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NUMBER OF ZEROS OF SOLUTIONS TO SINGULAR INITIAL VALUE PROBLEMS

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Abstract. The behavior of solutions of singular initial value problems is studied for a second order ordinary differential equation. The main purpose of this paper is to obtain sharp sufficient conditions so that any solution has a finite number of zeros or infinitely many zeros. We treat them systematically and generalize previous results by using the Pohozaev identity. As an application, we investigate the number of zeros of radially symmetric solutions to generalized Laplace equations.

1. Introduction. The asymptotic behavior of solutions is one of the main topics in the theory of ordinary differential equations. In particular, the finiteness of the number of zeros of solutions is a fundamental question. In this paper we consider the behavior of solutions to an equation of the form

$$(\varphi(v_t))_t + k(t)f(v) = 0\,, \tag{1.1}$$

where

$$\varphi(\xi) = |\xi|^{m-1}\operatorname{sgn}\xi\,, \qquad m > 1\,.$$

Here, we introduce the following assumptions on $f(v)$:

$$\begin{cases} f(v) \in C(\boldsymbol{R}) \cap C^1(\boldsymbol{R} \setminus \{0\})\,, \\ vf(v) > 0 \quad \text{for} \quad v \neq 0\,, \\ \limsup\limits_{v \to 0} \dfrac{v|f'(v)|}{f(v)} < \infty\,, \end{cases} \tag{f.0}$$

$$q_1 := \inf_{v \neq 0} \frac{vf'(v)}{f(v)} > m-1\,, \tag{f.1}$$

$$q_2 := \sup_{v \neq 0} \frac{vf'(v)}{f(v)} < \infty\,. \tag{f.2}$$

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The conditions $(f.1)$ and $(f.2)$ imply the "super-linearity" and "polynomial growth" of f, respectively. A typical example satisfying $(f.0)$, $(f.1)$ and $(f.2)$ is $f(v)=|v|^{q-1}v$ with $q>m-1$.

Concerning $k(t)$, we introduce the hypotheses

$$(k.0) \qquad k(t)>0 \quad \text{on} \quad (0,\infty)\,, \quad k(t)\in C^1((0,\infty))\,,$$

$$(k.1) \qquad \int_1^\infty \varphi^{-1}\left(\int_s^\infty k(\tau)d\tau\right)ds<\infty\,,$$

$$(k.2) \qquad \int_0^1 k(\tau)|f(c\tau)|d\tau<\infty \qquad \text{for any} \quad c\neq 0\,,$$

where $\varphi^{-1}(\zeta)=|\zeta|^{1/(m-1)}\operatorname{sgn}\zeta$. Note that $(k.1)$ implies $k(t)\in L^1(1,\infty)$ under the condition $(k.0)$.

Under the standing assumptions $(f.0)$ and $(k.0)$, we will consider the following singular initial value problems

$$(\mathrm{P}) \qquad \begin{cases} (\varphi(v_t))_t+k(t)f(v)=0\,, & t\in(0,\infty)\,,\\ v(t_1)=a\,, \quad v_t(t_1)=b\,, \end{cases}$$

$$(\mathrm{P}_\alpha) \qquad \begin{cases} (\varphi(v_t))_t+k(t)f(v)=0\,, & t\in(0,\infty)\,,\\ \lim\limits_{t\to\infty} v(t)=\alpha>0\,, \end{cases}$$

$$(\tilde{\mathrm{P}}_\beta) \qquad \begin{cases} (\varphi(v_t))_t+k(t)f(v)=0\,, & t\in(0,\infty)\,,\\ \lim\limits_{t\downarrow 0} \dfrac{v(t)}{t}=\beta>0\,, \end{cases}$$

where $t_1>0$ and the initial data a, b, α, β are arbitrarily given. Our interest here is to classify solutions of the above problems according to their asymptotic behavior and their zeros.

We note that the above problems are closely related to radial solutions of m-Laplace equations in $\boldsymbol{R}^n$. Let us consider the m-Laplace equation of the form

$$(1.2) \qquad \operatorname{div}(|\nabla u|^{m-2}\nabla u)+K(|x|)|u|^{q-1}u=0\,.$$

Any radial solution to this equation satisfies

$$(1.3) \qquad \frac{1}{r^{n-1}}(r^{n-1}|u_r|^{m-2}u_r)_r+K(r)|u|^{q-1}u=0\,.$$

Changing the variables by

$$v(t)=\left(\frac{m-1}{n-m}\right)^{q/(q-m+1)}u(r)\,, \quad k(t)=r^{m(n-1)/(m-1)}K(r)\,, \quad r=t^{-(m-1)/(n-m)}\,,$$

we can reduce the above equation to (1.1).

When $m=2$, the structure of positive solutions to (P_α) which correspond to radial solutions to (1.2) is precisely investigated by many authors (see, e.g., Li-Ni [12], Yanagida-Yotsutani [17] and the references therein). Concerning solutions with zeros, there are a lot of existence and non-existence results such as Atkinson [1], Coffman-Uhllich [2], Kiguradze [8], Ni [13], Ding-Ni [3], Kusano-Naito [10] and Yanagida [16].

On the other hand, when $m\neq 2$, a sharp structure theorem of positive solutions to (P_α) is obtained for (1.3) by Kawano-Yanagida-Yotsutani [7], which is a generalization of Ni-Serrin [14], Kawano-Ni-Yotsutani [5] and Kawano-Yanagida-Yotsutani [6]. Concerning solutions with an infinite number of zeros, however, there are few results for the existence and non-existence of solutions to (P_α) except for Kusano-Ogata-Usami [11]. It seems that there is no systematic treatments to $(\tilde{P}_\beta)$, which corresponds to the problem of finding radial solutions with $u \sim |x|^{-(n-m)/(m-1)}$ at $x=\infty$ to m-Laplace equations (1.2) in $\boldsymbol{R}^n \setminus \{0\}$.

The main purpose of this paper is to give sharp sufficient conditions so that any solution to (1.1) has a finite number of zeros or infinitely many zeros near $t=0$ or $t=\infty$. We systematically use the Pohozaev type identity by which we not only give comprehensive proofs but also generalize the previous results.

The Pohozaev type identity was first introduced in Pohozaev [15], in which the non-existence of positive solutions to some class of nonlinear elliptic equations was shown. This identity is a fundamental and sharp energy equality, and has been used to investigate properties of solutions after suitable rearrangement of the equality. We show that the identity is also very effective to give answers to our problem by introducing its various rearrangements.

In order to classify solutions, we define the type of a solution according to its behavior at $t=0$ and $t=\infty$.

At $t=0$ we say that

(i) $v(t)$ is of type R if $v(t)$ has at most a finite number of zeros in (0, 1) and $\lim_{t\downarrow 0} v(t)/t$ exists and is finite,

(ii) $v(t)$ is of type S if $v(t)$ has at most a finite number of zeros in (0, 1) and $\lim_{t\downarrow 0} |v(t)|/t=\infty$,

(iii) $v(t)$ is of type O if $v(t)$ has infinitely many zeros in (0, 1).

Similarly, at $t=\infty$, we say that

(i) $v(t)$ is of type $\tilde{R}$ if $v(t)$ has at most a finite number of zeros in $(1, \infty)$ and $\lim_{t\to\infty} v(t)$ exists and is finite,

(ii) $v(t)$ is of type $\tilde{S}$ if $v(t)$ has at most finite number of zeros in $(1, \infty)$ and $\lim_{t\to\infty} |v(t)|=\infty$,

(iii) $v(t)$ is of type $\tilde{O}$ if $v(t)$ has infinitely many zeros in $(1, \infty)$.

As we will see in Proposition 2.1 in the next section, there is no other type of solutions.

First of all, we consider the existence and uniqueness of solutions to (P), and investigate their possible behavior at $t=0$ and $t=\infty$. The first theorem is a modification of the results by Coffman-Uhllich [2], Kawano-Ni-Yotsutani [5], Kawano-Yanagida-Yotsutani [7], Kitano-Kusano [9] and Ni-Serrin [14].

THEOREM 1. *Suppose that* $(f.0)$ *and* $(k.0)$ *hold. Then for any* $t_1>0$, *a and b, there exists a unique solution* $v=v(t)$ *to* (P) *satisfying*

$$v(t)\in\begin{cases} C^{1,1/(m-1)}((0,\infty)) & \text{if} \quad m>2\,, \\ C^2((0,\infty)) & \text{if} \quad 1<m\leq 2\,, \end{cases} \tag{1.4}$$

and

$$\varphi(v_t)\in C^1((0,\infty))\,. \tag{1.5}$$

In general, it is not easy to determine the type of solutions to (P). However, if the singularity of $k(t)$ is sufficiently "strong" at $t=0$ (resp. $t=\infty$), then any solution must be of type O (resp. type $\tilde{O}$). The following result is a generalization of Atkinson [1] and Ni [13], who treated the case $m=2$.

THEOREM 2. *Suppose that* $(f.0)$, $(f.1)$ *and* $(k.0)$ *hold.*

(i) *If* $\int_0^1 k(\tau)|f(c\tau)|d\tau=\infty$ *for all* $c\neq 0$, *then any solution v to* (P) *is of type O.*

(ii) *If* $\int_1^\infty \varphi^{-1}(\int_s^\infty k(\tau)d\tau)ds=\infty$, *then any solution* $\tilde{v}$ *to* (P) *is of type* $\tilde{O}$.

In view of (i) (resp. (ii)) of Theorem 2, any solution to (P) is type O (resp. type $\tilde{O}$), if $(k.2)$ (resp. $(k.1)$) does not hold in the case where $f(v)\sim|v|^{q-1}v$ at $v=0$ for some q.

As an immediate consequence of Theorem 2, we obtain the following result.

COROLLARY 1. *Suppose that* $f(v)=|v|^{q-1}v$ *with* $q>m-1$ *and that* $k(t)$ *satisfies* $(k.0)$. *Let v be a solution to* (P) *with* $(a,b)\neq(0,0)$.

(i) *If* $k(t)\sim t^\sigma$ *at* $t=0$ *with some* $\sigma\leq-(q+1)$, *then v is of* type O.

(ii) *If* $k(t)\sim t^\rho$ *at* $r=\infty$ *with some* $\rho\geq -m$, *then v is of* type $\tilde{O}$.

By virtue of the above corollary, the type of solutions is determined only by the asymptotic behavior of $k(t)$ if $\sigma\leq-(q+1)$ and $\rho\geq -m$. The numbers $-(q+1)$ and $-m$ are optimal in the sense that if $\sigma>-(q+1)$ and $\rho<-m$, the types of solutions delicately depend on the property of $k(t)$ and initial values (see Corollary 3 and [4, 6, 7, 16, 17, 18]).

On the contrary to Theorem 2, if the singularity of $k(t)$ is sufficiently "weak" at $t=0$ (resp. $t=\infty$), then any solution must be of type S or of type R (resp. of type $\tilde{S}$ or of it type $\tilde{R}$). Let us define

$$\mu(q):=\frac{(m-1)q+(2m-1)}{m}\,. \tag{1.6}$$

We note that $m<\mu(q)<q+1$ if $q>m-1$ with $m>1$. The following theorem is a generalization of Kiguradze [8].

THEOREM 3. *Suppose that* $(f.0)$, $(f.1)$, $(f.2)$ *and* $(k.0)$ *hold. Let* v *be a solution to* (P) *with* $(a, b) \neq (0, 0)$.

(i) *If* $\sup_{v\neq 0}\{|f(v)|/|v|^{q_1}\}<\infty$ *and* $\liminf_{t\to 0}\{tk_t(t)/k(t)\}>-\mu(q_1)$, *then* v *is of* type S *or* type R.

(ii) *If* $\sup_{v\neq 0}\{|f(v)|/|v|^{q_2}\}<\infty$ *and* $\limsup_{t\to\infty}\{tk_t(t)/k(t)\}<-\mu(q_2)$, *then* v *is of* type $\tilde{S}$ *or* type $\tilde{R}$.

As a consequence of Theorem 3, we obtain the following result.

COROLLARY 2. *Suppose that* $f(v)=|v|^{q-1}v$ *with* $q>m-1$ *and that* $k(t)$ *satisfies* $(k.0)$. *Let* v *be a solution to* (P) *with* $(a, b)\neq(0, 0)$.

(i) *If* $(t^{-\sigma}k(t))_t\geq 0$ *near* $t=0$ *with some* $\sigma>-\mu(q)$, *then* v *is of* type S *or* type R.

(ii) *If* $(t^{-\rho}k(t))_t\leq 0$ *near* $t=\infty$ *with some* $\rho<-\mu(q)$, *then* v *is of* type $\tilde{S}$ *or* type $\tilde{R}$.

The number $-\mu(q)$ is "critical" in the sense that the structure of solutions changes sensitively at $k(t)=ct^{-\mu(q)}$ with a constant $c>0$ when $f(v)=|v|^{q-1}v$ (see, Corollaries 3 and 4).

Now we focus on the initial value problems (P_α) and $(\tilde{\mathrm{P}}_\beta)$. In view of a remark after Theorem 2, the assumptions $(k.1)$ and $(k.2)$ are necessary for the solvability of the initial value problems (P_α) and $(\tilde{\mathrm{P}}_\beta)$, respectively, when $f(v)\sim|v|^{q-1}v$ at $v=0$ for some q. We will show that $(k.1)$ and $(k.2)$ are sufficient for the existence of solutions to the problems (P_α) and $(\tilde{\mathrm{P}}_\beta)$, respectively. The next result is a generalization of [5], [7] and [14].

THEOREM 4. *Suppose that* $(f.0)$ *and* $(k.0)$ *hold.*

(i) *If* $(k.1)$ *holds, then for any* $\alpha>0$, *there exists a unique solution* $v(t;\alpha)$ *to* (P_α) *satisfying* (1.4) *and* (1.5).

(ii) *If* $(k.2)$ *holds, then for any* $\beta>0$, *there exists a unique solution* $\tilde{v}(t;\beta)$ *to* $(\tilde{\mathrm{P}}_\beta)$ *satisfying* (1.4) *and* (1.5).

Let us consider the behavior of solutions to (P_α) and $(\tilde{\mathrm{P}}_\beta)$. A solution to (P_α) is of type O if the assumption of (i) of Theorem 2 holds, and a solution to $(\tilde{\mathrm{P}}_\beta)$ is of type $\tilde{O}$ if the assumption of (ii) of Theorem 2 holds. However, if the assumptions of Theorem 2 do not hold, the structure of solutions becomes more complicated.

In fact, when $f(v)=|v|^{q-1}v$ with $q>m-1$, the structure of solutions crucially depends on m, q and $k(t)$. We can give sharp sufficient conditions so that any solution is of type O, type S or type R. To state the sufficient conditions, we introduce some auxiliary functions

$$G(t):=\frac{1}{q+1}\,tk(t)-\frac{m-1}{m}\int_t^\infty k(s)ds \tag{1.7}$$

and

$$H(t):=\frac{1}{q+1}\,t^{q+2}k(t)-\frac{1}{m}\int_0^t s^{q+1}k(s)ds\,. \tag{1.8}$$

We note that $G(t)$ (resp. $H(t)$) is well-defined under the conditions (k.0) and (k.1) (resp. (k.2)). We also note that

$$G_t(t)=\frac{1}{q+1}\left\{\frac{tk_t(t)}{k(t)}+\mu\right\}k(t) \tag{1.9}$$

and

$$H_t(t)=\frac{1}{q+1}\left\{\frac{tk_t(t)}{k(t)}+\mu\right\}t^{q+1}k(t)=t^{q+1}G_t(t)\,, \tag{1.10}$$

where $\mu=\mu(q)$ is defined by (1.6).

If $G(t)$ (resp. $H(t)$) is identically equal to zero in $(0, \infty)$, then

$$k(t)=ct^{-\mu(q)}$$

for some positive constant c and $v(t;\alpha)$ (resp. $\tilde{v}(t;\beta)$) is of type R (resp. $\tilde{R}$) for any α (resp. β). In fact, all solutions to (P_α) and (P_β) are explicitly obtained as

$$v(t;\alpha)=\alpha t\{t^\gamma+l(\alpha)\}^{-1/\gamma}\,,\quad \tilde{v}(t;\beta)=\beta t\{t^\gamma+l(\beta)\}^{-1/\gamma}\,, \tag{1.11}$$

where $\gamma=(q-m+1)/m$ and

$$l(\alpha)=\alpha^{(q-m+1)/(m-1)}\left\{\frac{mc}{(m-1)(q+1)}\right\}^{1/(m-1)}.$$

The following result is a generalization of [3], [5], [7] and [10].

THEOREM 5. *Suppose that* $f(v)=|v|^{q-1}v$ *with* $q>m-1$, (k.0), (k.1) (*resp.* (k.2)), *and that* $G(t)$ (*resp.* $H(t)$) *is not identically equal to zero in* $(0, \infty)$. *Let* $v(t;\alpha)$ (*resp.* $\tilde{v}(t;\beta)$) *be the unique solution to* (P_α).

(i) *If* $G(t)\ge 0$ *on* $(0, \infty)$, *then* $v(t;\alpha)$ *has at least one zero in* $(0, \infty)$. *Moreover, if* $G_t(t)\le 0$ *on* $(0, \infty)$, *then* $v(t;\alpha)$ *is of* type O.

(ii) *If* $G(t)\le 0$ *on* $(0, \infty)$, *then* $v(t;\alpha)$ *is positive and of* type S.

(iii) *If* $H(t)\ge 0$ *on* $(0, \infty)$, *then* $\tilde{v}(t;\beta)$ *has at least one zero in* $(0, \infty)$. *Moreover, if* $H_t(t)\ge 0$ *on* $(0, \infty)$ *and* $H_t(t)$ *is not identically equal to zero, then* $\tilde{v}(t;\beta)$ *is of* type $\tilde{O}$.

(iv) *If* $H(t)\le 0$ *on* $(0, \infty)$, *then* $\tilde{v}(t;\beta)$ *is positive and of* type $\tilde{S}$.

The above theorems will play important roles in generalizing the results in [17] and [18] for the existence and the structure of radial solutions to the m-Laplace equations. We discuss these problems in [4].

As an easy application of Theorem 5, we consider the case where $f(v)=|v|^{q-1}v$ and $k(t)=t^\sigma$. In this case, $G(t)$ and $H(t)$ are explicitly expressed as

$$G(t)=\frac{1}{(q+1)(\sigma+1)}\{\sigma+\mu(q)\}t^{\sigma+1}\qquad\text{for}\quad \sigma<-m\,,$$

$$H(t)=\frac{1}{(q+1)(\sigma+q+2)}\{\sigma+\mu(q)\}t^{\sigma+q+2}\qquad\text{for}\quad \sigma>-(q+1)\,.$$

Hence, by applying Theorems 1, 2, 3, 4 and 5, we can completely classify the behavior of solutions.

COROLLARY 3. *Suppose that* $f(v)=|v|^{q-1}v$ *with* $q>m-1$. *Let* $k(t)=t^{\sigma}$.

(I) *The structure of solution to* (P_α) *is as follows.*

(i) *If* $\sigma\geq -m$, *then* (P_α) *has no solutions for any* $\alpha>0$.

(ii) *If* $-\mu(q)<\sigma<-m$, *then* $v(t;\alpha)$ *is positive and of* type S *for any* $\alpha>0$.

(iii) *If* $\sigma=-\mu(q)$, *then* $v(t;\alpha)$ *is positive and of* type R *for any* $\alpha>0$.

(iv) *If* $\sigma<-\mu(q)$, *then* $v(t;\alpha)$ *is of* type O *for any* $\alpha>0$.

(II) *The structure of solutions to* $(\tilde{\mathrm{P}}_\beta)$ *is as follows.*

(i) *If* $\sigma\leq -(q+1)$, *then* $(\tilde{\mathrm{P}}_\beta)$ *has no solutions for any* $\beta>0$.

(ii) *If* $-(q+1)<\sigma<-\mu(q)$, *then* $\tilde{v}(t;\beta)$ *is positive and of* type $\tilde{S}$ *for any* $\beta>0$.

(iii) *If* $\sigma=-\mu(q)$, *then* $\tilde{v}(t;\beta)$ *is positive and of* type $\tilde{R}$ *for any* $\beta>0$.

(iv) *If* $\sigma>-\mu(q)$, *then* $\tilde{v}(t;\beta)$ *is of* type $\tilde{O}$ *for any* $\beta>0$.

We see from this corollary that the case $k(t):=ct^{-\mu(q)}$ with a constant $c>0$ is critical. Let us consider the perturbation around $k(t):=ct^{-\mu(q)}$. Let $\eta(t)$ be a smooth bounded positive function, and $k(t):=\eta(t)t^{-\mu(q)}$. In this situation, $G(t)$ and $H(t)$ are well-defined by virtue of $1<m<\mu(q)<q+1$, and expressed as

$$G(t)=\frac{1}{q+1}\,t^{1-\mu(q)}\eta(t)-\frac{m-1}{m}\int_t^\infty s^{-\mu(q)}\eta(s)ds$$

$$H(t)=\frac{1}{q+1}\,t^{q+2-\mu(q)}\eta(t)-\frac{1}{m}\int_0^t s^{q+1-\mu(q)}\eta(s)ds\,.$$

We see that $G(t)\to 0$ as $t\to\infty$, $H(t)\to 0$ as $t\to 0$, and that

$$G_t(t)=\frac{1}{q+1}\,t^{-1-\mu(q)}\eta_t(t)=t^{1-q}H_t(t)\,.$$

By Theorem 5, we obtain the following result, which implies that the type of solutions drastically changes according to the sign of η_t (see also [3] and [14]).

COROLLARY 4. *Suppose that* $f(v)=|v|^{q-1}v$ *with* $q>m-1$. *Let* $k(t)=\eta(t)t^{-\mu(q)}$, *where* $\eta(t)$ *is a positive non-constant function such that* $\eta(t)\in C^1(0,\infty)$ *and* $\eta(t)$ *converges to some positive number as* $t\to 0$ *and* $t\to\infty$.

(I) *The structure to solutions of* (P_α) *is as follows.*

(i) *If* $\eta_t(t)\leq 0$, *then* $v(t;\alpha)$ *is of* type O *for any* $\alpha>0$.

(ii) *If* $\eta_t(t)\geq 0$, *then* $v(t;\alpha)$ *is positive and of* type S *for any* $\alpha>0$.

(II) *The structure to solutions of* $(\tilde{\mathrm{P}}_\beta)$ *is as follows.*

(i) *If* $\eta_t(t)\geq 0$, *then* $\tilde{v}(t;\beta)$ *is of* type $\tilde{O}$ *for any* $\beta>0$.

(ii) *If* $\eta_t(t)\leq 0$, *then* $\tilde{v}(t;\beta)$ *is positive and of* type $\tilde{S}$ *for any* $\beta>0$.

This paper consists of six sections: Section 2 contains basic properties of solutions to the initial value problems (P_α) and ($\tilde{\mathrm{P}}_\beta$). Section 3 is devoted to the proof of Theorem 2. In Section 4 we prove Theorem 3. Proofs of (i) and (ii) of Theorem 5 are given in Section 5. A proof of (iii) and (iv) of Theorem 5 is given in Section 6. Characterizations of solutions in terms of the Pohozaev identity play an essential role in the proofs in Sections 5 and 6. The outline of proofs of the existence of solutions to the initial value problems (i.e., Theorems 1 and 4) is given in the Appendix.

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2. Preliminaries. In this section we show basic properties of (P), (P_α) and ($\tilde{\mathrm{P}}_\beta$). First, we collect basic properties of a solution to (1.1).

LEMMA 2.1. *Suppose that* (f.0), (k.0) *hold and let* v *be a solution to* (1.1) *satisfying* (1.4) *and* (1.5). *Then the following hold*:

(a) *If* $v(t)>0$ (*resp.* $v(t)<0$) *near* $t=0$, *then* $v_t(t)$ *is decreasing* (*resp. increasing*) *and* $v_t(t)\neq 0$ *near* $t=0$.

(b) *If* $v(t)>0$ (*resp.* $v(t)<0$) *near* $t=\infty$, *then* $v_t(t)$ *is positive and decreasing* (*resp. negative and increasing*) *near* $t=\infty$.

(c) *If* $v(t)\to\alpha$ *as* $t\to\infty$ *for some* α, *then* $v_t(t)\to 0$ *as* $t\to\infty$.

(d) $v(t)$ *satisfies the Pohozaev identity*

$$\frac{d}{dt}P(t;v)=\left\{\frac{tk_t(t)}{k(t)}+\left(1+\frac{m-1}{m}\frac{f(v)v}{F(v)}\right)\right\}k(t)F(v)\,, \tag{2.1}$$

where

$$P(t;v)=\frac{m-1}{m}\varphi(v_t)\{tv_t-v\}+tk(t)F(v)\,,\quad F(v)=\int_0^v f(\xi)d\xi\,. \tag{2.2}$$

REMARK 2.1. In particular, if $f(v)=|v|^{q-1}v$ with $q>m-1$, then

$$\frac{d}{dt}P(t;v)=G_t(t)|v|^{q+1}=t^{-(q+1)}H_t(t)|v|^{q+1}\,, \tag{2.3}$$

where $G_t(t)$, $H_t(t)$ are defined by (1.9), (1.10) and

$$P(t;v)=\frac{m-1}{m}|v_t|^{m-2}v_t\{tv_t-v\}+\frac{1}{q+1}tk(t)|v|^{q+1}\,. \tag{2.4}$$

PROOF OF LEMMA 2.1. First, we may assume that $v(t)>0$ near $t=0$. From (1.1), we have $(\varphi(v_t))_t=-k(t)f(v)<0$. Thus v_t is decreasing. Hence $v_t<0$ near $t=0$ or $v_t>0$ near $t=0$, which implies (a).

As for (b), similarly to the proof of (a), v_t is decreasing. Hence $v_t>0$ near $t=\infty$ in view of $v>0$ near $t=\infty$. Thus we get (b).

We obtain (c) by the positivity and the monotonicity of v_t.

Finally we show (d). We obtain (2.1) for t with $v_t(t)\neq 0$ by differentiating (2.2) and using (1.1). On the other hand, $P(t; v)$ is written as

$$P(t;v)=\frac{m-1}{m}\,t\,|\varphi(v_t)|^{m/(m-1)}-\frac{m-1}{m}\,\varphi(v_t)v+tk(t)F(v)\,.$$

We see that $P(t;v)\in C^1$ by virtue of $\varphi(v_t)\in C^1$. Thus (2.1) holds for all t. ∎

Now we show several properties of solutions to (P), (P_α) and ($\tilde{P}_\beta$).

Proposition 2.1. *Suppose that* (f.0) *and* (k.0) *hold. Let $v(t)$ be a solution to* (P) *satisfying* (1.4) *and* (1.5). *Then $v(t)$ is classified into one of* type R, type S *and* type O at $t=0$, *and $v(t)$ is classified into one of* type $\tilde{R}$, type $\tilde{S}$ *and* type $\tilde{O}$ *at $t=\infty$. Moreover, the following hold.*

(a) *If $v(t)>0$ (resp. $v(t)<0$) near $t=0$, then $(v(t)t^{-1})_t<0$ (resp. $(v(t)t^{-1})_t>0$) near $t=0$. In particular, if $v(t)>0$ on $(0,\infty)$, then $(v(t)t^{-1})_t<0$ on $(0,\infty)$.*

(b) *If $\lim_{t\downarrow 0}v(t)=0$ and $v(t)\neq 0$ near $t=0$, then $tv_t(t)\to 0$ as $t\downarrow 0$, and*

$$\lim_{t\downarrow 0}\frac{v(t)}{t}=\lim_{t\downarrow 0}v_t(t)=\varphi^{-1}\left(\varphi(v_t(1))+\int_0^1 k(s)f(v(s))ds\right).$$

(c) *Suppose that $\int_0^1\varphi^{-1}(\int_s^1 k(\tau)d\tau)ds=\infty$. If $v\neq 0$ near $t=0$, then $vv_t>0$ near $t=0$.*

(d) *Suppose that $\int_1^\infty k(\tau)|f(c\tau)|d\tau=\infty$ for any $c\neq 0$. If $v(t)\neq 0$ near $t=\infty$, then $v(t)(v(t)t^{-1})_t<0$ near $t=\infty$.*

Remark 2.2. If $v_t(t)\neq 0$, then $v_{tt}(t)$ exists and

$$v_{tt}(t)=-\frac{k(t)f(v)}{\varphi'(v_t)}=-\frac{k(t)f(v)}{(m-1)|v_t|^{m-2}}\,. \tag{2.5}$$

Remark 2.3. If $v(t)=0$, then $v_t(t)\neq 0$. This comes from Lemma A.2 in the Appendix.

Proof of Proposition 2.1. To prove (a), (b) and (c), we may assume that $v>0$ near $t=0$ since $vf(v)>0$ for $v\neq 0$. Then

$$\varphi'(v_t)\frac{d}{dt}\left\{t^2\frac{d}{dt}\left(\frac{v}{t}\right)\right\}=t\varphi'(v_t)v_{tt}=-tk(t)f(v)<0 \tag{2.6}$$

by (1.1), $t^2(v/t)_t$ is monotone decreasing and $t^2(v/t)_t>0$ or $t^2(v/t)_t<0$ near $t=0$.

Now we prove (a). Suppose that $v(t)>0$ and $t^2(v/t)_t>0$ near $t=0$. Then there exist $c>0$ and $t^*>0$ such that $t^2(v/t)_t>c$ for $t\in(0,t^*]$. Hence we have

$$\frac{v(t^*)}{t^*}>\frac{v(t^*)}{t^*}-\frac{v(t)}{t}\geq c\int_t^{t^*}\frac{ds}{s^2}=c\left(\frac{1}{t}-\frac{1}{t^*}\right)\to\infty$$

as $t\to 0$. This is a contradiction. Thus the first part of (a) is proved. The second part is proved in a similar way.

As for (b), it follows from the assumption $\lim_{t\downarrow 0} v(t)=0$ that $v_t(t)>0$ near $t=0$. Combining this with (a), there exists $t_*>0$ such that $0<tv_t(t)<v(t)$ on $(0, t_*]$. Since $v\to 0$ as $t\downarrow 0$, $tv_t\to 0$ as $t\downarrow 0$. The last part of (b) comes from L'Hospital's rule and (1.1).

Next we prove (c). It follows from (2.6) that v_t never vanishes near $t=0$. If $v_t<0$ in an interval $(0, t_1]$ with some $t_1>0$, then there exist constants $c>0$ and $t_*>0$ such that $v(t)\geq c$ on $(0, t_*]$. Hence we see from $(f.1)$ and the assumption on $k(t)$ that

$$0<v(t)=v(t_*)-\int_t^{t_*}\varphi^{-1}\left(v_t(t_*)+\int_s^{t_*}k(\tau)f(v)d\tau\right)ds$$

$$\leq v(t_*)-\int_t^{t_*}\varphi^{-1}\left(v_t(t_*)+f(c)\int_s^{t_*}k(\tau)d\tau\right)ds\to-\infty$$

as $t\downarrow 0$. This is a contradiction. Thus $v_t>0$ near $t=0$.

As for (d), we may assume that $v>0$ near $t=\infty$. It follows from (2.6) that $(v/t)_t>0$ or $(v/t)_t<0$ near $t=\infty$. If the former holds, then there exist $\delta>0$ and $T_1>0$ such that $v/t\geq\delta>0$ on $[T_1, \infty)$. Integrating (1.1) over $[T_1, t]$, we have

$$\varphi(v_t(T_1))=\varphi(v_t(t))+\int_{T_1}^t k(s)f(v)ds\geq\int_{T_1}^t k(s)f(\delta s)ds\to\infty$$

as $t\to\infty$ by assumption. This is a contradiction. Thus the latter holds.

Finally we prove the classification of solutions. Let v be not of type O. We may assume that $v(t)>0$ near $t=0$. Then we have $(v/t)_t<0$ near $t=0$ by (a). Hence $\lim_{t\downarrow 0} v/t$ exists or $\lim_{t\downarrow 0} v/t=\infty$. Thus $v(t)$ is of type R or type S. We can also obtain the classification of solutions near $t=\infty$ in view of (b) of Lemma 2.1. ∎

As for (P_α) and $(\tilde{\mathrm{P}}_\beta)$, we have the following propositions in view of Lemma 2.1.

PROPOSITION 2.2. *Suppose that* $(f.0)$, $(k.0)$ *and* $(k.1)$ *hold. Let* $v(t)$ *be a solution to* (P_α) *satisfying* (1.4) *and* (1.5). *Then the following properties hold*:

(a) $\lim_{t\to\infty} v_t(t)=0$.

(b) $v_t(t)=\varphi^{-1}(\int_t^\infty k(s)f(v(s))ds)$.

(c) $v(t)$ *is increasing near* $t=\infty$ *and* $\int_1^\infty |v_t(t)|dt<\infty$.

(d) $v(t)$ *is a solution to* (P_α) *satisfying* (1.4) *and* (1.5) *if and only if* $v(t)\in C((0, \infty))\cap L^\infty(1, \infty)$ *satisfies*

$$v(t)=\alpha-\int_t^\infty\varphi^{-1}\left(\int_s^\infty k(\tau)f(v(\tau))d\tau\right)ds\,.$$

PROPOSITION 2.3. *Suppose that* $(f.0)$, $(k.0)$ *and* $(k.2)$ *hold. Let* $\tilde{v}(t)$ *be a solution to* $(\tilde{\mathrm{P}}_\beta)$ *satisfying* (1.4) *and* (1.5). *Then the following properties hold*:

(a) $\tilde{v}_t(0)=\beta$.

(b) *If* $\tilde{v}_t(t)\neq 0$ *on* $(0, \tau_1)$ *for some* $\tau_1>0$, *then for* $t\in(0, \tau_1)$

$$\left(\frac{\tilde{v}}{t}\right)_t=-\frac{1}{t^2}\int_0^t \frac{sk(s)f(\tilde{v})}{\varphi(\tilde{v}_t)}\,ds\,.$$

(c) *$\tilde{v}(t)$ is increasing near $t=0$ and $\int_0^1|\tilde{v}_t|dt<\infty$.*

(d) *$\tilde{v}(t)$ is a solution to $(\tilde{\mathrm{P}}_\beta)$ satisfying* (1.4) *and* (1.5), *if and only if $\tilde{v}\in C((0,\infty))$ satisfies*

$$\tilde{v}(t)=\int_0^t \varphi^{-1}\left(\varphi(\beta)-\int_0^s k(\tau)f(\tilde{v}(\tau))d\tau\right)ds\,.$$

3. Proof of Theorem 2. In this section we give a proof of Theorem 2.

(i) Employing the idea of the proof of Theorem 2.2 of [11], we will prove (i) by contradiction. We may suppose that $v>0$ near $t=0$. Since $(v/t)_t<0$ near $t=0$ by (a) of Proposition 2.1, we have $tv_t(t)<v(t)$ on $(0,t_0)$ and $v\geq ct$ on $(0,t_0)$ for some $t_0>0$ and $c>0$. We note that $f(v)$ and $f(v)/v^q$ are increasing in $v>0$ for some $q>m-1$ by virtue of the assumption $(f.1)$. Integrating (1.1) over $[t,t_0]$, we have

$$\varphi(v_t(t))=\varphi(v_t(t_0))+\int_t^{t_0}k(s)f(v)ds\geq\varphi(v_t(t_0))+\int_t^{t_0}k(s)f(cs)ds\,,$$

which implies that $\varphi(v_t(t))>0$ near $t=0$ by assumption. Hence we have

$$0<tv_t<v\quad\text{in}\quad(0,t_1) \tag{3.1}$$

for some $t_1>0$. Thus we get

$$\begin{aligned}\frac{m-1}{q-m+1}((\varphi(v_t))^{-q/(m-1)+1})_t&=\left(\frac{v}{v_t}\right)^q k(t)\frac{f(v)}{v^q}\\&\geq t^q k(t)\frac{f(ct)}{(ct)^q}=\frac{1}{c^q}k(t)f(ct)\end{aligned} \tag{3.2}$$

by (1.1), (3.1) and the monotonicity of $f(v)/v^q$. Integrating (3.2) over $[\varepsilon,t]$ and letting $\varepsilon\to 0$, we obtain

$$\frac{(m-1)c^q}{q-m+1}(\varphi(v_t(t)))^{-q/(m-1)+1}\geq\int_0^t k(s)f(cs)ds=\infty$$

by assumption. This is a contradiction. ∎

(ii) We prove (ii) by contradiction. Suppose that $\tilde{v}>0$ on $[T_0,\infty)$. We note that

$$\tilde{v}_t(t)>0\quad\text{on}\quad[T_0,\infty)$$

by (b) of Lemma 2.1. Integrating (1.1) over $[t,T]$ $(\subset[T_0,\infty))$, we have

$$\int_t^T k(\tau)d\tau=-\int_t^T\frac{(\varphi(\tilde{v}_t))_t}{f(\tilde{v})}\,d\tau\,. \tag{3.3}$$

The right-hand side of (3.3) yields

$$-\int_t^T \frac{(\varphi(\tilde{v}_t))_t}{f(\tilde{v})}\,d\tau=-\left[\frac{\varphi(\tilde{v}_t)}{f(\tilde{v})}\right]_t^T-\int_t^T \frac{\varphi(\tilde{v}_t)\tilde{v}_t f'(\tilde{v})}{(f(\tilde{v}))^2}\,d\tau$$

$$<-\frac{\varphi(\tilde{v}_t(T))}{f(\tilde{v}(T))}+\frac{\varphi(\tilde{v}_t(t))}{f(\tilde{v}(t))}<\frac{\varphi(\tilde{v}_t(t))}{f(\tilde{v}(t))}\leq\frac{\tilde{v}(T_0)^q}{f(\tilde{v}(T_0))}\,\frac{\varphi(\tilde{v}_t)}{\tilde{v}(t)^q}$$

for some $q>m-1$ in view of the assumption $(f.1)$. Using the above inequality in (3.3) and letting $T\to\infty$, we have

$$0<\varphi^{-1}\left(\int_t^\infty k(\tau)d\tau\right)<c\,\frac{\tilde{v}_t}{\tilde{v}^{q/(m-1)}}\,, \tag{3.4}$$

where $c=\varphi^{-1}(\tilde{v}(T_0)^q/f(\tilde{v}(T_0)))$. Integrating (3.4) over $[T_0,\infty)$, we have

$$\infty=\int_{T_0}^\infty \varphi^{-1}\left(\int_s^\infty k(\tau)d\tau\right)ds<\frac{(m-1)c}{q-m+1}\,(\tilde{v}(T_0))^{-(q-m+1)/(m-1)}<\infty\,.$$

This is a contradiction. ∎

4. Proof of Theorem 3. (i) For simplicity, we put $\mu=\mu(q_1)$. We note that $\mu>1$ since $q_1>1$ from $(f.1)$.

By assumption, there exist sufficiently small numbers $\varepsilon\in(0,\mu-1)$ and $t^*>0$ such that

$$\frac{tk_t}{k}+\mu\geq\varepsilon \quad \text{on} \quad (0,t^*)\,. \tag{4.1}$$

By (d) of Lemma 2.1 and the definition of q_1 we have

$$\begin{aligned}\frac{d}{dt}P(t;v)&=\left\{\frac{tk_t(t)}{k(t)}+\left(1+\frac{m-1}{m}\,\frac{f(v)v}{F(v)}\right)\right\}k(t)F(v)\\&\geq\left\{\frac{tk_t(t)}{k(t)}+\mu\right\}k(t)F(v)\geq\varepsilon k(t)F(v)\,.\end{aligned} \tag{4.2}$$

Now we suppose that v has infinitely many zeros on $(0,t^*)$ and let $\{t_j\}$ be a sequence of zeros of v with $0<\cdots<t_j<\cdots<t_2<t_1<t^*$. Integrating (4.2) over $[t_j,t_0]$, we have

$$\varepsilon\int_{t_j}^{t_0}k(\tau)F(v)d\tau\leq P(t_0;v(t_0))-P(t_j;v(t_j))\leq P(t_0;v(t_0))\,,$$

since $P(t_j;v(t_j))=((m-1)/m)t_j|v_t(t_j)|^m>0$. Letting $t_j\downarrow 0$, we have

$$\int_0^{t_0}k(\tau)F(v)d\tau<\frac{1}{\varepsilon}P(t_0,v(t_0))\,,$$

which implies that

$$\int_0^{t_0} k(\tau)f(v)v\,d\tau<\infty \tag{4.3}$$

in view of $(f.2)$. Now let $\xi\le t_0$ be a point such that $v(\xi)=\gamma>0$ and $v_t(\xi)=0$, and let $z<\xi$ be a zero of v such that $v>0$ on $(z,\xi]$ and $v(z)=0$. Integrating (1.1), we have

$$v(t)=\gamma-\int_t^{\xi}\varphi^{-1}\left(\int_s^{\xi}k(\tau)f(v(\tau))d\tau\right)ds\,. \tag{4.4.}$$

We see from Hölder's inequality, the boundedness of $f(v)/v^q$ and (4.3) that

$$\begin{aligned}\int_s^{\xi}k(\tau)f(v)d\tau&=\int_s^{\xi}\left(\frac{vf(v)}{v^{q+1}}\right)^{m/(q+1)}v^{m-1}k(\tau)^{m/(q+1)}\{k(\tau)vf(v)\}^{1-m/(q+1)}d\tau\\&\le C_1\gamma^{m-1}\left(\int_s^{\xi}k(\tau)d\tau\right)^{m/(q+1)},\end{aligned} \tag{4.5}$$

where

$$C_1=\left\{\sup_{0\le v\le\gamma}\frac{vf(v)}{v^{q+1}}\right\}^{m/(q+1)}\left\{\int_0^{t_0}k(\tau)vf(v)d\tau\right\}^{(q-m+1)/(q+1)}.$$

Moreover, we have

$$\{t^{\mu-\varepsilon}k(t)\}_t\ge 0\quad\text{on}\quad(0,t_0)$$

by (4.1). Thus we get

$$\int_s^{t_0}k(\tau)d\tau\le\frac{C_2}{\mu-\varepsilon-1}s^{-(\mu-\varepsilon-1)}$$

by noting $\varepsilon-\mu<-1$, where $C_2=(t_0)^{\mu-\varepsilon}k(t_0)$. Hence we have

$$\int_t^{t_0}\varphi^{-1}\left(\left(\int_s^{t_0}k(\tau)d\tau\right)^{m/(q+1)}\right)ds<C_3\varphi^{-1}(t^{\varepsilon m/(q+1)})$$

for some positive constant C_3 independent of t. Consequently, we obtain

$$0=v(z)=\gamma-\int_z^{\xi}\varphi^{-1}\left(\int_s^{\xi}k(\tau)f(v(\tau))d\tau\right)ds\ge\gamma-\gamma C_4z^{\varepsilon m/\{(m-1)(q+1)\}}$$

from (4.4) and (4.5), where C_4 is a positive constant independent of γ and z. If $z>0$ is small enough, we get a contradiction. ■

(ii) We can show (ii) by an argument similar to that of (i). ■

5. Proof of Theorem 5 (i), (ii). The purpose of this section is to prove (i) and (ii) of Theorem 5, in which the special case $f(u)=|u|^{q-1}u$ is treated.

Before giving the proofs, we state characterization of solutions in terms of the Pohozaev identity.

LEMMA 5.1. *Suppose that* $(f.0)$, $(k.0)$, *and* $(k.1)$ *are satisfied. Then for any solution* $v(t;\alpha)$ *to* (P_α), *there exists a sequence* $\{T_j\}$ *such that* $T_j\to\infty$, $T_jk(T_j)\to 0$ *and* $P(T_j;v)\to 0$ *as* $j\to\infty$, *where* $P(t;v)$ *is defined by* (2.4).

PROOF. By using $(k.0)$, $(k.1)$ and (c) of Proposition 2.2, we get $k(t)+|v_t|\in L^1[1,\infty)$. Thus we can choose a sequence $\{T_j\}$ such that $T_j\to\infty$, $T_jk(T_j)\to 0$ and $T_j|v_t(T_j)|\to 0$ as $j\to\infty$. Hence we have $P(T_j;v)\to 0$ as $j\to\infty$. ■

The following characterizations of solutions to (P_α) in terms of $P(t;v)$ are useful.

PROPOSITION 5.1. *Suppose that* $(f.0)$, $(f.1)$, $(k.0)$ *and* $(k.1)$ *are satisfied. Then the following hold*:

(a) *If* $v=v(t;\alpha)$ *is of* type R, *then there exists a positive sequence* $\{\bar t_j\}$ *such that* $\bar t_j\to 0$ *and* $P(\bar t_j;v)\to 0$ as $j\to\infty$.

(b) *If* $v=v(t;\alpha)$ *is of* type S *and* $vv_t>0$ *in a neighborhood of* $t=0$, *then there exists a positive sequence* $\{\hat t_j\}$ *such that* $\hat t_j\to 0$ *as* $j\to\infty$ *and* $P(\hat t_j;v)<0$ *for every* j.

(c) *If* $v=v(t;\alpha)$ *is of* type O, *then there exists a positive sequence* $\{\check t_j\}$ *such that* $\check t_j\to 0$ *as* $j\to\infty$ *and* $P(\check t_j;v)>0$ *for every* j.

PROOF. If $v=v(t;\alpha)$ is of type R, then it follows from (b) of Proposition 2.1 that

$$\int_0^1 k(s)f(v)ds<\infty\ .$$

Moreover, by $(f.1)$, we have $F(v)\le vf(v)$ for $v\ge 0$. Because v is of type R, we have $v(0)=0$ and there exists $\varepsilon_0>0$ such that $v(t)\le 1$ on $[0,\varepsilon_0]$. Hence we have

$$\int_0^{\varepsilon_0} k(s)F(v)ds\le\int_0^{\varepsilon_0} k(s)f(v)ds<\infty\ .$$

Thus there exists a positive sequence $\{\bar t_j\}$ such that $\bar t_j\to 0$ and $\bar t_jk(\bar t_j)F(v(\bar t_j;\alpha))\to 0$ as $j\to\infty$. On the other hand, using (b) of Proposition 2.1 we have

$$\lim_{t\downarrow 0}|v_t|^{m-2}v_t\{-tv_t+v\}=0\ .$$

This implies that $P(\bar t_j;v)\to 0$ as $j\to\infty$, and (a) is proved.

As for (b), we may assume that $v>0$ and $v_t>0$ near $t=0$. The other case is proved in the same way. If $t>0$ is sufficiently small, we have

$$\begin{aligned}
P(t; v) &= \frac{m-1}{m} t^2 \varphi(v_t)\left(\frac{v}{t}\right)_t + t(k(t)F(v)) \\
&= \frac{m-1}{m} t^2 \varphi(v_t)\left(\frac{v}{t}\right)_t + \frac{tF(v)}{f(v)}(k(t)f(v)) \\
&= \frac{m-1}{m} t^2 \varphi(v_t)\left(\frac{v}{t}\right)_t - \frac{tF(v)}{f(v)}(\varphi(v_t))_t \\
&= tv\varphi(v_t)\left\{\frac{m-1}{m}\frac{(v/t)_t}{(v/t)} - \frac{F(v)}{f(v)v}\frac{(\varphi(v_t))_t}{\varphi(v_t)}\right\} \\
&= (m-1)tv\varphi(v_t)\left\{\frac{1}{m}\frac{(v/t)_t}{(v/t)} - \frac{F(v)}{f(v)v}\frac{v_{tt}}{v_t}\right\}
\end{aligned}$$

by (1.1) and the definition of φ. From $(f.1)$, there exists $q > m-1$ such that

$$\frac{F(v)}{f(v)v} \le \frac{1}{q+1}.$$

Noting that $v_{tt} < 0$ near $t = 0$ by (2.5), we have

$$\begin{aligned}
P(t; v) &\le (m-1)tv\varphi(v_t)\frac{d}{dt}\left\{\frac{1}{m}\log\left(\frac{v}{t}\right) - \frac{1}{q+1}\log(v_t)\right\} \\
&= (m-1)tv\varphi(v_t)\frac{d}{dt}\left\{\left(\frac{1}{m} - \frac{1}{q+1}\right)\log\left(\frac{v}{t}\right) + \frac{1}{q+1}\log\left(\frac{v}{tv_t}\right)\right\}.
\end{aligned}$$

By (a) of Proposition 2.1, we get

$$\frac{v}{tv_t} > 1$$

for small $t > 0$. Since v is of type S and $q+1 > m$, we obtain

$$\left(\frac{1}{m} - \frac{1}{q+1}\right)\log\left(\frac{v}{t}\right) + \frac{1}{q+1}\log\left(\frac{v}{tv_t}\right) \to \infty$$

as $t \downarrow 0$. Thus we can choose a sequence $\{\hat{t}_j\}$ such that $\hat{t}_j \to 0$ as $j \to \infty$ and $P(\hat{t}_j, v) < 0$ for all j. The assertion (b) is proved.

Finally, let $v = v(t; \alpha)$ be of type O and let $z_j(\alpha)$ be the jth zero of $v(t; \alpha)$. If $v(z_j(\alpha); \alpha) = 0$, then $v_t(z_j(\alpha); \alpha) \neq 0$ by virtue of Remark 2.3. Then

$$P(z_j(\alpha); v(z_j(\alpha); \alpha)) = \frac{m-1}{m} z_j(\alpha)|v_t|^m > 0.$$

Thus (c) holds by taking $\check{t}_j = z_j(\alpha)$. ∎

PROOF OF THEOREM 5 (i). Consider the case $G(t)\geq 0$. If $v>0$ on $(0, \infty)$, then we have $v_t>0$ on $(0, \infty)$ by (b) of Proposition 2.2. Thus there exist $c>0$ and $\tau>0$ such that

$$P(t; v)=G(t)|v|^{q+1}+(q+1)\int_t^\infty G(s)|v|^{q-1}vv_t ds\geq c$$

for $t\in(0, \tau]$ in view of the assumption on G. Hence v can be neither of type R nor of type S. Thus v must have a zero on $(0, \infty)$.

Now we consider the case $G_t(t)\leq 0$ and G_t is not identically equal to zero. It follows from (2.3) and Lemma 5.1 that $P(t; v)$ is non-increasing and not identically equal to zero. Thus there exist δ and $\tau>0$ such that

$$P(t; v)>\delta \quad \text{on} \quad (0, \tau)\,. \tag{5.1}$$

Hence v is not of type R by Proposition 5.1. On the other hand, $G_t\leq 0$ implies that

$$\frac{tk_t}{k}\leq -\mu \quad \text{on} \quad (0, \infty)\,.$$

Thus we get

$$k(t)\geq ct^{-\mu} \quad \text{on} \quad (0, 1]$$

with some constant $c>0$. Hence we have

$$\varphi^{-1}\left(\int_s^\infty k(\tau)d\tau\right)\geq\varphi^{-1}(c)\varphi^{-1}\left(\int_s^1 \tau^{-\mu}d\tau\right)\geq\varphi^{-1}(c)\frac{1}{\varphi^{-1}((1-\mu))}\varphi^{-1}(s^{1-\mu}-1)\,,$$

which implies that

$$\int_0^1 \varphi^{-1}\left(\int_s^1 k(\tau)d\tau\right)ds=\infty$$

in view of $(\mu-1)/(m-1)=(q+1)/m>1$. By (c) of Proposition 2.1, if $v\neq 0$ near $t=0$, then $vv_t>0$ near $t=0$. Thus v is not of type S by virtue of (5.1) and (b) of Proposition 5.1. Consequently, v must be of type O. ■

PROOF OF THEOREM 5 (ii). Suppose that v has a zero. Let z be the largest zero of v. It holds that $v_t>0$ on $[z, \infty)$ by (b) of Proposition 2.2. Hence we see from the assumption $G\leq 0$ that

$$P(z; v)=(q+1)\int_z^\infty G(s)|v|^{q-1}vv_t ds\leq 0\,,$$

which contradicts $P(z; v)=(m-1)z^n|v_t|^m/m>0$. Therefore we have $v>0$ and $v_t>0$ on $(0, \infty)$. Thus there exist $\delta>0$ and $\tau>0$ such that, for any $t\in(0, \tau)$,

$$P(t; v)=G(t)|v|^{q+1}+(q+1)\int_t^\infty G(s)|v|^{q-1}vv_t ds$$

$$\leq (q+1)\int_t^\infty G(s)|v|^{q-1}vv_t ds < -\delta < 0$$

in view of the assumption that $G \leq 0$ and G is not identically equal to zero. By Proposition 5.1, v is neither of type R nor of type O. Hence v must be of type S. ■

6. Proof of Theorem 5 (iii), (iv). In the case $m=2$, we can obtain (iii) and (iv) of Theorem 5 by virtue of (i) and (ii) of Theorem 5 and the Kelvin transformation $W(s)=v(t)/t$ and $s=1/t$. However, in the case $m \neq 2$, the transformation does not work well. Thus we need the following proposition similar to Proposition 5.1.

LEMMA 6.1. *Suppose that* (f.0), (f.2), (k.0), *and* (k.2) *hold. Then for any solution* $\tilde{v}(t; \beta)$ *to* $(\tilde{\mathrm{P}}_\beta)$, *there exists a positive sequence* $\{\varepsilon_j\}$ *such that* $\varepsilon_j \to 0$, $\varepsilon_j k(\varepsilon_j)F(\tilde{v}(\varepsilon_j)) \to 0$, *and* $P(\varepsilon_j; \tilde{v}) \to 0$ *as* $j \to \infty$.

PROOF. Let $w(t)=\tilde{v}(t)/t$. From (f.2), we have

$$k(t)F(\tilde{v}) \leq k(t)t|wf(tw)| \leq Mtk(t)|f(Mt)|$$

near $t=0$, where $M=\sup_{t\in[0,1]}|w(t)|$. Using (k.2), we have $k(t)|f(Mt)| \in L^1(0, 1)$, which implies $k(t)F(\tilde{v}) \in L^1(0, 1)$. Thus, we get $k(t)F(\tilde{v})+|\tilde{v}_t| \in L^1(0, 1)$ by virtue of (k.0), (k.2) and (c) of Proposition 2.3. Thus there exists a positive sequence $\{\varepsilon_j\}$ such that $\varepsilon_j \to 0$, $\varepsilon_j k(\varepsilon_j)F(\tilde{v}(\varepsilon_j)) \to 0$ and $\varepsilon_j|\tilde{v}_t(\varepsilon_j)| \to 0$ as $j \to \infty$. Hence we have $P(\varepsilon_j; \tilde{v}) \to 0$ as $j \to \infty$. ■

Similarly to Proposition 5.1, there are characterizations of solutions at $t=0$.

PROPOSITION 6.1. *Suppose that* (f.0), (f.1), (k.0), *and* (k.2) *hold. Then the following hold*:

(a) *If* $\tilde{v}=\tilde{v}(t; \beta)$ *is of* type $\tilde{R}$, *then there exists a sequence* $\{\bar{T}_j\}$ *such that* $\bar{T}_j \to \infty$ *and* $P(\bar{T}_j; \tilde{v}) \to 0$ *as* $j \to \infty$.

(b) *If* $\tilde{v}=\tilde{v}(t; \beta)$ *is of* type $\tilde{S}$ *and* $\tilde{v}\{\tilde{v}/t\}_t < 0$ *near* $t=\infty$, *then there exists a sequence* $\{\hat{T}_j\}$ *such that* $\hat{T}_j \to \infty$ *as* $j \to \infty$ *and* $P(\hat{T}_j; \tilde{v}) < 0$ *for every* j.

(c) *If* $\tilde{v}=\tilde{v}(t; \beta)$ *is of* type $\tilde{O}$, *then there exists a sequence* $\{\check{T}_j\}$ *such that* $\check{T}_j \to \infty$ *as* $j \to \infty$ *and* $P(\check{T}_j; \tilde{v}) > 0$ *for every* j.

PROOF. By (2.10), we have

$$\varphi(\tilde{v}_t(t))=\varphi(\beta)-\int_0^t k(s)f(\tilde{v})ds,$$

which implies that

$$\int_0^\infty k(s)f(\tilde{v})ds=\varphi(\beta)$$

by (c) of Lemma 2.1. On the other hand, from (b) of Lemma 2.1 and the fact that $\tilde{v}$ is of type $\tilde{R}$, we have

$$\int_T^\infty |\tilde{v}_t|ds=\int_T^\infty \tilde{v}_t ds=\lim_{t\to\infty}\tilde{v}(t)-\tilde{v}(T)<\infty$$

for some $T>0$. Hence we can choose $\{\bar{T}_j\}$ with $\bar{T}_j\to\infty$ as $j\to\infty$ so that

$$\bar{T}_j k(\bar{T}_j)\to 0 \quad \text{and} \quad \bar{T}_j\tilde{v}_t(\bar{T}_j;\beta)\to 0$$

as $j\to\infty$. Consequently (a) is proved.

As for (b), suppose that $\tilde{v}(t)>0$ near $t=\infty$. We see from (b) of Lemma 2.1 that $\tilde{v}_t(t)>0$ near $t=\infty$. By assumption we have

$$\left(\frac{\tilde{v}}{t}\right)_t<0$$

near $t=\infty$, which implies

$$0<\frac{t\tilde{v}_t(t)}{\tilde{v}(t)}<1 \tag{6.1}$$

near $t=\infty$. Thus, we have

$$\begin{aligned}
P(t;\tilde{v})&=-\frac{m-1}{m}\varphi(\tilde{v}_t)(-t\tilde{v}_t+\tilde{v})+tk(t)F(\tilde{v})\\
&=-\frac{m-1}{m}\varphi(\tilde{v}_t)(-t\tilde{v}_t+\tilde{v})+t\frac{F(\tilde{v})}{f(\tilde{v})}k(t)f(\tilde{v})\\
&=-\frac{m-1}{m}|\tilde{v}_t|^{m-2}\tilde{v}_t(-t\tilde{v}_t+\tilde{v})-(m-1)t\frac{F(\tilde{v})}{f(\tilde{v})}|\tilde{v}_t|^{m-2}\tilde{v}_{tt}\\
&=-(m-1)|\tilde{v}_t|^{m-2}\tilde{v}(-t\tilde{v}_t+\tilde{v})\left\{\frac{\tilde{v}_t}{m\tilde{v}}-\frac{F(\tilde{v})}{f(\tilde{v})\tilde{v}}\frac{\{-t\tilde{v}_t+\tilde{v}\}_t}{-t\tilde{v}_t+\tilde{v}}\right\}
\end{aligned}$$

by (2.5). From $(f.1)$, there exists $\bar{q}>m-1$ such that

$$\frac{F(\tilde{v})}{f(\tilde{v})\tilde{v}}\leq\frac{1}{\bar{q}+1}.$$

Noting that $-t\tilde{v}_t+\tilde{v}>0$, we have

$$\begin{aligned}
P(t;\tilde{v})&\leq-(m-1)|\tilde{v}_t|^{m-2}\tilde{v}^2\left(-\frac{t\tilde{v}_t}{\tilde{v}}+1\right)\frac{d}{dt}\left\{\frac{1}{m}\log\tilde{v}-\frac{1}{\bar{q}+1}\log(-t\tilde{v}_t+\tilde{v})\right\}\\
&=-(m-1)|\tilde{v}_t|^{m-2}\tilde{v}^2\left(-\frac{t\tilde{v}_t}{\tilde{v}}+1\right)\frac{d}{dt}\left\{\left(\frac{1}{m}-\frac{1}{\bar{q}+1}\right)\log\tilde{v}-\frac{1}{\bar{q}+1}\log\left(1-\frac{t\tilde{v}_t}{\tilde{v}}\right)\right\}.
\end{aligned}$$

From (6.1), $1/m-1/(\bar{q}+1)>0$ and $\lim_{t\to\infty}\tilde{v}(t)=\infty$, we can take a sequence $\{\hat{T}_j\}$ with $\hat{T}_j\to\infty$ as $j\to\infty$ such that $P(\hat{T}_j;\tilde{v})<0$ for all j.

The proof of (c) is similar to that of Proposition 5.1 (c). ■

PROOF OF THEOREM 5 (iii). Consider the case $H(t)\geq 0$ on $(0,\infty)$. Suppose that $\tilde{v}(t;\beta)>0$ on $(0,\infty)$. Then we have $(\varphi(\tilde{v}_t))_t<0$ on $(0,\infty)$ by (1.1). Thus $\varphi(\tilde{v}_t)$ is strictly decreasing and $\varphi(\tilde{v}_t)>0$ on $(0,\infty)$, which implies $\tilde{v}_t>0$ on $(0,\infty)$. Then by (b) of Proposition 2.3, $(\tilde{v}/t)_t<0$ on $(0,\infty)$. Hence by the assumption $H\geq 0$, there exists a constant $c>0$ such that

$$P(t;\tilde{v})=t^{-(q+1)}H(t)|\tilde{v}|^{q+1}-(q+1)\int_0^t H(s)\left|\frac{\tilde{v}}{s}\right|^q\left(\frac{\tilde{v}}{s}\right)_t ds\geq c>0$$

near $t=\infty$. Consequently, $\tilde{v}(t;\beta)$ is neither of type $\tilde{R}$ nor $\tilde{S}$ by Proposition 6.1. Thus $\tilde{v}$ has a zero on $(0,\infty)$.

Consider the case $H_t(t)\geq 0$ and H_t is not identically equal to zero. It follows from (2.3) and Lemma 6.1 that $P(t,\tilde{v})$ is non-decreasing and $P(t;\tilde{v})\geq 0$ and not identically equal to zero. Thus there exist $\delta>0$ and $T>0$ such that

$$P(t;\tilde{v})\geq\delta>0 \tag{6.2}$$

on (T,∞). Hence $\tilde{v}$ is not of type $\tilde{R}$ by (a) of Proposition 6.1. On the other hand, $H_t(t)\geq 0$ implies that $k(t)\geq ct^{-\mu}$ on $[1,\infty)$. Hence, we have

$$\int_0^\infty s^q k(s)ds\geq c\int_0^\infty s^{-2+(q+1)/m}ds=\infty$$

in view of $-2+(q+1)/m>-1$. Therefore we get $\tilde{v}\{\tilde{v}/t\}_t<0$ by (d) of Proposition 2.1. Hence $\tilde{v}$ is not of type $\tilde{S}$ by (6.2) and (b) of Proposition 6.1. Consequently $\tilde{v}$ is of type $\tilde{O}$. ■

PROOF OF THEOREM 5 (iv). Suppose that $\tilde{v}(t;\beta)$ has a zero. Then there must exist a critical point. Let $\tilde{z}_0$ be the smallest critical point of $\tilde{v}$. Then we have

$$P(\tilde{z}_0;\tilde{v})=\frac{1}{q+1}\tilde{z}_0k(\tilde{z}_0)|\tilde{v}(\tilde{z}_0;\beta)|^{q+1}>0\,.$$

On the other hand, we get

$$P(\tilde{z}_0;\tilde{v})=\tilde{z}_0^{-(q+1)}H(\tilde{z}_0)|\tilde{v}|^{q+1}-(q+1)\int_0^{\tilde{z}_0} H(t)\left|\frac{\tilde{v}}{t}\right|^q\left(\frac{\tilde{v}}{t}\right)_t dt\leq 0$$

by $H(t)\leq 0$ and (b) of Proposition 2.3. This is a contradiction. Hence $\tilde{v}$ has no critical point on $(0,\infty)$ and thus $\tilde{v}_t>0$ on $(0,\infty)$ by $\tilde{v}_t(0)=\beta>0$. Consequently, $\tilde{v}>0$ on $(0,\infty)$. Thus there exist $\delta>0$ and $T>0$ such that, for any $t\in[T,\infty)$,

$$P(t;\tilde{v})=t^{-(q+1)}H(t)|\tilde{v}|^{q+1}-(q+1)\int_0^t H(s)\left|\frac{\tilde{v}}{s}\right|^q\left(\frac{\tilde{v}}{s}\right)_t ds$$
$$\leq -(q+1)\int_0^t H(s)\left|\frac{\tilde{v}}{s}\right|^q\left(\frac{\tilde{v}}{s}\right)_t ds \leq -\delta<0$$

in view of the assumption that $H\leq 0$ and H is not identically equal to zero. By Proposition 6.1, $\tilde{v}$ can be neither of type $\tilde{R}$ nor type $\tilde{O}$. Consequently, $\tilde{v}$ is of type $\tilde{S}$. ■

Appendix. Here we give the outline of the proofs of Theorems 1 and 4 since they are based on a standard argument. See Coffman-Ullrich [2] and Kitano-Kusano [9].

LEMMA A.1. *Suppose that* (f.0) *and* (k.0) *hold. Then for any* $t_1>0$ *there exists* $\varepsilon>0$ *such that* (P) *has a unique solution satisfying* (1.4) *and* (1.5) *on* $[t_1-\varepsilon, t_1+\varepsilon]$.

To show the global existence of solutions, we need a priori estimate. We put

$$E(t):=\frac{m-1}{m}|v_t(t)|^m+k(t)F(v(t)).$$

LEMMA A.2. *Let* $v(t)$ *be a solution to* (P). *Then it holds that*

$$E(t)\leq E(t_0)\exp\left(-\int_t^{t_0}\min\left\{0, \frac{k_t(s)}{k(s)}\right\}ds\right) \qquad t\in(0, t_0],$$

and

$$E(t)\leq E(t_1)\exp\left(\int_{t_0}^t\max\left\{0, \frac{k_t(s)}{k(s)}\right\}ds\right) \qquad t\in[t_0, \infty)$$

for any $t_0>0$.

Now we consider (P_α). We see the following fact by (c) of Lemma 2.1.

LEMMA A.3. *Suppose that* (f.0), (k.0) *and* (k.1) *hold. Then the following conditions are equivalent*:

(i) v *is a solution of* (P_α) *satisfying* (1.4), (1.5).

(ii) $v\in C((0, \infty))\cap L^\infty(1, \infty)$ *satisfies*

$$v(t;\alpha)=\alpha-\int_t^\infty \varphi^{-1}\left(\int_s^\infty k(\tau)f(v(\tau;\alpha))d\tau\right)ds.$$

We obtain the following lemma similar to Lemma A.1.

LEMMA A.4. *Suppose that* (f.0), (k.0) *and* (k.1) *hold. Then there exists* $T_0>0$ *such that* (P_α) *has a unique solutions on* $[T_0, \infty)$.

On the other hand, we consider ($\tilde{P}_\beta$).

LEMMA A.5. *Suppose that* $(f.0)$, $(k.0)$ *and* $(k.2)$ *hold. Then the following conditions are equivalent*:

(i) $\tilde{v}$ *is a unique solution of* $(\tilde{\mathrm{P}}_\beta)$ *satisfying* (1.4), (1.5) *with* $v=\tilde{v}$.

(ii) $\tilde{v}\in C((0,\infty))$ *satisfies*

$$\tilde{v}(t;\beta)=\int_0^t \varphi^{-1}\left(\varphi(\beta)-\int_0^s k(\tau)f(\tilde{v}(\tau;\beta))d\tau\right)ds\,.$$

LEMMA A.6. *Suppose that* $(f.0)$, $(k.0)$ *and* $(k.2)$ *hold. Then there exists* $t_0>0$ *such that* $(\tilde{\mathrm{P}}_\beta)$ *has a unique solution on* $[0, t_0]$.

Now we are in a position to prove Theorems 1 and 4.

PROOF OF THEOREM 1. By Lemma A.1, (P) has a local unique solution satisfying (1.4) and (1.5). The solution can be uniquely prolonged to $(0,\infty)$ by Lemma A.2. Thus the existence and the uniqueness of solutions are proved. ■

PROOF OF THEOREM 4. The local solvability and uniqueness of (P_α) (resp. $(\tilde{\mathrm{P}}_\beta)$) are ensured by Lemma A.1 and Lemma A.4 (resp. Lemma A.1 and Lemma A.6). The global solvability is obtained by Lemma A.2. ■

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