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ON A GENERALIZED BESSEL FUNCTION OF TWO VARIABLES II. CASE OF COALESCING SADDLE POINTS

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Abstract. A generalized Bessel function of two variables satisfies a system of partial differential equations. Two of the singular loci of the system are of irregular type. Near one of them we study the asymptotic behavior of suitably chosen linearly independent solutions. In our calculation coalescing saddle points are treated.

Introduction. Let z_-^* be a function of $(x, y)\in \boldsymbol{C}^2$ defined by

$$z_-^* = I_\alpha(x, y) = \int_{C_-^*} \exp\left(-\frac{t^2}{2} - xt - \frac{y}{t}\right) t^{-\alpha-1} dt\,, \tag{0.1}$$

where α ($\notin \boldsymbol{Z}$) is a complex constant and

$$C_-^* = (t:\ \infty \xrightarrow{(0)} \infty;\ \arg t:\ -\pi \longrightarrow \pi)$$

is a loop starting from $t=\infty$, encircling $t=0$ in the positive sense and returning to $t=\infty$ along which $\arg t$ varies from $-\pi$ to π. It is known that (0.1) is a solution of a system of partial differential equations

$$\begin{gathered} \partial_x^2 u = x\partial_x u - y\partial_y u - \alpha u\,, \\ \partial_x\partial_y u = u\,, \\ y\partial_y^2 u = -\partial_x u - (\alpha+1)\partial_y u + xu \end{gathered} \tag{0.2}$$

(see [6]), which is equivalent to a completely integrable Pfaffian system of the form

$$dV = (P(x, y)dx + Q(x, y)dy/y)V \tag{0.3}$$

with

$$P(x, y) = \begin{pmatrix} 0 & 1 & 0 \\ -\alpha & x & -y \\ 1 & 0 & 0 \end{pmatrix}, \quad Q(x, y) = \begin{pmatrix} 0 & 0 & y \\ y & 0 & 0 \\ x & -1 & -(\alpha+1) \end{pmatrix}, \quad V = \begin{pmatrix} u \\ \partial_x u \\ \partial_y u \end{pmatrix}.$$

System (0.2) or (0.3) possesses the singular loci $x=\infty$, $y=\infty$ of irregular type, and $y=0$ of regular type. For a fixed point $(x_0, y_0)\in \boldsymbol{C}\times\boldsymbol{C}^\times$ ($\boldsymbol{C}^\times = \boldsymbol{C}-\{0\}$), the

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solutions analytic in a neighborhood of (x_0, y_0) are continued analytically to the region $C\times\mathscr{R}(C^\times)$ and constitute a three-dimensional vector space, where $\mathscr{R}(C^\times)$ is the universal covering of $C^\times$. In fact every solution $u(x, y)$ of (0.2) is written in the form $u(x, y)=u(x_0, y_0)v_0(x, y)+\partial_x u(x_0, y_0)v_1(x, y)+\partial_y u(x_0, y_0)v_2(x, y)$, where $v_j(x, y)$ $(j=0, 1, 2)$ are the first entries of the solution vectors $V_j(x, y)$ of (0.3) satisfying $V_j(x_0, y_0)={}^t(\delta_{0j}, \delta_{1j}, \delta_{2j})$. Consider the power series expansion $u(x, y)=\sum_{m,n\geq 0} c_{m,n}\xi^m\eta^n/(m!n!)$ with $(\xi, \eta)=(x-x_0, y-y_0)$, $c_{0,0}=u(x_0, y_0)$, $c_{1,0}=\partial_x u(x_0, y_0)$, $c_{0,1}=\partial_y u(x_0, y_0)$, which converges for $|\xi|<\infty$, $|\eta|<|y_0|$. From the second equation of (0.2), we have $c_{m+1,n+1}=c_{m,n}$ for every pair (m, n) of nonnegative integers. This implies that $u(x, y)$, and hence (0.1), is written in the form

$$c_{0,0}\Lambda_0(\xi\eta)+\sum_{p=1}^{\infty}(c_{p,0}\xi^p+c_{0,p}\eta^p)\Lambda_p(\xi\eta)\,. \tag{0.4}$$

Here $\Lambda_\mu(\tau)$ $(\mu\in C)$ is a power series expressible in terms of the modified Bessel function (see [5, §7.2]):

$$\begin{aligned}\Lambda_\mu(\tau)&=\sum_{k=0}^{\infty}\frac{\tau^k}{k!\Gamma(k+\mu+1)}\\&=\tau^{-\mu/2}I_\mu(2\tau^{1/2})=\tau^{-\mu/2}e^{-\mu\pi i/2}J_\mu(2e^{\pi i/2}\tau^{1/2})\,.\end{aligned} \tag{0.5}$$

Furthermore it is easy to see that

$$\begin{aligned}\lim_{\substack{\lambda\to\infty\\ \lambda>0}}\lambda^{-\alpha}I_\alpha(-\lambda, s^2/(4\lambda))&=\int_{C_-^*}\exp\left(t-\frac{s^2}{4t}\right)t^{-\alpha-1}dt\\&=2\pi i(s/2)^{-\alpha}J_\alpha(s)\,.\end{aligned} \tag{0.6}$$

Considering the facts above and comparing the integrands of (0.1) and (0.6), we can regard the function $z_-^*=I_\alpha(x, y)$ as a generalization of the Bessel function $(x/2)^{-\alpha}J_\alpha(x)$. (Other generalizations of the Bessel function $J_\alpha(x)$ are found in [1], [2], [3], [4].) In [7] we studied the behavior of linearly independent solutions of (0.2) near $y=0$ and $y=\infty$. In addition to z_-^*, recall solutions of (0.2) expressed as

$$z_0=\int_{C_0}f dt\,,\qquad z_+=\int_{C_+}f dt\,,\qquad z_-=\int_{C_-}f dt\,,$$

$$z_0^*=\int_{C_0^*}f dt\,,\qquad z'_-=\int_{C'_-}f dt\,,$$

in which the integrand and the paths of integration are given by

(0.7)
$$f=f(x,y,t)=\exp\left(-\frac{t^2}{2}-xt-\frac{y}{t}\right)t^{-\alpha-1},$$

$$C_0=(t\colon 0\longrightarrow\infty;\ \arg t\colon \arg y\longrightarrow 0),$$
$$C_+=(t\colon 0\longrightarrow\infty;\ \arg t\colon \arg y\longrightarrow \pi),$$
$$C_-=(t\colon 0\longrightarrow\infty;\ \arg t\colon \arg y\longrightarrow -\pi),$$
$$C_0^*=(t\colon 0\xrightarrow{(0)}0;\ \arg t\colon \arg y\longrightarrow \arg y+2\pi),$$
$$C'_-=(t\colon \infty\longrightarrow\infty;\ \arg t\colon -\pi\longrightarrow 0)$$

(cf. [7, §1]). Here, for example, C_+ denotes a path starting from $t=0$ and tending to $t=\infty$ along which $\arg t$ varies from $\arg y$ to π, and C_0^* a loop starting from $t=0$, encircling $t=0$ in the positive sense and returning to $t=0$ along which $\arg t$ varies from $\arg y$ to $\arg y+2\pi$. There exist relations of the form

(0.8)
$$z_0^*=z_+-e^{-2\pi i\alpha}z_-,\qquad z_-^*=z_+-z_-,\qquad z'_-=z_0-z_-.$$

The triples of linearly independent solutions (z_0^*, z_+^*, z_-^*) and (z_0, z_+, z_-) (with $z_+^*=(1-e^{2\pi i\alpha})z_0+e^{2\pi i\alpha}z_+-z_-$) are expressed respectively by convergent power series in the domain $\boldsymbol{C}\times\mathscr{R}(\boldsymbol{C}^\times)$ and by asymptotic expansions near the singular locus $y=\infty$ (see [7]). By (0.8) we know the asymptotic behavior of the generalized Bessel function z_-^* near $y=\infty$.

The purpose of this paper is to study the asymptotic behavior of linearly independent solutions of (0.2) in the domain $0<|y/x|<R$ around another singular locus $x=\infty$ of irregular type, where R denotes an arbitrary fixed positive constant. We write (0.7) in the form

(0.9)
$$f(x,y,t)=t^{-\alpha-1}\exp g(t)=\exp h(t),$$
$$g(t)=-\frac{t^2}{2}-xt-\frac{y}{t},\qquad h(t)=g(t)-(\alpha+1)\log t,$$

where $\operatorname{Im}\log t=\arg t$. In the calculation of asymptotic expansions, we treat the saddle points of $g(t)$ or of $h(t)$, namely the zeroes of $g'(t)$ or of $h'(t)$, which are approximately equal to $-x$, $-(y/x)^{1/2}$, $(y/x)^{1/2}$. Since two points $-(y/x)^{1/2}$, $(y/x)^{1/2}$ coalesce as $x\to\infty$, we consider the two cases where they are close to each other and where they are separated. To do so we define two domains

$$D_-=\{(x,y)\in\boldsymbol{C}\times\mathscr{R}(\boldsymbol{C}^\times)\,|\,|xy|<2R_0,\ |y/x|<R,\ |x|>R_\infty\},$$
$$D_+=\{(x,y)\in\boldsymbol{C}\times\mathscr{R}(\boldsymbol{C}^\times)\,|\,|xy|>R_0,\ |y/x|<R,\ |x|>R_\infty\},$$

the union of which covers the domain $0<|y/x|<R$, $|x|>R_\infty$ around $x=\infty$. Here R_0 and R_∞ are sufficiently large positive constants. In each domain we choose a suitable triple of linearly independent solutions and discuss the asymptotic behavior of it. We state the main results on asymptotic expansions in Section 1, and the ones on Stokes multipliers in D_- and D_+ in Section 2. By the use of preparatory lemmas in Section 3, they are proved in Sections 4 and 5. The constants R_0 and $R_\infty = \max\{R_\infty^{(1)}, R_\infty^{(2)}, R_\infty^{(3)}\}$, which depend on an arbitrary small positive constant δ in the main results, are chosen in the proofs of Lemmas 4.1, 4.3 and 4.5. Throughout this paper, we assume that $\alpha \in \boldsymbol{C} - \boldsymbol{Z}$ and use the following notation:

$$e^{(\gamma)} = \exp(2\pi i\gamma)\,, \qquad (\gamma)_k = \Gamma(\gamma+k)/\Gamma(\gamma)\,,$$

for $\gamma \in \boldsymbol{C}$, $k \in \boldsymbol{Z}$.

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1. Asymptotic expansions. By $H_n(\xi)$ and by $L_\nu^{(\alpha)}(\xi)$ we denote the Hermite polynomial

$$H_n(\xi) = (-1)^n e^{\xi^2}\left(\frac{d}{d\xi}\right)^n e^{-\xi^2} = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2\xi)^{n-2m}}{m!(n-2m)!}$$

and the Laguerre polynomial

$$L_\nu^{(\alpha)}(\xi) = \frac{1}{\nu!} e^{\xi}\xi^{-\alpha}\left(\frac{d}{d\xi}\right)^\nu (e^{-\xi}\xi^{\nu+\alpha}) = \sum_{j=0}^{\nu} \binom{\nu+\alpha}{\nu-j} \frac{(-\xi)^j}{j!}$$

(cf. [5, §§10.12, 10.13]). Let δ be an arbitrary small positive constant.

THEOREM 1.1. *The solution z_+ admits an asymptotic expression*

$$z_+ \simeq W_+(x, y)$$

with

$$W_+(x,y) = \sqrt{\pi}\, i e^{-\alpha\pi i}\left(\frac{y}{x}\right)^{-\alpha/2} (xy)^{-1/4} \exp\left(2(xy)^{1/2} - \frac{y}{2x}\right) \sum_{m=0}^{\infty} h_m(y/(2x))(xy)^{-m/2}$$

uniformly for $|y/x| < R$, as x and xy tend to ∞ through the sector $|\arg x - \pi| < 3\pi/4 - \delta$, $|\arg(xy) - 2\pi| < 3\pi - \delta$. Here the sum on the right-hand side is a formal power series in $(xy)^{-1/2}$, and $h_m(u)$ is a polynomial in u of degree $2m$ expressed as

$$h_m(u) = \sum_{k=0}^{2m} \frac{(-1)^{k+m}(1/2)_{k+m}}{k!} \sum_{n=0}^{2m-k} \frac{(k+\alpha+1)_{2m-k-n}}{(2m-k-n)!\,n!} u^{n/2} H_n(u^{1/2})\,.$$

THEOREM 1.2. *The solution z'_- admits an asymptotic expression*

$$z'_- \simeq W_1(x, y)$$

with

$$W_1(x, y) = -\sqrt{2\pi}e^{\alpha\pi i}x^{-\alpha-1}\exp\left(\frac{x^2}{2}+\frac{y}{x}\right)\sum_{m=0}^{\infty}\frac{(2m)!}{2^m m!}L_{2m}^{(\alpha)}(-y/x)x^{-2m}$$

uniformly for $|y/x| < R$, *as* x *tends to* ∞ *through the sector* $|\arg x - \pi/2| < 3\pi/4 - \delta$, *where the sum on the right-hand side is a formal power series in* x^{-2}.

THEOREM 1.3. *The solution* z_- *admits an asymptotic expression*

$$z_- \simeq W_-(x, y)$$

with

$$W_-(x, y) = e^{(\alpha)}W_+(x, e^{2\pi i}y)$$

uniformly for $|y/x| < R$, *as* x *and* xy *tend to* ∞ *through the sector* $|\arg x - \pi| < 3\pi/4 - \delta$, $|\arg(xy)| < 3\pi - \delta$.

THEOREM 1.4. *The solution* z_-^* *admits an asymptotic expression*

$$z_-^* \simeq W_*(x, y)$$

with

$$W_*(x, y) = 2\pi i e^{-\alpha\pi i}x^{\alpha}\sum_{m=0}^{\infty}\frac{\Lambda_{\alpha-2m}(xy)}{m!}(-2x^2)^{-m}$$

uniformly for $|xy| < 2R_0$, *as* x *tends to* ∞ *through the sector* $|\arg x - \pi| < 3\pi/4 - \delta$. *Here the sum on the right-hand side is a formal power series in* $(-2x^2)^{-1}$, *and* $\Lambda_\mu(\tau)$ *is the function defined by* (0.5).

2. Stokes multipliers.

2.1. Stokes multipliers in the domain D_+. By Theorems 1.1, 1.2 and 1.3, in the domain D_+,

$$z'_- \simeq W_1(x, y), \qquad z_+ \simeq W_+(x, y), \qquad z_- \simeq W_-(x, y)$$

uniformly for $|y/x| < R$, as x and xy tend to ∞ through the sector $|\arg x - 3\pi/4| < \pi/2 - \delta$, $|\arg(xy) - \pi| < 2\pi - \delta$. It is easy to see that z'_-, z_+, z_- are linearly independent solutions of (0.2). Let $S = S_+(\theta_1, \theta_2)$ denote a sector defined by

$$S_+(\theta_1, \theta_2) = \{(x, y) \in D_+ \mid |\arg x - \theta_1| < \pi/2 - \delta, |\arg(xy) - \theta_2| < 2\pi - \delta\}.$$

We call a matrix $T(S)$ ($\in GL(3, \mathbf{C})$) *a Stokes multiplier corresponding to the sector S with respect to* (z'_-, z_+, z_-), if linearly independent solutions $z_S^{(1)}, z_S^{(2)}, z_S^{(3)}$ such that

$$^t(z'_-, z_+, z_-) = T(S)\,^t(z_S^{(1)}, z_S^{(2)}, z_S^{(3)})$$

satisfy

$$z_S^{(1)} \simeq W_1(x, y), \qquad z_S^{(2)} \simeq W_+(x, y), \qquad z_S^{(3)} \simeq W_-(x, y)$$

uniformly for $|y/x|<R$, as x and xy tend to ∞ through the sector S. In this sector we have

$$^t(z'_-, z_+, z_-) \simeq T(S)^t(W_1(x, y), W_+(x, y), W_-(x, y)).$$

THEOREM 2.1. *We have Stokes multipliers corresponding to the sectors* $S_+((2j-1)\pi/4, \pi)$ $(j=1, 2, 3, 4)$ *with respect to* (z'_-, z_+, z_-) *written in the form*

$$T(S_+(\pi/4, \pi)) = \begin{pmatrix} 1 & 0 & 0 \\ -e^{(-\alpha)} & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad T(S_+(3\pi/4, \pi)) = I,$$

$$T(S_+(5\pi/4, \pi)) = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T(S_+(7\pi/4, \pi)) = \begin{pmatrix} e^{(\alpha)} & 1 & -1 \\ -1 & 1 & 0 \\ -e^{(\alpha)} & 0 & 1 \end{pmatrix}.$$

Moreover, $T(S_+((2j-1)\pi/4, -\pi)) = M_0^{-1} T(S_+((2j-1)\pi/4, \pi))\Omega$ $(j=1, 2, 3, 4)$, *where*

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{(-\alpha)} \\ 0 & -1 & 1+e^{(-\alpha)} \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{(-\alpha)} \\ 0 & -1 & 0 \end{pmatrix}.$$

2.2. Stokes multipliers in the domain D_-. In the domain D_-, we choose linearly independent solutions z'_-, z^*_-, z^*_0. It is known that z^*_0 is represented by the convergent power series

$$z^*_0 = 2\pi i e^{-\alpha\pi i} y^{-\alpha} \sum_{m=0}^{\infty} \frac{\Lambda_{2m-\alpha}(xy)}{m!} \left(-\frac{y^2}{2}\right)^m$$

in D_- (cf. (0.5) and [7, Theorem 2.1]). Hence it is sufficient to consider the solutions z'_-, z^*_-. By Theorems 1.2 and 1.4, in the domain D_-,

$$z'_- \simeq W_1(x, y), \qquad z^*_- \simeq W_*(x, y)$$

uniformly for $|xy|<2R_0$, $|y/x|<R$, as x tends to ∞ through the sector $|\arg x - 3\pi/4| < \pi/2 - \delta$. For a sector $S = S_-(\theta)$ expressed as

$$S_-(\theta) = \{x \in \boldsymbol{C} \mid |\arg x - \theta| < \pi/2 - \delta\}$$

we call a matrix $U(S)$ $(\in GL(2, \boldsymbol{C}))$ *a Stokes multiplier corresponding to the sector S with respect to* (z'_-, z^*_-), if linearly independent solutions $z_S^{(1)}$, $z_S^{(2)}$ such that

$$^t(z'_-, z^*_-) = U(S)^t(z_S^{(1)}, z_S^{(2)})$$

satisfy

$$z_S^{(1)} \simeq W_1(x, y), \qquad z_S^{(2)} \simeq W_*(x, y)$$

uniformly for $|xy|<2R_0$, $|y/x|<R$, as x tends to ∞ through the sector S. In this sector, we have

$$^t(z'_-, z^*_-) \simeq U(S)^t(W_1(x, y), W_*(x, y)) .$$

THEOREM 2.2. *We have Stokes multipliers corresponding to the sectors* $S_-((2j-1)\pi/4)$ $(j=1, 2, 3, 4)$ *with respect to* (z'_-, z^*_-) *written in the form*

$$U(S_-(\pi/4)) = \begin{pmatrix} 1 & 0 \\ 1-e^{(-\alpha)} & 1 \end{pmatrix}, \qquad U(S_-(3\pi/4)) = I ,$$

$$U(S_-(5\pi/4)) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad U(S_-(7\pi/4)) = \begin{pmatrix} e^{(\alpha)} & 1 \\ e^{(\alpha)}-1 & 1 \end{pmatrix}.$$

3. Preliminaries. Consider the functions

$$g_0(t) = -\frac{t^2}{2} - xt , \qquad g_1(t) = -xt - \frac{y}{t} . \tag{3.1}$$

Clearly the zero of $g_0'(t)$ is $-x$, and those of $g_1'(t)$ are

$$\mu_0 = -(y/x)^{1/2} , \qquad \mu_1 = (y/x)^{1/2} .$$

Simple computation leads us to the following lemma concerning the saddle points of $g(t)$ and $h(t)$.

LEMMA 3.1. *Under the condition* $|y/x| < R$, $|x| > R'$, *the zeroes of* $g'(t)$ *are given by*

$$\xi_0 = -x + O(x^{-1}) , \qquad \eta_j = \mu_j(1 + O(x^{-1})) = \mu_j + O(x^{-1})$$

$(j=0, 1)$, *and those of* $h'(t)$ *by*

$$\xi_0^* = -x + O(x^{-1}) , \qquad \eta_j^* = \mu_j(1 + O(|xy|^{-1/2})) = \mu_j + O(x^{-1})$$

$(j=0, 1)$, *where* R' *is a sufficiently large positive constant.*

In the series of lemmas stated below, r denotes an arbitrary constant satisfying $0 < r \le r_0$ (<1).

LEMMA 3.2. *Under the condition* $|y/x| < R$, $|x| > R'$, *we have the following*:

$$g_1(t) - 2(xy)^{1/2} = x^{3/2} y^{-1/2} (1 + O(r))(t - \mu_0)^2 \tag{3.2}$$

for $|t - \mu_0| < r|y/x|^{1/2}$;

$$h(t) - h(\xi_0^*) = -(1/2)(1 + O(x^{-2}))(t - \xi_0^*)^2 , \tag{3.3}$$

$$h'(t) = -(1 + O(x^{-2}))(t - \xi_0^*) \tag{3.4}$$

for $|t - \xi_0^*| < r|x|$; *and*

$$h(\xi_0^*) = x^2/2 - (\alpha + 1)\log x + O(1) . \tag{3.5}$$

LEMMA 3.3. *Under the condition* $|y/x| < R$, $|xy| > R_1$, $|x| > R'$, *we have the following*:

(3.6) $$h(t)-h(\eta_j^*)=x^{3/2}y^{-1/2}((-1)^j+O(|x|^{-1}+|xy|^{-1/2}+r))(t-\eta_j^*)^2\,,$$

(3.7) $$h'(t)=2x^{3/2}y^{-1/2}((-1)^j+O(|x|^{-1}+|xy|^{-1/2}+r))(t-\eta_j^*)$$

for $|t-\eta_j^*|<r|y/x|^{1/2}$; *and*

(3.8) $$h(\eta_j^*)-2(-1)^j(xy)^{1/2}=-(\alpha+1)\log\mu_j+O(1)=O(\log x)\,.$$

Here R_1 *is a sufficiently large positive constant.*

LEMMA 3.4. *Under the condition* $|y/x|<R$, $|x|>R'$, *we have the following*:

(3.9) $$g(t)-g(\xi_0)=-(1/2)(1+O(x^{-2}))(t-\xi_0)^2\,,$$

(3.10) $$g'(t)=-(1+O(x^{-2}))(t-\xi_0)$$

for $|t-\xi_0|<r|x|$;

(3.11) $$g(\xi_0)=x^2/2+O(1)\,;$$

(3.12) $$g(\eta_j)=2(-1)^j(xy)^{1/2}+O(1)\,;$$

and

(3.13) $$g(t)=-xt(1+O(x^{-1/2}))\,,\qquad g'(t)=-x(1+O(x^{-1/2}))$$

on the circle $|t|=|x|^{1/2}$.

4. Proofs of the theorems in Section 1. In the calculation of an asymptotic series of z_+ (or z'_-), we modify the path C_+ (or C'_-) in such a way that it passes through the point μ_0 (or $-x$). A major contribution comes from the integral along a part of C_+ (or C'_-) near μ_0 (or $-x$). To evaluate the integral along the remaining part of C_+, we have to use $h(t)$ because of the multiplier $(y/x)^{-\alpha/2}$ (cf. Theorem 1.1 and Lemma 4.2). On the other hand, in the corresponding evaluation concerning C'_-, we need $g(t)$, because (3.8) is not always valid without the condition $|xy|>R_1$. For the same reason we use $g(t)$ in the proof of Theorem 1.4 as well.

4.1. Modification of the path C_+. We need to modify the path C_+ in such a way that it has the following properties.

(a) C_+ consists of three curves Γ_-, Γ_0, Γ_+ such that

(a.1) Γ_0 is an arc passing through $t=\mu_0$ and lying inside the circle K_0 defined by $|t-\mu_0|=\varepsilon_{xy}|y/x|^{1/2}$, where $\varepsilon_{xy}=|xy|^{-1/6}$;

(a.2) Both ends a_+, a_- of Γ_0 are located on K_0;

(a.3) Γ_- (or Γ_+) is a curve starting from a_- (or a_+) and tending to ∞ (or 0).

(b) C_+ lies outside the circles $|t-\eta_1^*|=\varepsilon_{xy}|y/x|^{1/2}$, $|t-\xi_0^*|=\varepsilon_{xy}|x|$, and Γ_- and Γ_+ outside the circle $|t-\eta_0^*|=\varepsilon_{xy}|y/x|^{1/2}/4$.

(c) $\mathrm{Re}(g_1(t)-2(xy)^{1/2})\le 0$, $\mathrm{Im}(g_1(t)-2(xy)^{1/2})=0$ for $t\in\Gamma_0$.

(d) $(d/d\rho)\mathrm{Re}\,h(t)\le -c$ for $t\in\Gamma_-$ (or $t\in\Gamma_+$), in which c is a positive constant and $\rho=\rho(t)$ denotes the length of a part of $h(\Gamma_-)$ (or $h(\Gamma_+)$) from $h(a_-)$ (or $h(a_+)$) to $h(t)$.

LEMMA 4.1. *As long as* (x, y) $(\in \boldsymbol{C}\times\mathscr{R}(\boldsymbol{C}^{\times}))$ *satisfies* $|y/x|<R$, $|xy|>R_0$, $|x|>R_{\infty}^{(1)}$ *and*

$$|\arg x-\pi|<3\pi/4-\delta\,, \qquad |\arg(xy)-2\pi|<3\pi-\delta\,, \tag{4.1}$$

we can modify the path C_+ *continuously with respect to* (x, y) *preserving the properties above, where* $R_0=R_0(\delta)$ *and* $R_{\infty}^{(1)}=R_{\infty}^{(1)}(\delta)$ *are sufficiently large positive constants.*

PROOF. Consider (x, y) $(\in \boldsymbol{C}\times\mathscr{R}(\boldsymbol{C}^{\times}))$ satisfying $|y/x|<R$, $|xy|>R_0$, $|x|>R_{\infty}^{(1)}$, where the constants R_0 and $R_{\infty}^{(1)}$ are chosen in the following argument. We may assume that $R_0>R_1$, $R_{\infty}^{(1)}>R'$ and that Lemmas 3.1, 3.2 and 3.3 are applicable. We begin with the special case where $\arg x=\arg y=\pi$ and $\alpha\in\boldsymbol{R}-\boldsymbol{Z}$. By Lemma 3.1, if R_0 and $R_{\infty}^{(1)}$ are sufficiently large, the saddle points of $h(t)$ and $g_1(t)$ are so located that $\eta_0^*<0<\eta_1^*<\xi_0^*$ and $\mu_0<0<\mu_1$. Now we take the path $C_+=\Gamma_-\cup\Gamma_0\cup\Gamma_+$ to be the negative real axis with $\Gamma_-: t\le a_-$, $\Gamma_0: a_-\le t\le a_+$, $\Gamma_+: a_+\le t<0$, where $a_-=a_-^0=\mu_0-\varepsilon_{xy}|y/x|^{1/2}$, $a_+=a_+^0=\mu_0+\varepsilon_{xy}|y/x|^{1/2}$. Then the images $S_0=g_1(\Gamma_0)$, $T_-^0=h(\Gamma_-)$, $T_+^0=h(\Gamma_+)$ are included in the negative real axis in the τ-plane, and are written as S_0: $\min\{g_1(a_-^0), g_1(a_+^0)\}\le\tau\le 2(xy)^{1/2}$ (<0), $T_-^0: \tau\le h(a_-^0)$, $T_+^0: \tau\le h(a_+^0)$, respectively. It is easy to see that conditions (a), (b), (c) and (d) are satisfied.

In the case where the condition $\arg x=\arg y=\pi$ is not necessarily satisfied and $\alpha\in\boldsymbol{C}-\boldsymbol{Z}$, the path C_+ is constructed in the following way. Take the segment S defined by $-2|xy|^{1/6}\le\mathrm{Re}(\tau-2(xy)^{1/2})\le 0$, $\mathrm{Im}(\tau-2(xy)^{1/2})=0$ in the τ-plane. By (3.2) the inverse image $g_1^{-1}(S)$ passes through μ_0 and intersects the circle $|t-\mu_0|=\varepsilon_{xy}|y/x|^{1/2}$ at a_-, a_+, which, in case $\arg x=\arg y=\pi$, coincide with a_-^0, a_+^0, respectively. Thus we obtain an arc

$$\Gamma_0=\{t\in g_1^{-1}(S)\,|\,|t-\mu_0|\le\varepsilon_{xy}|y/x|^{1/2}\}$$

with the properties (a.1), (a.2) and (c). Then $G_0=g_1(a_\pm)-g_1(\mu_0)=g_1(a_\pm)-2(xy)^{1/2}=-|xy|^{1/6}(1+O(\varepsilon_{xy}))$. Note that $a_\pm-\eta_0^*=(a_\pm-\mu_0)(1+O(\varepsilon_{xy}^2))$. We obtain, from (3.2) and (3.6), that $h(a_\pm)-h(\eta_0^*)=G_0(h(a_\pm)-h(\eta_0^*))G_0^{-1}=-|xy|^{1/6}(1+O(\varepsilon_{xy}+|x|^{-1}+|xy|^{-1/2}))$, and, from (3.8), that $h(\eta_0^*)-h(\eta_1^*)=4(xy)^{1/2}(1+O(|xy|^{-1/2}))$. In view of (3.5) and these estimates, we can take $R_0=R_0(\delta)$ and $R_{\infty}^{(1)}=R_{\infty}^{(1)}(\delta)$ so large that, for $|y/x|<R$, $|xy|>R_0$, $|x|>R_{\infty}^{(1)}$,

$$|\arg(h(\eta_0^*)-h(\eta_1^*))-\arg(xy)^{1/2}|<\delta/4\,,$$

$$|\arg h(\xi_0^*)-\arg(x^2/2)|<\delta/2\,,$$

$$h(a_\pm)-h(\eta_0^*)=-|xy|^{1/6}(1+\theta(x, y))\,, \qquad |\theta(x, y)|<1/2\,. \tag{4.2}$$

Hence, as long as (4.1) is satisfied, we can draw curves T_- and T_+ in the τ-plane which are continuous modifications of T_-^0 and T_+^0, respectively, with properties below (see Figures 4.1 and 4.2):

(i) T_- (or T_+) is a curve starting from $h(a_-)$ (or $h(a_+)$) and tending to ∞, and

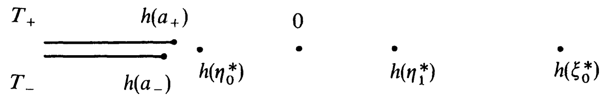

FIGURE 4.1. $\arg h(\xi_0^*)=2\pi$, $\arg h(\eta_0^*)=\pi$

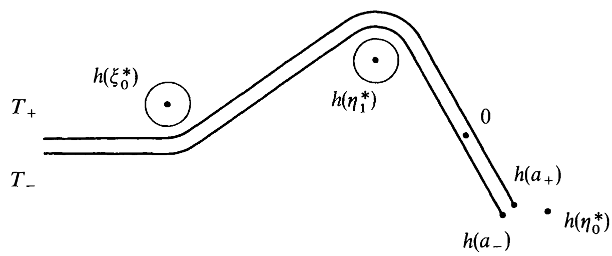

FIGURE 4.2. $5\pi/2<\arg h(\xi_0^*)<7\pi/2$, $-\pi/2<\arg h(\eta_0^*)<0$

lies outside the circles $|\tau-h(\xi_0^*)|=\varepsilon_{xy}^2|x|^2$, $|\tau-h(\eta_1^*)|=2|xy|^{1/6}$;

(ii) $(d/d\tilde{\rho})\operatorname{Re}\tau\leq -c$ for $\tau\in T_-$ (or $\tau\in T_+$), where $\tilde{\rho}=\tilde{\rho}(\tau)$ denotes the length of the part of T_- (or T_+) from $h(a_-)$ (or $h(a_+)$) to τ.

Note that T_- and T_+ lie outside the circle $|\tau-h(\eta_0^*)|=|xy|^{1/6}/2$. The function $\tau=h(t)$ is biholomorphic at each point $t=t_0$ ($\neq\xi_0^*$, η_j^*, 0, ∞), and is continuous in α. Now take the inverse images $\Gamma_-=h^{-1}(T_-)$ and $\Gamma_+=h^{-1}(T_+)$ tending to $t=\infty$ and $t=0$, respectively, and put $C_+=\Gamma_-\cup\Gamma_0\cup\Gamma_+$. Then this new path satisfies the desired conditions. □

LEMMA 4.2. *We have*

$$\int_{\Gamma_-\cup\Gamma_+}\exp h(t)dt=(x/y)^{\alpha/2}(xy)^{-1/3}\exp(2(xy)^{1/2})E(x,y)$$

with

$$E(x,y)=O(\exp(-|xy|^{1/6}/2)).$$

PROOF. Note that $1/h'(t)$ is analytic at $t\neq\xi_0^*$, η_j^*. From (b), (3.4), (3.7) combined with the maximal modulus principle, it follows that $|dt|=|1/h'(t)||dh/d\rho|d\rho=O(\varepsilon_{xy}^{-1}|x|^{-1})d\rho$ for $t\in\Gamma_-$. The property (d) yields $\operatorname{Re}(h(t)-h(a_-))\leq -c\rho$. Using (3.8), (4.2) and this inequality, we obtain

$$\begin{aligned}|\exp h(t)|&\leq e^{-c\rho}|\exp h(a_-)|=e^{-c\rho}|\exp(h(\eta_0^*)-|xy|^{1/6}(1+\theta(x,y)))|\\&\leq e^{-c\rho}|(x/y)^{(\alpha+1)/2}\exp(2(xy)^{1/2})|\exp(-|xy|^{1/6}/2),\end{aligned}$$

namely

$$[(x/y)^{\alpha/2}(xy)^{-1/3}\exp(2(xy)^{1/2})]^{-1}\exp h(t)=O(\varepsilon_{xy}|x|\exp(-|xy|^{1/6}/2)e^{-c\rho})$$

for $t \in \Gamma_-$. From this estimate and a similar one for $t \in \Gamma_+$, the lemma immediately follows. $\square$

4.2. Proof of Theorem 1.1. Under condition (4.1), take the path C_+ with the properties (a), ..., (d). To calculate the asymptotic expansion we divide the integral z_+ into two parts

$$I_1 = \int_{\Gamma_0} f\,dt\,, \qquad I_2 = \int_{\Gamma_- \cup \Gamma_+} f\,dt\,.$$

By Lemma 4.2, it is sufficient to show that I_1 admits the same asymptotic expansion as that of Theorem 1.1. Observing that $\arg t \to \pi$ as $t\ (\in C_+) \to \infty$, and considering the case where $\arg x = \arg y = \pi$, we have $\arg \mu_0 = \pi + (1/2)\arg(y/x)$. In I_1 we put $t = e^{\pi i}(y/x)^{1/2}(1+\sigma)$ ($|\sigma| \le \varepsilon_{xy}$), where $|\arg(1+\sigma)| < \pi/2$ for $t \in \Gamma_0$. Then $g_1(t) = 2(xy)^{1/2} + (xy)^{1/2}\sigma^2 - (xy)^{1/2}\sigma^3/(1+\sigma)$. By (c) the variable $\tau = (xy)^{1/2}\sigma^2$ moves along arcs which are tangent to the negative real axis at $\tau = 0$ and are contained in the left half plane $\operatorname{Re}\tau \le 0$. Further change of variables $s = e^{-\pi i/2}(xy)^{1/4}\sigma$ yields

$$I_1 = ie^{-\alpha\pi i}\left(\frac{y}{x}\right)^{-\alpha/2}(xy)^{-1/4}\exp\left(2(xy)^{1/2} - \frac{y}{2x}\right)J_1$$

with

$$J_1 = \int_{|s| \le |xy|^{1/12}} \exp(w(1-(1+v^{-1}s)^2))\exp\left(\frac{v^{-1}s^3}{1+v^{-1}s}\right)(1+v^{-1}s)^{-\alpha-1}e^{-s^2}ds\,, \tag{4.3}$$

where $w = y/(2x)$, $v = -i(xy)^{1/4}$. The path of integration is a curve passing through $s = 0$, on which $|\operatorname{Im} s|/|\operatorname{Re} s| = O(|xy|^{-1/6})$. The variable s moves along it in such a way that $\operatorname{Re} s$ monotonically increases. Putting $u = w^{1/2}$, $z = -w^{1/2}v^{-1}s$ in the generating function $\sum_{n=0}^{\infty} H_n(u)z^n/n! = \exp(2uz - z^2)$ (see [5, §10.13, (19)]), we have

$$\exp(w(1-(1+v^{-1}s)^2)) = \sum_{n=0}^{\infty} \varphi_n(w)(-1)^n v^{-n}s^n/n!\,, \qquad \varphi_n(w) = w^{n/2}H_n(w^{1/2})\,.$$

Hence, by the estimates $|w| < R/2$, $v^{-1}s^3 = O(1)$ and $v^{-1}s = O(|xy|^{-1/6})$, the integrand is written in the form

$$\begin{aligned}
&\left(\sum_{n=0}^{N} \frac{\varphi_n(w)(-1)^n v^{-n}s^n}{n!} + O((v^{-1}s)^{N+1})\right) \\
&\qquad \times \left(\sum_{k=0}^{N} \frac{v^{-k}s^{3k}}{k!}(1+v^{-1}s)^{-k-\alpha-1} + O((v^{-1}s^3)^{N+1})\right)e^{-s^2} \\
&= \sum_{n=0}^{N}\sum_{k=0}^{N} \frac{(-1)^n\varphi_n(w)}{n!k!} v^{-n-k}s^{n+3k}(1+v^{-1}s)^{-k-\alpha-1}e^{-s^2} + E_N^{(1)}(v,w,s) \\
&= \sum_{n=0}^{N}\sum_{k=0}^{N}\sum_{l=0}^{N} \frac{(-1)^{n+l}(k+\alpha+1)_l}{n!k!l!}\varphi_n(w)v^{-n-k-l}s^{n+3k+l}e^{-s^2} + E_N^{(2)}(v,w,s)
\end{aligned}$$

with

$$E_N^{(j)}(v, w, s) = O(v^{-(N+1)}(1+|s|^{2(N+1)})s^{N+1}e^{-s^2}) \qquad (j=1, 2),$$

N being an arbitrary positive integer. Substitute this into (4.3). Note that, for each $q \in N$, $\int_{-X}^{X} s^q e^{-s^2} ds = \sqrt{\pi}(1/2)_{q/2} + O(X^{q-1}e^{-X^2})$ (if q is even), $= O(X^{q-1}e^{-X^2})$ (if q is odd), as $\operatorname{Re} X \to +\infty$ ($|\operatorname{Im} X|/|\operatorname{Re} X| = O(1)$). If $n+k+l$ is odd, the coefficient of v^{-n-k-l} vanishes. Putting $n+k+l=2m$, $M=[N/2]$, we have

$$J_1 = \sqrt{\pi} \sum_{m=0}^{M} \left(\sum_{k=0}^{2m} \frac{(-1)^{m+k}(1/2)_{m+k}}{k!} \sum_{n=0}^{2m-k} \frac{(k+\alpha+1)_{2m-k-n}}{(2m-k-n)!n!} \varphi_n(w) \right)(xy)^{-m/2} + O((xy)^{-(M+1)/2}).$$

Thus we arrive at the asymptotic expansion of z_+.

4.3. Proof of Theorem 1.2. In the proof of Theorem 1.2, we use the path C'_- modified in such a way that it has the following properties.

(a′) C'_- consists of three curves Γ'_-, Γ'_0, Γ'_+. Here Γ'_0 is a segment defined by $|\operatorname{Re}(t-(-x))| \le |x|/2$, $\operatorname{Im}(t-(-x))=0$, and Γ'_- (or Γ'_+) is a curve starting from $t = b_- = -x-|x|/2$ (or $t=b_+ = -x+|x|/2$), tending to $t=\infty$, and satisfying $\arg t \to -\pi$ (or $\arg t \to 0$) as $t \to \infty$.

(b′) C'_- lies outside the circle $|t| = |x|^{1/2}$, and Γ'_- and Γ'_+ outside the circle $|t-\xi_0| = |x|/4$.

(c′) $\operatorname{Re}(g_0(t)-x^2/2) \le 0$, $\operatorname{Im}(g_0(t)-x^2/2)=0$ for $t \in \Gamma'_0$.

(d′) $(d/d\rho)\operatorname{Re} g(t) \le -c$ for $t \in \Gamma'_-$ (or $t \in \Gamma'_+$), in which c is a positive constant and $\rho = \rho(t)$ denotes the length of a part of $g(\Gamma'_-)$ (or $g(\Gamma'_+)$) from $g(b_-)$ (or $g(b_+)$) to $g(t)$.

LEMMA 4.3. *As long as* (x, y) $(\in \boldsymbol{C} \times \mathscr{R}(\boldsymbol{C}^{\times}))$ *satisfies* $|y/x| < R$, $|x| > R_\infty^{(2)}$ *and* $|\arg x - \pi/2| < 3\pi/4 - \delta$, *we can modify the path* C'_- *continuously with respect to* (x, y) *preserving the properties above, where* $R_\infty^{(2)} = R_\infty^{(2)}(\delta)$ *is a sufficiently large positive constant.*

PROOF. Assume that $|y/x| < R$, $|x| > R_\infty^{(2)} > R'$. The constant $R_\infty^{(2)}$ is chosen in the following argument. Since $g_0(t) = x^2/2 - (t+x)^2/2$, the image $g_0(\Gamma'_0)$ is expressed as $-|x|^2/8 \le \operatorname{Re}(\tau - x^2/2) \le 0$, $\operatorname{Im}(\tau - x^2/2) = 0$, which means the property (c′). Observing that $\xi_0 = -x + O(x^{-1})$, by the same argument as in the verification of (4.2), we derive from (3.9) that $g(b_\pm) - g(\xi_0) = -|x|^2/8 + O(1)$, where $g(\xi_0) = x^2/2 + O(1)$ (cf. (3.11)). Denote by $T'_{0,-}$ (or $T'_{0,+}$) the half line defined by $\operatorname{Re}(\tau - g(b_-)) \le 0$, $\operatorname{Im}(\tau - g(b_-)) = 0$ (or $\operatorname{Re}(\tau - g(b_+)) \le 0$, $\operatorname{Im}(\tau - g(b_+)) = 0$). We can take $R_\infty^{(2)} = R_\infty^{(2)}(\delta)$ so large that, as long as $|g(\xi_0) - \pi| < 3\pi/2 - \delta$, $|x| > R_\infty^{(2)}$, $|y/x| < R$, there exist curves T'_- and T'_+ in the τ-plane which are continuous modifications of $T'_{0,-}$ and $T'_{0,+}$, respectively, with the properties below:

(i) T'_- (or T'_+) is a curve starting from $g(b_-)$ (or $g(b_+)$) and tending to ∞, and lies outside the circle $|\tau| = 2|x|^{3/2}$. In particular, when $\arg x = \pi/2$, T'_- (or T'_+) coincides with $T'_{0,-}$ (or $T'_{0,+}$);

(ii) $(d/d\tilde{\rho})\mathrm{Re}\,\tau \le -c$ for $\tau \in T'_-$ (or $\tau \in T'_+$), where $\tilde{\rho} = \tilde{\rho}(\tau)$ denotes the length of the part of T'_- (or T'_+) from $g(b_-)$ (or $g(b_+)$) to τ.
Observe that $\eta_j = O(1)$, $g(\eta_j) = O(x)$ $(j = 0, 1)$ (cf. (3.12)), and that $\tau = g(t)$ is biholomorphic at each point in the domain $|x|^{1/2} < |t| < +\infty$, $t \ne \xi_0$. Let Γ'_- (or Γ'_+) be a curve in the t-plane satisfying $g(\Gamma'_-) = T'_-$ (or $g(\Gamma'_+) = T'_+$), along which $\arg t \to -\pi$ (or $\arg t \to 0$) as $t \to \infty$. When $\arg x = \pi/2$, we may assume that the original path C'_- is given by $C'_{-,0}$: $-\infty < \mathrm{Re}(t-(-x)) < +\infty$, $\mathrm{Im}(t-(-x)) = 0$. We put $C'_- = \Gamma'_- \cup \Gamma'_0 \cup \Gamma'_+$. Then it is a continuous modification of $C'_{-,0}$, and has the desired properties. □

By Cardano's formula, $t^{-\alpha-1} = O((|x| + |g(t)|^{1/2})^{|\mathrm{Re}\,\alpha|})$ for $t \in C'_-$. Using (3.9), (3.10), (3.11), (3.13), (b′), (d′), and the maximal modulus principle, we have

$$\int_{\Gamma'_- \cup \Gamma'_+} t^{-\alpha-1} \exp g(t)dt = x^{\alpha} \exp(x^2/2) E_1(x, y)$$

with

$$E_1(x, y) = O(\exp(-|x|^2/16)) .$$

Hence it is sufficient to show that the integral

$$I_0 = \int_{\Gamma'_0} f dt$$

admits the asymptotic expansion of the theorem. In view of the case where $\arg x = \pi/2$ combined with the fact that $\arg t$ varies from $-\pi$ to 0 along C'_-, we have $\arg(-x) = -\pi + \arg x$. Put $t = e^{-\pi i}x + s$ in I_0. By (c′) the integral I_0 becomes

$$-e^{\alpha\pi i} x^{-\alpha-1} e^{x^2/2} \int_{-|x|/2}^{|x|/2} \exp\left(-\frac{s^2}{2} + \frac{y/x}{1-s/x}\right)(1-s/x)^{-\alpha-1} ds , \tag{4.4}$$

where $|\arg(1-s/x)| < \pi/2$ for $|s| < |x|/2$. The generating function $\sum_{l=0}^{\infty} L_l^{(\alpha)}(u) z^l = e^u (1-z)^{-\alpha-1} \exp(-u/(1-z))$ $(|z| < 1)$ (see [5, §10.12, (17)]) yields

$$(1-s/x)^{-\alpha-1} \exp\left(\frac{y/x}{1-s/x}\right) = e^{y/x}\left(\sum_{l=0}^{N} L_l^{(\alpha)}(-y/x) s^l x^{-l} + O(x^{-N-1} s^{N+1})\right)$$

for $|s| \le |x|/2$, where N is an arbitrary positive integer. Substituting this into (4.4), and using $\int_{-\infty}^{\infty} s^{2m} e^{-s^2/2} ds = \sqrt{2\pi}\, 2^m (1/2)_m = \sqrt{2\pi}\, 2^{-m} (2m)!/m!$ $(m \in N \cup \{0\})$, we obtain the asymptotic expansion of z'_-.

4.4. Proof of Theorem 1.3. Theorem 1.3 immediately follows from Theorem 1.1 and the relations below.

LEMMA 4.4 (cf. [7, Proposition 1.2]). *We have*

$$z_+(x, y) = e^{-\alpha\pi i} z_0(-x, e^{-\pi i} y) , \qquad z_-(x, y) = e^{\alpha\pi i} z_0(-x, e^{\pi i} y) .$$

4.5. Proof of Theorem 1.4. We wish to modify the path $C^*_{\pm}$ in such a way that it fulfills the following conditions.

(a*) $C^*_{\pm}$ consists of three curves $-K$, Γ_1, $+K$ with the properties:

(a*.1) $-K$ is a curve starting from $t=\infty$ and ending at $t=t_1=\exp(-i\arg x)$;

(a*.2) Γ_1 is a circle expressed as $t=e^{i\sigma}$, where σ varies from $\sigma=-\arg x$ to $\sigma=2\pi-\arg x$;

(a*.3) $+K$ is a curve, congruent with $-K$, starting from $t=e^{2\pi i}t_1$ and ending at $t=\infty$;

(a*.4) $\arg t\to-\pi$ as $t\to\infty$ along $-K$, and $\arg t\to\pi$ as $t\to\infty$ along $+K$.

(b*) C^*_{-} lies outside the circle $|t-\xi_0|=|x|^{1/2}$, and $-K$ and $+K$ outside the circle $|t|=1/2$.

(c*) The points η_j $(j=0,1)$ are located inside the circle $|t|=1/2$.

(d*) $(d/d\rho)\operatorname{Re} g(t)\le -c$ for $t\in -K$ (or $t\in +K$), where c is a positive constant and ρ denotes the length of a part of $g(-K)$ (or $g(+K)$) from $g(t_1)$ (or $g(e^{2\pi i}t_1)$) to $g(t)$.

LEMMA 4.5. *As long as* (x,y) $(\in \boldsymbol{C}\times\mathscr{R}(\boldsymbol{C}^\times))$ *satisfies* $|xy|<2R_0$, $|x|>R^{(3)}_\infty$ *and* $|\arg x-\pi|<3\pi/4-\delta$, *we can modify the path* $C^*_{\pm}$ *continuously with respect to* (x,y) *preserving the conditions above, where* $R^{(3)}_\infty=R^{(3)}_\infty(\delta)$ *is a sufficiently large positive constant.*

PROOF. Since $\eta_j=O(|y/x|^{1/2})=O(R_0^{1/2}|x|^{-1})$ $(R_0=R_0(\delta),\ j=0,1)$, the condition (c*) is satisfied for $|x|>R^{(3)}_\infty$, provided that $R^{(3)}_\infty=R^{(3)}_\infty(\delta)>R'$ is sufficiently large. We first consider the special case where $\arg x=\arg y=\pi$. Take a curve $-K$ (or $+K$) to be the interval $-\infty<t\le-1$. The image $g(-K)$ (or $g(+K)$) coincides with the interval T_0: $-\infty<\tau\le x-1/2+y$ contained in the negative real axis. Then all the conditions above are fufilled.

Next consider the general case. Note that $g(t_1)=g(e^{2\pi i}t_1)=-|x|(1+O(x^{-1}))$, $g(\eta_j)=O(1)$ $(j=0,1)$, and $g(\xi_0)=(x^2/2)(1+O(x^{-2}))$ (cf. (3.11), (3.12)). We can retake $R^{(3)}_\infty=R^{(3)}_\infty(\delta)$ so large that, as long as $|\arg g(\xi_0)-2\pi|<3\pi/2-\delta$, $|xy|<2R_0$, $|x|>R^{(3)}_\infty$, there exists a curve T in the τ-plane which is a continuous modification of T_0 with the properties:

(1) T starts from $\tau=g(t_1)$ and tends to $\tau=\infty$;

(2) T is located outside the circles $|\tau|=|g(t_1)|$, $|\tau-g(\xi_0)|=|x|$;

(3) $(d/d\rho)\operatorname{Re}\tau\le -c$ along T, where ρ denotes the length of a part of T from $g(t_1)$ to τ.

Taking the inverse images of T, we obtain the curves $-K$, $+K$. It is easy to see that the curve $C^*_{\pm}=(-K)\cup\Gamma_1\cup(+K)$ fulfills the conditions (a*), ..., (d*). □

Using (3.10), (b*), (d*), Cardano's formula, and the maximal modulus principle, we have

$$\int_{\pm K} f\,dt=O\left(|x|^{-1/2}|\exp g(t_1)|\int_0^{+\infty}(|x|+\rho^{1/2})^{|\operatorname{Re}\alpha|}e^{-c\rho}d\rho\right)=O(e^{-|x|/2}).$$

We calculate the asymptotic expansion of

$$(4.5) \qquad I_* = \int_{\Gamma_1} f\,dt = e^{-\alpha\pi i} x^\alpha \int_{-x\Gamma_1} \exp\left(s + \frac{xy}{s}\right) e^{-s^2/(2x^2)} s^{-\alpha-1} ds$$

($s = e^{-\pi i} xt$), where $-x\Gamma_1$ denotes the circle defined by $s = |x| e^{i\sigma}$, $-\pi \le \sigma \le \pi$. Now replace the path $-x\Gamma_1$ by the curve Γ_2 consisting of the segment $|x| e^{-\pi i} \le s \le e^{-\pi i}$, the circle $s = e^{i\sigma}$ $(-\pi \le \sigma \le \pi)$ and the segment $|x| e^{\pi i} \le s \le e^{\pi i}$. For an arbitrary integer $N > |\mathrm{Re}\,\alpha|$, substitution of $e^{-s^2/(2x^2)} = \sum_{m=0}^N (-2x^2)^{-m} s^{2m}/m! + O(x^{-2(N+1)} s^{2(N+1)})$ $(s \in \Gamma_2)$ into (4.5) yields

$$I_* = e^{-\alpha\pi i} x^\alpha \left(\sum_{m=0}^N \frac{(-2x^2)^{-m}}{m!} \int_{\Gamma_2} s^{2m-\alpha-1} \exp\left(s + \frac{xy}{s}\right) ds + R_N(x, y) \right)$$

with

$$R_N(x, y) = O\left(x^{-2(N+1)} \int_{\Gamma_2} |s^{2N-\alpha+1}| \left| \exp\left(s + \frac{xy}{s}\right) \right| |ds| \right).$$

Observing that, for $|xy| < 2R_0$,

$$\begin{aligned} R_N(x, y) &= O\left(x^{-2(N+1)} \left(\int_{|s|=1} \left| \exp\left(s + \frac{xy}{s}\right) \right| |ds| + \int_{-|x|}^{-1} |s|^{2N - \mathrm{Re}\,\alpha + 1} e^s ds \right) \right) \\ &= O(x^{-2(N+1)}), \end{aligned}$$

and substituting

$$\int_{\Gamma_2} s^{2m-\alpha-1} \exp\left(s + \frac{xy}{s}\right) ds = 2\pi i (i(xy)^{1/2})^{2m-\alpha} J_{-(2m-\alpha)}(2i(xy)^{1/2}) + O(x^{2m-\alpha} e^{-|x|})$$

(cf. [5, p. 15]), we obtain the asymptotic expansion of $z_\pm^*$.

5. Proofs of the theorems in Section 2.

5.1. Preliminaries. In this section we use Lemma 4.4 and the following.

LEMMA 5.1 (cf. [7, Proposition 2.2]). *For* $\boldsymbol{z}(x, y) = {}^t(z_0, z_+, z_-)$, *we have* $\boldsymbol{z}(x, e^{2\pi i} y) = M \boldsymbol{z}(x, y)$, *where*

$$M = \begin{pmatrix} 1 & -1 & e^{(-\alpha)} \\ 0 & 0 & e^{(-\alpha)} \\ 0 & -1 & 1 + e^{(-\alpha)} \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} 1 & e^{(\alpha)} & -1 \\ 0 & 1 + e^{(\alpha)} & -1 \\ 0 & e^{(\alpha)} & 0 \end{pmatrix}.$$

LEMMA 5.2. *We have*

$$W_+(e^{\pi i}x, e^{-\pi i}y)=e^{\alpha\pi i}W_+(x, y)\,, \qquad W_+(x, e^{2\pi i}y)=e^{(-\alpha)}W_-(x, y)\,,$$
$$W_-(e^{\pi i}x, e^{-\pi i}y)=e^{\alpha\pi i}W_-(x, y)\,, \qquad W_-(x, e^{2\pi i}y)=-W_+(x, y)\,,$$
$$W_1(e^{\pi i}x, e^{-\pi i}y)=-e^{-\alpha\pi i}W_1(x, y)\,, \quad W_1(x, e^{2\pi i}y)=W_1(x, y)\,,$$
$$W_*(e^{\pi i}x, e^{-\pi i}y)=e^{\alpha\pi i}W_*(x, y)\,.$$

For simplicity, to indicate sectors, we use the notation below;

$$\Sigma(\theta_1, \theta_2)=\{(x, y)\in D_+\,|\,|\arg x-\theta_1|<3\pi/4-\delta,\ |\arg(xy)-\theta_2|<3\pi-\delta\}\,,$$
$$\Sigma_0(\theta)=\{x\in \boldsymbol{C}\,|\,|\arg x-\theta|<3\pi/4-\delta\}\,.$$

5.2. Proof of Theorem 2.1. By Theorems 1.1, 1.2 and 1.3, for $|y/x|<R$,

(5.1) $$z'_-\simeq W_1(x, y) \qquad (\text{in } \Sigma_0(\pi/2))\,,$$

(5.2) $$z_+\simeq W_+(x, y) \qquad (\text{in } \Sigma(\pi, 2\pi))\,,$$

(5.3) $$z_-\simeq W_-(x, y) \qquad (\text{in } \Sigma(\pi, 0))\,.$$

In addition to these relations, we need the following formulas.

Lemma 5.3. *For* $|y/x|<R$,

(5.4) $$z'_- - z_+ + z_- \simeq W_1(x, y) \qquad (\textit{in } \Sigma_0(3\pi/2))\,,$$

(5.5) $$e^{(-\alpha)}z'_- + z_+ \simeq W_+(x, y) \qquad (\textit{in } \Sigma(0, 2\pi))\,,$$

(5.6) $$z'_- + z_- \simeq W_+(x, y) \qquad (\textit{in } \Sigma(2\pi, 2\pi))\,,$$

(5.7) $$z'_- + z_- \simeq W_-(x, y) \qquad (\textit{in } \Sigma(0, 0))\,,$$

(5.8) $$e^{(\alpha)}z'_- - e^{(\alpha)}z_+ + (1+e^{(\alpha)})z_- \simeq W_-(x, y) \qquad (\textit{in } \Sigma(2\pi, 0))\,.$$

Proof. We show (5.4) and (5.8). The others are similarly derived. Suppose that $|\arg(e^{-\pi i}x)-\pi/2|=|\arg x-3\pi/2|<3\pi/4-\delta$. By (5.1) and Lemma 5.2,

(5.9) $$z'_-(e^{-\pi i}x, e^{\pi i}y)\simeq W_1(e^{-\pi i}x, e^{\pi i}y)=-e^{\alpha\pi i}W_1(x, y)\,.$$

On the other hand, by (0.8), Lemmas 4.4 and 5.1,

$$\begin{aligned} z'_-(e^{-\pi i}x, e^{\pi i}y)&=z_0(e^{-\pi i}x, e^{\pi i}y)-z_-(e^{-\pi i}x, e^{\pi i}y)\\ &=e^{-\alpha\pi i}z_-(x, y)-e^{\alpha\pi i}z_0(x, e^{2\pi i}y)=e^{\alpha\pi i}z_+(x, y)-e^{\alpha\pi i}z_0(x, y)\\ &=e^{\alpha\pi i}(-z'_-(x, y)+z_+(x, y)-z_-(x, y))\,. \end{aligned}$$

The formula (5.4) follows from this and (5.9). Note that $\Sigma(2\pi, 0)$ is written in the form $|\arg(e^{-\pi i}x)-\pi|<3\pi/4-\delta$, $|\arg(e^{-\pi i}xe^{\pi i}y)|<3\pi-\delta$. By (5.3) and Lemma 5.2,

(5.10) $$z_-(e^{-\pi i}x, e^{\pi i}y)\simeq W_-(e^{-\pi i}x, e^{\pi i}y)=e^{-\alpha\pi i}W_-(x, y)\,.$$

By (0.8), Lemmas 4.4 and 5.1,

$$z_-(e^{-\pi i}x, e^{\pi i}y) = e^{\alpha\pi i}z_0(x, e^{2\pi i}y) = e^{\alpha\pi i}(z_0(x, y) - z_+(x, y) + e^{(-\alpha)}z_-(x, y))$$
$$= e^{\alpha\pi i}(z'_-(x, y) - z_+(x, y) + (1 + e^{(-\alpha)})z_-(x, y)) .$$

Combining this with (5.10), we obtain (5.8). □

Using (5.4), (5.6) and (5.8), we have

$$S_1{}^t(z'_-, z_+, z_-) \simeq {}^t(W_1(x, y), W_+(x, y), W_-(x, y))$$

in the sector $S_+(7\pi/4, \pi)$, where

$$S_1 = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ e^{(\alpha)} & -e^{(\alpha)} & 1+e^{(\alpha)} \end{pmatrix} .$$

Thus $T(S_+(7\pi/4, \pi)) = S_1^{-1}$ follows. Similarly we derive $T(S_+(\pi/4, \pi))$ from (5.1), (5.5) and (5.7), and $T(S_+(5\pi/4, \pi))$ from (5.2), (5.3) and (5.4). Next consider the case where $(x, y) \in S_+((2j-1)\pi/4, -\pi)$ $(j = 1, 2, 3, 4)$. Since $(x, e^{2\pi i}y) \in S_+((2j-1)\pi/4, \pi)$, we have

$$\boldsymbol{u}(x, e^{2\pi i}y) \simeq T(S_+((2j-1)\pi/4, \pi))\boldsymbol{W}(x, e^{2\pi i}y) \tag{5.11}$$

with $\boldsymbol{u}(x, y) = {}^t(z'_-(x, y), z_+(x, y), z_-(x, y))$, $\boldsymbol{W}(x, y) = {}^t(W_1(x, y), W_+(x, y), W_-(x, y))$. By (0.8), Lemmas 5.1 and 5.2, we have $\boldsymbol{u}(x, e^{2\pi i}y) = M_0\boldsymbol{u}(x, y)$, $\boldsymbol{W}(x, e^{2\pi i}y) = \Omega\boldsymbol{W}(x, y)$. From these relations combined with (5.11), $T(S_+((2j-1)\pi/4, -\pi))$ $(j = 1, 2, 3, 4)$ immediately follow.

5.3. Proof of Theorem 2.2. Note that (5.1) and (5.4) are valid in D_- as well. Theorem 1.4 implies that

$$z^*_- \simeq W_*(x, y) \qquad (\text{in } \Sigma_0(\pi)) . \tag{5.12}$$

Furthermore we have the following relations.

LEMMA 5.4. *For* $|y/x| < R$, $|xy| < 2R_0$,

$$z'_- - z^*_- \simeq W_1(x, y) \qquad (\textit{in } \Sigma_0(3\pi/2)) , \tag{5.13}$$

$$(e^{(-\alpha)} - 1)z'_- + z^*_- \simeq W_*(x, y) \qquad (\textit{in } \Sigma_0(0)) , \tag{5.14}$$

$$(1 - e^{(\alpha)})z'_- + e^{(\alpha)}z^*_- \simeq W_*(x, y) \qquad (\textit{in } \Sigma_0(2\pi)) . \tag{5.15}$$

PROOF. Putting $z^*_- = z_+ - z_-$ in (5.4) we obtain (5.13). Since $|\arg x| = |\arg(e^{\pi i}x) - \pi| < 3\pi/4 - \delta$, it follows from (5.12) and Lemma 5.2 that

$$z^*_-(e^{\pi i}x, e^{-\pi i}y) \simeq W_*(e^{\pi i}x, e^{-\pi i}y) = e^{\alpha\pi i}W_*(x, y) .$$

By Lemmas 4.4 and 5.1,

$$\begin{aligned} z_-^*(e^{\pi i}x, e^{-\pi i}y) &= z_+(e^{\pi i}x, e^{-\pi i}y) - z_-(e^{\pi i}x, e^{-\pi i}y) \\ &= e^{-\alpha\pi i}z_0(x, e^{-2\pi i}y) - e^{\alpha\pi i}z_0(x, y) \\ &= e^{\alpha\pi i}((e^{(-\alpha)} - 1)z'_-(x, y) + z_+(x, y) - z_-(x, y)) \\ &= e^{\alpha\pi i}((e^{(-\alpha)} - 1)z'_-(x, y) + z_-^*(x, y)) . \end{aligned}$$

Hence we have (5.14). The formula (5.15) is obtained by an analogous argument. $\square$

If $(x, y) \in S_-(\pi/4)$, then, by (5.1) and (5.14), we have $U_1{}^t(z'_-, z_-^*) \simeq {}^t(W_1(x, y), W_*(x, y))$ with

$$U_1 = \begin{pmatrix} 1 & 0 \\ e^{(-\alpha)} - 1 & 1 \end{pmatrix},$$

from which $U(S_-(\pi/4)) = U_1^{-1}$ follows. Similarly we obtain $U(S_-(5\pi/4))$ from (5.12) and (5.13), and $U(S_-(7\pi/4))$ from (5.13) and (5.15).

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