{K}_{\min,\perp}(M, r) = \min_{v \in S_r M} \min_{w \in T_{\pi(v)} M, w \perp v} \frac{g(\tilde{K}_{v,\perp}(w), w)}{g(w, w)} < 0 .$$

Let $\delta(M, r)$ denote $\{-\tilde{K}_{\min,\perp}(M, r)\}^{1/2}$. From Lemma 4.4,

$$|(\mathcal{U}_{v,\infty}(t)x, x)| \le \delta(M, r)$$

on c_v for all unit vectors $x \in \boldsymbol{R}^{n-1}$ and all $v \in S_r M$. Since $\mathcal{U}_{v,\infty}(t)$ is symmetric,

$$\|\mathcal{U}_{v,\infty}(t)x\| \le \delta(M, r)$$

on c_v for all unit vectors $x \in \boldsymbol{R}^{n-1}$ and all $v \in S_r M$. Therefore, we obtain the following result:

LEMMA 7.4. *There exists some* $\alpha_2(M, r) > 0$ *such that*

1. *for all* $\xi \in E^s(v)$

$$g\left(\frac{D}{dt} J_\xi(t), \frac{D}{dt} J_\xi(t)\right) \le \alpha_2(M, r)\exp(-2\gamma(M, r)t) g(d\pi(\xi), d\pi(\xi)) \qquad (t \ge 0),$$

2. *for all* $\xi \in E^u(v)$

$$g\left(\frac{D}{dt} J_\xi(t), \frac{D}{dt} J_\xi(t)\right) \le \alpha_2(M, r)\exp(2\gamma(M, r)t) g(d\pi(\xi), d\pi(\xi)) \qquad (t \le 0).$$

PROOF. Let $\xi \in E^s(v) \oplus E^u(v)$, and let $J_\xi = \sum_{i=1}^n f_{\xi,i} V_i$. First,

$$\frac{D}{dt} J_\xi = \sum_{i=1}^n \left(\dot{f}_{\xi,i} + \sum_{j=1}^n \Omega^i_{\dot{c}_v,j} f_{\xi,j} \right) V_i = \sum_{i=2}^n \left(\dot{f}_{\xi,i} + \sum_{j=1}^n \Omega^i_{\dot{c}_v,j} f_{\xi,j} \right) V_i ,$$

since $(1/r) g((D/dt) J_\xi, \dot{c}_v) = \dot{f}_{\xi,1} + \sum_{j=2}^n \Omega^1_{\dot{c}_v,j} f_{\xi,j} \equiv 0$. Then,

$$\begin{aligned}
g\left(\frac{D}{dt} J_\xi(t), \frac{D}{dt} J_\xi(t) \right) &= \sum_{i=2}^n \left(\dot{f}_{\xi,i} + \sum_{j=1}^n \Omega^i_{\dot{c}_v,j} f_{\xi,j} \right)^2 \\
&\leq \sum_{i=2}^n \left\{ 2 \left(\dot{f}_{\xi,i} + \sum_{j=2}^n \Omega^i_{\dot{c}_v,j} f_{\xi,j} \right)^2 + 2 (\Omega^i_{\dot{c}_v,1} f_{\xi,1})^2 \right\} \\
&= 2 \left\| \frac{d}{dt} \{ \exp(-\theta_{v,\perp}) \mathcal{Y}_{v,\infty} \boldsymbol{f}_{\xi,\perp}(0) \} \right\|^2 + 2 f_{\xi,1}^2 \sum_{i=2}^n (\Omega^i_{\dot{c}_v,1})^2 \\
&\leq \| \Omega_{\dot{c}_v,\perp} \exp(-\theta_{v,\perp}) \mathcal{Y}_{v,\infty} \boldsymbol{f}_{\xi,\perp}(0) \|^2 + 4 \| \mathcal{U}_{v,\infty} \mathcal{Y}_{v,\infty} \boldsymbol{f}_{\xi,\perp}(0) \|^2 + 2 f_{\xi,1}^2 \sum_{i=2}^n (\Omega^i_{\dot{c}_v,1})^2 \\
&\leq \left\{ \max_{w \in S_1 M} g(\Omega(w), \Omega(w)) + 4 \delta(M, r)^2 \right\} \| \mathcal{Y}_{v,\infty} \boldsymbol{f}_{\xi,\perp}(0) \|^2 + 2 f_{\xi,1}^2 \max_{w \in S_1 M} g(\Omega(w), \Omega(w)) \\
&\leq \left\{ 3 \max_{w \in S_1 M} g(\Omega(w), \Omega(w)) + 4 \delta(M, r)^2 \right\} g(J_\xi(t), J_\xi(t)) .
\end{aligned}$$

Therefore, if $\alpha_2(M, r)$ is defined by

$$\alpha_2(M, r) = \left\{ 3 \max_{w \in S_1 M} g(\Omega(w), \Omega(w)) + 4 \delta(M, r)^2 \right\} \alpha_1(M, r) .$$

We are done by Lemma 7.3. q.e.d.

LEMMA 7.5. *There exists some* $\alpha(M, r) > 0$ *such that*

1. *for all* $\xi \in E^s(v)$

$$\tilde{g}(d\varphi_t(\xi), d\varphi_t(\xi)) \leq \alpha(M, r)^2 \exp(-2\gamma(M, r) t) \tilde{g}(\xi, \xi) \qquad (t \geq 0) ,$$

2. *for all* $\xi \in E^u(v)$

$$\tilde{g}(d\varphi_t(\xi), d\varphi_t(\xi)) \leq \alpha(M, r)^2 \exp(2\gamma(M, r) t) \tilde{g}(\xi, \xi) \qquad (t \leq 0) .$$

PROOF. Let $\xi \in E^s(v)$. By Lemma 7.3 and 7.4, we find

$$\begin{aligned}
\tilde{g}(d\varphi_t(\xi), d\varphi_t(\xi)) &\leq (\alpha_1(M, r) + \alpha_2(M, r)) \exp(-2\gamma(M, r) t) g(d\pi(\xi), d\pi(\xi)) \\
&\leq (\alpha_1(M, r) + \alpha_2(M, r)) \exp(-2\gamma(M, r) t) \tilde{g}(\xi, \xi)
\end{aligned}$$

for $t \geq 0$. Let us set $\alpha(M, r) = \{ \alpha_1(M, r) + \alpha_2(M, r) \}^{1/2} > 1$. Then, the first inequality is obtained. The second inequality is proved in the same way. q.e.d.

LEMMA 7.6. *There exists some* $\beta(M, r) > 0$ *such that for all* $\xi \in E^s(v) \oplus E^u(v)$

$$g(\mathrm{pr}_v(d\pi(\xi)), \mathrm{pr}_v(d\pi(\xi))) \geq \beta(M, r)^2 \tilde{g}(\xi, \xi) .$$

PROOF. Setting $t=0$ in Lemmas 7.3 and 7.4, we have

$$\tilde{g}(\xi, \xi) \leq \alpha(M, r)^2 g(d\pi(\xi), d\pi(\xi)) .$$

From the proof of Lemma 7.3,

$$g(d\pi(\xi), d\pi(\xi)) \leq \alpha_1(M, r) g(\mathrm{pr}_v(d\pi(\xi)), \mathrm{pr}_v(d\pi(\xi))) .$$

Then set $\beta(M, r) = 1/\alpha(M, r)^{3/2} < 1$. q.e.d.

COROLLARY 7.7. 1. *For all* $\xi \in E^s(v)$

$$\tilde{g}(d\varphi_t(\xi), d\varphi_t(\xi)) \geq \beta(M, r)^2 \exp(-2\gamma(M, r)t) \tilde{g}(\xi, \xi) \qquad (t \leq 0) .$$

2. *For all* $\xi \in E^u(v)$

$$\tilde{g}(d\varphi_t(\xi), d\varphi_t(\xi)) \geq \beta(M, r)^2 \exp(2\gamma(M, r)t) \tilde{g}(\xi, \xi) \qquad (t \geq 0) .$$

PROOF. Let $\xi \in E^s(v)$, and let $J_\xi = \sum_{i=1}^n f_{\xi,i} V_i$. By Lemma 4.6,

$$\begin{aligned} \tilde{g}(d\varphi_t(\xi), d\varphi_t(\xi)) &\geq g(J_\xi(t), J_\xi(t)) \\ &\geq \|\exp(-\theta_{v,\perp}) \mathscr{Y}_{v,+\infty} \boldsymbol{f}_{\xi,\perp}(0)\|^2 \\ &\geq \exp(-2\gamma(M, r)t) g(\mathrm{pr}_v(d\pi(\xi)), \mathrm{pr}_v(d\pi(\xi))) \end{aligned}$$

for $t \leq 0$. By Lemma 7.6, the first inequality is obtained. The second inequality is proved in the same way. q.e.d.

By Lemma 7.5 and 7.7, the magnetic flow $\varphi_t: S_r M \to S_r M$ satisfies the second condition of Definition 7.1. Therefore, the proof of Theorem 7.2 is completed.

COROLLARY 7.8. *Let (M, g) be a compact Riemannian manifold with negative sectional curvature of dimension $n \geq 2$, and let $\kappa_{\max}(M)$ denote the maximum of sectional curvature of M. If*

$$\max_{u,w \in S_1 M} \{ r g(u, (\nabla \Omega)(w; w)) + g(\Omega(w), \Omega(w)) \} < -r^2 \kappa_{\max}(M) ,$$

then the magnetic flow $\varphi_t: S_r M \to S_r M$ associated with B is of Anosov type.

COROLLARY 7.9. *Let (M, g) be a compact orientable surface with constant curvature κ, and let $B = b\,\mathrm{vol}_M$ $(b \in \boldsymbol{R})$. If $r^2\kappa + b^2 < 0$, then the magnetic flow $\varphi_t: S_r M \to S_r M$ associated with B is of Anosov type.*

COROLLARY 7.10. *Let (M, g) be a compact Kähler manifold with constant holomorphic sectional curvature κ. Let B_M denote the Kähler form, and let $B = bB_M$ $(b \in \boldsymbol{R})$. If $r^2\kappa + b^2 < 0$, then the magnetic flow $\varphi_t: S_r M \to S_r M$ associated with B is of Anosov type.*

REFERENCES

[1] T. ADACHI, Kähler magnetic flows for a manifold of constant holomorphic sectional curvature, Tokyo J. Math. 18 (1995), 473–483.

[2] V. I. ARNOLD AND A. AVEZ, Problems ergodiques de la mecanique classique, Ganthier-Villars, Paris, 1967.

[3] J. CHEEGER AND D. G. EBIN, Comparison Theorems in Riemannian Geometry, North-Holland, 1975.

[4] A. COMTET, On the Landau levels on the hyperbolic plane, Ann. Phys. 173 (1987), 185–209.

[5] P. EBERLEIN, When is a geodesic flow of Anosov type? 1. J. Differential Geom. 8 (1973), 437–463.

[6] L. W. GREEN, A theorem of E. Hopf, Michigan Math. J. 5 (1958), 31–34.

[7] E. HOPF, Closed surfaces without conjugate points, Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 47–51.

[8] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, Vol. 2, John Wiley and Sons, New York, 1969.

[9] J. MILNOR, Morse Theory, Ann. of Math. Study 51, Princeton Univ. Press, 1962.

[10] T. NIWA, N. OTSUKI AND T. MIYAHARA, Ergodic problems of classical mechanics, Seminar on Probability 30 (1969) (in Japanese).

[11] T. SUNADA AND P. W. SY, Geometry of magnetic fields, preprint (1993).

[12] T. SUNADA, Magnetic flows on a Riemannian surface, Proc. of KAIST Math. Workshop 8 (1993), Analysis and Geometry, 93–108.

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