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ALMOST PERIODIC SOLUTIONS OF A COMPETITION SYSTEM WITH DOMINATED INFINITE DELAYS

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Abstract. In this paper, we consider an n-species almost periodic Lotka-Volterra competition system with dominated infinite delays. By constructing suitable Lyapunov functionals, we are able to show that, under a set of algebraic conditions, the system has a unique positive almost periodic solution which is globally attractive.

1. Introduction. In this paper, we consider an almost periodic Lotka-Volterra system

$$\dot{x}_i(t)=x_i(t)\left[b_i(t)-\sum_{j=1}^{n}a_{ij}(t)\int_{-\infty}^{t}K_{ij}(t-s)x_j(s)ds\right],\qquad i=1,\dots,n, \tag{1.1}$$

which describes a model of the dynamics of an n-species competition in mathematical ecology. When the system (1.1) has delay-independent dominated terms, it takes the form

$$\dot{x}_i(t)=x_i(t)\left[b_i(t)-a_{ii}(t)x_i(t)-\sum_{j=1,j\neq i}^{n}a_{ij}(t)\int_{-\infty}^{t}K_{ij}(t-s)x_j(s)ds\right],\qquad i=1,\dots,n. \tag{1.2}$$

Recently, Gopalsamy [3] discussed the system (1.2) with ω-periodic coefficients b_i, a_{ij} $(i,j=1,\dots,n)$ and proved that, under a set of delay-independent algebraic conditions, the system (1.2) has a unique globally attractive ω-periodic solution. Murakami [10] generalized the discussion to the system (1.2) with almost periodic parameters b_i, a_{ij} $(i,j=1,\dots,n)$. By investigating the stability properties of the solutions of the system (1.2), Murakami [10] was able to show that (1.2) has an almost periodic solution. We also refer to Hamaya [7] and Hamaya and Yoshizawa [8] for further discussion on the periodic and almost periodic system (1.2), respectively. As one can see easily, when such delay-independent dominated terms are not present, the argument used in Gopalsamy [3], Hamaya [7], Hamaya and Yoshizawa [8] and Murakami [10] cannot be used for (1.1). For (1.1), when $n=1$, the related problem has been studied recently by Gopalsamy et al. [5] in the periodic case and Gopalsamy and He [4] and Seifert [11] in the almost periodic case. We also refer to He and Gopalsamy [9] for the

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discussion on the periodic system (1.1) with $n=2$. However, when $n \geq 2$, it has been an open problem whether the system (1.1) has a unique globally attractively positive almost periodic solution (see also Gopalsamy and He [6] and He and Gopalsamy [9]). It is the purpose of this paper to solve this problem. Motivated by recent work of Gopalsamy and He [4], [6] and Murakami [10], we first give estimates for the uniform upper and lower bounds of positive solutions of (1.1). Then, by constructing some Lyapunov functionals, we obtain a set of algebraic conditions, under which the systems (1.1) has a unique positive almost periodic solution which is globally attractive. As in the case $n=1$ (see Gopalsamy and He [4]), the sufficient conditions are delay-dependent, which characterizes the competition systems with delay-dominated terms, while the conditions for (1.2) are often delay-independent (see Gopalsamy [3], Hamaya [7], Hamaya and Yoshizawa [8] and Murakami [10]).

In what follows, we denote by R^n the n-dimensional real Euclidean space and by $|x|$ the norm of $x \in R^n$. Given $x=(x_1, \ldots, x_n) \in R^n$ and $y=(y_1, \ldots, y_n) \in R^n$, we put $x>y$ if $x_i > y_i$ and $x \geq y$ if $x_i \geq y_i$ for all $i \in I=\{1, 2, \ldots, n\}$. R^n_+ will denote the nonnegative cone of R^n. Throughout this paper, we assume that the functions b_i, a_{ij} and K_{ij} in (1.1) are real-valued functions on R and that the following conditions are satisfied:

(H1) a_{ij} and b_i are continuous, almost periodic functions, and $\inf_{t \in R} a_{ij}(t) \geq 0$ for $i \neq j$, $\inf_{t \in R} a_{ii}(t) > 0$ and $\inf_{t \in R} b_i(t) > 0$ for $i, j \in I$.

(H2) K_{ij} is nonnegative piecewise continuous, $\int_0^\infty K_{ij}(s)ds = 1$, $\int_0^\infty sK_{ij}(s)ds < \infty$ and $\int_0^\infty s^2 K_{ij}(s)ds < \infty$ for $i, j \in I$.

Consequently, define constants b_i^l, b_i^u, a_{ij}^l, a_{ij}^u $(i, j \in I)$ by

$$b_i^l = \inf_{t \in R} b_i(t)\,, \quad b_i^u = \sup_{t \in R} b_i(t)\,, \quad a_{ij}^l = \inf_{t \in R} a_{ij}(t)\,, \quad a_{ij}^u = \sup_{t \in R} a_{ij}(t)\,.$$

Let BC^+ be the set of all bounded nonnegative continuous functions from $R_- = (-\infty, 0]$ into R^n_+ satisfying $\phi(0) > 0$. Set $\|\phi\| = \sup_{s \in R_-} \phi(s)$ for $\phi(s) \in \mathrm{BC}^+$. We assume that the system (1.1) is supplemented with the initial condition

$$x(s) = \phi(s) \in \mathrm{BC}^+ \qquad \text{for} \quad s \in R_-\,. \tag{1.3}$$

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2. Uniform upper and lower bounds.

In this section, following the idea in Gopalsamy and He [4], we obtain a priori upper and lower bounds of the positive solutions of (1.1) and (1.3). One can see that, under the conditions (H1) and (H2), the solutions of (1.1) and (1.3) exist for all $t \in [0, r)$ $(r \leq +\infty)$ and remain positive. From the following Lemma 2.1, we know that the solutions of (1.1) and (1.3) are continuable to $t=\infty$. It follows from the positivity of the solution $x(t)$ of (1.1) and (1.3) that

$$\dot{x}_i(t) \le x_i(t)\left[b_i(t) - a_{ii}(t)\int_0^\infty K_{ii}(s)x_i(t-s)ds\right], \qquad i \in I, \tag{2.1}$$

from which, using the same argument as in Theorem 2.1 in Gopalsamy and He [4], we can derive the following estimate for the uniform upper bound of the solutions of (1.1) and (1.3).

LEMMA 2.1. *Under the assumptions* (H1) *and* (H2), *the solutions* $x(t)=(x_1(t), \dots, x_n(t))$ *of* (1.1) *and* (1.3) *satisfy* $x(t)>0$ *for all* $t \ge 0$, *and furthermore*

$$\limsup_{t\to\infty} x_i(t) \le M_i := \frac{b_i^u}{a_{ii}^l \int_0^\infty K_{ii}(s)\exp(-b_i^u s)ds} \qquad \text{for} \quad i \in I. \tag{2.2}$$

Using Lemma 2.1 and the idea of Theorem 2.2 in Gopalsamy and He [4], we now have the following estimate for the uniform lower bound of the solutions of (1.1) and (1.3).

LEMMA 2.2. *Assume the system* (1.1) *satisfies* (H1), (H2) *and*

(H3) $b_i^l > \sum_{j=1, j\ne i}^n a_{ij}^u M_j$ *with* M_i *defined by* (2.2) *and* $i \in I$;

(H4) *there exists* $\delta > 0$ *such that*

$$\int_0^\infty K_{ii}(s)\exp\left[-\left(b_i^l - \sum_{j=1}^n a_{ij}^u M_j\right)s + \delta s\right]ds < \infty .$$

Then the solution $x(t)$ *of* (1.1) *and* (1.3) *satisfies* $\liminf_{t\to\infty} x_i(t) \ge m_i$ *with*

$$m_i = \frac{b_i^l - \sum_{j=1, j\ne i}^n a_{ij}^u M_j}{a_{ii}^u \int_0^\infty K_{ii}(s)\exp\left[-\left(b_i^l - \sum_{j=1}^n a_{ij}^u M_j\right)s\right]ds} \qquad \text{for} \quad i \in I. \tag{2.3}$$

PROOF. The proof is similar to that of Theorem 2.2 in Gopalsamy and He [4] and we indicate it briefly.

Let $x(t)=(x_1(t), \dots, x_n(t))$ be any solution of (1.1) and (1.3). It follows from Lemma 2.1 and the condition (H4) that, for $\varepsilon < \delta$, there exists a $t_1 > 0$ such that

$$x_i(t) \le M_i + \varepsilon \qquad \text{for} \quad t \ge t_1 \quad \text{and} \quad i \in I \tag{2.4}$$

with M_i defined by (2.2). Then, following the positivity of $x(t)$ and (1.1) and (2.4), one can see that, for $t \ge t_1$,

$$\dot{x}_i(t) \geq x_i(t)\left[b_i^l - \sum_{j=1}^{n} a_{ij}^u\left(\int_0^{t-t_1} K_{ij}(s)x_j(t-s)ds + \int_{t-t_1}^{\infty} K_{ij}(s)x_j(t-s)ds\right)\right] \tag{2.5}$$
$$\geq x_i(t)\left[b_i^l - \sum_{j=1}^{n} a_{ij}^u\left([M_j+\varepsilon]\int_0^{t-t_1} K_{ij}(s)ds + \int_{t-t_1}^{\infty} K_{ij}(s)x_j(t-s)ds\right)\right]$$
$$= x_i(t)c_i(t)$$

with

$$c_i(t) = b_i^l - \sum_{j=1}^{n} a_{ij}^u\left([M_j+\varepsilon]\int_0^{t-t_1} K_{ij}(s)ds + \int_{t-t_1}^{\infty} K_{ij}(s)x_j(t-s)ds\right), \quad t \geq t_1, \quad i \in I.$$

By the boundedness of $x(t)$ and the assumption (H2), we have

$$\lim_{t\to\infty} c_i(t) = C_i := b_i^l - \sum_{j=1}^{n} a_{ij}^u(M_j+\varepsilon), \qquad i \in I. \tag{2.6}$$

It follows from (2.5) that

$$x_i(t-s) \leq x_i(t)\exp\left[-\int_{t-s}^{t} c_i(s)ds\right] \qquad \text{for} \quad t-s > t_1,$$

which, together with (2.5), implies that

$$\dot{x}_i(t) \geq x_i(t)\left[b_i^l - a_{ii}^u\left(\int_0^{t-t_1} K_{ii}(s)\exp\left[-\int_{t-s}^{t} c_i(r)dr\right]ds\right)x_i(t)\right. \tag{2.7}$$
$$\left. - \sum_{j=1, j\neq i}^{n} a_{ij}^u(M_j+\varepsilon)\int_0^{t-t_1} K_{ij}(s)ds - \sum_{j=1}^{n} a_{ij}^u \int_{t-t_1}^{\infty} K_{ij}(s)x_j(t-s)ds\right].$$

Note that

$$\lim_{t\to\infty} \sum_{j=1, j\neq i}^{n} a_{ij}^u(M_j+\varepsilon)\int_0^{t-t_1} K_{ij}(s)ds = \sum_{j=1, j\neq i}^{n} a_{ij}^u(M_j+\varepsilon), \qquad i \in I,$$
$$\lim_{t\to\infty} \sum_{j=1}^{n} a_{ij}^u \int_{t-t_1}^{\infty} K_{ij}(s)x_j(t-s)ds = 0$$

and

$$a_{ii}^u\left(\int_0^{t-t_1} K_{ii}(s)\exp\left[-\int_{t-s}^{t} c_i(r)dr\right]ds\right) \leq a_{ii}^u \int_0^{\infty} K_{ii}(s)\exp(-C_i s)\exp(\varepsilon_1 s)ds$$

for $t \geq t_2 \geq t_1$ and some ε_1 with $0 < \varepsilon_1 < \delta$. Then we can see from (2.7) that

$$\dot{x}_i(t) \geq x_i(t)\left[(b_i^l - \varepsilon_2) - \sum_{j=1, j\neq i}^{n} a_{ij}^u(M_j+\varepsilon)\right. \tag{2.8}$$
$$\left. - a_{ii}^u\left(\int_0^{\infty} K_{ii}(s)\exp(-C_i s)\exp(\varepsilon_1 s)ds\right)x_i(t)\right]$$

for some $\varepsilon_2 > 0$ sufficiently small and all large t. Then, similar to the proof of Theorem 2.2 in Gopalsamy and He [4], one can show from (2.8) that $\liminf_{t\to\infty} x_i(t) \geq m_i$ with m_i defined by (2.3). This completes the proof.

It is noticed that the upper and lower bounds obtained in Lemmas 2.1 and 2.2 depend on the diagonal delay terms only. As a special case of (1.1), it can take the following form with finite discrete delays:

$$\dot{x}_i(t) = x_i(t)\left[b_i(t) - a_{ii}(t)x_i(t-\tau_i) - \sum_{j=1, j\neq i}^{n} a_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds\right], \quad i \in I, \tag{2.9}$$

where $0 \leq \tau_i$ ($i \in I$) are finite constants. Consequently, from Lemmas 2.1 and 2.2, we have the following bound estimate for the solutions of (2.9), which can also be found in Gopalsamy and He [6]. This result will be used in our later discussion.

COROLLARY 2.3. *Under the assumptions* (H1) *and* (H2), *if*

$$b_i^l > \sum_{j=1, j\neq i}^{n} a_{ij}^u \frac{b_j^u}{a_{jj}^l} \exp(b_j^u \tau_j) \qquad \text{for} \quad i \in I, \tag{2.10}$$

then the positive solutions $x(t)$ *of* (2.9) *satisfy*

$$0 < m_i \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x_i(t) \leq M_i$$

with

$$\begin{aligned} M_i &= \frac{b_i^u}{a_{ii}^l} \exp(b_i^u \tau_i) \\ m_i &= \frac{b_i^l - \sum_{j=1, j\neq i}^{n} a_{ij}^u M_j}{a_{ii}^u} \exp\left[\left(b_i^l - \sum_{j=1}^{n} a_{ij}^u M_j\right)\tau_i\right]. \end{aligned} \qquad (i \in I). \tag{2.11}$$

3. Extreme stability. In this section we will show that, under a set of algebraic conditions, the system (1.1) is extremely stable (see Yoshizawa [12], [13]) in the sense that, for any two positive solutions $x(t)$ and $y(t)$ of (1.1) and (1.3), we have

$$\lim_{t\to\infty} [x(t) - y(t)] = 0.$$

LEMMA 3.1. *Suppose the system* (1.1) *satisfies* (H1)–(H4). *Then there exists a solution* $x(t) = (x_1(t), \ldots, x_n(t))$ *of* (1.1) *and* (1.3) *on* R *such that* $0 < m_i - \varepsilon \leq x_i(t) \leq M_i + \varepsilon$ ($i \in I$) *for* $t \in R$ *and sufficiently small* $\varepsilon > 0$, *where* M_i *and* m_i ($i \in I$) *are defined by* (2.2) *and* (2.3), *respectively.*

Lemma 3.1 can be proved by repeating almost the same argument as in Lemma 2 in Murakami [10] and Lemma 4 in Seifert [11], so we omit the details.

THEOREM 3.2. *Assume* (H1)–(H4) *are satisfied. Suppose that*

$$(M_i a_{ii}^u)^2 \sigma_i < a_{ii}^l \int_0^\infty K_{ii}(s)\exp(-b_i^u s)ds \qquad (i=1,\dots,n)$$

with $\sigma_i = \int_0^\infty sK_{ii}(s)ds$ *and* $E=(e_{ij})_{n\times n}$ *is an M-matrix, where*

$$e_{ij} = \begin{cases} a_{ii}^l \int_0^\infty K_{ii}(s)\exp(-b_i^u s)ds - (M_i a_{ii}^u)^2\sigma_i & \text{for } i=j \\ -[1+M_i^2 a_{ii}^u \sigma_i]a_{ij}^u & \text{for } i\neq j. \end{cases}$$

Then the system (1.1) *is extremely stable.*

PROOF. Since the matrix $E=(e_{ij})_{n\times n}$ is an M-matrix, we know that (see [2], [6]) there exist $\alpha=(\alpha_1,\dots,\alpha_n)>0$ and $\varepsilon_0>0$ such that

$$\alpha_i(e_{ii}-\varepsilon_o) > \sum_{j=1, j\neq i}^{n} \alpha_j(|e_{ji}|+\varepsilon_o), \qquad i\in I. \tag{3.1}$$

Clearly there exists an $\varepsilon_1\in(0,\varepsilon_o)$ such that

$$b_i^l-\varepsilon_1 \le b_i(t) \le b_i^u+\varepsilon_1, \quad 0<a_{ii}^l-\varepsilon_1\le a_{ii}(t), \quad a_{ij}(t)\le a_{ij}^u+\varepsilon_1$$

for $t\in R$ and

$$e_{ii}-\varepsilon_o < e_{ii}(\varepsilon_1), \quad e_{ij}-\varepsilon_o < e_{ij}(\varepsilon_1) \tag{3.2}$$

with

$$\begin{cases} e_{ii}(\varepsilon_1) = (a_{ii}^l-\varepsilon_1)\left[\int_0^\infty K_{ii}(s)\exp(-(b_i^u+\varepsilon_1)s)ds - \varepsilon_1\right] - [(a_{ii}^u+\varepsilon_1)(M_i+\varepsilon_1)]^2\sigma_i \\ e_{ij}(\varepsilon_1) = -(a_{ij}^u+\varepsilon_1)[1+(M_i+\varepsilon_1)^2(a_{ii}^u+\varepsilon_1)\sigma_i] \qquad i\neq j. \end{cases} \tag{3.3}$$

We first know from Lemma 3.1 that there exists a solution, say $y(t)=(y_1(t),\dots,y_n(t))$ of (1.1) and (1.3) satisfying

$$0<m_i-\varepsilon_1 \le y_i(t) \le M_i+\varepsilon_1 \qquad \text{for} \quad t\in R \quad \text{and} \quad i\in I. \tag{3.4}$$

To prove the extreme stability of (1.1), it is enough to show that for any positive solution $x(t)=(x_1(t),\dots,x_n(t))$ of (1.1) with $x(s)=\phi(s)\in \mathrm{BC}^+$ for $s\in R_-$ and the solution $y(t)$ satisfying (3.4), we have

$$\lim_{t\to\infty}[x_i(t)-y_i(t)]=0, \qquad i\in I. \tag{3.5}$$

Define $u(t)=(u_1(t),\dots,u_n(t))$, $v(t)=(v_1(t),\dots,v_n(t))$ and $w(t)=(w_1(t),\dots,w_n(t))$ as follows:

$$u_i(t)=\ln[x_i(t)], \quad v_i(t)=\ln[y_i(t)], \quad w_i(t)=u_i(t)-v_i(t), \qquad i\in I. \tag{3.6}$$

Then, from (1.1) and (3.6),

(3.7) $$\frac{d}{dt}[u_i(t)-v_i(t)]=-\sum_{j=1}^{n}a_{ij}(t)\int_0^\infty K_{ij}(s)[\exp(u_j(t-s))-\exp(v_j(t-s))]ds\,.$$

For $i\in I$ and $t\geq 0$, the equation (3.7) can be written as

(3.8) $$\begin{aligned}\dot{w}_i(t)=&-a_{ii}(t)\int_0^t K_{ii}(s)y_i(t-s)[\exp(w_i(t-s))-1]ds\\&-a_{ii}(t)\int_t^\infty K_{ii}(s)[\exp(u_i(t-s))-\exp(v_i(t-s))]ds\\&-\sum_{j=1,j\neq i}^{n}a_{ij}(t)\int_0^\infty K_{ij}(s)y_j(t-s)[\exp(w_j(t-s))-1]ds\\=&-a_{ii}(t)\left[\int_0^t K_{ii}(s)y_i(t-s)ds\right][\exp(w_i(t))-1]\\&+a_{ii}(t)\int_0^t K_{ii}(s)y_i(t-s)\left(\int_{t-s}^t \exp(w_i(s_1))\dot{w}_i(s_1)ds_1\right)ds\\&-a_{ii}(t)\int_t^\infty K_{ii}(s)[\exp(u_i(t-s))-\exp(v_i(t-s))]ds\\&-\sum_{j=1,j\neq i}^{n}a_{ij}(t)\int_0^\infty K_{ij}(s)y_j(t-s)[\exp(w_j(t-s))-1]ds\\=&-a_{ii}(t)\left[\int_0^t K_{ii}(s)y_i(t-s)ds\right][\exp(w_i(t))-1]\\&-a_{ii}(t)\int_t^\infty K_{ii}(s)[\exp(u_i(t-s))-\exp(v_i(t-s))]ds\\&-\sum_{j=1,j\neq i}^{n}a_{ij}(t)\int_0^\infty K_{ij}(s)y_j(t-s)[\exp(w_j(t-s))-1]ds\\&-a_{ii}(t)\sum_{j=1}^{n}\int_0^t K_{ii}(s)y_i(t-s)\left[\int_{t-s}^t \exp(w_i(s_1))a_{ij}(s_1)\right.\\&\times\left.\left(\int_0^\infty K_{ij}(s_2)y_j(s_1-s_2)[\exp(w_j(s_1-s_2))-1]ds_2\right)ds_1\right]ds\,.\end{aligned}$$

Let

(3.9) $$V_{i1}(w(t))=|w_i(t)|$$

and

(3.10) $$J_i(w(t))=-a_{ii}(t)\int_t^\infty K_{ii}(s)[\exp(u_i(t-s))-\exp(v_i(t-s))]ds\,.$$

Then, it follows from (3.9) and (3.8) that the upper right derivative $D^+/(Dt)V_{i1}(w)$ of $V_{i1}(w)$ along the solutions of (3.8) is given by

$$\begin{aligned}(3.11)\quad \frac{D^+}{Dt}V_{i1}(w(t)) \le & -a_{ii}(t)\left[\int_0^t K_{ii}(s)y_i(t-s)ds\right]|\exp(w_i(t))-1|+|J_i(w(t))| \\ & +\sum_{j=1,j\ne i}^n a_{ij}(t)\int_0^\infty K_{ij}(s)y_j(t-s)|\exp(w_j(t-s))-1|ds \\ & +a_{ii}(t)\sum_{j=1}^n\int_0^\infty K_{ii}(s)y_i(t-s)\left[\int_{t-s}^t \exp(w_i(s_1))a_{ij}(s_1)\right. \\ & \left.\times\left(\int_0^\infty K_{ij}(s_2)y_j(s_1-s_2)|\exp(w_j(s_1-s_2))-1|ds_2\right)ds_1\right]ds\,.\end{aligned}$$

By (3.10),

$$(3.12)\qquad |J_i(w_i(t))|\le(a_{ii}^u+\varepsilon_1)\left(\int_t^\infty K_{ii}(s)ds\right)\sup_{-\infty<s\le 0}|x_i(s)-y_i(s)|\,.$$

Denote $E_i=(a_{ii}^u+\varepsilon_1)\sup_{-\infty<s\le 0}|x_i(s)-y_i(s)|$. Then, for $t\ge 0$, we have from (3.12) that

$$(3.13)\qquad |J_i(w(t))|\le E_i\int_t^\infty K_{ii}(s)ds\,.$$

Let

$$\begin{aligned}(3.14)\quad V_{i2}(w)(t)= & \sum_{j=1,j\ne i}^n\int_0^\infty K_{ij}(s)\int_{t-s}^t a_{ij}(s_1+s)y_j(s_1)|\exp(w_j(s_1))-1|ds_1ds \\ & +\sum_{j=1}^n\int_0^\infty K_{ii}(s)\int_{t-s}^t a_{ii}(s_3+s)y_i(s_3)\int_{s_3}^t \exp(w_i(s_1))a_{ij}(s_1) \\ & \times\int_0^\infty K_{ij}(s_2)y_j(s_1-s_2)|\exp(w_j(s_1-s_2))-1|ds_2ds_1ds_3ds\,.\end{aligned}$$

Then, from (3.11) and (3.14),

$$\begin{aligned}(3.15)\quad \frac{D^+}{Dt}[V_{i1}+V_{i2}](w(t)) \le & -a_{ii}(t)\left[\int_0^t K_{ii}(s)y_i(t-s)ds\right]|\exp(w_i(t))-1|+|J_i(w(t))| \\ & +\sum_{j=1,j\ne i}^n\left(\int_0^\infty K_{ij}(s)a_{ij}(t+s)ds\right)y_j(t)|\exp(w_j(t))-1| \\ & +\sum_{j=1}^n\left(\int_0^\infty K_{ii}(s)\int_{t-s}^t a_{ii}(s_1+s)y_i(s_1)ds_1ds\right) \\ & \times\exp(w_i(t))a_{ij}(t)\int_0^\infty K_{ij}(s)y_j(t-s)|\exp(w_j(t-s))-1|ds\,.\end{aligned}$$

Denote

$$d_i(t)=\int_0^\infty K_{ii}(s)\int_{t-s}^t a_{ii}(s_1+s)y_i(s_1)ds_1 ds\,.$$

By (3.4),

$$d_i(t)\le(M_i+\varepsilon_1)(a_{ii}^u+\varepsilon_1)\int_0^\infty sK_{ii}(s)ds=(M_i+\varepsilon_1)(a_{ii}^u+\varepsilon_1)\sigma_i\,.$$

Also, for $\varepsilon_1>0$, there exists a $T_1\ge 0$ such that

$$\exp(w_i(t))\le\max\{x_i(t),\, y_i(t)\}\le M_i+\varepsilon_1$$

and

$$\int_t^\infty K_{ii}(s)\exp[-(b^u+\varepsilon_1)s]ds\le\int_t^\infty K_{ii}(s)ds<\varepsilon_1$$

for $t\ge T_1$. It then follows from (3.15) that, for $t\ge T_1$,

(3.16)
$$\begin{aligned}\frac{D^+}{Dt}[V_{i1}+V_{i2}](w(t))\le&-(a_{ii}^l-\varepsilon_1)\left[\int_0^t K_{ii}(s)y_i(t-s)ds\right]|\exp(w_i(t))-1|\\&+\sum_{j=1,j\ne i}^n(a_{ij}^u+\varepsilon_1)y_j(t)|\exp(w_j(t))-1|\\&+|J_i(w(t))|+(a_{ii}^u+\varepsilon_1)(M_i+\varepsilon_1)^2\sigma_i\sum_{j=1}^n(a_{ij}^u+\varepsilon_1)\\&\times\int_0^\infty K_{ij}(s)y_j(t-s)|\exp(w_j(t-s))-1|ds\,.\end{aligned}$$

For $i, j\in I$, let

$$b_{ij}=(a_{ii}^u+\varepsilon_1)(M_i+\varepsilon_1)^2\sigma_i(a_{ij}^u+\varepsilon_1)$$

and

(3.17)
$$V_i(w)(t)=[V_{i1}+V_{i2}+V_{i3}](w(t))\,,\qquad i\in I$$

with

$$V_{i3}(w(t))=\sum_{j=1}^n b_{ij}\int_0^\infty K_{ij}(s)\int_{t-s}^t y_j(s_1)|\exp(w_j(s_1))-1|ds_1 ds\,.$$

Then, one can derive from (3.17) and (3.16) that, for $t\ge T_1$,

$$\text{(3.18)}\qquad \frac{D^+}{Dt}V_i(w)(t)\leq -(a_{ii}^l-\varepsilon_1)\left[\int_0^t K_{ii}(s)y_i(t-s)ds\right]|\exp(w_i(t))-1|$$
$$+\sum_{j=1,j\neq i}^{n}(a_{ij}^u+\varepsilon_1)y_j(t)|\exp(w_j(t))-1|+|J_i(w(t))|$$
$$+\sum_{j=1}^{n}b_{ij}y_j(t)|\exp(w_j(t))-1|\,.$$

On the other hand, one has from (1.1) that

$$y_i'(t)\leq b_i(t)y_i(t)\qquad\text{for}\quad t\geq 0\quad\text{and}\quad i\in I,$$

which implies

$$\text{(3.19)}\qquad y_i(t)\leq y_i(t-s)\exp\left(\int_{t-s}^t b_i(s_1)ds_1\right)\qquad\text{for}\quad t\geq s\,.$$

By (3.19), for $t\geq T_1$,

$$\text{(3.20)}\qquad -\int_0^t K_{ii}(s)y_i(t-s)ds\leq -\left[\int_0^t K_{ii}(s)\exp\left(-\int_{t-s}^t b_i(s_1)ds_1\right)ds\right]y_i(t)$$
$$\leq -\left[\int_0^\infty K_{ii}(s)\exp[-(b_i^u+\varepsilon_1)s]ds\right.$$
$$\left.-\int_t^\infty K_{ii}(s)\exp[-(b_i^u+\varepsilon_1)s]ds\right]y_i(t)$$
$$\leq -\left[\int_0^\infty K_{ii}(s)\exp[-(b_i^u+\varepsilon_1)s]ds-\varepsilon_1\right]y_i(t)\,.$$

Note that $y_i(t)|\exp(w_i(t))-1|=|x_i(t)-y_i(t)|$. Therefore, it follows from (3.18) and (3.20) that, for $t\geq T_1$,

$$\text{(3.21)}\quad \frac{D^+}{Dt}V_i(w)(t)\leq -(a_{ii}^l-\varepsilon_1)\left[\int_0^\infty K_{ii}(s)\exp[-(b_i^u+\varepsilon_1)s]ds-\varepsilon_1\right]|x_i(t)-y_i(t)|$$
$$+|J_i(w(t))|+\sum_{j=1,j\neq i}^{n}(a_{ij}^u+\varepsilon_1)|x_j(t)-y_j(t)|+\sum_{j=1}^{n}b_{ij}|x_j(t)-y_j(t)|$$
$$=-\sum_{j=1}^{n}e_{ij}(\varepsilon_1)|x_j(t)-y_j(t)|+|J_i(w(t))|$$

with $e_{ij}(\varepsilon_1)$ defined by (3.3). Now, let

$$V(w(t))=\sum_{j=1}^{n}\alpha_i V_i(w(t))\,.$$

Then, from (3.21), (3.2) and (3.1), for $t \geq T_1$,

$$\begin{aligned}
(3.22) \qquad \frac{D^+}{Dt} V(w(t)) &\leq -\sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} e_{ij}(\varepsilon_1)|x_j(t)-y_j(t)| + \sum_{i=1}^{n} \alpha_i |J_i(w(t))| \\
&= -\sum_{i=1}^{n} \left[\sum_{j=1}^{n} \alpha_j e_{ji}(\varepsilon_1) \right] |x_i(t)-y_i(t)| + J(w(t)) \\
&\leq -\sum_{i=1}^{n} \left[\alpha_i(e_{ii}-\varepsilon_o) - \sum_{j=1, j\neq i}^{n} \alpha_j(|e_{ji}|+\varepsilon_o) \right] |x_i(t)-y_i(t)| + J(w(t)) \\
&= -\sum_{i=1}^{n} \beta_i |x_i(t)-y_i(t)| + J(w(t)),
\end{aligned}$$

where

$$J(w(t)) = \sum_{i=1}^{n} \alpha_i |J_i(w(t))|$$

and

$$\beta_i = \alpha_i[e_{ii}-\varepsilon_o] - \sum_{j=1, j\neq i}^{n} \alpha_j(|e_{ji}|+\varepsilon_o) > 0 \qquad \text{(by (3.1))} .$$

Note that $V(w(t)) \geq 0$ and also, from (3.13),

$$J(w(t)) \leq \sum_{i=1}^{n} \alpha_i E_i \int_t^{\infty} K_{ii}(s)ds .$$

Hence, for $t \geq T_1$,

$$\begin{aligned}
\int_{T_1}^{t} J(w(s))ds &\leq \sum_{i=1}^{n} \alpha_i E_i \int_{T_1}^{t} \int_s^{\infty} K_{ii}(p)dpds \\
&\leq \sum_{i=1}^{n} \alpha_i E_i \int_{T_1}^{\infty} \int_{T_1}^{s} K_{11}(s)dpds \\
&\leq \sum_{i=1}^{n} \alpha_i E_i \int_{T_1}^{\infty} sK_{ii}(s)ds < \infty .
\end{aligned}$$

Integrating (3.22) from T_1 to $t \geq T_1$, we have

$$\sum_{i=1}^{n} \beta_i \int_{T_1}^{t} |x_i(s)-y_i(s)|ds < \infty .$$

Consequently, $\sum_{i=1}^{n} \beta_i \int_{T_1}^{\infty} |x_i(t)-y_i(t)|ds < \infty$. Hence, by the uniform continuity of $\sum_{i=1}^{n} \beta_i |x_i(t)-y_i(t)|$ on $[0, \infty)$, we have $|x_i(t)-y_i(t)| \to 0$ as $t \to \infty$. This completes the proof.

4. Existence of an almost periodic solution. In this section, we shall use the stability properties established in section 3 and employ Murakami's idea (see [10]) to derive the existence of a positive almost periodic solution of the system (1.1). For convenience in the following discussion, we rename the system (1.1) as (E), that is,

$$\text{(E)} \qquad \dot{x}_i(t)=x_i(t)\left[b_i(t)-\sum_{j=1}^{n} a_{ij}(t)\int_0^{\infty} K_{ij}(s)x_i(t-s)ds\right], \qquad i=1,\dots,n.$$

For completeness, we include the following notation, lemma and definitions introduced by Murakami [10]. We denote by S(E) the set of all solutions $x(t)=(x_1(t),\dots,x_n(t))$ of the system (E) on R satisfying $0<m_i-\varepsilon\le x_i(t)\le M_i+\varepsilon$ for $i=1,2,\dots,n$, $t\in R$ and sufficiently small $\varepsilon>0$. Let BC be the set of all bounded continuous functions from R_- into R^n. For any $\phi,\psi\in\mathrm{BC}$ we set

$$\rho_k(\phi,\psi)=\sup_{-k\le s\le 0}|\phi(s)-\psi(s)|,$$

$$\rho(\phi,\psi)=\sum_{k=1}^{\infty}\rho_k(\phi,\psi)/[2^k(1+\rho_k(\phi,\psi))].$$

Clearly, $\rho(\phi_m,\phi)\to 0$ as $m\to\infty$ if and only if $\phi_m(s)\to\phi(s)$ as $n\to\infty$ uniformly on each bounded subset of $(-\infty,0]$. For any function $x\colon R\to R^n$ and any $t\in R$, we define a function $x^t\colon(-\infty,0]\to R^n$ by $x^t(s)=x(t+s)$ for $s\le 0$. Similarly to Lemma 3 in Murakami [10], we can conclude:

LEMMA 4.1. *Let a* $p\in\mathrm{S(E)}$ *and a sequence* $\{t_n\}$, $t_n\ge 0$, *be given. If*

(H5) $a_{ij}(t+t_n)\to\bar{a}_{ij}(t)$ *and* $b_i(t+t_n)\to\bar{b}_i(t)$ *as* $n\to\infty$ *on* R *for all* $i,j=1,\dots,n$, *and* $p(t+t_n)\to\bar{p}(t)$ *as* $n\to\infty$ *uniformly on each bounded subset of* R *for some functions* $\bar{a}_{ij}$, $\bar{b}_i$ *and* $\bar{p}$,

then $\bar{p}\in\mathrm{S}(\bar{\mathrm{E}})$, *where* $\mathrm{S}(\bar{\mathrm{E}})$ *denotes the set of all solutions* $y(t)=(y_1(t),\dots,y_n(t))$ *of the system*

$$(\bar{\mathrm{E}}) \qquad \dot{y}_i(t)=y_i(t)\left[\bar{b}_i(t)-\sum_{j=1}^{n}\bar{a}_{ij}(t)\int_0^{\infty}K_{ii}(s)y_i(t-s)ds\right], \qquad i=1,\dots,n,$$

on R *satisfying* $0<m_i-\varepsilon\le y_i(t)\le M_i+\varepsilon$ *for* $i=1,2,\dots,n$, $t\in R$ *and sufficiently small* $\varepsilon>0$. (*Henceforth, we denote* $(\bar{p},\bar{\mathrm{E}})\in\Omega(p,\mathrm{E})$ *when* (H5) *holds*).

DEFINITION 4.2. A function $p\in\mathrm{S(E)}$ is said to be relatively uniformly stable in Ω(E) (RUS in Ω(E), for short) if for any $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ with the property that for any $t_o\ge 0$, any $(\bar{p},\bar{\mathrm{E}})\in\Omega(p,\mathrm{E})$ and any $\bar{z}\in\mathrm{S}(\bar{\mathrm{E}})$ satisfying $\rho(\bar{p}^{t_o},\bar{z}^{t_o})<\delta(\varepsilon)$ we have $\rho(\bar{p}^t,\bar{z}^t)<\varepsilon$ for all $t\ge t_o$.

DEFINITION 4.3. A function $p\in\mathrm{S(E)}$ is said to be relatively weakly uniformly asymptotically stable in Ω(E) (RWUAS in Ω(E), for short) if p is RUS in Ω(E), and if $\rho(\bar{p}^t,\bar{z}^t)\to 0$ as $t\to\infty$ for all $(\bar{p},\bar{\mathrm{E}})\in\Omega(p,\mathrm{E})$ and all $\bar{z}\in\mathrm{S}(\bar{\mathrm{E}})$.

DEFINITION 4.4. A function $p \in S(E)$ is said to be relatively totally stable for (E) (RTS for (E), for short) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ with the property that if $t_o \geq 0$, $\rho(x^{t_o}, p^{t_o}) < \delta(\varepsilon)$ and $g(t) = (g_1(t), \ldots, g_n(t)) : R \to R^n$ is any continuous function satisfying $\sup_{t \in R} |g(t)| < \delta(\varepsilon)$, then we have $\rho(x^t, p^t) < \varepsilon$ for all $t \geq t_o$, where x is any solution of the system

$$(\bar{E}_g) \qquad \dot{x}_i(t) = x_i(t)\left[b_i(t) - \sum_{j=1}^{n} a_{ij}(t) \int_0^\infty K_{ii}(s) x_i(t-s) ds \right] + g_i(t), \qquad i = 1, \ldots, n,$$

on R satisfying $0 < m_i - \varepsilon \leq x_i(t) \leq M_i + \varepsilon$ for $i = 1, 2, \ldots, n$, $t \in R$ and sufficiently small $\varepsilon > 0$.

By repeating the same argument as in the proof of Lemma 4 in Murakami [10], we have the following conclusion.

LEMMA 4.5. *If $p \in S(E)$ is RWUAS in $\Omega(E)$, then it is RTS for (E).*

We now state our main result on the existence and global attractivity of the positive almost periodic solution of (E).

THEOREM 4.6. *Under the assumptions of Theorem 3.2, the system (1.1) has a positive almost periodic solution, which is globally attractive.*

PROOF. From Theorem 3.2, one can see that it is enough to show the existence of an almost periodic solution of (E). The proof is essentially the same as the one for Theorem in Murakami [10]. For the completeness, we indicate it briefly. By Lemma 3.1, there exists a $p \in S(E)$. We shall prove that p is asymptotically almost periodic.

Let $\{t_m\}$ be any sequence satisfying $t_m \to \infty$ as $m \to \infty$. We may assume that the sequence $\{p(t + t_m)\}_{m=1}^\infty$ is uniformly convergent on each bounded subset of R and that the sequences $\{a_{ij}(t + t_m)\}_{m=1}^\infty$ and $\{b(t + t_m)\}_{m=1}^\infty$ are uniformly convergent on R. Set $p^k(t) = p(t + t_k)$, $t \in R$, for each positive integer k. Clearly, p^k is a solution of the system

$$(E^k) \qquad \dot{x}_i(t) = x_i(t)\left[b_i(t + t_k) - \sum_{j=1}^{n} a_{ij}(t + t_k) \int_0^\infty K_{ii}(s) x_i(t-s) ds \right], \qquad i = 1, \ldots, n,$$

on R.

We first prove that p^k is RTS for the system (E^k). By Lemma 4.5 it suffices to show that p^k is RWUAS for the system (E^k).

CLAIM A. For arbitrary $(\bar{p}^k, \bar{E}^k) \in \Omega(p^k, E^k)$ and $\bar{z}^k \in S(\bar{E}^k)$, we have $\rho((\bar{p}^k)^t, (\bar{z}^k)^t) \to 0$ as $t \to \infty$.

The proof of this claim is essentially the same as the proof of Theorem 3.2. However, for convenience in the proof of the next claim, we describe the modified Lyapunov functionals, which will be used in the proof of the next claim.

Let

$$u_i(t)=\ln[\bar{p}_i^k(t)]\,,\quad v_i(t)=\ln[\bar{z}_i^k(t)]\,,\quad w_i(t)=u_i(t)-v_i(t)$$

for $i=1,\ldots,n$ and $w(t)=(w_1(t),\ldots,w_n(t))$. Then, for $i=1,\ldots,n$,

$$\begin{aligned}(4.1)\qquad \dot{w}_i(t)&=-\sum_{j=1}^{n}\bar{a}_{ij}(t+t_k)\int_0^{\infty}K_{ij}(s)[\exp(u_j(t-s))-\exp(v_j(t-s))]ds\\ &=-\bar{a}_{ii}(t+t_k)\Bigg[\left(\int_0^{t+t_k}K_{ii}\bar{z}_i^k(t-s)ds\right)[\exp(w_i(t))-1]\\ &\quad+\int_0^{t+t_k}K_{ii}(s)\bar{z}_i^k(t-s)\int_{t-s}^{t}\exp(w_i(s_1))\dot{w}_i(s_1)ds_1ds\\ &\quad+\int_{t+t_k}^{\infty}K_{ii}(s)[\bar{p}_i^k(t-s)-\bar{z}_i^k(t-s)]ds\Bigg]\\ &\quad-\sum_{j=1,j\neq i}^{n}\bar{a}_{ij}(t+t_k)\int_0^{\infty}K_{ii}(s)\bar{z}_j^k(t-s)s\,[\exp(w_j(t-s))-1]ds\,.\end{aligned}$$

Clearly, for the $\varepsilon_o>0$ satisfying (3.1), there exists an $\varepsilon_1\in(0,\varepsilon_o)$ such that for $t\in R$,

$$(4.2)\qquad \begin{gathered}b_i^l-\varepsilon_1\le\bar{b}_i(t+t_k)\le b_i^u+\varepsilon_1\,,\quad 0<a_{ii}^l-\varepsilon_1\le\bar{a}_{ii}(t+t_k)\,,\quad \bar{a}_{ij}(t+t_k)\le a_{ij}^u+\varepsilon_1\,,\\ 0<m_i-\varepsilon_1\le\bar{p}_i^k(t)\,,\quad \bar{z}_i^k(t)\le M_i+\varepsilon_1\end{gathered}$$

and (3.2) holds with $e_{ij}(\varepsilon_1)$ defined by (3.3). Also, we can select a large positive integer k_o such that, for $k\ge k_o$ and $t\ge 0$,

$$(4.3)\qquad \int_{t+t_k}^{\infty}K_{ii}(s)ds<\varepsilon_1\,.$$

Let V_{i1} be defined by (3.9). It then follows from (4.1), (4.2) and (3.11) that, for $t\ge 0$,

$$\begin{aligned}(4.4)\qquad \frac{D^+}{Dt}V_{i1}(w(t))&\le-(a_{ii}^l-\varepsilon_1)\left[\int_0^{t+t_k}K_{ii}(s)\bar{z}_i^k(t-s)ds\right]|\exp(w_i(t))-1|\\ &\quad+\sum_{j=1,j\neq i}^{n}(a_{ij}^u+\varepsilon_1)\int_0^{\infty}K_{ij}(s)\bar{z}_j^k(t-s)|\exp(w_j(t-s))-1|ds\\ &\quad+(a_{ii}^u+\varepsilon_1)\bar{J}_i^k(t)+\sum_{j=1}^{n}c_{ij}\int_0^{\infty}K_{ii}(s)\int_{t-s}^{t}\int_0^{\infty}K_{ij}(s_2)\bar{z}_j^k(s_1-s_2)\\ &\quad\times|\exp(w_j(s_1-s_2))-1|ds_2ds_1ds\end{aligned}$$

with

$$c_{ij}=(a_{ii}^u+\varepsilon_1)(M_i+\varepsilon_1)^2(a_{ij}^u+\varepsilon_1)$$

and

(4.5) $$\bar{J}_i^k(t)=\int_{t+t_k}^{\infty} K_{ii}(s)\,|\bar{p}_i^k(t-s)-\bar{z}_i^k(t-s)|\,ds\,.$$

Let

(4.6) $$W_i(w(t))=(V_{i1}+W_{i1})(w(t))\,,$$

where

(4.7) $$\begin{aligned} W_{i1}(w(t))= & \sum_{j=1,j\neq i}^{n}(a_{ij}^u+\varepsilon_1)\int_0^{\infty}K_{ij}(s)\int_{t-s}^{t}\bar{z}_j^k(s_1)|\exp(w_j(s_1))-1|\,ds_1 ds \\ & +\sum_{j=1}^{n}c_{ij}\left\{\int_0^{\infty}K_{ii}(s)\int_{t-s}^{t}\int_{s_3}^{t}\int_0^{\infty}K_{ij}(s_2)\bar{z}_j^k(s_1-s_2)\right. \\ & \times|\exp(w_j(s_1-s_2))-1|\,ds_2 ds_1 ds_3 ds \\ & \left.+\sigma_i\int_0^{\infty}K_{ij}(s)\int_{t-s}^{t}\bar{z}_j^k(s_1)|\exp(w_j(s_1))-1|\,ds_1 ds\right\}. \end{aligned}$$

Then, from (4.4)–(4.7),

(4.8) $$\begin{aligned} \frac{D^+}{Dt}W_i(w(t))\leq & -(a_{ii}^l-\varepsilon_1)\left[\int_0^{t+t_k}K_{ii}(s)\bar{z}_i^k(t-s)ds\right]|\exp(w_i(t))-1| \\ & +\sum_{j=1,j\neq i}^{n}(a_{ij}^u+\varepsilon_1)\bar{z}_j^k(t)|\exp(w_j(t))-1| \\ & +(a_{ii}^u+\varepsilon_1)\bar{J}_i^k(t)+\sum_{j=1}^{n}c_{ij}\sigma_i\bar{z}_j^k(t)|\exp(w_j(t))-1|\,. \end{aligned}$$

Similarly to (3.20), we have

(4.9) $$-\int_0^{t+t_k}K_{ii}(s)\bar{z}_i^k(t-s)ds\leq-\left[\int_0^{\infty}K_{ii}(s)\exp[-(b_u+\varepsilon_1)s]ds-\varepsilon_1\right]\bar{z}_i^k(t)\,.$$

Thus, from (4.8), (4.9), (4.3) and (3.3),

(4.10) $$\frac{D^+}{Dt}W_i(w(t))\leq-\sum_{j=1}^{n}e_{ij}(\varepsilon_1)|\bar{p}_j^k(t)-\bar{z}_j^k(t)|+(a_{ii}+\varepsilon_1)\bar{J}_i^k(t)\,.$$

Denote

(4.11) $$W(w(t))=\sum_{j=1}^{n}\alpha_i W_i(w(t))\,.$$

Then, for $t\geq 0$,

(4.12) $$\frac{D^+}{Dt}W(w(t))\leq-\sum_{j=1}^{n}\beta_i|\bar{p}^k(t)-\bar{z}_k(t)|+\bar{J}(t)$$

with

$$\bar{J}(t) = \sum_{i=1}^{n} \alpha_i (a_{ii} + \varepsilon_1) \bar{J}_i^k(t) . \tag{4.13}$$

Note that (from (4.5) and (4.2))

$$\bar{J}(t) \leq \sum_{i=1}^{n} \alpha_i (a_{ii} + \varepsilon_1)(M_i + \varepsilon_1) \int_{t+t_k}^{\infty} K_{ii}(s) ds \in L_1[0, \infty) .$$

Then, similarly to the last part of the proof of Theorem 3.2, we can conclude from (4.12) that $|\bar{p}_i^k(t) - \bar{z}_i^k(t)| \to 0$ as $t \to \infty$, which leads to $\rho((\bar{p}^k)^t, (\bar{z}^k)^t) \to 0$ as $t \to \infty$.

CLAIM B. p^k is RUS in Ω(E).

It follows from (4.12) that, for $t \geq t_o \geq 0$,

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i |\ln[\bar{p}_i^k(t)] - \ln[\bar{z}_i^k(t)]| &\leq W(w(t)) \leq W(w(t_o)) + \int_{t_o}^{t} \bar{J}(s) ds \\ &= \sum_{i=1}^{n} \alpha_i \left[V_{i1}(w(t_o)) + W_{i1}(w(t_o)) + (a_{ii} + \varepsilon_1) \int_{t_o}^{t} \bar{J}_i^k(s) ds \right] . \end{aligned} \tag{4.14}$$

Note that, for $i, j = 1, \ldots, n$ and all $L \geq 0$,

$$V_{i1}(w(t_o)) = |\ln[\bar{p}_i^k(t_o)] - \ln[\bar{z}_i^k(t_o)]| ,$$

$$\begin{aligned} &\int_0^{\infty} K_{ij}(s) \int_{t_o - s}^{t_o} |\bar{p}_j^k(s_1) - \bar{z}_j^k(s_1)| ds_1 ds \\ &\quad \leq (M_j + \varepsilon_1) \int_L^{\infty} s K_{ij}(s) ds + \left(\int_0^{\infty} s K_{ij}(s) ds \right) \sup_{t_o - L \leq s \leq t_o} |\bar{p}_j^k(s) - \bar{z}_j^k(s)| \end{aligned}$$

and

$$\begin{aligned} &\int_0^{\infty} K_{ii}(s) \int_{t_o - s}^{t_o} \int_{s_3}^{t_o} \int_0^{\infty} K_{ij}(s_2) |\bar{p}_j^k(s_1 - s_2) - \bar{z}_j^k(s_1 - s_2)| ds_2 ds_1 ds_3 ds \\ &\quad \leq \int_0^{\infty} K_{ii}(s) \int_{t_o - s}^{t_o} \int_{t_o - s}^{t_o} \int_0^{\infty} K_{ij}(s_2) |\bar{p}_j^k(s_1 - s_2) - \bar{z}_j^k(s_1 - s_2)| ds_2 ds_1 ds_3 ds \\ &\quad = \int_0^{\infty} s K_{ii}(s) \left\{ \int_0^{\infty} K_{ij}(s_2) \int_{t_o - s}^{t_o} |\bar{p}_j^k(s_1 - s_2) - \bar{z}_j^k(s_1 - s_2)| ds_1 ds_2 \right\} ds \\ &\quad = R_1 + R_2 , \end{aligned}$$

where

$$R_1=\int_0^L sK_{ii}(s)\left\{\int_0^\infty K_{ij}(s_2)\int_{t_o-s}^{t_o}|\bar{p}_j^k(s_1-s_2)-\bar{z}_j^k(s_1-s_2)|ds_1ds_2\right\}ds$$

$$=\int_0^L sK_{ii}(s)\left\{\int_0^L K_{ij}(s_2)\int_{t_o-s}^{t_o}|\bar{p}_j^k(s_1-s_2)-\bar{z}_j^k(s_1-s_2)|ds_1ds_2\right\}ds$$

$$+\int_0^L sK_{ii}(s)\left\{\int_L^\infty K_{ij}(s_2)\int_{t_o-s}^{t_o}|\bar{p}_j^k(s_1-s_2)-\bar{z}_j^k(s_1-s_2)|ds_1ds_2\right\}ds$$

$$\leq\left(\int_0^\infty s^2K_{ii}(s)ds\right)\sup_{t_o-2L\leq s\leq t_o}|\bar{p}_j^k(s)-\bar{z}_j^k(s)|$$

$$+(M_j+\varepsilon_1)\left(\int_0^\infty s^2K_{ii}(s)ds\right)\int_L^\infty K_{ij}(s)ds$$

and

$$R_2=\int_L^\infty sK_{ii}(s)\left\{\int_0^\infty K_{ij}(s_2)\int_{t_o-s}^{t_o}|\bar{p}_j^k(s_1-s_2)-\bar{z}_j^k(s_1-s_2)|ds_1ds_2\right\}ds$$

$$\leq(M_j+\varepsilon_1)\int_L^\infty s^2K_{ii}(s)ds\,.$$

Also,

$$G_i(t)=\int_{t_o}^t \bar{J}_i^k(u)du$$

$$=\int_{t_o}^t\int_{u+t_k}^\infty K_{ii}(s)|\bar{p}_j^k(u-s)-\bar{z}_j^k(u-s)|dsdu$$

$$\leq\int_{t_o}^\infty\int_{u+t_k}^\infty K_{ii}(s)|\bar{p}_j^k(u-s)-\bar{z}_j^k(u-s)|dsdu$$

$$=\int_{t_o+t_k}^\infty K_{ii}(s)\int_{t_o}^{s-t_k}|\bar{p}_j^k(u-s)-\bar{z}_j^k(u-s)|duds$$

$$=\int_{t_o+t_k}^{2(t_o+t_k)} K_{ii}(s)\int_{t_o}^{s-t_k}|\bar{p}_j^k(u-s)-\bar{z}_j^k(u-s)|duds$$

$$+\int_{2(t_o+t_k)}^\infty K_{ii}(s)\int_{t_o}^{s-t_k}|\bar{p}_j^k(u-s)-\bar{z}_j^k(u-s)|duds$$

$$\leq\sigma_i\sup_{-t_o-2t_k\leq s\leq t_o}|\bar{p}_j^k(s)-\bar{z}_j^k(s)|+(M_i+\varepsilon_1)\int_{t_k}^\infty K_{ii}(s)ds\,.$$

Thus, for $i=1,\ldots,n$ and all $L\geq 0$,

$$W_{i1}(w(t_o)) \leq \sum_{j=1}^{n} d_{ij}\left[\left(\int_L^\infty sK_{ij}(s)ds\right)\left(\sup_{t_o-2L\leq s\leq t_o} |\bar{p}_j^k(s)-\bar{z}_j^k(s)|\right.\right.$$
$$\left.\left.+(M_j+\varepsilon_1)\int_L^\infty K_{ij}(s)ds\right)+(M_j+\varepsilon_1)\int_L^\infty s^2K_{ii}(s)ds\right],$$

where

$$d_{ii}=c_{ii}\sigma_i\,,\quad d_{ij}=(a_{ij}^u+\varepsilon_1)+c_{ij}\sigma_i\quad (i\neq j)\,,\qquad i,j=1,\ldots,n\,.$$

Now, using the argument in [10, p. 77], one can show that, for each $\varepsilon>0$ we can select large k_o, $L>0$ and small $\delta(\varepsilon)>0$ such that $|\bar{p}^k(t)-\bar{z}^k(t)|<\varepsilon$ for all $t\geq t_o\geq 0$, provided $\rho((\bar{p}^k)^{t_o},(\bar{z}^k)^{t_o})<\delta(\varepsilon)$. This implies that, $\rho((\bar{p}^k)^{t_o},(\bar{z}^k)^{t_o})<\delta(\varepsilon)$ leads to $\rho((\bar{p}^k)^t,(\bar{z}^k)^t)<\delta(\varepsilon)$ for all $t\geq t_o$. Therefore, for $k\geq k_o$, $p^k\in S(\mathrm{E}^k)$ is RUS in $\Omega(\mathrm{E}^k)$.

By the above claims, we conclude that p^k is RWUAS for system (E^k) and hence p^k is RTS for (E^k). Then, following the same argument as in [10, p. 78], we conclude that $p(t)$ is asymptotically almost periodic, and thus, its almost periodic part is a solution of (E). This completes the proof.

5. Discussion. We conclude this paper with the following remark. The conditions of Theorem 3.2 depend only on the size of the diagonal delays, which are measured by σ_i $(i\in I)$. For (2.9), which is a special case of (1.1), $\sigma_i=\tau_i$ and the conditions of Theorem 3.2 become (2.10), $(M_i a_{ii}^u)^2\tau_i<a_{ii}^l\exp(-b_i^u\tau_i)$ and that $E^*=(e_{ij}^*)_{n\times n}$ is an M-matrix with $e_{ii}^*=a_{ii}^l\exp(-b_i^u\tau_i)-(M_i a_{ii}^u)^2\tau_i$ and $e_{ij}^*=-[1+M_i^2a_{ii}^u\tau_i]a_{ij}^u$ for $i\neq j$, $i,j\in I$, where M_i $(i\in I)$ are defined by (2.11). In particular, when $\tau_i=0$ $(i\in I)$, the system (2.9) becomes (1.2) and the corresponding conditions become

$$b_i^l>\sum_{j=1,j\neq i}^{n} a_{ij}^u\frac{b_j^u}{a_{jj}^l}\qquad (i\in I) \tag{5.1}$$

and that $E=(e_{ij})_{n\times n}$ with $e_{ii}=a_{ii}^l$ and $e_{ij}=-a_{ij}^u$ $(i\neq j)$ is an M-matrix. Using the properties of an M-matrix (see Gopalsamy and He [6]), one can verify that the condition (5.1) implies that E is an M-matrix. In fact, in addition to (5.1), under the condition

$$a_{ii}^l>\sum_{j=1,j\neq i}^{n} a_{ji}^u\qquad (i\in I)\,, \tag{5.2}$$

the existence of a strictly positive almost periodic solution was shown by Gopalsamy [3] in the periodic case in (1.2) and by Murakami [10] in the almost periodic case in (1.2). It was shown by Hamaya and Yoshizawa [8] that, for the almost periodic system (1.2), the condition (5.2) is not necessary. Therefore, when (1.1) takes the form (1.2), our conditions are reduced to the one for (1.2). It is in this sense that our result is a significant generalization of the known results.

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