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THE FUNCTOR OF A TORIC VARIETY WITH ENOUGH INVARIANT EFFECTIVE CARTIER DIVISORS

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Abstract. The homogeneous coordinate ring of a toric variety was first introduced by Cox. In this paper, we study that of a toric variety with enough invariant effective Cartier divisors in detail. Here a toric variety is said to have enough invariant effective Cartier divisors if, for each nonempty affine open subset stable under the action of the torus, there exists an effective Cartier divisor whose support equals its complement. Both quasi-projective toric varieties and simplicial toric varieties have enough invariant effective Cartier divisors. In terms of the homogeneous coordinate ring, we describe the data needed to specify a morphism from a scheme to such a toric variety. As a consequence, we generalize a result of Cox, one of Oda and Sankaran, and one of Guest concerning data on morphisms.

Introduction. Let k be a field, N a free $\boldsymbol{Z}$-module of rank r, M the $\boldsymbol{Z}$-module dual to N, $T:=\boldsymbol{G}_m\otimes N$ the algebraic torus of dimension r corresponding to N, and Δ a (finite) fan of $N_{\boldsymbol{Q}}$. Let X_Δ be the toric variety associated to Δ, D_ρ the closure of the T-orbit corresponding to a one-dimensional cone $\rho\in\Delta$, $\sigma(1)$ the set of one-dimensional cones contained in a cone $\sigma\in\Delta$, and $\mathrm{Pic}(\Delta)_{\geq 0}$ the monoid of linear equivalence classes of invariant effective Cartier divisors. A toric variety X_Δ is said to have enough invariant effective Cartier divisors if, for each cone $\sigma\in\Delta$, there exists an effective T-invariant Cartier divisor D with $\mathrm{Supp}\,D=\bigcup_{\rho\notin\sigma(1)}D_\rho$. Both quasi-projective toric varieties and simplicial toric varieties have enough invariant effective Cartier divisors (cf. Remark 1.6(3)).

Cox [1] introduced two homogeneous coordinate rings of a toric variety X_Δ: one is the monoid algebra S of the monoid of effective T-invariant Weil divisors with Chow-grading, while the other is the subring S_Δ of S with Pic-grading (see [1, p. 19, p. 35]). He constructed in [1] the toric variety X_Δ as the quotient of an open subscheme of $\mathrm{Spec}\,S$, and described in [2, Theorem 1.1] the data needed to specify a map from a scheme to an arbitrary *smooth* toric variety in terms of its homogeneous coordinate ring.

The purpose of this paper is to generalize Cox's description to one for an arbitrary toric variety with enough invariant effective Cartier divisors by studying the latter homogeneous coordinate ring in detail (cf. Theorem 3.4 and Theorem 4.3).

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In Section 1, we study the latter homogeneous coordinate ring S_Δ in detail and prove that a toric variety X_Δ with enough invariant effective Cartier divisors is the geometric quotient of an open subscheme of $\operatorname{Spec} S_\Delta$. In Section 2, we study quasi-coherent modules on X_Δ associated to graded S_Δ-modules in the same way as that in EGA [9, II §2]. In Section 3, we prove a one-to-one correspondence between the set of morphisms from a scheme to a closed subscheme of X_Δ and the set of graded algebra homomorphisms satisfying a nondegeneracy condition (Theorem 3.4). In Section 4, applying the above correspondence to a toric variety with enough invariant effective Cartier divisors, we generalize all known results on morphisms from a scheme to a toric variety.

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Convention: A ring means a commutative ring with unity. A monoid means a commutative semigroup with unity. For a ring A, we denote by $A^\times$ the multiplicative group of units in A.

1. The homogeneous coordinate ring of a toric variety with enough invariant effective Cartier divisors. In this section, we study the homogeneous coordinate ring S_Δ of a toric variety X_Δ with enough invariant effective Cartier divisors in detail (see the Introduction and Definition 1.5), which Cox [1, p. 35] studied only in the case of simplicial toric varieties. We prove that such a toric variety X_Δ is the geometric quotient of an open subscheme of $\operatorname{Spec} S_\Delta$ (cf. [2, Theorem 2.1]).

Throughout this paper, except in Section 2, we let k be a field, N a free $\boldsymbol{Z}$-module of rank r, M the $\boldsymbol{Z}$-module dual to N, $T:=\boldsymbol{G}_m\otimes N$ the algebraic torus of dimension r corresponding to N, Δ a (finite) fan of $N_{\boldsymbol{Q}}$, $\Delta_{\max}$ the set of maximal cones in Δ, $\Delta(1)$ the set of one-dimensional cones in Δ, $\langle\,,\rangle\colon M_{\boldsymbol{Q}}\times N_{\boldsymbol{Q}}\to\boldsymbol{Q}$ the duality pairing, and X_Δ the toric variety associated to Δ.

We first recall T-invariant Cartier divisors on the toric variety X_Δ and the Picard group $\operatorname{Pic}(X_\Delta)$ (cf., e.g., [5, 3.4], [10, §2.1]). It is well-known that the following three groups are canonically isomorphic to one another:

(a) The group $T\mathrm{CDiv}(X_\Delta)$ of T-invariant Cartier divisors on X_Δ;

(b) The group $\mathrm{SF}(N,\Delta)$ of Δ-linear support functions on $|\Delta|:=\bigcup_{\sigma\in\Delta}\sigma$;

(c) The kernel of the homomorphism

$$\text{(1.0.a)}\qquad \bigoplus_{\sigma\in\Delta_{\max}} M/(M\cap\sigma^\perp)\to\bigoplus_{(\sigma,\tau)\in\Delta^2_{\max}} M/(M\cap(\sigma\cap\tau)^\perp);$$
$$([m_\sigma];\,\sigma\in\Delta_{\max})\mapsto([m_\tau-m_\sigma];\,\sigma,\tau\in\Delta_{\max}),$$

where $\sigma^{\perp}$ (resp. $[m_\sigma]$) denotes the set $\{m \in M; \langle m, n\rangle = 0 \text{ for all } n \in \sigma\}$ (resp. the equivalence class of $m_\sigma \in M$ in $M/(M \cap \sigma^{\perp})$). The above isomorphism maps a T-invariant effective Cartier divisor to an $\boldsymbol{R}_{\leq 0}$-valued Δ-linear support function and to an element in the intersection of the above kernel with $\bigoplus_{\sigma \in \Delta_{\max}} (M \cap \sigma^{\vee})/(M \cap \sigma^{\perp})$ respectively. Here $\sigma^{\vee}$ is the cone dual to σ and we adopt an isomorphism between (b) and (c) which maps $f \in \mathrm{SF}(N, \Delta)$ to $(-f|_\sigma; \sigma \in \Delta_{\max}) \in \bigoplus_{\sigma \in \Delta_{\max}} M/(M \cap \sigma^{\perp})$.

The Picard group $\mathrm{Pic}(X_\Delta)$ is the quotient of $T\,\mathrm{CDiv}(X_\Delta)$ modulo the subgroup of those principal divisors which are of the form $\mathrm{div}(m) := \sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle D_\rho$ for $m \in M$. Here D_ρ (resp. n_ρ) is the *Weil* divisor corresponding to $\rho \in \Delta(1)$ (resp. the unique generator of $\rho \cap N$). Hence $\mathrm{Pic}(X_\Delta)$ is isomorphic to $T\,\mathrm{CDiv}(X_\Delta)/\mathrm{div}(M)$. Since both $T\,\mathrm{CDiv}(X_\Delta)$ and $\mathrm{Pic}(X_\Delta)$ are described only in terms of a fan Δ, we define $\mathrm{CDiv}(\Delta)$ and $\mathrm{Pic}(\Delta)$ as follows:

DEFINITION 1.1. Let Δ be a fan.

(1) A *Cartier divisor on* Δ is defined to be an element in the kernel of the homomorphism in (1.0.a). A Cartier divisor $([m_\sigma]; \sigma \in \Delta_{\max})$ on Δ is said to be *effective* if $m_\sigma \in M \cap \sigma^{\vee}$ for each $\sigma \in \Delta_{\max}$. We denote by $\mathrm{CDiv}(\Delta)$ (resp. $\mathrm{CDiv}(\Delta)_{\geq 0}$) the group of Cartier divisors on Δ (resp. the monoid of effective Cartier divisors on Δ).

(2) A Cartier divisor $([m_\sigma]; \sigma \in \Delta_{\max})$ on Δ is said to be *principal* if there exists $m \in M$ with $[m_\sigma] = [m]$ in $M/(M \cap \sigma^{\perp})$ for each $\sigma \in \Delta_{\max}$. We denote by $\mathrm{PDiv}(\Delta)$ (resp. $\mathrm{div}(m)$) the group of principal divisors on Δ (resp. the principal divisor $([m]; \sigma \in \Delta_{\max})$).

(3) The quotient $\mathrm{CDiv}(\Delta)/\mathrm{PDiv}(\Delta)$ is said to be the *Picard group of* Δ, denoted by $\mathrm{Pic}(\Delta)$. We also denote by $\mathrm{Pic}(\Delta)_{\geq 0}$ the image of the monoid $\mathrm{CDiv}(\Delta)_{\geq 0}$.

REMARK 1.2. (1) The group $\mathrm{CDiv}(\Delta)$ of Cartier divisors on Δ is a free $\boldsymbol{Z}$-module of finite rank because it is a subgroup of $\mathrm{Hom}(\boldsymbol{Z}^{\Delta(1)}, \boldsymbol{Z})$. Hence $\mathrm{Pic}(\Delta)$ is finitely generated.

(2) The Picard group $\mathrm{Pic}(\Delta)$ of a fan Δ is isomorphic to the first cohomology group of the following cochain complex defined in a natural way:

$$0 \to \bigoplus_{\sigma \in \Delta_{\max}} M \cap \sigma^{\perp} \to \bigoplus_{(\sigma,\tau) \in \Delta^2_{\max}} M \cap (\sigma \cap \tau)^{\perp} \to \bigoplus_{(\sigma,\tau,\nu) \in \Delta^3_{\max}} M \cap (\sigma \cap \tau \cap \nu)^{\perp} \to \cdots,$$

where the first direct sum is the group of degree zero cochains. In particular, $\mathrm{Pic}(\Delta)$ is free if $\dim \sigma = r$ for each $\sigma \in \Delta_{\max}$. See Lemma 1.13 in general.

We introduce some useful notation as follows:

DEFINITION 1.3. Let Δ be a fan.

(1) For a Cartier divisor $D = ([m_\sigma]; \sigma \in \Delta_{\max})$, the *support* $\mathrm{Supp}\, D$ *of* D is defined to be the subset $\{\rho \in \Delta(1); \langle m_\sigma, n_\rho \rangle \neq 0 \text{ for some } \sigma \in \Delta_{\max} \text{ containing } \rho\}$ of $\Delta(1)$.

(2) For a cone $\sigma \in \Delta$, we denote by $\mathrm{CDiv}(\hat{\sigma})^{+}$ the set of effective Cartier divisors on Δ whose supports equal exactly $\hat{\sigma}(1) := \Delta(1) \setminus \sigma(1)$. We also denote by $\mathrm{CDiv}(\hat{\sigma})$ the

subgroup of Cartier divisors on Δ whose supports are contained in $\hat{\sigma}(1) := \Delta(1) \setminus \sigma(1)$. A submonoid $\mathrm{CDiv}(\Delta)_{\sigma \geq 0}$ of $\mathrm{CDiv}(\Delta)$ is defined to be the submonoid $\{D = ([m_\tau]; \tau \in \Delta_{\max}) \in \mathrm{CDiv}(\Delta); \langle m_\sigma, n_\rho \rangle \geq 0 \text{ for all } \rho \in \sigma(1)\}$.

(3) A monoid ideal $\mathrm{CDiv}(\Delta)^+$ of $\mathrm{CDiv}(\Delta)_{\geq 0}$ is defined to be the ideal generated by $\bigcup_{\sigma \in \Delta} \mathrm{CDiv}(\hat{\sigma})^+$.

REMARK 1.4. It is easy to see that given $\sigma \in \Delta$, we have $\mathrm{CDiv}(\hat{\sigma})^+ \ni \{0\}$ if and only if Δ is affine with $|\Delta| = \sigma$. Hence $\mathrm{CDiv}(\Delta)^+ = \mathrm{CDiv}(\Delta)_{\geq 0}$ if and only if Δ is affine.

The following notion on a fan is important in constructing the associated toric variety as a geometric quotient (Theorem 1.9).

DEFINITION 1.5. Let Δ be a fan.

(1) A cone σ is said to be *good in* Δ if $\mathrm{CDiv}(\hat{\sigma})^+$ is not empty. We denote by Δ^{good} the set of good cones in Δ.

(2) A fan Δ is said to be *good* if $\Delta^{\mathrm{good}} = \Delta$.

(3) The toric variety associated to a good fan Δ is said to *have enough invariant effective Cartier divisors.*

REMARK 1.6. (1) For a good cone σ, any face τ of σ is good because for $m \in M \cap \sigma^\vee$ with $m^\perp \cap \sigma = \tau$ and $D \in \mathrm{CDiv}(\hat{\sigma})^+$, we have $lD + \mathrm{div}(m) \in \mathrm{CDiv}(\hat{\tau})^+$ for $l \gg 0$. This shows that the above set Δ^{good} forms a subfan of Δ. This argument also shows that Δ is good if $\mathrm{CDiv}(\hat{\sigma})^+ \neq \varnothing$ for each maximal cone $\sigma \in \Delta_{\max}$.

(2) Although such a fan in Definition 1.5 should be said to have enough support functions, we adopt the terminology in Definition 1.5 for simplicity.

(3) Both simplicial fans and quasi-projective fans (i.e., the associated toric varieties are quasi-projective) are easily seen to be good. Hence both simplicial toric varieties and quasi-projective toric varieties have enough invariant effective Cartier divisors.

(4) There exists a complete fan Δ with $\mathrm{Pic}(\Delta) = (0)$ (cf. [4]). For such a fan, we have $\Delta^{\mathrm{good}} = \varnothing$.

Throughout this paper (except in Corollary 1.11 and Remarks 4.4 and 4.6), we assume that the set $\Delta(1)$ of one-dimensional cones in a fan Δ spans $N_{\boldsymbol{Q}}$. Then we have a fundamental exact sequence (cf., e.g., [5, 3.4]):

$$0 \longrightarrow M \xrightarrow{\mathrm{div}} \mathrm{CDiv}(\Delta) \xrightarrow{\deg} \mathrm{Pic}(\Delta) \longrightarrow 0 . \tag{1.6.b}$$

The proof of the following elementary but useful lemma is left to the reader.

LEMMA 1.7. *Let Δ be a fan.*

(1) *For a good cone $\sigma \in \Delta$, the monoid $\mathrm{CDiv}(\Delta)_{\sigma \geq 0}$ is generated by $\mathrm{CDiv}(\Delta)_{\geq 0} \cup (-\mathrm{CDiv}(\hat{\sigma})^+)$ as a submonoid. Here $-\mathrm{CDiv}(\hat{\sigma})^+$ is the set $\{D \in \mathrm{CDiv}(\Delta); -D \in \mathrm{CDiv}(\hat{\sigma})^+\}$. Moreover, $\mathrm{CDiv}(\hat{\sigma})$ is exactly the group of invertible elements in $\mathrm{CDiv}(\Delta)_{\sigma \geq 0}$.*

(2) *For every cone $\sigma \in \Delta$, the image of $\mathrm{CDiv}(\hat{\sigma})$ by* deg *in* (1.6.b) *is equal to* $\mathrm{Pic}(\Delta)$.

(3) *For every cone* $\sigma \in \Delta$, *we have*

$$M \cap \sigma^{\vee} \cong \operatorname{div}(M) \cap \operatorname{CDiv}(\Delta)_{\sigma \geq 0}$$
$$\cong \text{the kernel of } \deg : \operatorname{CDiv}(\Delta)_{\sigma \geq 0} \to \operatorname{Pic}(\Delta) .$$

We now introduce the homogeneous coordinate ring of a toric variety defined in [1, p. 35].

DEFINITION 1.8. Let Δ be a fan.

(1) The *homogeneous coordinate ring of* Δ is defined to be the monoid ring $S = S_{\Delta} := k[\xi^D ; D \in \operatorname{CDiv}(\Delta)_{\geq 0}]$ of $\operatorname{CDiv}(\Delta)_{\geq 0}$, where ξ^D is a symbol, with the multiplication defined by $\xi^D \cdot \xi^{D'} := \xi^{D+D'}$ for $D, D' \in \operatorname{CDiv}(\Delta)_{\geq 0}$.

(2) The *exceptional ideal* $B = B_{\Delta}$ *of* Δ (resp. B_{σ} of a cone $\sigma \in \Delta$) is defined to be the ideal generated by $\{\xi^D ; D \in \operatorname{CDiv}(\Delta)^+\}$ (resp. $\{\xi^D ; D \in \operatorname{CDiv}(\hat{\sigma})^+\}$).

From now on, we regard the homogeneous coordinate ring S_{Δ} as a $\operatorname{Pic}(\Delta)_{\geq 0}$-graded ring with $\deg \xi^D := \deg D \in \operatorname{Pic}(\Delta)_{\geq 0}$. A $\operatorname{Pic}(\Delta)_{\geq 0}$-graded ring is called a *Δ-graded* ring for simplicity.

The main theorem in this section is as follows:

THEOREM 1.9. *Let* Δ *be a fan with* $\Delta^{\text{good}} \neq \emptyset$.

(1) *The algebraic group* $G_0 := \operatorname{Hom}(\operatorname{CDiv}(\Delta), \boldsymbol{G}_m)$ *canonically acts on* $\widetilde{X}_{\Delta} := \operatorname{Spec} S_{\Delta} \setminus V(B_{\Delta})$. *Here* $V(B_{\Delta})$ *denotes the closed subset* $\{\mathfrak{p} \in \operatorname{Spec} S_{\Delta} ; B_{\Delta} \subset \mathfrak{p}\}$ *of* $\operatorname{Spec} S_{\Delta}$.

(2) *There exists the universal geometric quotient* $(Y, \pi) = (Y_{\Delta}, \pi_{\Delta})$ *of* $\widetilde{X}_{\Delta}$ *with respect to* $G := \operatorname{Hom}(\operatorname{Pic}(\Delta), \boldsymbol{G}_m)$, *and* Y *has a canonical action of* $T = G_0/G$.

(3) *The above quotient* Y *is canonically isomorphic to the toric variety associated to* Δ^{good}.

PROOF. We first define a morphism $\pi = \pi_{\Delta} : \widetilde{X}_{\Delta} \to X_{\Delta^{\text{good}}}$. Let σ be a good cone in Δ. For each $D, D' \in \operatorname{CDiv}(\hat{\sigma})^+$ and $m \gg 0$, we have $mD - D' \in \operatorname{CDiv}(\hat{\sigma})^+$. This shows that the closed subset $V(B_{\sigma})$ equals $V(\xi^D)$ for any $D \in \operatorname{CDiv}(\hat{\sigma})^+$. Hence $\widetilde{U}_{\sigma} = \operatorname{Spec} S_{\Delta} \setminus V(B_{\sigma})$ is an affine scheme $\operatorname{Spec} S_{\Delta, \xi^D} = \operatorname{Spec} k[\xi^D ; D \in \operatorname{CDiv}(\Delta)_{\sigma \geq 0}]$. See the last equation for Lemma 1.7 (1). For $\sigma \in \Delta^{\text{good}}$, a morphism π_{σ} is defined by the injective homomorphism $\operatorname{div} : M \cap \sigma^{\vee} \to \operatorname{CDiv}(\Delta)_{\sigma \geq 0}$. Using an argument similar to that in Remark 1.6 (1), we have the following commutative diagram for each cone σ and each face $\tau \prec \sigma$:

$$\begin{array}{ccc} \widetilde{U}_{\tau} & \xrightarrow{\subset} & \widetilde{U}_{\sigma} \\ {\scriptstyle \pi_{\tau}}\downarrow & & \downarrow{\scriptstyle \pi_{\sigma}} \\ U_{\tau} & \xrightarrow{\subset} & U_{\sigma} . \end{array}$$

Gluing π_{σ} ($\sigma \in \Delta^{\text{good}}$), we have the morphism $\pi = \pi_{\Delta} : \widetilde{X}_{\Delta} \to X_{\Delta^{\text{good}}}$.

To complete the proof, it suffices to show that $(\widetilde{U}_{\sigma}, \pi|_{\widetilde{U}_{\sigma}} : \widetilde{U}_{\sigma} \to U_{\sigma})$ is the universal

geometric quotient of $\widetilde{U_\sigma}$ with respect to G. Since $M\cap\sigma^\perp$ is a direct summand of M, we take a section $s: M/(M\cap\sigma^\perp)\to M$. We remark that the section s maps $(M\cap\sigma^\vee)/(M\cap\sigma^\perp)$ to $M\cap\sigma^\vee$ because $M\cap\sigma^\perp$ is a *subgroup* of the monoid $M\cap\sigma^\vee$. By Lemma 1.7 (2) and the snake lemma, the group $M/(M\cap\sigma^\perp)$ is isomorphic to $\mathrm{CDiv}(\Delta)/\mathrm{CDiv}(\hat{\sigma})$ via div. Moreover, this gives an isomorphism $\iota: (M\cap\sigma^\vee)/(M\cap\sigma^\perp)\to \mathrm{CDiv}(\Delta)_{\sigma\geq 0}/\mathrm{CDiv}(\hat{\sigma})$. Hence we have a commutative diagram

$$\begin{array}{ccc} \mathrm{CDiv}(\hat{\sigma})\oplus(\mathrm{CDiv}(\Delta)_{\sigma\geq 0}/\mathrm{CDiv}(\hat{\sigma})) & \xrightarrow{\mathrm{incl}\oplus(\mathrm{div}\circ s\circ \iota^{-1})} & \mathrm{CDiv}(\Delta)_{\sigma\geq 0} \\ {\scriptstyle \mathrm{div}\oplus\iota}\uparrow & & \uparrow{\scriptstyle \mathrm{div}} \\ (M\cap\sigma^\perp)\oplus((M\cap\sigma^\vee)/(M\cap\sigma^\perp)) & \xrightarrow[\mathrm{incl}\oplus s]{} & M\cap\sigma^\vee\,. \end{array}$$

Here both of the horizontal arrows are isomorphisms. This shows that π_σ is the base change of $\mathrm{Spec}\,k[\mathrm{CDiv}(\hat{\sigma})]\to\mathrm{Spec}\,k[M\cap\sigma^\perp]$ by the first projection

$$U_\sigma\cong\mathrm{Spec}\,k[M\cap\sigma^\perp]\times_k\mathrm{Spec}\,k[(M\cap\sigma^\vee)/(M\cap\sigma^\perp)]\to\mathrm{Spec}\,k[M\cap\sigma^\perp]\,.$$

Therefore we have only to show that the exact sequence

$$0\longrightarrow M\cap\sigma^\perp\xrightarrow{\mathrm{div}}\mathrm{CDiv}(\hat{\sigma})\xrightarrow{\deg}\mathrm{Pic}(\Delta)\longrightarrow 0$$

induces an isomorphism between $\mathrm{Spec}\,k[M\cap\sigma^\perp]$ and the geometric quotient of $\mathrm{Spec}\,k[\mathrm{CDiv}(\hat{\sigma})]$ of with respect to G. This follows from the fact that $\mathrm{Spec}\,k[\mathrm{CDiv}(\hat{\sigma})]$ is a G-torsor over $\mathrm{Spec}\,k[M\cap\sigma^\perp]$ with respect to the fppf topology. □

REMARK 1.10. One can easily prove that a commutative diagram of monoids

$$\begin{array}{ccc} \mathrm{CDiv}(\hat{\sigma}) & \longrightarrow & \mathrm{CDiv}(\Delta)_{\sigma\geq 0} \\ \uparrow & & \uparrow \\ M\cap\sigma^\perp & \longrightarrow & M\cap\sigma^\vee \end{array}$$

is a push-out in the category of (commutative) monoids. This gives another proof that a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\,k[\mathrm{CDiv}(\hat{\sigma})] & \longleftarrow & \mathrm{Spec}\,k[\mathrm{CDiv}(\Delta)_{\sigma\geq 0}] \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,k[M\cap\sigma^\perp] & \longleftarrow & \mathrm{Spec}\,k[M\cap\sigma^\vee] \end{array}$$

is Cartesian.

COROLLARY 1.11. *Let Δ be a good fan whose set of one-dimensional cones $\Delta(1)$ may not span $N_{\mathbf{Q}}$, and $\Delta(1)_{\mathbf{Q}}$ the subspace of $N_{\mathbf{Q}}$ generated by $\Delta(1)$. Let us denote by $N_0 := N \cap \Delta(1)_{\mathbf{Q}}$ (resp. $\Delta_0 := N_0 \cap \Delta$) the sublattice of N contained in $\Delta(1)_{\mathbf{Q}}$ (resp. the fan of N_0 induced by Δ). Then, in the notation of Theorem* 1.9, *the associated toric variety X_Δ is isomorphic to the universal geometric quotient of $\widetilde{X}_{\Delta_0} \times_k (\boldsymbol{G}_m \otimes N/N_0)$ with respect to $G = \mathrm{Hom}(\mathrm{Pic}(\Delta), \boldsymbol{G}_m)$. This isomorphism depends on the choice of a section $N/N_0 \to N$.*

PROPOSITION 1.12. *Let Δ be a good fan. For a homogeneous ideal I of $S := S_\Delta$ (with respect to the Δ-grading), we denote by $V_+(I) := \pi_\Delta(V(I) \cap \widetilde{X}_\Delta)$ the image of the G-stable closed subset $V(I) \cap \widetilde{X}_\Delta$ under π_Δ.*

(1) *(The toric Nullstellensatz.) For any homogeneous ideal $I \subset S$, we have $V_+(I) = \varnothing$ if and only if $B^n \subset I$ for some integer n.*

(2) *(The toric ideal-variety correspondence.) The map $I \mapsto V_+(I)$ induces a one-to-one correspondence between the set of radical homogeneous ideals of S contained in B and the set of closed subsets of X.*

The proof is similar to that in [1, 2.4], and left to the reader.

Finally, we make a few remarks on the homogeneous spectrum of S_Δ with respect to the Δ-grading. If Δ is good and if $\mathrm{Pic}(\Delta)$ is free, one can define the associated toric variety X_Δ as the homogeneous spectrum of S_Δ consisting of homogeneous prime ideals with respect to the Δ-grading as in [3] and in [9, II §2]. The freeness of $\mathrm{Pic}(\Delta)$ implies every G-orbit is irreducible, so each closed orbit corresponds to a homogeneous prime ideal of S_Δ. However, $\mathrm{Pic}(\Delta)$ may not be free even if $\Delta(1)$ spans $N_{\mathbf{Q}}$ and X_Δ is smooth. (The assertion otherwise in [5, 3.4] is to be corrected in the next printing.) For instance, look at the toric variety $X_\Delta = \mathrm{Spec}(k[x, y, z]/(xy - z^2)) \setminus \{(0, 0, 0)\}$ associated to $\Delta = \{(0, 0), \boldsymbol{Q}_{\geq 0}(1, 0), \boldsymbol{Q}_{\geq 0}(1, 2)\}$. The following is a necessary and sufficient condition for $\mathrm{Pic}(\Delta)$ to be free:

LEMMA 1.13. *The Picard group $\mathrm{Pic}(\Delta)$ of a fan Δ is free if and only if so is N/N', where N' denotes the subgroup of N generated by $N \cap |\Delta|$.*

REMARK 1.14. The abelian group N/N' is isomorphic to the fundamental group $\pi_1(X_\Delta)$ of X_Δ if $k = \boldsymbol{C}$ (cf. [5, p. 57]). Hence $\mathrm{Pic}(\Delta)$ is free if and only if so is $\pi_1(X_\Delta)$.

PROOF. Consider the exact sequence

$$0 \longrightarrow \mathrm{Hom}(N/N', \boldsymbol{Z}) \longrightarrow \mathrm{Hom}(N, \boldsymbol{Z}) \xrightarrow{\mathrm{res}} \mathrm{Hom}(N', \boldsymbol{Z}) \longrightarrow \mathrm{Ext}^1(N/N', \boldsymbol{Z}) \longrightarrow 0 .$$

It is easy to see that $\mathrm{Hom}(N', \boldsymbol{Z})$ is equal to the group of Cartier divisors whose image in $\mathrm{Pic}(\Delta)$ is of finite order. Hence $\mathrm{Pic}(\Delta)$ is free if and only if the homomorphism res is surjective. □

2. Quasi-coherent modules associated to graded modules. Let Δ be a good fan, $S:=S_\Delta$ the Δ-graded homogeneous coordinate ring of Δ, and $X:=X_\Delta$ the associated toric variety over k as in Section 1. In this section we introduce the quasi-coherent $\mathcal{O}_X$-module associated to a Δ-graded S-module and state its properties. We omit proofs if they are similar to those in [3], [8] and [9, II §2.5]. Let M and N be Δ-graded S-modules in an obvious sense. (Although we have already used the notation M and N for free abelian groups in Section 1, we adopt this notation only in this section without fear of confusion.) Let $S_{(f)}$ (resp. $M_{(f)}$) be the subring (resp. the $S_{(f)}$-submodule) of elements of degree zero in $S_f:=S[1/f]$ (resp. $M_f:=M\otimes_S S_f$) for each homogeneous element $f\in B:=B_\Delta$. Let U_σ be the open affine toric subvariety of X associated to a cone $\sigma\in\Delta$.

PROPOSITION 2.1. *For a Δ-graded S-module M, there exists a unique quasi-coherent $\mathcal{O}_X$-module $\tilde{M}$ satisfying the following*:

(a) $\tilde{M}|_{U_\sigma}=(M_{(\xi^D)})^\sim$ *for each* $D\in\mathrm{CDiv}(\hat{\sigma})^+$;

(b) *for a cone σ and its face τ, the canonical homomorphism $M_{\xi^{D_\sigma}}\to M_{\xi^{D_\tau}}$ with $D_\sigma\in\mathrm{CDiv}(\hat{\sigma})^+$ and $D_\tau\in\mathrm{CDiv}(\hat{\tau})^+$ induces an isomorphism $(M_{(\xi^{D_\sigma})})^\sim|_{U_\tau}\to(M_{(\xi^{D_\tau})})^\sim$.*

PROOF. See [9, II (2.5.2)]. In the notation of Theorem 1.9, we remark that $\pi_\Delta:\tilde{X}_\Delta\to Y=X_\Delta$ is an affine morphism (cf. the proof of Theorem 1.9). □

DEFINITION 2.2. The quasi-coherent $\mathcal{O}_X$-module $\tilde{M}$ in Proposition 2.1 is said to be the quasi-coherent $\mathcal{O}_X$-module *associated to* the Δ-graded S-module M.

PROPOSITION 2.3. *The map $M\mapsto\tilde{M}$ gives a covariant additive exact functor from the category of Δ-graded S-modules to that of quasi-coherent $\mathcal{O}_X$-modules, and commutes with direct limits and direct sums.*

PROOF. The question is local on X. This immediately follows from Proposition 2.1. □

PROPOSITION 2.4. (1) *If a Δ-graded S-module M is of finite type, then $\tilde{M}$ is a coherent $\mathcal{O}_X$-module.*

(2) *Let M be a Δ-graded S-module of finite type. Then $\tilde{M}=0$ if and only if $B^mM=0$ for $m\gg 0$.*

PROOF. (1) follows from Proposition 2.3 and a surjective homomorphism $S^{\oplus n}\to M$ for some n. We remark that $\tilde{S}$ is canonically isomorphic to $\mathcal{O}_X$.

(2) Note that X is Noetherian and that $\mathrm{CDiv}(\hat{\sigma})^+$ is finitely generated for each cone $\sigma\in\Delta$. One can verify (2) by an argument similar to that in [9, (2.7.3)]. □

DEFINITION 2.5. (1) For $\alpha\in\mathrm{Pic}(\Delta)$ and a Δ-graded S-module M, we define $M(\alpha)$ to be the Δ-graded module with $M(\alpha)_\beta=M_{\alpha+\beta}$ for each $\beta\in\mathrm{Pic}(\Delta)$.

(2) $\mathcal{O}_X(\alpha)$ denotes the quasi-coherent $\mathcal{O}_X$-module $S(\alpha)^\sim$ for $\alpha\in\mathrm{Pic}(\Delta)$.

(3) For $\alpha \in \mathrm{Pic}(\Delta)$ and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the $\mathcal{O}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha)$ is denoted by $\mathcal{F}(\alpha)$.

PROPOSITION 2.6. (1) *For each $\alpha \in \mathrm{Pic}(\Delta)$, the quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X(\alpha)$ is an invertible sheaf.*

(2) *For each $D \in \mathrm{CDiv}(\Delta)$, the invertible sheaf $\mathcal{O}_X(D)$ is canonically isomorphic to $\mathcal{O}_X(\alpha)$, where $\alpha \in \mathrm{Pic}(\Delta)$ is the isomorphism class of $\mathcal{O}_X(D)$.*

PROOF. Let us take $D = ([m_\sigma]; \sigma \in \Delta_{\max}) \in \mathrm{CDiv}(\Delta)$ with $\deg D = \alpha$.

(1) For each cone $\sigma \in \Delta$, one can easily show that $\mathcal{O}_X(\alpha)|_{U_\sigma}$ is a free $\mathcal{O}_{U_\sigma}$-module of rank one with $\xi^{D - \mathrm{div}(m_\sigma)}$ as a basis.

(2) Let us regard $\mathcal{O}_X(D)$ as a locally principal $\mathcal{O}_X$-submodule of the function field $k(X)$ of X. By multiplication of ξ^D, we have a canonical isomorphism from $\mathcal{O}_X(D)$ to $\mathcal{O}_X(\alpha)$. □

Let M and N be Δ-graded S-modules. Glueing canonical homomorphisms on U_σ's, we obtain a functorial homomorphism of $\mathcal{O}_X$-modules

$$\lambda : \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \to (M \otimes_S N)^{\sim} .$$

See [9, II (2.5.11.2)].

PROPOSITION 2.7. *The above homomorphism λ is an isomorphism.*

PROOF. We have only to show that for each cone $\sigma \in \Delta$, the restriction λ_σ of λ to U_σ is an isomorphism. Set $D \in \mathrm{CDiv}(\hat{\sigma})^+$. We first remark that $M_{\xi^D} \otimes_{S_{\xi^D}} N_{\xi^D}$ is isomorphic to the quotient of the bigraded $\boldsymbol{Z}$-module $M_{\xi^D} \otimes_{\boldsymbol{Z}} N_{\xi^D}$ modulo the $\boldsymbol{Z}$-submodule generated by

$$\{am \otimes n - m \otimes an ; \text{homogeneous elements } m \in M_{\xi^D},\ n \in N_{\xi^D}, \text{ and } a \in S_{\xi^D}\} .$$

Let us take a set-theoretic section $s : \mathrm{Pic}(\Delta) \to \mathrm{CDiv}(\hat{\sigma})$ with $s(0) = 0$. Then we have $\{\xi^{s(\alpha)} \in (S_{\xi^D})^\times ; \alpha \in \mathrm{Pic}(\Delta)\}$ and $\{c_{\alpha,\beta} \in (S_{(\xi^D)})^\times ; \alpha, \beta \in \mathrm{Pic}(\Delta)\}$ with $\xi^{s(\alpha+\beta)} = c_{\alpha,\beta} \xi^{s(\alpha)+s(\beta)}$. Simple calculation shows $c_{\alpha+\beta,\gamma} c_{\alpha,\beta} = c_{\alpha,\beta+\gamma} c_{\beta,\gamma}$. We define an $S_{(\xi^D)}$-module homomorphism ε as

$$m \otimes n \in (M_{\xi^D})_\alpha \otimes_{S_{\xi^D}} (N_{\xi^D})_\beta \mapsto c_{-\alpha,-\beta} \xi^{s(-\alpha)} m \otimes \xi^{s(-\beta)} n \in M_{(\xi^D)} \otimes_{S_{(\xi^D)}} N_{(\xi^D)} ,$$

where is well-defined because of the above formula. (See the beginning of this section for the notation.) Here we denote by $(M_{\xi^D})_\alpha$ (resp. $(N_{\xi^D})_\beta$) the S_{ξ^D}-submodule of elements of degree α in M_{ξ^D} (resp. of degree β in N_{ξ^D}). One can easily verify that $\lambda \circ \varepsilon$ is an isomorphism and that $\varepsilon \circ \lambda = \mathrm{id}$. □

COROLLARY 2.8. *For every $\alpha, \beta \in \mathrm{Pic}(\Delta)$ and $n \in \boldsymbol{Z}$, we have the following canonical isomorphisms*:

(2.8.i) $$\mathcal{O}_X(\alpha) \otimes_{\mathcal{O}_X} \mathcal{O}_X(\beta) \cong \mathcal{O}_X(\alpha + \beta) ;$$

(2.8.ii) $$\mathcal{O}_X(\alpha)^{\otimes n} \cong \mathcal{O}_X(n\alpha) .$$

COROLLARY 2.9. *For $\alpha \in \mathrm{Pic}(\Delta)$ and a Δ-graded S-module M, the quasi-coherent $\mathcal{O}_X$-module $M(\alpha)^{\sim}$ is canonically isomorphic to $\tilde{M}(\alpha) = \tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha)$.*

Let us denote by $\Gamma_*(\mathcal{F})$ the direct sum $\bigoplus_{\alpha \in \mathrm{Pic}(\Delta)} \Gamma(X, \mathcal{F}(\alpha))$ for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. The module $\Gamma_*(\mathcal{F})$ has a natural structure of a Δ-graded $\Gamma_*(\mathcal{O}_X)$-module by Corollary 2.9.

We now define two homomorphisms $v = v_M$ and $\mu = \mu_{\mathcal{F}}$ for a Δ-graded S-module M and for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ as follows. See [9, II §2.6].

Let M be a Δ-graded S-module. By a canonical homomorphism $v_\alpha : M_\alpha \to \Gamma(X, \tilde{M}(\alpha))$ for each $\alpha \in \mathrm{Pic}(\Delta)$, we have a homomorphism $v = \bigoplus v_\alpha : M \to \Gamma_*(\tilde{M})$ of Δ-graded modules. See [9, (2.6.2)]. One can verify that $v : S \to \Gamma_*(\tilde{S}) = \Gamma_*(\mathcal{O}_X)$ is an isomorphism as Δ-graded k-algebras (cf., e.g., [1, Proof of 3.1]).

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Using Corollary 2.9, the homomorphism

$$\mu|_{U_\sigma} : \Gamma_*(\mathcal{F})_{(\xi^D)} \to \Gamma(U_\sigma, \mathcal{F}) ; \qquad m/\xi^{D'} \mapsto (m|_{U_\sigma})/\xi^{D'}$$

is well-defined for each $\sigma \in \Delta$. Since these homomorphisms are compatible with the restriction homomorphisms for these sheaves, we have a homomorphism $\mu = \mu_{\mathcal{F}} : \Gamma_*(\mathcal{F})^{\sim} \to \mathcal{F}$.

PROPOSITION 2.10. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the homomorphism $\mu : \Gamma_*(\mathcal{F})^{\sim} \to \mathcal{F}$ is an isomorphism. In particular, every quasi-coherent module $\mathcal{F}$ is of the form $\tilde{M}$ for some Δ-graded S-module M.*

REMARK 2.11. Cox [1, 3.2] proved Proposition 2.10 for a simplicial toric variety X.

One can easily prove the above proposition by [8, II.5.14] and the following lemma:

LEMMA 2.12. *For each cone $\sigma \in \Delta$ and each $D \in \mathrm{CDiv}(\hat{\sigma})^+$, we have*

(2.12.i) $$U_\sigma = \{x \in X ; \xi^D \notin \mathfrak{m}_{X,x} \mathcal{O}_X(\deg D)_x\} .$$

The proof is straightforward, and left to the reader.

COROLLARY 2.13. *For every coherent $\mathcal{O}_X$-module $\mathcal{F}$, there exists a finitely generated Δ-graded S-module N with $\tilde{N} \cong \mathcal{F}$.*

PROOF. See [9, II (2.7.8)]. □

PROPOSITION 2.14. *Let M be a Δ-graded S-module and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then both of the following two composites are the identity homomorphisms:*

(2.14.i) $$\tilde{M} \xrightarrow{\tilde{v}} \Gamma_*(\tilde{M})^{\sim} \xrightarrow{\mu} \tilde{M} ;$$

(2.14.ii) $$\Gamma_*(\mathscr{F}) \xrightarrow{\nu} \Gamma_*(\Gamma_*(\mathscr{F})^{\sim}) \xrightarrow{\Gamma_*(\mu)} \Gamma_*(\mathscr{F}).$$

PROOF. The question is local on X. It is straightforward to show that the above composites are the identity homomorphisms on U_σ for each $\sigma \in \Delta$. The detail is left to the reader. □

COROLLARY 2.51. *Let the notation be as in Proposition* 2.14.

(1) $\tilde{\nu}_M : \tilde{M} \to \Gamma_*(\tilde{M})^{\sim}$ *is an isomorphism for each* Δ*-graded* S*-module* M.

(2) $\mu_{\mathscr{F}} : \Gamma_*(\mathscr{F}) \to \Gamma_*(\Gamma_*(\mathscr{F})^{\sim})$ *is an isomorphism for each quasi-coherent* $\mathcal{O}_X$*-module* $\mathscr{F}$.

PROOF. This immediately follows from Propositions 2.10 and 2.14. □

COROLLARY 2.16. (1) *Let* Y *be a closed subscheme of* X *with the defining ideal* $\mathscr{J}$. *Then there exists a homogeneous ideal* I *of* S *contained in* B *with* $\tilde{I} = \mathscr{J}$. *Moreover, in the notation of Theorem* 1.9, Y *is the geometric quotient of the closed subscheme* $\tilde{X}_\Delta \cap \operatorname{Spec} S/I$ *with respect to* G.

(2) *Two homogeneous ideals* I *and* J *of* S *contained in* B *define the same closed subscheme of* X *if and only if* $(I : B^m) = (J : B^m)$ *for* $m \gg 0$.

PROOF. (1) Clearly $I := \Gamma_*(\mathscr{J})$ is a homogeneous ideal of $\Gamma_*(\mathcal{O}_X) = S$. By Propositions 2.3 and 2.4, and Corollary 2.15 (2), we have $(I \cap B)^{\sim} = \tilde{I} = \mathscr{J}$. The last assertion is verified in the same way as in Theorem 1.9.

(2) We have only to show the assertion when I is contained in J, say, $J = \Gamma_*(I)$. It follows from Propositions 2.3 and 2.4, and Corollary 2.15 (2) that $\tilde{I}$ is equal to $(I : B^m)^{\sim}$ for each $m > 0$, and that $(I : B^m)$ contains J for some integer $m > 0$. By the same argument, $(I : B^{m+m'})$ contains $(J : B^m)$ for some $m' > 0$. Since S is Noetherian, we have $(I : B^m) = (J : B^m)$ for $m \gg 0$. □

3. Main Result. Throughout this section, let Δ be a good fan, S the homogeneous coordinate ring of Δ, B the ideal generated by $\mathrm{CDiv}(\Delta)^+$, I a homogeneous ideal of S contained in B, $X := X_\Delta$ the associated toric variety, and Z the closed subscheme defined by I (see Corollary 2.16). In this section, we first define on a k-scheme Y a Δ-graded $\mathcal{O}_Y$-algebra with invertible components, which is a generalization of the tensor algebra associated to an invertible sheaf on X. We next prove a one-to-one correspondence between the set of morphisms of Y to Z and the set of equivalence classes of those Δ-graded $\mathcal{O}_Y$-algebra homomorphisms which satisfy a nondegeneracy condition from $\mathcal{O}_Y \otimes_k S/I$ to a Δ-graded algebra with invertible components (Theorem 3.4).

DEFINITION 3.1. Let Δ be a good fan, and Y a k-scheme. A Δ-graded $\mathcal{O}_Y$-algebra $\mathscr{L} = \bigoplus_{\alpha \in \mathrm{Pic}(\Delta)_{\geq 0}} \mathscr{L}_\alpha$ is said to be a *Δ-graded $\mathcal{O}_Y$-algebra with invertible components* if it satisfies the following conditions:

(1) each homogeneous component $\mathscr{L}_\alpha$ ($\alpha \in \mathrm{Pic}(\Delta)_{\geq 0}$) is a locally free $\mathcal{O}_Y$-module

of rank one;

(2) the $\mathcal{O}_Y$-module $\mathscr{L}_\alpha \otimes_{\mathcal{O}_Y} \mathscr{L}_\beta$ is isomorphic to $\mathscr{L}_{\alpha+\beta}$ via the product of $\mathscr{L}$ for each pair of α and β in $\mathrm{Pic}(\Delta)_{\geq 0}$.

We denote by $B(\Delta)$ (resp. $\mathscr{B}_{\mathscr{L}}$) the image of $\mathrm{CDiv}(\Delta)^+$ by deg (resp. the homogeneous ideal $\bigoplus_{\alpha \in B(\Delta)} \mathscr{L}_\alpha \subset \mathscr{L}$). Note that $B(\Delta)$ is a monoid ideal of $\mathrm{Pic}(\Delta)_{\geq 0}$ since $\mathrm{CDiv}(\Delta)_{\geq 0} \to \mathrm{Pic}(\Delta)_{\geq 0}$ is surjective by definition.

EXAMPLE 3.2. We illustrate Definition 3.1 by looking at the r-dimensional projective space $\boldsymbol{P}^r$. Let Δ be the fan defining the projective space $\boldsymbol{P}^r$ (cf., e.g., [5, p. 22]). By Definition 3.1, every Δ-graded algebra $\mathscr{L}$ with invertible components is canonically isomorphic to $\bigoplus_{n \in \boldsymbol{Z}_{\geq 0}} (\mathscr{L}_1)^{\otimes n}$. Hence giving a Δ-graded $\mathcal{O}_{\boldsymbol{P}^r}$-algebra with invertible components is equivalent to giving an invertible sheaf $\mathscr{L}_1$ on $\boldsymbol{P}^r$.

EXAMPLE 3.3. For the associated toric variety X_Δ, we have a canonical Δ-graded $\mathcal{O}_X$-algebra $\bigoplus_{\alpha \in \mathrm{Pic}(\Delta)_{\geq 0}} \mathcal{O}_X(\alpha)$ with invertible components, and a canonical Δ-graded algebra homomorphism $\mathcal{O}_X \otimes_k S \to \bigoplus_{\alpha \in \mathrm{Pic}(\Delta)_{\geq 0}} \mathcal{O}_X(\alpha)$ using Corollary 2.8 and a homomorphism v in Section 2. Here a Δ-graded algebra homomorphism is an algebra homomorphism preserving the Δ-grading. In the case of Example 3.2, this Δ-graded $\mathcal{O}_{\boldsymbol{P}^r}$-algebra is nothing but $\bigoplus_{n \geq 0} \mathcal{O}_{\boldsymbol{P}^r}(n)$. For a closed subscheme Z of X, we define a canonical Δ-graded $\mathcal{O}_Z$-algebra $\bigoplus_{\alpha \in \mathrm{Pic}(\Delta)} \mathcal{O}_Z(\alpha)$ (resp. a canonical Δ-graded algebra homomorphism $\mathcal{O}_Z \otimes_k S/I \to \bigoplus_{\alpha \in \mathrm{Pic}(\Delta)_{\geq 0}} \mathcal{O}_Z(\alpha)$) to be the restriction to Z, where $I \subset B$ is a homogeneous ideal of S defining Z.

THEOREM 3.4. *Let Δ be a good fan, Y a k-scheme, and Z a closed subscheme of $X = X_\Delta$ defined by a homogeneous ideal $I \subset B$ of S. Then there exists a canonical one-to-one correspondence between the following two sets*:

(a) *the set of k-morphisms from Y to Z;*

(b) *the set of equivalence classes of pairs $(\mathscr{L}, \varphi)$ of a Δ-graded $\mathcal{O}_Y$-algebra $\mathscr{L}$ with invertible components, and a Δ-graded $\mathcal{O}_Y$-algebra homomorphism $\varphi: \mathcal{O}_Y \otimes_k S/I \to \mathscr{L}$ satisfying the following nondegeneracy condition:*

> (nondegeneracy): *For every point $y \in Y$, there exists a homogeneous element $f \in B/I$ with $\varphi_y(1 \otimes f) \notin \mathfrak{m}_y \mathscr{L}_y$.*

Here, $(\mathscr{L}, \varphi)$ and $(\mathscr{L}', \varphi')$ are said to be equivalent if there exists an isomorphism $\iota: \mathscr{L} \to \mathscr{L}'$ of Δ-graded $\mathcal{O}_Y$-algebras with $\varphi' = \iota \circ \varphi$.

PROOF. We first define a correspondence between (a) and (b).

(a)$\to$(b): Let $r: Y \to Z$ be a k-morphism. By the pull-back of the canonical Δ-graded $\mathcal{O}_Z$-algebra homomorphism $\mathcal{O}_Z \otimes_k S/I \to \bigoplus_{\alpha \in \mathrm{Pic}(\Delta)_{\geq 0}} \mathcal{O}_Z(\alpha)$ in Example 3.3, we have a Δ-graded $\mathcal{O}_Y$-algebra homomorphism $\varphi_r: \mathcal{O}_Y \otimes_k S/I = r^* \mathcal{O}_Z \otimes_k S/I \to \mathscr{L}_r := \bigoplus r^* \mathcal{O}_Z(\alpha)$. This homomorphism satisfies the nondegeneracy condition because $Y = \bigcup_\sigma r^{-1}(U_\sigma \cap Z)$.

(b)$\to$(a): We may assume that for any $y \in Y$, there exists a cone $\sigma_y \in \Delta$ such that $\varphi_y(\xi^D) \notin \mathfrak{m}_y \mathscr{L}_y$ for each $D \in \mathrm{CDiv}(\hat{\sigma}_y)^+$. Since $\mathrm{Pic}(\Delta)_{\geq 0}$ is finitely generated, we can take

an open affine neighborhood V_y of y such that all of the invertible sheaves $\mathscr{L}_\alpha$ on V_y are trivial with $\{\varphi(\xi^{D'}); D' \in \mathrm{CDiv}(\hat{\sigma}_y)^+\}$ as a basis. By restricting φ to V_y, we have a k-algebra homomorphism $(S/I)_{(\xi^D)} \to (\mathscr{L}|_{V_y})_0 = \mathscr{O}_{V_y}$. Hence we have a morphism $r_y: V_y \to U_\sigma \cap Z$. Using φ and the nondegeneracy condition, we glue these morphisms r_y to get a k-morphism $r_{\mathscr{L},\varphi}: Y \to Z$.

Finally, we show the bijectivity of this correspondence modulo the above equivalence relation. Given $(\mathscr{L}, \varphi)$, we first prove that $(\mathscr{L}_{r_{\mathscr{L},\varphi}}, \varphi_{r_{\mathscr{L},\varphi}})$ is equivalent to $(\mathscr{L}, \varphi)$. Locally on Y, we have a unique isomorphism ι which makes commutative the following diagram:

$$\begin{array}{ccc} & \mathscr{O}_Y \otimes_k S/I & \\ \text{canonical} \swarrow & & \searrow \varphi_{r_{\mathscr{L},\varphi}} \\ \bigoplus r_{\mathscr{L},\varphi}^* \mathscr{O}_Z(\alpha) & \xrightarrow{\iota} & \mathscr{L}. \end{array}$$

One can easily glue these isomorphisms, and hence the above two pairs are equivalent. On the other hand, it is straightforward to show that each morphism $r: Y \to Z$ is exactly the morphism defined above by $(\mathscr{L}_r, \varphi_r)$. □

EXAMPLE 3.5. We illustrate that Theorem 3.4 is a generalization of the classical result [9, (4.2.3)] for projective spaces. Let Δ be the fan defining the r-dimensional projective space $\boldsymbol{P}^r$. As in Example 3.2, every Δ-graded algebra on Y with invertible components is of the form $\bigoplus_{n \geq 0} \mathscr{L}^{\otimes n}$ for an invertible sheaf $\mathscr{L}$. In this case, the set in Theorem 3.4 (b) is exactly the set of equivalence classes of surjective homomorphisms $(\mathscr{O}_X)^{r+1} \to \mathscr{L}$.

4. Applications. In this section, we restrict ourselves to the case $Z = X_\Delta$ in Theorem 3.4, study the nondegeneracy condition in Theorem 3.4 in more detail, and generalize all known results on morphisms from a scheme to a toric variety.

The following proposition gives useful conditions equivalent to the nondegeneracy condition in Theorem 3.4 in the case $Z = X_\Delta$.

PROPOSITION 4.1. *Let Y be a k-scheme, Δ a good fan, X_Δ the associated toric variety, $\mathscr{L}$ a Δ-graded $\mathscr{O}_Y$-algebra with invertible components, and φ a Δ-graded $\mathscr{O}_Y$-algebra homomorphism $\mathscr{O}_Y \otimes_k S_\Delta \to \mathscr{L}$. Let $V_{\mathscr{L}}(s)$ denote the closed subset $\{y \in Y; s_y \in \mathfrak{m}_y \mathscr{L}_y\}$ of Y for a global section $s \in \Gamma(Y, \mathscr{L})$. Then the following are equivalent:*

(1) *for each $y \in Y$, there exist a cone $\sigma \in \Delta$ and a divisor $D \in \mathrm{CDiv}(\hat{\sigma})^+$ with $\varphi(\xi^D) \notin \mathfrak{m}_y \mathscr{L}_y$;*

(1′) *for every $y \in Y$, there exists a cone $\sigma \in \Delta$ such that $\varphi(\xi^D) \notin \mathfrak{m}_y \mathscr{L}_y$ for each $D \in \mathrm{CDiv}(\hat{\sigma})^+$;*

(2) $\bigcap_{\sigma \in \Delta} \bigcap_{D \in \mathrm{CDiv}(\hat{\sigma})^+} V_{\mathscr{L}}(\varphi(\xi^D)) = \varnothing$;

(2′) $\bigcap_{\sigma \in \Delta} V_{\mathscr{L}}(\varphi(\xi^{D_\sigma})) = \varnothing$, *where D_σ denotes any element in* $\mathrm{CDiv}(\hat{\sigma})^+$;

(2″) $\bigcap_{\sigma \in \Delta_{\max}} V_{\mathscr{L}}(\varphi(\xi^{D_\sigma})) = \varnothing$, *where D_σ denotes any element in* $\mathrm{CDiv}(\hat{\sigma})^+$;

(3) *(when Δ is simplicial)* $V_{\mathscr{L}}(\varphi(\xi^{mD_{\rho_1}})) \cap \cdots \cap V_{\mathscr{L}}(\varphi(\xi^{mD_{\rho_l}})) = \varnothing$ *for each one-dimensional cones* $\rho_1, \ldots, \rho_l$ *which are not contained in any cone of* Δ, *and for each* $m>0$ *with* $mD_{\rho_i} \in \mathrm{CDiv}(\Delta)$.

PROOF. Clearly (1) and (2) are equivalent. It is easy to see that for D, $D' \in \mathrm{CDiv}(\Delta)_{\geq 0}$, we have $V_{\mathscr{L}}(\varphi(\xi^{D})) \subset V_{\mathscr{L}}(\varphi(\xi^{D'}))$ if $\mathrm{Supp}\, D \subset \mathrm{Supp}\, D'$. This shows that (1) and (1′) (resp. (2), (2′) and (2″)) are equivalent. The statement (2) implies (3) because $\bigcap_{\sigma \in \Delta_{\max}} V_{\mathscr{L}}(\varphi(\xi^{D_\sigma})) \supset \bigcap V_{\mathscr{L}}(\varphi(\xi^{D_{\rho_i}}))$ if no maximal cone contains all $\rho_1, \ldots, \rho_l$. We assume (3). Then for $y \in Y$, the set $\{\rho \in \Delta(1);\ y \in V_{\mathscr{L}}(\varphi(\xi^{D_\rho}))\}$ needs to be contained in some cone $\sigma \in \Delta$. Hence $y \notin V_{\mathscr{L}}(\varphi(\xi^{D_\sigma}))$ for $D \in \mathrm{CDiv}(\hat{\sigma})^{+}$. □

DEFINITION 4.2. Let Y be a k-scheme, Δ a good fan, X_Δ the associated toric variety, $\mathscr{L}$ a Δ-graded $\mathcal{O}_Y$-algebra with invertible components. A Δ-graded homomorphism $\varphi: \mathcal{O}_Y \otimes_k S_\Delta \to \mathscr{L}$ is said to be *base-point-free* if φ satisfies one of the equivalent conditions (1)–(3) in Proposition 4.1.

We state again Theorem 3.4 in the case $Z = X_\Delta$ for reference.

THEOREM 4.3. *Let Y be a k-scheme, Δ a good fan, and X_Δ the associated toric variety over k. Then there exists a one-to-one correspondence between the following two sets*:

(a) *the set of k-morphisms from Y to X_Δ;*

(b) *the set of equivalence classes of pairs $(\mathscr{L}, \varphi)$ of a Δ-graded $\mathcal{O}_Y$-algebra with invertible components and a base-point-free Δ-graded $\mathcal{O}_Y$-algebra homomorphism φ from $\mathcal{O}_Y \otimes_k S_\Delta$ to $\mathscr{L}$.*

Here, $(\mathscr{L}, \varphi)$ and $(\mathscr{L}', \varphi')$ are said to be equivalent *if there exists an isomorphism $\iota: \mathscr{L} \to \mathscr{L}'$ of Δ-graded $\mathcal{O}_Y$-algebras with $\varphi' = \iota \circ \varphi$.*

REMARK 4.4. When $\Delta(1)$ may not span $N_{\mathbf{Q}}$, there exists a (non-canonical) one-to-one correspondence between the following two sets:

(a) the set of k-morphisms from Y to X_Δ;

(b) the set of equivalence classes of pairs $(\mathscr{L}, \varphi)$ of a Δ-graded $\mathcal{O}_Y$-algebra with invertible components and a base-point-free Δ-graded $\mathcal{O}_Y$-algebra homomorphism φ from $\mathcal{O}_Y \otimes_k (S_\Delta \otimes k[M_0])$ to $\mathscr{L}$.

Here M_0 denotes the kernel of $\mathrm{div}: M \to \mathrm{CDiv}(\Delta)$. We remark that S_Δ is canonically isomorphic to the homogeneous coordinate ring of the associated toric variety X_{Δ_0} in the notation of Corollary 1.11.

COROLLARY 4.5 (Cox [2]). *Let the notation be as in Theorem* 4.3. *Assume that Δ is smooth, i.e., X_Δ is smooth over k. Then there exists a one-to-one correspondence between the following two sets*:

(a) *the set of k-morphisms from Y to X_Δ;*

(b) *the set of equivalence classes of Δ-collections on Y* (see [2, Definition 1.1]).

REMARK 4.6. (1) If $\Delta(1)$ may not span $N_{\mathbf{Q}}$, we choose an isomorphism $X_\Delta \to X_{\Delta_0} \times_k (\boldsymbol{G}_m \otimes N/N_0)$ to get a Δ-collection with c_m corresponding to $Y \to \boldsymbol{G}_m \otimes N/N_0$ for $m \in N_0^\perp$ in [2, Theorem 1.1]. Here the notation is the same as that in Corollary 1.11. It is straightforward to show that the set of equivalence classes of Δ-collections bijectively corresponds to the set of equivalence classes $(\mathscr{L}, \varphi)$ as in Remark 4.4.

(2) In the proof of [2, Theorem 1.1], Cox used the freeness of $\mathrm{Pic}(\Delta)$. As we mentioned at the end of Section 1, $\mathrm{Pic}(\Delta)$ may not be free even if X_Δ is smooth. Although his proof is thus incomplete, the result is nevertheless true.

PROOF. Let D_ρ be the divisor on X_Δ corresponding to $\rho \in \Delta(1)$ and s a set-theoretic section $\mathrm{Pic}(\Delta)_{\geq 0} \to \mathrm{CDiv}(\Delta)_{\geq 0}$ with $\deg \circ s = \mathrm{id}$. For a Δ-collection $\{(\mathscr{L}_\rho, u_\rho, c_m)\}$, we use the compatibility condition in [2, Definition 1.1] to get a Δ-graded $\mathscr{O}_Y$-algebra $\mathscr{L} := \bigoplus_{\alpha \in \mathrm{Pic}(\Delta)_{\geq 0}} \mathscr{L}^{s(\alpha)}$ with invertible components. Hence $\mathscr{L}^{s(\alpha)}$ is the invertible sheaf $\bigotimes_\rho \mathscr{L}_\rho^{\otimes a_\rho}$ if $s(\alpha) = \sum_\rho a_\rho D_\rho$. By the global sections u_ρ and the nondegeneracy condition, we have a base-point-free Δ-graded $\mathscr{O}_Y$-algebra homomorphism $\varphi : \mathscr{O}_Y \otimes_k S_\Delta \to \mathscr{L}$ with $\varphi(1 \otimes \xi^{D_\rho}) = u_\rho$.

On the other hand, for a given $(\mathscr{L}, \varphi)$ in Theorem 4.3 (b), it can be verified that

$$\left(\mathscr{L}_{\deg D_\rho},\ \varphi(D_\rho),\ \bigotimes_\rho \mathscr{L}_{\deg D_\rho}^{\otimes \langle m, n_\rho \rangle} \cong \mathscr{L}_0 = \mathscr{O}_Y \text{ via product} \right)$$

forms a Δ-collection. It is easy to show that this correspondence preserves their equivalence relations. □

COROLLARY 4.7. *Let the notation be as in Theorem* 4.3 *and T the algebraic torus as in Section* 1. *Assume that Y is an integral scheme. Then there exists a one-to-one correspondence between the following two sets*:

(a) *the set of k-morphisms* $f : Y \to X_\Delta$ *with* $f^{-1}(T) \neq \varnothing$;

(b) *the set of pairs* (φ, ψ) *of a monoid homomorphism* φ *from* $\mathrm{CDiv}(\Delta)_{\geq 0}$ *to the monoid* $\mathrm{CDiv}(Y)_{\geq 0}$ *of effective Cartier divisors and a group homomorphism* $\psi : M \to k(Y)^\times$ *which induce a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{\psi} & k(Y)^\times \\ {\scriptstyle \mathrm{div}}\downarrow & & \downarrow{\scriptstyle \mathrm{div}} \\ \mathrm{CDiv}(\Delta) & \xrightarrow[\varphi^{\mathrm{gp}}]{} & \mathrm{CDiv}(Y), \end{array}$$

and satisfies one of the equivalent conditions (2)–(3) *in Proposition* 4.1.

In particular, if Δ *is smooth, we can replace* (b) *by the following* (b′):

(b′) *the set of pairs of a homomorphism* $\psi : M \to k(Y)^\times$ *and a collection* $(D_\rho)_{\rho \in \Delta(1)}$ *of effective Cartier divisors* D_ρ *on Y satisfying both the condition* (3) *in Proposition* 4.1 *and the equation*

$$\operatorname{div}\psi(m)=\sum_{\rho\in\Delta(1)}\langle m, n_\rho\rangle D_\rho$$

for each $m\in M$.

Here $k(Y)$ *(resp.* n_ρ*, resp.* $\langle\ ,\ \rangle$*) is the rational function field of* Y *(resp. the unique generator of the monoid* $\rho\cap N$*, resp. the duality pairing* $M\times N\to \boldsymbol{Z}$*).*

REMARK 4.8. Oda and Sankaran (unpublished) proved Corollary 4.7 in the case where Y (resp. X_Δ) is normal (resp. smooth).

PROOF. It is obvious that a morphism $f: Y\to X_\Delta$ with $f^{-1}(T)\neq\varnothing$ gives data in (b).

On the other hand, let ψ (resp. φ) be as in (b). Let us identify invertible sheaves with locally principal coherent subsheaves of $k(X)$ (cf., e.g., [8, II §6]). Then the invertible sheaf $\mathcal{O}_Y(\varphi(D))$ for each effective divisor D has the global section $u_D:=1\in k(Y)^\times$ with $\varphi(D)=$the zero locus $(u_D)_0$ of u_D, which induces an $\mathcal{O}_Y$-algebra homomorphism

$$\varphi_1: \mathcal{O}_Y\otimes_k S_\Delta\to\bigoplus_{D\in\mathrm{CDiv}(\Delta)_{\geq 0}}\mathcal{O}_Y(\varphi(D))$$

with $\varphi_1(\xi^D)=u_D$. Let us take a set-theoretic section $s:\mathrm{Pic}(\Delta)_{\geq 0}\to\mathrm{CDiv}(\Delta)_{\geq 0}$ with $(\deg)\circ s=\mathrm{id}$. If two divisors D and D' are linearly equivalent, then the invertible sheaf $\mathcal{O}_Y(\varphi(D'))$ coincides with $\mathcal{O}_Y(\varphi(D))\psi(D-D')$. Multiplying $\mathcal{O}_Y(\varphi(D))$ by $\psi(D-s(\deg D))$, we have an $\mathcal{O}_Y$-algebra homomorphism

$$\varphi_2:\bigoplus_{D\in\mathrm{CDiv}(\Delta)_{\geq 0}}\mathcal{O}_Y(\varphi(D))\to\bigoplus_{\alpha\in\mathrm{Pic}(\Delta)_{\geq 0}}\mathcal{O}_Y(\varphi\circ s(\alpha))\,.$$

Hence a pair of the Δ-graded $\mathcal{O}_Y$-algebra $\bigoplus_\alpha\mathcal{O}_X(\varphi\circ s(\alpha))$ and the composite of φ_1 with φ_2 defines the morphism from Y to X_Δ in (a). Note that up to equivalence, the k-morphism does not depend on the choice of a section s. The latter assertion follows from the former and the fact that $\mathrm{CDiv}(\Delta)_{\geq 0}$ is isomorphic to $\bigoplus_{\rho\in\Delta(1)}\boldsymbol{Z}_{\geq 0}\cdot D_\rho$ if Δ is smooth. □

COROLLARY 4.9. *Let* Δ *(resp.* Δ'*) be a good fan (resp. a complete fan), and* X_Δ *(resp.* $X_{\Delta'}$*) the associated toric variety. Then there exists a one-to-one correspondence between the following two sets*:

(a) *the set* $\mathrm{Hom}_k(X_{\Delta'}, X_\Delta)$ *of not necessarily equivalent* k*-morphisms from* $X_{\Delta'}$ *to* X_Δ;

(b) *the set of equivalence classes of (not necessarily* Δ*-grade preserving)* k*-algebra homomorphisms* $\varphi: S_\Delta\to S_{\Delta'}$ *mapping each homogeneous component into a homogeneous component and satisfying the equivalent conditions in Proposition* 4.1.

Here φ and φ' *are said to be* equivalent *if there exists a homomorphism* g: $\mathrm{Pic}(\Delta)\to\boldsymbol{G}_m(k)$ *such that* $\varphi'(\xi^D)=g(\deg D)\varphi(\xi^D)$ *for each* $D\in\mathrm{CDiv}(\Delta)_{\geq 0}$. *Furthermore, if* Δ' *is good, we can replace* (b) *by the following*:

(b′) *the set of equivalence classes of* k*-algebra homomorphisms* $\varphi: S_\Delta\to S_{\Delta'}$ *mapping each homogeneous component into a homogeneous component such that* $B_{\Delta'}$ *is contained*

in the radical of an ideal generated by $\{\varphi(\xi^{D_\sigma}); \sigma \in \Delta_{\max}\}$ *for any* $D_\sigma \in \mathrm{CDiv}(\hat{\sigma})^+$.

PROOF. We first remark that S_Δ is canonically isomorphic to the Δ-graded k-algebra

$$\bigoplus_{\alpha \in \mathrm{Pic}(\Delta)} \Gamma(X_\Delta, \mathcal{O}_{X_\Delta}(\alpha))$$

for any fan Δ (cf., e.g., [1, p. 30]). Therefore, giving a pair $(\mathcal{L}, \varphi)$ on $X_{\Delta'}$ in Theorem 4.3 (b) is equivalent to giving a k-algebra homomorphism $S_\Delta \to S_{\Delta'}$ preserving homogeneous components and satisfying the conditions in Proposition 4.1. Moreover, giving an equivalence of two homomorphisms of Δ-graded $\mathcal{O}_{X_{\Delta'}}$-algebras with invertible components is equivalent to giving a homomorphism $\mathrm{Pic}(\Delta) \to \boldsymbol{G}_m(k) = k^\times$ as above. The last assertion is straightforward to prove by Proposition 1.12. □

REMARK 4.10. A homomorphism φ in Corollary 4.9 (b) may not map the ideal B_Δ into $B_{\Delta'}$. For instance, look at the first projection $X_{\Delta'} = \boldsymbol{P}^1 \times \boldsymbol{P}^1 \to X_\Delta = \boldsymbol{P}^1$.

COROLLARY 4.11. *Let* Δ *and* X_Δ *be as in Corollary* 4.9. *Then there exists a one-to-one correspondence between the following two sets*:

(a) $\mathrm{Hom}_k(\boldsymbol{P}^m, X_\Delta)$;

(b) *the set of equivalence classes of* k*-algebra homomorphisms* $S_\Delta \to k[T_0, \ldots, T_m]$ *mapping each homogeneous component into a homogeneous component such that* $(T_0, \ldots, T_m)$ *is contained in the radical ideal generated by* $\{\varphi(\xi^{D_\sigma}); \sigma \in \Delta_{\max}\}$ *of any* $D_\sigma \in \mathrm{CDiv}(\hat{\sigma})^+$.

The equivalence relation is the same as that in Corollary 4.9.

EXAMPLE 4.12. We calculate morphisms from the projective line $\boldsymbol{P}^1$ to the weighted projective plane $\boldsymbol{P}(1, 1, 2)$, using Corollary 4.11. Let $S := k[N^{\Delta(1)}] = k[v_1, v_2, v_3]$ be Cox's homogeneous coordinate ring of $\boldsymbol{P}(1, 1, 2)$. Here the degree of v_1 (resp. v_2, resp. v_3) is equal to 1 (resp. 1, resp. 2). It is easy to see that S_Δ equals the subring $k[v_1^2, v_1v_2, v_2^2, v_3] \cong k[x, y, z, w]/(xy - w^2)$ of S, where all of the variables x, y, z, w are of degree one. By Corollary 4.11, the set $\mathrm{Hom}(\boldsymbol{P}^1, \boldsymbol{P}(1, 1, 2))$ is equal to the set of quadruples (f_x, f_y, f_z, f_w) of homogeneous polynomials of the same degree with $f_x f_y = f_w^2$ and with $s^m, t^m \in (f_x, f_y, f_z)$ for some $m \in \boldsymbol{Z}_{>0}$. For instance, the algebra homomorphism $S_\Delta \to k[s, t]$ $(x, y, w \mapsto s, z \mapsto t)$ corresponds to the morphism

$$g : \boldsymbol{P}^1 \to \boldsymbol{P}(1, 1, 2) = (\boldsymbol{A}^3 \setminus \{0\})/k^\times, \quad (\alpha : \beta) \mapsto (\sqrt{\alpha} : \sqrt{\alpha} : \beta).$$

Here the action of $k^\times$ on $\boldsymbol{A}^3 \setminus \{0\}$ is defined by $g \cdot (\alpha, \beta, \gamma) = (g\alpha, g\beta, g^2\gamma)$ $(g \in k^\times, (\alpha, \beta, \gamma) \in \boldsymbol{A}^3 \setminus \{0\})$. This morphism g cannot be obtained by the morphism $\boldsymbol{A}^2 \setminus \{0\} \to \boldsymbol{A}^3 \setminus \{0\}$ corresponding to any *algebra* homomorphism $S \to k[s, t]$ because the latter cannot induce any isomorphism between their Picard groups (cf. [2, Remark 3.4]).

COROLLARY 4.13. *Let k be an algebraically closed field. Fix the point $\infty = (0:1) \in \boldsymbol{P}^1$ and a point p in the open dense torus orbit T in X_Δ. Then there exists a canonical one-to-one correspondence among the following three sets:*

(a) *the set* $\mathrm{Hom}^*(\boldsymbol{P}^1, X_\Delta)$ *of morphisms* $f: \boldsymbol{P}^1 \to X_\Delta$ *with* $f(\infty) = p$;

(b) *the set of monoid homomorphisms* ψ *from* $\mathrm{CDiv}(\Delta)_{\geq 0}$ *to the monoid of monic polynomials in the polynomial ring* $\in k[t]$ *in one variable* t *which preserve their degrees and such that the ideal generated by* $\{\psi(D_\sigma);\ D_\sigma \in \mathrm{CDiv}(\hat{\sigma})^+,\ \sigma \in \Delta_{\max}\}$ *is exactly* $k[t]$;

(c) *the set of homomorphisms in the kernel of the homomorphism*

$$\bigoplus_{z \in \boldsymbol{A}^1(k)} (\mathrm{CDiv}(\Delta)_{\geq 0})^\vee \to N; \quad (l_z) \mapsto \sum_z l_z \circ \mathrm{div}$$

such that the ideal generated by $\{\prod_{z \in \boldsymbol{A}^1(k)} (t-z)^{l_z(D_\sigma)};\ D_\sigma \in \mathrm{CDiv}(\hat{\sigma}),\ \sigma \in \Delta_{\max}\}$ *is exactly* $k[t]$.

Here $(\mathrm{CDiv}(\Delta)_{\geq 0})^\vee$ *denotes the cone dual to* $\mathrm{CDiv}(\Delta)_{\geq 0}$.

REMARK 4.14. Guest [7] proved Corollary 4.13 for a projective toric variety.

PROOF. Since k is algebraically closed, we can easily show the one-to-one correspondence between (b) and (c), using the roots of monic polynomials. We prove the one-to-one correspondence between (a) and (b). Using the homomorphism $M \to k^\times$ corresponding to $p \in T \subset X_\Delta$, every homomorphism $\varphi: S_\Delta \to k[T_0, T_1]$ as in Corollary 4.11 is uniquely equivalent to one with $\varphi(\xi^D)(0, T_1) = T_1^{\deg \varphi(\xi^D)}$ for each $D \in \mathrm{CDiv}(\Delta)_{\geq 0}$. Therefore $f_D(t) := \varphi(\xi^D)(1, t) \in k[t]$ is monic and uniquely determined by the zeros of f_D. Thus the homomorphism φ in Corollary 4.11 (b) gives the monoid homomorphism in the corollary (b). The converse is verified by the above argument. □

REFERENCES

[1] D. COX, The homogeneous coordinate ring of a toric variety, J. Algebraic Geom. 4 (1995), 17–50.

[2] D. COX, The functor of a smooth toric variety, Tôhoku Math. J. 47 (1995), 251–262.

[3] I. DOLGACHEV, Weighted projective varieties, in Group Actions and Vector Fields, Proceedings, Vancouver 1981 (J. B. Carrell, ed.), Lecture Notes in Math. 956, Springer-Verlag, Berlin, Heidelberg, New York, 1982, 34–71.

[4] M. EIKELBERG, The Picard group of a compact toric variety, Results in Math. vol. 22 (1992), 509–527.

[5] W. FULTON, Introduction to toric varieties, Princeton Univ. Press, 1993.

[6] D. MUMFORD, J. FOGARTY AND J. KIRWAN, Geometric Invariant Theory, 3rd ed., Ergebnisse der Math., 3. Folge, Band 34, Springer-Verlag, 1994.

[7] M. GUEST, The topology of the space of rational curves on a toric variety, Acta Math. 174 (1995), 119–145.

[8] R. HARTSHORNE, Algebraic geometry, Springer-Verlag, Berlin, Heidelberg, and New York, 1977.

[9] A. GROTHENDIECK AND J. DIEUDONNÉ, Eléments de Géométrie Algébrique II, Publ. Math. IHES, II 8 (1961).

[10] T. ODA, Convex Bodies and Algebraic Geometry—An Introduction to the Theory of Toric Varieties, Ergebnisse der Math. (3) 15, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo,

1989.

[11] T. Oda, Lectures on torus embeddings and applications, Tata Inst. of Fund. Research 58 (1978), Springer-Verlag.

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