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ROUGH ISOMETRY AND THE ASYMPTOTIC DIRICHLET PROBLEM

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Abstract. We propose a new asymptotic Dirichlet problem for harmonic functions via the rough isometry on a certain class of Riemannian manifolds. We prove that this problem is solvable for naturally defined class of functions. This result generalizes those of Schoen and Yau and of Cheng.

1. Introduction. The asymptotic Dirichlet problem for harmonic functions on a noncompact complete Riemannian manifold is to find the harmonic function satisfying the given Dirichlet boundary condition at infinity. It has a long history, and by now, it is well understood by the works of the first author, M. Anderson, D. Sullivan, R. Schoen and others, when M is a Cartan-Hadamard manifold with sectional curvature $-b^2 \leq K_M \leq -a^2 < 0$. (By a Cartan-Hadamard manifold, we mean a complete simply connected manifold of nonpositive sectional curvature.)

In [Ch], the first author posed the asymptotic Dirichlet problem and proved that it is solvable when a Cartan-Hadamard manifold M with sectional curvature $K_M \leq -a^2 < 0$ satisfies the convex conic neighborhood condition. In [A], Anderson constructed a convex neighborhood, thereby solving the problem when the sectional curvature satisfies $-b^2 \leq K_M \leq -a^2 < 0$. At the same time, Sullivan [S] also solved this problem using the probabilistic approach. In [A-S], Anderson and Schoen showed that the Martin boundary of M can be identified with $M(\infty)$ which is naturally defined to be the set of the asymptotic classes of geodesic rays. By constructing the Poisson kernel, they obtained the representation formula for harmonic functions, and proved the Fatou-type theorem. The essence of all of the above works is that the curvature assumption enables one to control the angle via the Toponogov comparison theorem and the convexity property near the boundary at infinity $M(\infty)$.

There have been many attempts to generalize the above results. A typical approach is to relax the curvature assumption to allow curvature decay at a certain rate. But the basic method of proof still remains the same and the improvements are mostly technical. One interesting generalization that does not directly involve the curvature bound

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is achieved by Schoen and Yau [S-Y]. They proved the following: Suppose that (M, ds^2) is complete and simply connected and its sectional curvature K_M satisfies $-b^2 \leq K_M \leq -a^2 < 0$. Let $d\tilde{s}^2$ be a new Riemannian metric on M which is uniformly equivalent to ds^2. When $(M, d\tilde{s}^2)$ has the bounded sectional curvature and the positive injectivity radius, they solved the asymptotic Dirichlet problem on $(M, d\tilde{s}^2)$ where $M(\infty)$ is defined with respect to the old metric ds^2. It is important to note this curious fact that $M(\infty)$ is still defined with respect to the old metric ds^2, whereas the Laplacian Δ is defined with respect to the new metric $d\tilde{s}^2$. Cheng [C] removed the assumptions on the sectional curvature and the injectivity radius of $(M, d\tilde{s}^2)$ from Schoen-Yau's result to obtain the following result:

THEOREM (Cheng). *Let (M, ds^2) be a complete simply connected Riemannian manifold with nonpositive sectional curvature and the Dirichlet eigenvalue $\lambda_1(M) > 0$. Assume that the local pinching condition holds on (M, ds^2), i.e., there exist a point $o \in M$ and a constant $C \geq 1$ such that at any $x \in (M, ds^2)$, $|K(\sigma)| \leq C|K(\sigma')|$, where σ, σ' are plane sections at x containing the tangent vector of the geodesic joining o to x and $K(\sigma)$, $K(\sigma')$ are the sectional curvatures of the plane sections σ, σ', respectively. Let $d\tilde{s}^2$ be a new metric on M which is uniformly equivalent to ds^2. Then for any continuous function f on $M(\infty)$, there exists $u \in C^\infty(M) \cap C^0(M \cup M(\infty))$ such that*

$$\begin{cases} \Delta u = 0 & \text{on } M \\ u = f & \text{on } M(\infty), \end{cases}$$

where Δ is the Laplacian of $(M, d\tilde{s}^2)$.

Again, it has to be emphasized that $M(\infty)$ is still defined with respect to the old metric ds^2. But it is only natural because one has difficulty in defining the asymptotic boundary at infinity of $(M, d\tilde{s}^2)$ if the assumption that the manifold is nonpositively curved is omitted. The results of Schoen and Yau and of Cheng indicate that the solvability of the asymptotic Dirichlet problem depends on some crude macroscopic property of M rather than more local geometric properties such as the curvature.

The main motivation of this paper is to validate this viewpoint using the rough isometry. The concept of rough isometry is introduced by Kanai [K1], [K2] and [K3]; and later, Coulhon and Saloff-Coste [C-S] gave some improvements. See the next section for the definition which we adopt. The rough isometry is a very crude equivalence relation on the class of Riemannian manifolds, and it is not even required to be continuous. Therefore even if M and N are roughly isometric, M and N may have completely different topology. But they share certain macroscopic geometric properties such as the volume growth, the positivity of the Sobolev or the Poincaré constants, etc.

In this paper, we propose and solve a new asymptotic Dirichlet problem via the rough isometry. It is stated as follows: Suppose $\varphi: M \to N$ is a rough isometry, and suppose M satisfies the conditions in Cheng's result with respect to the old metric.

We define a suitable function class $\mathscr{F}_\varphi$ on N, where $\varphi: M \to N$ is a rough isometry. For each $f \in \mathscr{F}_\varphi$, our problem is to find a function $\tilde{u} \in C^\infty(N)$ such that $\Delta\tilde{u}=0$ on N and $(\tilde{u}-f)(\varphi(x)) \to 0$ as $x \to \infty$. Note the curious way of stating the continuity of the solution at infinity in our formulation. But this is in fact in line with the approach of Schoen and Yau and of Cheng. First, since even the topology of N differs from that of M not to mention the nonpositivity of the curvature, there is no clear way of defining $N(\infty)$, the boundary at infinity, of N. This therefore forces one to state the boundary value in terms of $M(\infty)$. Note that Schoen and Yau's and Cheng's case is when M and N are the same manifolds with different, but uniformly equivalent, metrics, and φ is the identity map. Next simple situation is the case when M and N are the same manifolds equipped with different metrics such that the identity map is a rough isometry. Our result is new even in this simple case.

2. Rough isometry and net structure. A (not necessarily continuous) map φ: $X \to Y$ between two metric spaces X and Y is called a rough isometry, if the following conditions hold:

(1) there exists a constant $\tau > 0$ such that

$$Y = \bigcup_{x \in X} B_\tau(\varphi(x)),$$

where $B_\tau(\varphi(x))$ means the τ-neighborhood of $\varphi(x)$;

(2) there exist constants $a \geq 1$ and $b \geq 0$ such that

$$\frac{1}{a} d(x_1, x_2) - b \leq d(\varphi(x_1), \varphi(x_2)) \leq a d(x_1, x_2) + b$$

for all x_1, x_2 in X, where d denotes the distances of X and Y induced from their metrics, respectively.

It is easy to see that for such a rough isometry φ, there exists the inverse rough isometry φ^{-1}: $Y \to X$ such that $d(y, \varphi \circ \varphi^{-1}(y)) \leq \tau$ for all y in Y. Thus being roughly isometric is an equivalence relation. (See [K2].) But it is also important to note that two roughly isometric metric spaces may have completely different topology, since φ is not assumed to be continuous. For example, an infinite cylinder is roughly isometric to an infinite cylinder with infinitely many identical handles attached at equal distance going off to the infinity.

In [K1], [K2] and [K3], Kanai introduced the concept of rough isometry. He assumed that a rough isometry φ satisfies the conditions (1) and (2), and demanded the additional conditions on Riemannian manifolds as follows:

(K1) the Ricci curvature of manifold M is bounded below;

(K2) the injectivity radius is positive, i.e., $\mathrm{inj}(M) > 0$.

Later, Coulhon and Saloff-Coste [C-S] gave a slightly more general and convenient definition of the rough isometry between Riemannian manifolds. Let $\varphi: M \to N$ be a map satisfying the conditions (1) and (2). They assumed the following condition:

(3) there exists a constant $C \geq 1$ such that

$$\frac{1}{C} \operatorname{vol} B_1(x) \leq \operatorname{vol} B_1(\varphi(x)) \leq C \operatorname{vol} B_1(x)$$

for each $x \in M$.

They also assume that both M and N satisfy the local volume doubling condition as follows:

(4) for all $r > 0$, there exists a constant $C_r > 0$ depending only on r such that

$$\operatorname{vol} B_{2r}(x) \leq C_r \operatorname{vol} B_r(x)$$

for all x in M (in N, respectively).

But these assumptions are improvements of Kanai's assumptions. Thus we still call it a rough isometry between Riemannian manifolds. From now on, when we say φ is a rough isometry between Riemannian manifolds, it means that φ satisfies the conditions (1), (2) and (3), and the Riemannian manifolds satisfy the condition (4); and τ always means that which appears in (1).

We now collect relevant definitions concerning the rough isometry which we need in this paper. One of the key tools in combinatorially approximating a Riemannian manifold M is the concept of the net defined as below:

Let d be the distance function on M. A subset P of M is called μ-separated for some $\mu > 0$ if $d(p, p') \geq \mu$ for any distinct points p and p' of P.

A μ-separated subset is called maximal μ-separated if it is maximal with respect to the order relation of inclusion. Let P be a maximal μ-separated subset of M. Then we can define a net structure $\mathscr{N} = \{N(p) : p \in P\}$ by setting $N(p) = \{p' \in P : \mu \leq d(p, p') < 3\mu\}$. Note that this family $\mathscr{N}$ satisfies that for all $p, p' \in P$,

(i) $N(p)$ is a finite nonempty subset of P;

(ii) $p' \in N(p)$ if and only if $p \in N(p')$.

A maximal μ-separated subset P of M with the net structure described above is called the μ-net in M.

A sequence $\boldsymbol{p} = (p_0, \ldots, p_l)$ of points in P is called a path from p_0 to p_l with the length l if each p_k is an element of $N(p_{k-1})$ for $k = 1, 2, \ldots, l$. Then for two points p and p' of P, we can define $\delta(p, p')$ to be the minimum of the lengths of paths from p to p'. It is easy to check that δ defines a metric on P. In [K2], Kanai proved that a net P, with this metric δ, is roughly isometric to M, i.e., there exist constants $\alpha \geq 1$ and $\beta \geq 0$ such that

$$\frac{1}{\alpha} \delta(p_1, p_2) - \beta \leq d(p_1, p_2) \leq \alpha \delta(p_1, p_2) + \beta$$

for all $p_1, p_2 \in P$. For this metric δ, define an l-neighborhood $N_l(p) = \{p' \in P : \delta(p, p') \leq l\}$ for each $p \in P$ and for each $l \in \boldsymbol{N}$. A net P is said to be uniform if there exists a constant

λ such that $\sharp N(p) \leq \lambda < \infty$ for all $p \in P$, where $\sharp S$ denotes the cardinality of the set S. Either the condition (K1) or (4) guarantees that a μ-net P on M is uniform, and this uniformness plays a crucial role in the proof of the roughly isometric invariance of some analytic properties. Let us define the norm of gradient of functions on a net P; for a function f defined on a net P and for $p \in P$, set

$$|Df|(p) = \left(\sum_{p' \in N(p)} |f(p') - f(p)|^2 \right)^{1/2}.$$

3. Boundary value of a function under rough isometry. Let N be a complete Riemannian manifold which is roughly isometric to a Cartan-Hadamard manifold M via the rough isometry φ. We would like to pose and solve an asymptotic Dirichlet problem on N. The first difficulty one encounters in doing so is that there is no a priori good way of defining the boundary $N(\infty)$ at infinity of N. Our idea is to lean on $M(\infty)$ as a way to detect the infinity behavior of N. The rationale is that the structure of the infinity of N must be roughly equal to that of M, since the rough isometry is really on the infinity behavior of the respective manifolds. To be more specific, let $M(\infty)$ be the boundary at infinity of M which is the set of asymptotic classes of unit speed geodesic rays. It is topologized with the cone topology in the sense of Eberlein and O'Neill [E-O]. Let $\varphi : M \to N$ be a rough isometry and f be a function on N. We say that f is an element of the class $\mathscr{F}_\varphi$ if f satisfies the following conditions:

(i) $f \circ \varphi$ can be extended to $M \cup M(\infty)$ in such a way that $f \circ \varphi$ is continuous at every point of $M(\infty)$;

(ii) for given $\varepsilon > 0$, there exists $T > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in B_\tau(x)$ and $d(o', x) \geq T$ for some fixed point $o' \in N$.

In the above, the statement (i) means that for any $v \in M(\infty)$, $f \circ \varphi(x)$ converges to a number A_v as x converges to v, i.e., for any $\varepsilon > 0$, there exists a neighborhood K of v such that $|f(x) - A_v| < \varepsilon$ whenever $x \in K$. The statement (ii) is needed to control the wild variance of f within the τ-ball. (See Example 3.1.) It is perhaps worthwhile to understand better of the above conditions. First, as we do not know the boundary at infinity of N, the value of $f \circ \varphi$ on $M(\infty)$ is used as a surrogate value of f at the infinity of N. It is instructive to note that this is the approach that Schoen-Yau and Cheng took as explained in §1. Second, however, this definition depends on the choice of the particular rough isometry φ. What we need is to know under what circumstances the condition (i) above is independent of a particular choice of the rough isometry. With regard to this, it is interesting to recall the result of Li and Wang [L-W] in which they proved that if $\varphi : X \to Y$ is a rough isometry between two Cartan-Hadamard manifolds and if the sectional curvature of X or Y is bounded above by a negative constant, then φ extends to a map $\varphi : X \cup X(\infty) \to Y \cup Y(\infty)$ which induces a homeomorphism of $X(\infty)$ onto $Y(\infty)$.

In this paper, however, we do not assume that N is a Cartan-Hadamard manifold, and even though M is one, the sectional curvature is not assumed to be bounded above by a negative constant. Thus Li and Wang's result does not apply directly. But we still have some interesting invariance of the continuity of the boundary value at infinity. Suppose φ and ψ are both rough isometries of M to N. Then $\varphi^{-1}\circ\psi: M\to M$ is a rough isometry of the Cartan-Hadamard manifold M. It is not true in general that $\varphi^{-1}\circ\psi$ extends to a map $\varphi^{-1}\circ\psi: M\cup M(\infty)\to M\cup M(\infty)$ in such a way that it is continuous at every point of $M(\infty)$. However, if it does, we have the following result.

PROPOSITION 3.1. *Let $\varphi, \psi: M\to N$ be rough isometries such that $\varphi^{-1}\circ\psi$ extends to a map $\varphi^{-1}\circ\psi: M\cup M(\infty)\to M\cup M(\infty)$ in such a way that it is continuous at every point of $M(\infty)$. Then for each $f\in\mathscr{F}_\varphi$, $f\circ\psi$ can be extended to $M\cup M(\infty)$ so that $f\circ\psi$ is continuous at every point of $M(\infty)$.*

REMARK. Even though $\varphi^{-1}\circ\psi$ is continuous at every point of $M(\infty)$, we do not require $\varphi^{-1}\circ\psi$ to be continuous in the interior M.

PROOF. Fix a point $o\in M$. Then we can identify the unit tangent space U_oM at o with $M(\infty)$. We use the notation

$$K(v, \delta, R)=\{x\in M \mid \sphericalangle_o(v, \overrightarrow{ox})<\delta,\ d(o, x)>R\},$$

for $v\in U_oM$, where $\sphericalangle_o(v, \overrightarrow{ox})$ denotes the angle at o between v and the ray starting from o and passing through x. For any $\varepsilon>0$ and any $v\in U_oM$, we shall show that there exist $\delta>0$ and $R>0$ such that $|f\circ\psi(x)-f\circ\varphi(w)|<2\varepsilon$ if $x\in K(v, \delta, R)$, where $w=\varphi^{-1}\circ\psi(v)$. Since $f\circ\varphi$ is continuous on $M(\infty)$, there exist $\tilde{\delta}>0$ and $\tilde{R}>0$ such that

$$|f\circ\varphi(z)-f\circ\varphi(w)|<\varepsilon \qquad \text{for all} \quad z\in K(w, \tilde{\delta}, \tilde{R}).$$

Since $\varphi^{-1}\circ\psi$ is continuous on $M(\infty)$, we can choose $\delta>0$ so that

$$\varphi^{-1}\circ\psi(B_\delta(v))\subset B_{\tilde{\delta}/2}(w),$$

where $B_\delta(v)$ denotes the δ-neighborhood of v in U_oM and $B_{\tilde{\delta}/2}(w)$ denotes the $\tilde{\delta}/2$-neighborhood of w in U_oM. On the other hand, we can choose $R>0$ so that $\varphi^{-1}\circ\psi(M\setminus B_R(o))\subset M\setminus B_{2\tilde{R}}(o)$. Consequently, we have

$$\varphi^{-1}\circ\psi(K(v, \delta, R))\subset K\left(w, \frac{\tilde{\delta}}{2}, 2\tilde{R}\right).$$

For each $x\in K(v, \delta, R)$, there exists $y\in M$ such that $d(\varphi(y), \psi(x))\le\tau$, since $B_\tau(\varphi(M))=N$. Since $f\in\mathscr{F}_\varphi$, there exists $T>0$ such that $|f\circ\varphi(y)-f\circ\psi(x)|<\varepsilon$ whenever $d(\psi(o), \psi(x))\ge T$. We can choose a sufficiently large $R>0$ such that $d(\psi(o), \psi(x))\ge T$. This implies that $|f\circ\varphi(y)-f\circ\psi(x)|<\varepsilon$ whenever $d(o, x)\ge R$. On the other hand, since $\varphi^{-1}\circ\psi(x)\in K(w, \tilde{\delta}/2, 2\tilde{R})$ and

$$\begin{aligned} d(y, \varphi^{-1} \circ \psi(x)) &\leq ad(\varphi(y), \varphi \circ \varphi^{-1} \circ \psi(x)) + b \\ &\leq a\tau + ad(\varphi(y), \psi(x)) + b \\ &\leq a\tau + a\tau + b, \end{aligned}$$

we have $y \in K(w, \tilde{\delta}, \tilde{R})$ if $2a\tau + b \leq \tilde{R}$. This implies that $|f \circ \varphi(y) - f \circ \varphi(w)| < \varepsilon$. Thus for chosen $\delta > 0$ and $R > 0$ in the above, we have $|f \circ \psi(x) - f \circ \varphi(w)| < 2\varepsilon$ if $x \in K(v, \delta, R)$, where $w = \varphi^{-1} \circ \psi(v)$. □

EXAMPLE 3.1. We now give an example to illustrate why the condition (ii) of $\mathscr{F}_\varphi$ is needed. By the result of Li and Wang in [L-W], each rough isometry from the hyperbolic space H^2 into itself induces a homeomorphism from $H^2(\infty)$ onto itself. Let P be a μ-net in H^2. Define a rough isometry $\varphi: H^2 \to H^2$ by for each $x \in H^2$, $\varphi(x) = p_x$ for some $p_x \in P \cap B_\mu(x)$. Note that $H^2 = \bigcup_{p \in P} B_\mu(p)$.

Define another rough isometry $\psi: H^2 \to H^2$ by for each $x \in H^2$, $\psi(x) = y$, where y is a point on a unit speed minimal geodesic $\gamma: [0, \infty) \to H^2$ such that $\gamma(0) = p_x$, $\gamma(t) = x$ with $\gamma(\mu/3) = y$. Then for a function f on H^2 given by

$$f(x) = \begin{cases} 1 & x \in \bigcup_{p \in P} B_{\mu/4}(p) \\ 0 & \text{otherwise}, \end{cases}$$

we have $f \circ \varphi \equiv 1$ and $f \circ \psi \equiv 0$. This wild variance of f permits proving the existence of a rough isometry $\theta: H^2 \to H^2$ such that $f \circ \theta$ is discontinuous at infinity $H^2(\infty)$. To show this, we choose a suitable coordinate on H^2 as follows: Set $H^2 = (R^2_+, (dx^2 + dy^2)/y^2)$, where $R^2_+ = \{(x, y) \in R^2 \mid y > 0\}$. And define a rough isometry $\theta: H^2 \to H^2$ by

$$\theta(x, y) = \begin{cases} \varphi(x, y) & x \geq 0 \\ \psi(x, y) & x < 0, \end{cases}$$

where φ and ψ are given above. Then $f \circ \theta$ is discontinuous on $H^2(\infty)$. □

Note that the rough isometries $\varphi, \psi, \theta: H^2 \to H^2$ induce homeomorphisms $\varphi, \psi, \theta: H^2(\infty) \to H^2(\infty)$, respectively. Thus we need to add the local condition (ii) of $\mathscr{F}_\varphi$ in order to make the Proposition 3.1 valid.

4. The asymptotic Dirichlet problem. Let $\varphi: M \to N$ be a rough isometry of a Cartan-Hadamard manifold M into a complete Riemannian manifold N. Let f be a function on N such that $f \in \mathscr{F}_\varphi$. The asymptotic Dirichlet problem we propose is to find a harmonic function $\tilde{u}$ on N such that $\tilde{u} \circ \varphi$ has the same boundary value as $f \circ \varphi$ on $M(\infty)$.

For technical reasons, we redefine φ as follows: Let P be μ-net of M and Q be ν-net of N. Since both M and N satisfy the local condition (4), each of P and Q is a uniform net. And since M and N are roughly isometric, there exists a rough isometry $\psi: P \to Q$. In fact, we can choose ψ so that $\psi(p)=q$ for some $q \in Q$, where q is a point in Q satisfying $d(q, \varphi(p)) \leq \tau$. Let $\pi: M \to P$ be a rough isometry. Then $i \circ \psi \circ \pi: M \to N$ is a rough isometry, where $i: Q \to N$ is the inclusion map. It is easy to check that $\varphi^{-1} \circ (i \circ \psi \circ \pi)$ extends to a map $\varphi^{-1} \circ (i \circ \psi \circ \pi): M \cup M(\infty) \to M \cup M(\infty)$ such that the restriction map $M(\infty) \to M(\infty)$ is the identity map, where φ^{-1} is the inverse rough isometry of φ. Thus by the Proposition 3.1, $f \circ (i \circ \psi \circ \pi)$ is again continuous at every point of $M(\infty)$. For this reason, we may redefine φ by $i \circ \psi \circ \pi$. Then $\psi: P \to Q$ is simply the restriction of φ to P. It is easy to check that solving the asymptotic Dirichlet problem for this newly defined φ is enough to solve this original asymptotic Dirichlet problem.

To prove our main results, we need to add the following condition on M:

(5) there exists a constant $C>0$ depending only on $r>0$ such that

$$\int_{B_r(x)} |\nabla f| \geq C \int_{B_r(x)} |f-\bar{f}|$$

for all $x \in M$ and for all $f \in C^\infty(B_r(x))$, where $\bar{f}=(\operatorname{vol} B_r(x))^{-1} \int_{B_r(x)} f$.

Note that if the Ricci curvature is bounded below, then we have a constant $C>0$ satisfying the condition (5). (See Buser [B].)

THEOREM 4.1. *Let M be a Cartan-Hadamard manifold satisfying the local conditions* (4) *and* (5) *and let N be a complete Riemannian manifold satisfying the local condition* (4). *Suppose the Dirichlet eigenvalue $\lambda_1(M)>0$, and suppose there exist a point $o \in M$ and a constant $C \geq 1$ such that at any $x \in M$, we have $|K(\sigma)| \leq C|K(\sigma')|$, where σ, σ' are plane sections at x containing the tangent vector of the geodesic joining o to x and $K(\sigma)$, $K(\sigma')$ are the sectional curvatures of the plane sections σ, σ', respectively. Let $\varphi: M \to N$ be a rough isometry. Then for any $f \in \mathscr{F}_\varphi$, there exists a solution $\tilde{u} \in C^\infty(N)$ such that $\Delta \tilde{u}=0$ on N and $(\tilde{u}-f)(\varphi(x)) \to 0$ as $x \to \infty$.*

For each $f \in \mathscr{F}_\varphi$, we can define an extension h of $f \circ \varphi$ such that $h \in C^\infty(M) \cap C^0(M \cup M(\infty))$ and $h|_{M(\infty)}=f \circ \varphi|_{M(\infty)}$. We may define h to be radially constant outside some compact subset of M. In [C], Cheng imposed a local pinching condition for curvatures. From this condition, he obtained that $|\nabla h| \in L^s(M)$ for sufficiently large $s \geq 2$. (See [C, Theorem 3.1].) Without loss of generality, we may assume that h is positive and bounded.

Let P be a μ-net of M and Q be a ν-net of N. Define a function h_μ on P by

$$h_\mu(p)=\left(\frac{1}{\operatorname{vol} B_{4\mu}(p)} \int_{B_{4\mu}(p)} h^s\right)^{1/s}$$

for $p \in P$. Define a function k on Q by

$$k(q)=h_\mu \circ \varphi^{-1}(q)$$

for $q \in Q$, where φ^{-1} is the inverse rough isometry of φ such that $d(\varphi \circ \varphi^{-1}(q), q) \leq \tau$ for each $q \in Q$. Define a new function $g: N \to \boldsymbol{R}$ by

$$g(x) = \sum_{q \in Q} k(q) \eta_q(x),$$

where $\eta_q(x)$ is a partition of unity defined as follows: Let ξ_q be a Lipschitz function given by

$$\xi_q(x) = \begin{cases} 1 - (2/3v)d(x, q), & x \in B_{3v/2}(q) \\ 0, & \text{otherwise}. \end{cases}$$

We define

$$\eta_q(x) = \frac{\xi_q(x)}{\sum_{q' \in Q} \xi_{q'}(x)}.$$

Then it is easy to check that there exist $C_1 > 0$ and $C_2 > 0$ such that $\sum_{q' \in Q} \xi_{q'}(x) \geq C_1 > 0$ and $|\nabla \eta_q|(x) \leq C_2$ for all $x \in N$ and for all $q \in Q$, where $\nabla \eta_q$ is a weak derivative.

THEOREM 4.2. *For sufficiently large $s \geq 2$, we have the following*:

$$\int_N |\nabla g|^s < \infty,$$

where g is defined as above.

This fact is useful in showing that the solution of Theorem 4.1 converges to the boundary data at infinity. The proof of Theorem 4.2 is divided into several steps: In proving Lemma 4.1, Lemma 4.2, Lemma 4.3 and Lemma 4.4, we follow Kanai's program in [K1], [K2] and [K3] and in proving Lemma 4.5 and 4.6, we follow Cheng's program in [C], respectively.

LEMMA 4.1. *For each $p \in P$, there exists a constant $C > 0$ depending only on $\mu > 0$ and $s \geq 2$ such that*

$$C \int_{B_{4\mu}(p)} |h^s - h_\mu^s(p)| \leq \operatorname{vol} B_{4\mu}^{1-1/s}(p) h_\mu^{s-1}(p) \left(\int_{B_{4\mu}(p)} |\nabla h|^s \right)^{1/s}.$$

PROOF. Using the condition (5), we find a constant $C > 0$ depending only on μ such that

$$C \int_{B_{4\mu}(p)} |h^s - h_\mu^s(p)| \leq \int_{B_{4\mu}(p)} |\nabla h^s|.$$

Since $|\nabla h^s| = s \cdot h^{s-1} |\nabla h|$, we have the conclusion by the Hölder inequality and the definition of $h_\mu(p)$. $\square$

LEMMA 4.2. *There exists a constant $C>0$ depending only on $\mu>0$ and $s\geq 2$ such that*

$$C\sum_{p\in P}\operatorname{vol}B_\mu(p)\,|Dh_\mu|^s(p)\leq\int_M|\nabla h|^s.$$

PROOF. For $p'\in N(p)$, we have

$$\{h_\mu^{s-1}(p)+h_\mu^{s-1}(p')\}\left(\int_{B_{7\mu}(p)}|\nabla h|^s\right)^{1/s}$$
$$\geq h_\mu^{s-1}(p)\left(\int_{B_{4\mu}(p)}|\nabla h|^s\right)^{1/s}+h_\mu^{s-1}(p')\left(\int_{B_{4\mu}(p')}|\nabla h|^s\right)^{1/s}.$$

Using the Lemma 4.1 and the condition (4), we have

$$\{h_\mu^{s-1}(p)+h_\mu^{s-1}(p')\}\left(\int_{B_{7\mu}(p)}|\nabla h|^s\right)^{1/s}$$
$$\geq C\operatorname{vol}B_{4\mu}^{-1+1/s}(p)\int_{B_{4\mu}(p)}|h^s-h_\mu^s(p)|+C\operatorname{vol}B_{4\mu}^{-1+1/s}(p')\int_{B_{4\mu}(p')}|h^s-h_\mu^s(p')|$$
$$\geq C\operatorname{vol}B_{7\mu}^{-1+1/s}(p)\int_{B_{4\mu}(p)\cap B_{4\mu}(p')}|h_\mu^s(p)-h_\mu^s(p')|$$
$$\geq C\operatorname{vol}B_{7\mu}^{-1+1/s}(p)\operatorname{vol}B_\mu(p)|h_\mu(p)-h_\mu(p')|\{h_\mu^{s-1}(p)+h_\mu^{s-1}(p')\}$$
$$\geq C\operatorname{vol}B_\mu^{1/s}(p)|h_\mu(p)-h_\mu(p')|\{h_\mu^{s-1}(p)+h_\mu^{s-1}(p')\}$$

where C depends only on μ and s. Therefore we have

$$C\operatorname{vol}B_\mu^{1/s}(p)|h_\mu(p)-h_\mu(p')|\leq\left(\int_{B_{7\mu}(p)}|\nabla h|^s\right)^{1/s}$$

for all $p'\in N(p)$. This implies that

$$C\operatorname{vol}B_\mu(p)|Dh_\mu|^s(p)\leq\int_{B_{7\mu}(p)}|\nabla h|^s.$$

Since $\sup_{x\in M}\#\{p\in P: x\in B_{7\mu}(p)\}<\infty$, we have the conclusion. □

LEMMA 4.3. *There exists a constant $C>0$ depending only on μ, ν, τ and $s\geq 2$ such that*

$$\sum_{q\in Q}\operatorname{vol}B_\nu(q)|Dk|^s(q)\leq C\sum_{p\in P}\operatorname{vol}B_\mu(p)\,|Dh_\mu|^s(p).$$

PROOF. For $q'\in N(q)\subset Q$, we can choose a minimal path $\boldsymbol{p}=(p_0,p_1,\dots,p_l)$ and $p_0=\varphi^{-1}(q)$ where $p_l=\varphi^{-1}(q')$. Then we have

$$\begin{aligned}|k(q)-k(q')|&=|h_\mu(p_0)-h_\mu(p_l)|\\&\leq|h_\mu(p_0)-h_\mu(p_1)|+\cdots+|h_\mu(p_{l-1})-h_\mu(p_l)|\\&\leq\sum_{p\in N_{l-1}(\varphi^{-1}(q))}|Dh_\mu|(p)\,.\end{aligned}$$

By the uniformness of the net P, we can choose an integer L satisfying $l\leq L$. Thus we have

$$|Dk|^s(q)\leq C\sum_{p\in N_{L-1}(\varphi^{-1}(q))}|Dh_\mu|^s(p)$$

for each $q\in Q$. By the conditions (3) and (4), we have

$$\begin{aligned}\mathrm{vol}\,B_\nu(q)|Dk|^s(q)&\leq C\,\mathrm{vol}\,B_\nu(q)\sum_{p\in N_{L-1}(\varphi^{-1}(q))}|Dh_\mu|^s(p)\\&\leq C\sum_{p\in N_{L-1}(\varphi^{-1}(q))}\mathrm{vol}\,B_\mu(p)|Dh_\mu|^s(p)\,.\end{aligned}$$

By the uniformness of the net P, we have the conclusion. □

LEMMA 4.4. *There exists a constant $C>0$ depending only on μ, ν, τ and $s\geq 2$ such that*

$$\int_N|\nabla g|^s\leq C\sum_{q\in Q}\mathrm{vol}\,B_\nu(q)|Dk|^s(q)\,.$$

PROOF. In the weak sense, we have

$$\sum_{q'\in N(q)\cup\{q\}}\nabla\eta_{q'}(x)=0\qquad\text{for}\quad x\in B_\nu(q)\,,\quad\text{since}\quad\sum_{q'\in N(q)\cup\{q\}}\eta_{q'}(x)\equiv 1\,.$$

Since for each $x\in B_\nu(q)$,

$$\nabla g(x)=\sum_{q'\in N(q)\cup\{q\}}k(q')\nabla\eta_{q'}(x)=\sum_{q'\in N(q)}(k(q')-k(q))\nabla\eta_{q'}(x)\,,$$

we have

$$|\nabla g|(x)\leq C\sum_{q'\in N(q)}|k(q')-k(q)|\leq C|Dk|(q)\,.$$

Raising to the s-th power and integrating both sides on $B_\nu(q)$, we have

$$\int_{B_\nu(q)}|\nabla g|^s\leq C\,\mathrm{vol}\,B_\nu(q)|Dk|^s(q)\,.$$

By the uniformness, we have the conclusion. □

Therefore, Theorem 4.2 is proved.

We now solve the following Dirichlet problem, in the weak sense, on $B_R(o)$

$$\text{(4.1)} \qquad \begin{cases} \Delta u_R = -\Delta g & \text{on } B_R(o) \\ u_R = 0 & \text{on } \partial B_R(o) . \end{cases}$$

Define a functional E by

$$E(v) = \int_{B_R(o)} \frac{1}{2} |\nabla v|^2 + \nabla v \cdot \nabla g$$

for $v \in H_0^{1,2}(B_R(o))$. Since for some $s \geq 2$,

$$\begin{aligned} E(v) &\geq -\frac{1}{2} \int_{B_R(o)} |\nabla g|^2 \\ &\geq -\frac{1}{2} (\mathrm{vol}\, B_R(o))^{1-2/s} \left(\int_{B_R(o)} |\nabla g|^s \right)^{2/s} , \end{aligned}$$

we can take a minimizer of this functional $E(v)$. For this solution, we have the following result.

LEMMA 4.5. *Let N be a complete Riemannian manifold with the Dirichlet eigenvalue $\lambda_1(N) > 0$. Then for a solution u_R of the equation* (4.1), *we have*

$$\int_{B_R(o)} |u_R|^s \leq \left(\frac{s^2}{\lambda_1(N)} \right)^{s/2} \int_{B_R(o)} |\nabla g|^s$$

for any $s \geq 2$.

PROOF. Set $u = u_R$. Since $u|_{\partial B_R(o)} = 0$, we have

$$\int_{B_R(o)} \nabla((\mathrm{sgn}\, u)|u|^{s-1}) \cdot \nabla u = -\int_{B_R(o)} \nabla((\mathrm{sgn}\, u)|u|^{s-1}) \cdot \nabla g .$$

From this equation, we have

$$\begin{aligned} \int_{B_R(o)} |u|^{s-2} |\nabla |u||^2 &\leq \int_{B_R(o)} |u|^{s-2} |\nabla |u|| |\nabla g| \\ &\leq \frac{1}{2} \int_{B_R(o)} |u|^{s-2} |\nabla |u||^2 + \frac{1}{2} \int_{B_R(o)} |u|^{s-2} |\nabla g|^2 . \end{aligned}$$

Since $|\nabla |u|^{s/2}|^2 = (s^2/4)|u|^{s-2} |\nabla |u||^2$, we have

$$\frac{4}{s^2} \int_{B_R(o)} |\nabla |u|^{s/2}|^2 \leq \int_{B_R(o)} |u|^{s-2} |\nabla g|^2 .$$

Using the Hölder inequality and the hypothesis $\lambda_1(N) > 0$, we have

$$\int_{B_R(o)} |u|^s \leq \frac{s^2}{\lambda_1(N)} \left(\int_{B_R(o)} |u|^s \right)^{(s-2)/s} \left(\int_{B_R(o)} |\nabla g|^s \right)^{2/s} .$$

□

For any compact subset Ω in N, we have

$$\sup_{\Omega}|u_R+g|\leq\sup_{N}|g| \quad \text{and} \quad \sup_{N}|g|\leq\sup_{N}|h|.$$

By the standard Schauder estimates, we can choose a subsequence $\{u_{R_k}+g\}$ of $\{u_R+g\}$ converging uniformly to $\tilde{u}$ on any compact subset of N satisfying $\Delta\tilde{u}=0$ on N. Set $u=\tilde{u}-g$. Then by the standard Moser iteration, we have the following lemma.

LEMMA 4.6. *Let N be given in Lemma 4.5 with the Sobolev constant $S_1(N)>0$. Then $u(z)\to 0$ as $z\to\infty$.*

PROOF. By Lemma 4.5, we have $\int_N|u|^s<\infty$. Thus for any $\varepsilon>0$, there exists a sufficiently large $\tilde{R}>0$ such that $\int_{B_{R_0}(z)}|u|^s<\varepsilon$ if $d(o',z)\geq\tilde{R}$ for some fixed $R_0>0$. By the standard Moser iteration, for any $z\in N$ and $R_0>0$, there exist constants $\delta>0$ and $C>0$ such that

$$\sup_{B_{R_0/2}(z)}|u|\leq C\left(\frac{1+R_0^2}{R_0}\right)^{n/s}\left(\int_{B_{R_0}(z)}|u|^s\right)^{\delta/s},$$

where C depends on $\sup_N|\nabla g|$, $\sup_N|g|$ and $\int_N|\nabla g|^s$. Thus for any $\varepsilon>0$, there exists a sufficiently large $R>0$ such that $|u(z)|<\varepsilon$ if $d(o',z)\geq R$. □

In [K1] and [K2], Kanai proved that $\lambda_1>0$ and $S_1>0$ are preserved under the rough isometry. In [C-S], Coulhon and Saloff-Coste also proved these facts under the rough isometry with the conditions (1), (2), (3), (4) and (5). Since $\lambda_1(M)>0$ and $S_1(M)>0$, we have $\lambda_1(N)>0$ and $S_1(N)>0$.

PROOF OF THEOREM 4.1. We have only to show that

$$(g\circ\varphi(x)-f\circ\varphi(x))\to 0 \qquad \text{as} \quad x\to\infty.$$

Note that

$$\begin{aligned}|g\circ\varphi(x)-f\circ\varphi(x)|&=\left|\sum_{q\in Q}\eta_q(\varphi(x))\left(\frac{1}{\operatorname{vol}B_{4\mu}(\varphi^{-1}(q))}\int_{B_{4\mu}(\varphi^{-1}(q))}h^s\right)^{1/s}-f(\varphi(x))\right|\\&\leq\sum_{q\in Q}\eta_q(\varphi(x))\left(\frac{1}{\operatorname{vol}B_{4\mu}(\varphi^{-1}(q))}\int_{B_{4\mu}(\varphi^{-1}(q))}|h(y)-f(\varphi(x))|^s dy\right)^{1/s}.\end{aligned}$$

By the compactness of $M\cup M(\infty)$, for given $\varepsilon>0$, there exist $R>0$, $v_1,\dots,v_i\in M(\infty)$ and positive numbers $\delta_1,\dots,\delta_i$ such that $M(\infty)\subset\bigcup_{j=1}^{i}K(v_j,\delta_j,R)$ and

$$|f\circ\varphi(x)-h(v_j)|<\varepsilon, \quad |h(y)-h(v_j)|<\varepsilon$$

if $x,y\in K(v_j,2\delta_j,R)$ for each $j=1,2,\dots,i$. For any $y\in\bigcup_{q\in B_{3\nu/2}(\varphi(x))}B_{4\mu}(\varphi^{-1}(q))$,

$$\begin{aligned} d(x, y) &\leq d(x, \varphi^{-1}(q)) + d(\varphi^{-1}(q), y) \\ &\leq ad(\varphi(x), \varphi \circ \varphi^{-1}(q)) + b + 4\mu \\ &\leq ad(\varphi(x), q) + a\tau + b + 4\mu \\ &\leq \frac{3}{2} av + a\tau + b + 4\mu . \end{aligned}$$

Thus for sufficiently large $R > (3/2)av + a\tau + b + 4\mu$, we have $d(x, y) < R$. If $d(o, x) \geq 2R$, then for some $v_j \in M(\infty)$, $x \in K(v_j, \delta_j, R/2)$. Thus $y \in K(v_j, 2\delta_j, R)$. This implies $|f \circ \varphi(x) - h(y)| < 2\varepsilon$. Thus for any $\varepsilon > 0$, there exists $R > 0$ such that

$$|\tilde{u} \circ \varphi(x) - f \circ \varphi(x)| < 3\varepsilon \qquad \text{if} \quad d(o, x) \geq 2R ,$$

where $R \geq a\tilde{R} + b$ and $\tilde{R}$ is given in the proof of Lemma 4.6. □

As corollaries, we have some interesting new results on the usual asymptotic Dirichlet problem which is considered in [Ch], [A], [A-S] and [S].

COROLLARY 4.1. *Let M be a Cartan-Hadamard manifold. Then the usual asymptotic Dirichlet problem on M is solvable, provided that M is roughly isometric to another Cartan-Hadamard manifold with the sectional curvature pinched between two negative constants.*

PROOF. Let $\varphi : X \to M$ be a rough isometry such that X is a Cartan-Hadamard manifold with the sectional curvature pinched between two negative constants. Then Li and Wang's result implies that φ extends to a homeomorphism $\varphi : X(\infty) \to M(\infty)$. The rest of the proof is the same as that in the proof of Corollary 4.2. □

COROLLARY 4.2. *Let φ, M and N be as in Theorem* 4.1. *Suppose further that N is also a Cartan-Hadamard manifold such that $\varphi : M \to N$ extends to a map $\varphi : M \cup M(\infty) \to N \cup N(\infty)$ in such a way that it is continuous at every point of $M(\infty)$. Then the usual asymptotic Dirichlet problem on N is solvable.*

PROOF. Let f be a continuous function on $N \cup N(\infty)$. Since the rough isometry $\varphi : M \to N$ extends to a continuous map $\varphi : M(\infty) \to N(\infty)$, we have a continuous function $f \circ \varphi$ on $M(\infty)$. Let h be an extension of $f \circ \varphi$ such that $h|_{M(\infty)} = f \circ \varphi|_{M(\infty)}$ and it is radially constant outside some compact subset of M. Define a function g on N by

$$g(z) = \sum_{q \in Q} \eta_q(z) \left(\frac{1}{\operatorname{vol} B_{4\mu}(\varphi^{-1}(q))} \int_{B_{4\mu}(\varphi^{-1}(q))} h^s \right)^{1/s}$$

where $\eta_q(x)$ is a partition of unity as defined just above the statement of Theorem 4.2. By Theorem 4.2, we have $\int_N |\nabla g|^s < \infty$ for sufficiently large $s \geq 2$. Since $\lambda_1 > 0$ and $S_1 > 0$ are preserved under the rough isometry, by Lemma 4.6, we have a harmonic function $\tilde{u}$ on N such that $|\tilde{u}(z) - g(z)| \to 0$ as $z \to \infty$. We have only to show that $|g(z) - f(z)| \to 0$ as $z \to \infty$. Note that

$$
\begin{aligned}
|g(z)-f(z)| &= \left| \sum_{q\in Q} \eta_q(z) \left(\frac{1}{\operatorname{vol} B_{4\mu}(\varphi^{-1}(q))} \int_{B_{4\mu}(\varphi^{-1}(q))} h^s \right)^{1/s} - f(z) \right| \\
&\le \sum_{q\in Q} \eta_q(z) \left(\frac{1}{\operatorname{vol} B_{4\mu}(\varphi^{-1}(q))} \int_{B_{4\mu}(\varphi^{-1}(q))} |h(y)-f(z)|^s dy \right)^{1/s}.
\end{aligned}
$$

For any $y \in \bigcup_{q\in B_{3\nu/2}(z)} B_\mu(\varphi^{-1}(q))$, $d(z, \varphi(y)) \le C$ where C depends only on μ, ν, τ, a and b. Note that $|h(y)-f(z)| \le |h(y)-f(\varphi(y))| + |f(\varphi(y))-f(z)|$. Using the definition of h and the continuity of f, we have $|h(y)-f(z)| \to 0$ as $z \to \infty$. Thus we have $|g(z)-f(z)| \to 0$ as $z \to \infty$. □

Finally, Proposition 3.1 and Theorem 4.1 imply the invariance property of our new asymptotic Dirichlet problem.

COROLLARY 4.3. *Let φ, M and N be as in Theorem 4.1. Let $\psi: M \to N$ be another rough isometry such that $\varphi^{-1}\circ\psi$ extends to a map $\varphi^{-1}\circ\psi: M \cup M(\infty) \to M \cup M(\infty)$ in such a way that it is continuous at every point of $M(\infty)$. Then for any $f \in \mathscr{F}_\varphi$, we have a solution $\tilde{u} \in C^\infty(N)$ such that $\Delta\tilde{u} = 0$ on N and $(\tilde{u}-f)(\psi(x)) \to 0$ as $x \to \infty$.*

REMARK. Let id be the identity map of (M, ds^2) onto $(M, d\tilde{s}^2)$. Then for any continuous function f on $(M, d\tilde{s}^2)$, it is easy to check that $f \in \mathscr{F}_{\mathrm{id}}$. By the consequence of our result, we have the results of Schoen and Yau and of Cheng.

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