Tôhoku Math. J.
50 (1998), 597–611

A REMARK ON THE ASYMPTOTIC PROPERTIES OF EIGENVALUES AND THE LATTICE POINT PROBLEM

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(Received May 6, 1997)

Abstract. The asymptotic distribution of eigenvalues of elliptic self-adjoint operators on the flat torus is discussed. A relation between a geometrical property of the operators and the error terms in the distribution formulas is given in the case when the operators have constant coefficients. As a corollary, the error terms can be determined only by the order of the operators and the dimension of the torus. This result also gives an information on the number of lattice points inside convex or nonconvex bodies in $\boldsymbol{R}^n$.

1. Introduction. Let M be an n-dimensional compact Riemannian manifold without boundary and P a partial differential operator on M of order m. We assume that P is self-adjoint and elliptic, namely the principal symbol $p_m(x, \xi) \in C^\infty(T^*M \setminus 0)$ of P is strictly positive. The famous formula of Weyl says that the number $N(\lambda)$ of eigenvalues of P which are not greater than λ behaves like

$$N(\lambda)=c\lambda^{n/m}+O(\lambda^{(n-1)/m})\,; \qquad c=(2\pi)^{-n}\iint_{p_m(x,\xi)\le 1} dxd\xi\,,$$

as $\lambda \to +\infty$. The order $(n-1)/m$ of the error term cannot be improved if we take the sphere S^n as M. Indeed, the spectrum of the standard Laplacian $-\Delta$ on the sphere is well-known and the case $P=(-\Delta)^{m/2}$ yields a counterexample. Refer to Hörmander [4] for these matters.

On the other hand, we can obtain a better estimate $o(\lambda^{(n-1)/m})$ for the error term if M and P satisfy extra conditions. For example, this is true if the closed orbits of the Hamilton flow H_{p_m} generated by the principal symbol form a set of measure zero in $T^*M \setminus 0$ (Duistermaat-Guillemin [2]). If P is the Laplace-Beltrami operator, then H_{p_m} is the geodesic flow, and the torus $M=\boldsymbol{T}^n$ satisfies this condition if $n \ge 2$ while $M=S^n$ does not. We remark that $\boldsymbol{T}^1 \cong S^1$.

Then our next question is what the exact order of the error term is for a given Riemannian manifold M and an operator P which satisfies the global condition above. In other words, our objective is the numbers $d>0$ for $O(\lambda^{(n-1)/m-d})$ to be true for the error term and their maximum $d_{\max}$. Our results in this paper answer this

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1991 *Mathematics Subject Classification*. Primary 35P20; Secondary 52C07, 11H16.

The author is supported by the Grant-In-Aid of the Inamori Foundation. This work is partly supported by the Grant-In-Aid for the sciences of the Sumitomo Foundation.

question to some extent for the flat torus $M=\boldsymbol{T}^n=\boldsymbol{R}^n/\boldsymbol{Z}^n$ with $n\geq 2$ and P with constant coefficients, which satisfy the global condition. Our general answer is $\alpha/\{m(n-\alpha)\}\leq d_{\max}\leq \alpha/m$, where α is the index defined by a geometrical property of P (Theorems 1 and 2). As a corollary, we have $d_{\max}\geq d_{m,n}=(m^2n-m)^{-1}$, which we can improve if P has a kind of convexity (Theorem 3).

In the special case when P is homogeneous, that is, P has only the principal part and no lower terms, our answers can be translated into those for the problem of determining the asymptotic distribution of the number of lattice points inside the region $R\Omega=\{R\xi;\ \xi\in\Omega\}$ as $R=(2\pi)^{-1}\lambda^{1/m}\to+\infty$. Here $\Omega=\{\xi;\ p(\xi)\leq 1\}$, and $p(\xi)$ is the symbol of $P=P(D)$. Especially, in the case when P is the standard Laplacian on the 2-dimensional flat torus $\boldsymbol{T}^2$, this is known as Gauss's circle problem, and better results than our answer $d_{\max}\geq d_{2,2}=1/6$ have been shown from number theoretical aspects (Remark 3). However, we would like to emphasize here that we can treat more general n and P and the order $d_{m,n}=(m^2n-m)^{-1}$ can be determined only by the dimension of the manifold and the order of the operator.

Finally, the author expresses his gratitude to Professor Takao Watanabe for valuable conversations.

2. Main results. In the rest of this paper, we always assume that $n\geq 2$ and $P=P(D)$ is an elliptic self-adjoint partial differential operator on $\boldsymbol{T}^n$ of order m with constant coefficients. Then the symbol of P can be expressed as

$$p(\xi)=p_m(\xi)+p_{m-1}(\xi)+\cdots+p_0(\xi)\,,$$

where $p_j(\xi)$ is a real polynomial of $\xi=(\xi_1,\xi_2,\dots,\xi_n)$ of order j $(j=0,1,\dots,m)$. We remark that m must be even and $p_m(\xi)$ is identically positive or negative for $\xi\neq 0$. We assume here the positivity, otherwise take $-P$ as P. Now, we set

$$\Omega=\{\xi\in\boldsymbol{R}^n;\ p_m(\xi)\leq 1\}\,.$$

We shall call its boundary $\partial\Omega$ the *cosphere* of P, which is a compact real analytic hypersurface in $\boldsymbol{R}^n$, that is, a submanifold of codimension 1. Before stating our main results, we shall define the indices for hypersurfaces which were introduced in Sugimoto [12] and [13] for another purpose.

DEFINITION 1. Let Σ be a hypersurface in $\boldsymbol{R}^n$, p a point on Σ, and H a 2-dimensional plane which contains the normal line of Σ at p, that is,

$$H=\{p+s\vec{\mu}+t\vec{\nu}\in\boldsymbol{R}^n;\ s,t\in\boldsymbol{R}\}\,.$$

Here $\vec{\mu}$ is a tangent vector of Σ at p and $\vec{\nu}$ is a normal. Let T be the tangent hyperplane of Σ at p. We define the index $\gamma(\Sigma;p,H)$ to be the order of contact of the curve $\Sigma\cap H$ to the line $T\cap H$ at p, that is, the smallest number $l\in\boldsymbol{N}$ such that $\psi^{(l)}(0)\neq 0$ when we express the curve $\Sigma\cap H$ as $\{p+s\vec{\mu}+\psi(s)\vec{\nu}\in\boldsymbol{R}^n;\ |s|<\varepsilon\}$ near p with small $\varepsilon>0$. Furthermore, we define the indices $\gamma(\Sigma)$ and $\gamma_0(\Sigma)$ by

$$\gamma(\Sigma)=\sup_p \sup_H \gamma(\Sigma;\, p, H)\,, \qquad \gamma_0(\Sigma)=\sup_p \inf_H \gamma(\Sigma;\, p, H)\,.$$

REMARK 1. We have $2\leq\gamma_0(\Sigma)\leq\gamma(\Sigma)$ by definition. Equality $\gamma_0(\Sigma)=\gamma(\Sigma)$ holds when $n=2$.

Hereafter, Σ always denotes the cosphere of P, that is,

$$\Sigma=\{\xi\in \boldsymbol{R}^n;\, p_m(\xi)=1\}\,.$$

Then we have the following inequality:

PROPOSITION 1 ([13; Proposition 2]). $2\leq\gamma_0(\Sigma)\leq\gamma(\Sigma)\leq m\,.$

REMARK 2. In the case when $m=2$ and $P=-\Delta$ is the Laplacian for instance, we have $\gamma_0(\Sigma)=\gamma(\Sigma)=2$. Even in the higher order case $m\geq 3$, this is true when the Gaussian curvature of Σ never vanishes.

We shall state our main theorems. In the following, $N(\lambda)$ denotes the number of eigenvalues of P which are not greater than λ (counted with multiplicity), and $|\Omega|$ the Lebesgue measure of Ω.

THEOREM 1. *Let $d=\alpha/\{m(n-\alpha)\}$. Then we have the asymptotic distribution*

$$N(\lambda)=(2\pi)^{-n}|\Omega|\lambda^{n/m}+O(\lambda^{(n-1)/m-d}) \tag{1}$$

with $\alpha=1/\gamma_0(\Sigma)$, hence with $\alpha=1/m$.

In the case when P is homogeneous, that is, the case $p(\xi)=p_m(\xi)$, the asymptotic distribution (1) for the number of eigenvalues can be translated into the behavior of the number $N'(R)$ of lattice points inside the region $R\Omega=\{R\xi;\, \xi\in\Omega\}$ as $R\to+\infty$. In fact, we have

$$N'(R)=|\Omega|R^n+O(R^{n-1-md})\,, \tag{1'}$$

since the number λ is an eigenvalue of P if and only if the Diophantus equation $p_m(2\pi\xi)=\lambda$ has a solution $\xi\in\boldsymbol{Z}^n$. (See Lemma 1 in Section 3.)

REMARK 3. In the case when $n=2$ and $P=-\Delta$, Theorem 1 is an answer to Gauss's circle problem and claims $N'(R)=\pi R^2+O(R^{2/3})$, where $N'(R)$ denotes the number of lattice points inside the disk $\{\xi;\, |\xi|\leq R\}$. This corresponds to classical results of Sierpinski [10]. There has been a series of improvements to this result, replacing $O(R^{2/3})$ by $O(R^{\kappa})$, where $\kappa<2/3$. For example, Chen [1] proved it with $\kappa=24/37+\varepsilon$ for arbitrary $\varepsilon>0$, which had been the best result until 1987. Iwaniec-Mozzochi [5] obtained a better result $\kappa=7/11+\varepsilon$. It has also been shown that $\kappa=1/2$ is not possible (Landau [7]). The final result $\kappa=1/2+\varepsilon$ is conjectured but remains unsolved.

EXAMPLE 1. Suppose $n\geq 3$ and $l\in\boldsymbol{N}$. Let

$$p(\xi)=p_{4l}(\xi)=(\xi_1^2+\cdots+\xi_{n-1}^2-\xi_n^2)^{2l}+\xi_n^{4l}.$$

Then $\gamma(\Sigma)=4l$ and $\gamma_0(\Sigma)=2l$, hence we have the asymptotic distribution (1) with $d=\alpha/\{m(n-\alpha)\}$, where $\alpha=(2l)^{-1}$. For the proof of this, refer to Sugimoto [13; Example 1].

As suggested in Remark 3, Theorem 1 might also be true for greater d's. However, by computing the order of the error term for Example 1, we have an upper bound for them.

THEOREM 2. *Suppose $n\geq 3$, $l\in \boldsymbol{N}$, and $l>1/2+1/(n-2)$. Let $m=4l$. Suppose that the operator P defined by the symbol p in Example* 1 *admits the asymptotic distribution* (1). *Then $d\leq\alpha/m$, where $\alpha=1/\gamma_0(\Sigma)$.*

REMARK 4. If we formally take $n=m=2$, the upper bound $\alpha/m=1/(m\gamma_0(\Sigma))$ for d's in (1) equals 1/4, which corresponds to the fact that the number of lattice points inside the disk $\{\xi;|\xi|\leq R\}$ cannot behaves like $\pi R^2+O(R^\kappa)$ with $\kappa<1/2$ (Landau [7]). In this sense, Theorem 2 is an extension of the result of [7] to the general lattice point problem.

The cosphere Σ in Example 1 is not convex. But Theorem 1 can be improved if P has some convexity property. We set

$$\Sigma_\varepsilon=\{\xi\in\boldsymbol{R}^n; p_m(\xi)+\varepsilon_1 p_{m-1}(\xi)+\varepsilon_2 p_{m-2}(\xi)+\cdots+\varepsilon_m p_0(\xi)=1\},$$

where $\varepsilon=(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m)$. We always assume that $|\varepsilon|$ is sufficiently small so that Σ_ε is a compact real analytic hypersurface.

DEFINITION 2. The cosphere Σ of P is called stably convex if there is $\delta>0$ such that Σ_ε is convex for $|\varepsilon|\leq\delta$.

REMARK 5. The stable convexity implies the convexity. If P is homogeneous, that is, $p_{m-1}(\xi)=\cdots=p_0(\xi)=0$, the convexity is equivalent to the stable convexity. If the Gaussian curvature of the cosphere never vanishes, then it is stably convex. Indeed, the curvature condition implies the convexity (Kobayashi-Nomizu [6; Chap. 7]) and this condition is stable under the lower term perturbation. Accordingly, the cospheres of the elliptic operators of order 2 are always stably convex because they are ellipsoid.

THEOREM 3. *Let $d=\alpha/\{m(n-\alpha)\}$. Suppose that the cosphere of P is stably convex. Then we have the asymptotic distribution* (1) *with $\alpha=(n-1)/\gamma(\Sigma)$, hence with $\alpha=(n-1)/m$.*

REMARK 6. In the case when the Gaussian curvature of the cosphere Σ never vanishes (then $\gamma(\Sigma)=2$), Theorem 3 was essentially proved by Hlawka [3].

EXAMPLE 2. Suppose $n\geq 3$ and $l\in\boldsymbol{N}$. Let

$$p(\xi)=p_{2l}(\xi)=\xi_1^{2l}+\xi_2^{2l}+\cdots+\xi_n^{2l}.$$

Then $\gamma(\Sigma)=\gamma_0(\Sigma)=2l$, hence we have the asymptotic distribution (1) with $d=\alpha/\{m(n-\alpha)\}$, where $\alpha=(n-1)/2l$.

REMARK 7. In the special case of $n=2$, the result in Example 2 is not the best one. In fact, for $l=1$, which corresponds to Gauss's circle problem, better estimates for the error term have been obtained by many authors. (See Remark 3.) Furthermore, for $l\geq 2$, the exact order of the error term is given by Müller-Nowak [8; Corollary 3], which shows that (1) holds for the best possible order $d=\alpha/m$, where $n=2$, $m=2l$, and $\alpha=1/(2l)$.

3. Proofs of Theorems 1 and 3. We shall show Theorems 1 and 3 by proving the following sequence of lemmas. We remark that the capital "C" (with some suffix) in estimates always denotes a positive constant (depending on the suffix) which may be different in each occasion.

First of all, we notice that each $f\in L^2(\boldsymbol{T}^n)$ has the Fourier series expansion $f(x)=\sum_{k\in\boldsymbol{Z}^n}c_k e^{2\pi ik\cdot x}$, therefore $Pf(x)=\sum_{k\in\boldsymbol{Z}^n}c_k p(2\pi k)e^{2\pi ik\cdot x}$. Hence we have

LEMMA 1. *The number λ is an eigenvalue of $P=P(D)$ if and only if the Diophantus equation $p(2\pi\xi)=\lambda$ has a solution $\xi\in\boldsymbol{Z}^n$.*

Now, for $\varepsilon=(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$, we shall denote by Ω_ε the closed region surrounded by Σ_ε, that is,

$$\Omega_\varepsilon=\{\xi\in\boldsymbol{R}^n;\, p_m(\xi)+\varepsilon_1 p_{m-1}(\xi)+\varepsilon_2 p_{m-2}(\xi)+\cdots+\varepsilon_m p_0(\xi)\leq 1\}\,.$$

In particular, we have

$$\Omega_{\varepsilon(\lambda)}=\{\xi\in\boldsymbol{R}^n;\, p(R(\lambda)\xi)\leq\lambda\}\,,$$

where

$$R(\lambda)=\lambda^{1/m}\,,\qquad \varepsilon(\lambda)=(R(\lambda)^{-1}, R(\lambda)^{-2}, \dots, R(\lambda)^{-m})\,.$$

We may assume that λ is large enough so that $|\varepsilon(\lambda)|$ is sufficiently small. Let χ_ε be the characteristic function of Ω_ε. From Lemma 1, we easily obtain

LEMMA 2. $N(\lambda)=\sum_{k\in\boldsymbol{Z}^n}\chi_{\varepsilon(\lambda)}(2\pi R(\lambda)^{-1}k)$.

Let us fix a smooth positive function $\psi(x)$ which is supported in a sufficiently small ball $\{x;\,|x|\leq a\}$ and satisfies $\int\psi(x)dx=1$. We define, for $R, \tau>0$,

$$N_\varepsilon(R,\tau)=\sum_{k\in\boldsymbol{Z}^n}[\chi_\varepsilon(2\pi R^{-1}\cdot)*\tau^{-n}\psi(\tau^{-1}\cdot)](k)\,.$$

By an argument using Friedrichs' mollifier, we have easily

LEMMA 3. *For $\tau>0$, $N_{\varepsilon(\lambda)}(R(\lambda)-\tau, \tau)\leq N(\lambda)\leq N_{\varepsilon(\lambda)}(R(\lambda)+\tau, \tau)$.*

On the other hand, direct application of Poisson's summation formula to $N_\varepsilon(R,\tau)$

yields

LEMMA 4. $N_\varepsilon(R,\tau)=(2\pi)^{-n}|\Omega_\varepsilon|R^n+\sum_{k\in\boldsymbol{Z},k\neq 0}(2\pi)^{-n}R^n\hat{\chi}_\varepsilon(Rk)\hat{\psi}(2\pi\tau k)$.

In order to use Lemma 4 with $\varepsilon=\varepsilon(\lambda)$, we shall estimate the difference between $|\Omega_{\varepsilon(\lambda)}|$ and $|\Omega|$. By using Heaviside function $Y(t)$, we express them in the form of oscillatory integrals as

$$\begin{aligned}|\Omega_{\varepsilon(\lambda)}|&=\int\chi_{\varepsilon(\lambda)}(\xi)d\xi\\&=\int Y(1-\lambda^{-1}p(R(\lambda)\xi))d\xi\\&=\frac{1}{2\pi}\iint e^{it(1-p_m(\xi)-r_\lambda(\xi))}\hat{Y}(t)dtd\xi\,,\end{aligned}$$

where

$$r_\lambda(\xi)=R(\lambda)^{-1}p_{m-1}(\xi)+R(\lambda)^{-2}p_{m-2}(\xi)+\cdots+R(\lambda)^{-m}p_0(\xi)\,.$$

Similarly, we have

$$|\Omega|=\frac{1}{2\pi}\iint e^{it(1-p_m(\xi))}\hat{Y}(t)dtd\xi\,.$$

Then, by Taylor's formula,

$$\begin{aligned}&|\Omega_{\varepsilon(\lambda)}|-|\Omega|\\&=\frac{1}{2\pi}\iint e^{it(1-p_m(\xi))}\left(-itr_\lambda(\xi)+(itr_\lambda(\xi))^2\int_0^1(1-\theta)e^{-it\theta r_\lambda(\xi)}d\theta\right)\hat{Y}(t)dtd\xi\\&=-\int\delta(1-p_m(\xi))r_\lambda(\xi)d\xi+\int_0^1(1-\theta)\left\{\int\delta'(1-p_m(\xi)-\theta r_\lambda(\xi))(r_\lambda(\xi))^2d\xi\right\}d\theta\,,\end{aligned}$$

where $\delta(t)$ is the formal expression of Dirac's delta function. The first term on the right hand side is $O(R(\lambda)^{-2})$, since

$$\begin{aligned}&\int\delta(1-p_m(\xi))r_\lambda(\xi)d\xi=\int_\Sigma r_\lambda(\xi)d\Sigma\\&\qquad=R(\lambda)^{-2}\int_\Sigma(p_{m-2}(\xi)+R(\lambda)^{-1}p_{m-3}(\xi)+\cdots+R(\lambda)^{-(m-2)}p_0(\xi))d\Sigma\,.\end{aligned}$$

Here we have used the fact that m is even, which implies $\int_\Sigma p_{m-1}(\xi)d\Sigma=0$. Similarly, the second term is $O(R(\lambda)^{-2})$ as well, since the integrand with respect to θ is essentially an integral of derivatives of $r_\lambda(\xi)^2$ over the hypersurface $\Sigma_{\theta\varepsilon(\lambda)}$. Thus we have obtained

LEMMA 5. $|\Omega_{\varepsilon(\lambda)}|=|\Omega|+O(R(\lambda)^{-2})$.

In view of Lemma 4, an estimate for the Fourier transform of the characteristic function $\hat{\chi}_\varepsilon(\xi)$ is needed for estimating the error term. In fact, the following is true.

LEMMA 6. *Let $\alpha \leq n/2$. Suppose*

$$|\hat{\chi}_\varepsilon(\xi)| \leq C(1+|\xi|)^{-(1+\alpha)}, \tag{2}$$

where C is independent of ξ and small ε. Then we have (1).

To prove Lemma 6, we first note that $\hat{\psi}$ is rapidly decreasing. With this fact and the estimate (2), the summation part of the equality in Lemma 4 is estimated as

$$\begin{aligned}
&\left|\sum_{k\in\boldsymbol{Z},k\neq 0}(2\pi)^{-n}R^n\hat{\chi}_\varepsilon(Rk)\hat{\psi}(2\pi\tau k)\right| \\
&\quad \leq CR^n\sum_{k\in\boldsymbol{Z},k\neq 0}(1+|Rk|)^{-(1+\alpha)}(1+|\tau k|)^{-K} \\
&\quad \leq CR^n\int_{|\xi|\geq 1}(1+|R\xi|)^{-(1+\alpha)}(1+|\tau\xi|)^{-K}d\xi \\
&\quad \leq CR^n\int_{1\leq|\xi|\leq 1/\tau}|R\xi|^{-(1+\alpha)}d\xi+CR^n\int_{1/\tau\leq|\xi|}|R\xi|^{-(1+\alpha)}|\tau\xi|^{-K}d\xi \\
&\quad \leq C(R/\tau)^{n-(1+\alpha)},
\end{aligned}$$

where $K>n-(1+\alpha)$ and C is independent of ε, R, and τ. Here we have used $n\geq 1+\alpha$. Applying this estimate together with Lemma 5 to Lemma 4, we have

$$\begin{aligned}
&N_{\varepsilon(\lambda)}(R(\lambda)\pm\tau,\tau) \\
&\quad =(2\pi)^{-n}(|\Omega|+O(R(\lambda)^{-2}))(R(\lambda)\pm\tau)^n+O(((R(\lambda)\pm\tau)/\tau)^{n-(1+\alpha)}) \\
&\quad =(2\pi)^{-n}|\Omega|R(\lambda)^n+O(\tau R(\lambda)^{n-1}+(R(\lambda)/\tau)^{n-(1+\alpha)})+O(R(\lambda)^{n-2}) \\
&\quad =(2\pi)^{-n}|\Omega|R(\lambda)^n+O(R(\lambda)^{n-1-\alpha/(n-\alpha)})
\end{aligned}$$

if we take $\tau=R(\lambda)^{-\alpha/(n-\alpha)}$. Here we have used the binomial expansion $(R\pm\tau)^n=R^n+O(\tau R^{n-1})$ and $n-1-\alpha/(n-\alpha)\geq n-2$. Then we have (1) by Lemma 3, which completes the proof of Lemma 6.

Thus all we have to show is the estimate (2). Let Γ be a conic neighborhood of $x=(0,\ldots,0,1)$ and $\varphi(x)$ a smooth function which is positive, homogeneous of order 0 for large $|\xi|$, and is supported in Γ. It suffices to show the same estimate for $\widehat{\chi_\varepsilon\varphi}(\xi)$ instead of $\hat{\chi}_\varepsilon(\xi)$, and we may take Γ sufficiently small if necessary. We shall express

$$\Sigma_\varepsilon\cap\Gamma=\{(y,h_\varepsilon(y));\ y=(y_1,y_2,\ldots,y_{n-1})\in U\},$$

where $U\subset\boldsymbol{R}^{n-1}$ is a neighborhood of the origin and $h_\varepsilon(y)\in C^\omega(U)$. We remark that h_ε is real analytic with respect to ε as well. Then we have, by the change of variables $x=(ty, th_\varepsilon(y))$ and integration by parts,

$$\begin{aligned}
\widehat{\chi_\varepsilon\varphi}(\xi)&=\int_{\Omega_\varepsilon} e^{-ix\cdot\xi}\varphi(x)dx \\
&=\int_0^1\int_U e^{-it\omega_n|\xi|(y\cdot\eta+h_\varepsilon(y))}g_\varepsilon(t, y)dtdy \\
&=\frac{i}{\omega_n|\xi|}\int_U e^{-i\omega_n|\xi|(y\cdot\eta+h_\varepsilon(y))}\frac{g_\varepsilon(1, y)}{y\cdot\eta+h_\varepsilon(y)}dy \\
&\quad-\frac{i}{\omega_n|\xi|}\int_0^1\left\{\int_U e^{-i\omega_n t|\xi|(y\cdot\eta+h_\varepsilon(y))}\frac{(\partial g_\varepsilon/\partial t)(t, y)}{y\cdot\eta+h_\varepsilon(y)}dy\right\}dt\,,
\end{aligned}$$

where $g_\varepsilon(t, y)=\varphi(ty, th_\varepsilon(y))t^{n-1}|h_\varepsilon(y)-y\cdot h_\varepsilon'(y)|$, $\omega=\xi/|\xi|=(\omega', \omega_n)$, $\omega'=(\omega_1, \omega_2, \dots, \omega_{n-1})$, and $\eta=\omega'/\omega_n$. We remark that $g_\varepsilon(t, y)$ vanishes identically for small t. We may assume here that ω_n is away from 0, since integration by parts argument yields a better estimate than we need in the direction $\omega=(\omega', 0)$. By all of this, the estimate (2) is reduced to that for the oscillatory integral of the type

$$I_\varepsilon(t;\eta)=\int_U e^{it(y\cdot\eta+h_\varepsilon(y))}g(y)dy\,;\qquad g\in C_0^\infty(U)\,.$$

That is, we have

LEMMA 7. *Suppose, for some $K\in\boldsymbol{N}$,*

$$|I_\varepsilon(t;\eta)|\le C_g|t|^{-\alpha}\,, \tag{3}$$

where $C_g=C\sum_{|\alpha|\le K}\|\partial^\alpha g/\partial y^\alpha\|_{L^\infty(U)}$ and $C>0$ is independent of t, η, and small ε. Then we have the estimate (2), *hence the asymptotic distribution* (1).

In order to obtain (3), we shall use the following *scaling principle* for oscillatory integrals:

LEMMA 8. *Let $f(t)\in C^\infty(\boldsymbol{R})$ be real-valued and let $\zeta(t)\in C_0^\infty(\boldsymbol{R})$. Suppose, for $\nu\ge 2$ and $\mu>0$,*

$$|f^{(\nu)}(t)|\ge\mu\qquad on\quad \operatorname{supp}\zeta\,.$$

Then we have

$$\left|\int e^{itf(t)}\zeta(t)dt\right|\le C(\|\zeta\|_{L^\infty}+\|\zeta'\|_{L^1})|t|^{-1/\nu}\,,$$

where the constant $C>0$ depends only on ν and μ.

For the proof of this lemma, refer to Stein [11, Chapter VIII, 1.2]. By an appropriate change of coordinates and taking U sufficiently small if necessary, we may assume $|(\partial^{\nu}h_{\varepsilon}/\partial y_1^{\nu})(y)|\geq\mu>0$ for $y\in U$ and small ε, where $\nu=\gamma_0(\Sigma)$. Hence, from Lemma 8, we obtain

$$\left|\int e^{it(y\cdot\eta+h_{\varepsilon}(y))}g(y)dy_1\right|\leq C\left(\sum_{|\alpha|\leq 1}\left\|\frac{\partial^{\alpha}g}{\partial y^{\alpha}}\right\|_{L^{\infty}(U)}\right)|t|^{-1/\gamma_0(\Sigma)}$$

for all $y'=(y_2,\ldots,y_{n-1})$, and hence

$$\begin{aligned}|I_{\varepsilon}(t;\eta)|&\leq\int\left|\int e^{it(y\cdot\eta+h_{\varepsilon}(y))}g(y)dy_1\right|dy'\\&\leq C_g|t|^{-1/\gamma_0(\Sigma)}.\end{aligned}$$

We have thus proved the following lemma which implies Theorem 1 by Lemma 7 and Proposition 1.

LEMMA 9. *The estimate* (3) *is true for* $\alpha=1/\gamma_0(\Sigma)$.

On the other hand, when Σ is stably convex, the map h_{ε}': $U\to -h_{\varepsilon}'(U)\subset\boldsymbol{R}^{n-1}$ is a homeomorphism because of the compactness and the real analyticity of Σ_{ε}. Then we can define $z_{\varepsilon}=z_{\varepsilon}(\eta)$ by the relation $\eta+h_{\varepsilon}'(z_{\varepsilon})=0$, otherwise I_{ε} has a better estimate than we need by integration by parts argument again. We set

$$\tilde{I}_{\varepsilon}(t;z)=\int_U e^{itE_{\varepsilon}(y;z)}g(y)dy\,;\qquad E(y;z)=h_{\varepsilon}(y)-h_{\varepsilon}(z)-h_{\varepsilon}'(z)\cdot(y-z)\,.$$

Now, we shall estimate $\tilde{I}_{\varepsilon}$ instead of I_{ε}, since $|I_{\varepsilon}(t;\eta)|=|\tilde{I}_{\varepsilon}(t;z_{\varepsilon})|$. For this purpose, we rewrite it with the polar coordinates as

$$\tilde{I}_{\varepsilon}(t;z)=\int_{S^{n-2}}G_{\varepsilon}(t;z,\omega)d\omega\,;\qquad G_{\varepsilon}(t;z,\omega)=\int_0^{\infty}e^{itF_{\varepsilon}(\rho;z,\omega)}\beta(\rho;z,\omega)d\rho\,,$$

where

$$F_{\varepsilon}(\rho;z,\omega)=h_{\varepsilon}(\rho\omega+z)-h_{\varepsilon}(z)-\rho h_{\varepsilon}'(z)\cdot\omega\,,\qquad \beta(\rho;z,\omega)=g(\rho\omega+z)\rho^{n-2}\,.$$

For the sake of simplicity, we shall often omit parameters z and ω. We split the function $G_{\varepsilon}(t)$ into the following two parts:

$$G_{\varepsilon}^1(t)=\int_0^{\infty}e^{itF_{\varepsilon}(\rho)}\beta_1(\rho,t)d\rho:\qquad \beta_1(\rho,t)=\beta(\rho)\Psi(|t|^{1/\gamma(\Sigma)}\rho)\,,$$

$$G_{\varepsilon}^2(t)=\int_0^{\infty}e^{itF_{\varepsilon}(\rho)}\beta_2(\rho,t)d\rho\,;\qquad \beta_2(\rho,t)=\beta(\rho)(1-\Psi)(|t|^{1/\gamma(\Sigma)}\rho)\,,$$

where the function $\Psi(\rho)\in C^{\infty}(\boldsymbol{R})$ equals to 1 for large ρ and vanishes near the origin. The estimate for the part $G_{\varepsilon}^2(t)$ is easy. In fact, we have

$$\begin{aligned}|G_\varepsilon^2(t)| &\leq \int_0^\infty |\beta_2(\rho, t)|d\rho \\ &\leq C_g \int_0^\infty |\rho^{n-2}(1-\Psi)(|t|^{1/\gamma(\Sigma)}\rho)|d\rho \\ &\leq C_g |t|^{-(n-1)/\gamma(\Sigma)} .\end{aligned}$$

On the other hand, integration by parts yields

$$G_\varepsilon^1(t) = \int_0^\infty e^{itF_\varepsilon(\rho)}(L^*)^l \beta_1(\rho, t)d\rho$$

for $l=0, 1, 2, \dots$. Here

$$L = \frac{1}{itF_\varepsilon'(\rho)} \frac{\partial}{\partial\rho}$$

and L^* is the transpose of L. By induction, we can easily have

$$(L^*)^l = \left(\frac{i}{t}\right)^l \sum C_{r,q,s_1,\dots,s_q} \frac{F_\varepsilon^{(s_1)} \cdots F_\varepsilon^{(s_q)}}{(F_\varepsilon')^{l+q}}(\rho) \frac{\partial^r}{\partial\rho^r} ,$$

where the summation $\sum$ is a finite sum over $r, q, s_1, \dots, s_q \geq 0$ which satisfy $r+s_1+\cdots+s_q=l+q$. The derivatives of F_ε have the following estimate.

LEMMA 10. *Suppose Σ is stably convex. Then there exist constants C, C_ν, $a>0$ such that the estimates*

$$|F_\varepsilon'(\rho)| \geq C\rho^{\gamma(\Sigma)-1} ,$$

$$|F_\varepsilon^{(\nu)}(\rho)| \leq C_\nu \rho^{1-\nu} |F_\varepsilon'(\rho)|$$

hold for $0 \leq \rho$, $|z|$, $|\varepsilon| \leq a$, $\omega \in S^{n-2}$, *and* $\nu = 0, 1, 2, \dots$.

If we use Lemma 10 and the estimate

$$\left|\frac{\partial^r \beta_1}{\partial\rho^r}(\rho, t)\right| \leq C_g \rho^{n-2-r} ,$$

we have, for a large number l and a constant $b>0$,

$$\begin{aligned}|G_\varepsilon^1(t)| &\leq \frac{C}{|t|^l} \sum \int_0^\infty \left| \frac{F_\varepsilon^{(s_1)} \cdots F_\varepsilon^{(s_q)}}{(F_\varepsilon')^{l+q}}(\rho) \frac{\partial^r \beta_1}{\partial\rho^r}(\rho, t)\right| d\rho \\ &\leq \frac{C_g}{|t|^l} \int_{b|t|^{-1/\gamma(\Sigma)}}^\infty \rho^{n-2-l\gamma(\Sigma)} d\rho \\ &\leq C_g |t|^{-(n-1)/\gamma(\Sigma)} .\end{aligned}$$

We have thus proved

LEMMA 11. *If Σ is stably convex, the estimate* (3) *is true for* $\alpha=(n-1)/\gamma(\Sigma)$.

From Lemma 7, Lemma 11 and Proposition 1, we obtain Theorem 3 if we prove Lemma 10. Although the proof of it is carried out essentially by the same argument as in Randol [9] and Sugimoto [12], we shall provide it for the sake of completeness.

First we note that the function $F_\varepsilon(\rho)$ is real analytic for fixed z, ω, and ε. For the expansion $F_\varepsilon(\rho)=\sum_{j=2}^{\infty} b_j(z, \omega, \varepsilon)\rho^j$, we set

$$\pi(\rho)=\sum_{j=2}^{\gamma(\Sigma)} |jb_j(z, \omega, \varepsilon)|\rho^{j-1}.$$

Since the definition of the order $\gamma(\Sigma)$ yields $\sum_{j=2}^{\gamma(\Sigma)} |b_j(z, \omega, \varepsilon)|\neq 0$, we have the estimate

$$\pi(\rho)\geq C\rho^{\gamma(\Sigma)-1} \tag{4}$$

for $0\leq\rho$, $|z|$, $|\varepsilon|\leq a$, and $\omega\in S^{n-2}$. Here $a>0$ is sufficiently small and the constant C is independent of ρ, z, ε, and ω. Accordingly, all we have to show is the estimates

$$|F'_\varepsilon(\rho)|\geq C\pi(\rho), \tag{5}$$

$$|F^{(\nu)}_\varepsilon(\rho)|\leq C_\nu\rho^{1-\nu}\pi(\rho). \tag{6}$$

We write $F^{(\nu)}_\varepsilon(\rho)=\sum_{j=\nu}^{\infty} j!(j-\nu)!^{-1}b_j(z, \omega, \varepsilon)\rho^{j-\nu}$. Then we can easily have

$$\left|\sum_{j=\nu}^{\gamma(\Sigma)} \frac{j!}{(j-\nu)!} b_j(z, \omega, \varepsilon)\rho^{j-\nu}\right|\leq C\rho^{1-\nu}\pi(\rho).$$

As for the remainder term, we use Cauchy's estimate, that is,

$$\begin{aligned}\left|\frac{j!}{(j-\nu)!} b_j(z, \omega, \varepsilon)\right| &\leq(2a)^{-(j-\nu)} \max_{|\sigma|=2a} |F^{(\nu)}_\varepsilon(\sigma)| \\ &\leq C_\nu(2a)^{-(j-\nu)}.\end{aligned}$$

Here the constant C_ν is independent of z, ε, ω, and j. Then we have

$$\begin{aligned}\left|\sum_{j=\gamma(\Sigma)+1}^{\infty} \frac{j!}{(j-\nu)!} b_j(z, \omega, \varepsilon)\rho^{j-\nu}\right| &\leq C_\nu \sum_{j=\gamma(\Sigma)+1}^{\infty} \left(\frac{\rho}{2a}\right)^{j-\nu} \\ &\leq C_\nu\rho^{\gamma(\Sigma)+1-\nu} \\ &\leq C_\nu\rho^{2-\nu}\pi(\rho)\end{aligned} \tag{7}$$

for $0\leq\rho\leq a$. Here we have used estimate (4). Combining these estimates, we have estimate (6). On the other hand, by the concavity of the function $h_\varepsilon(y)$ and the equality $F'_\varepsilon(0)=0$, we can see that the function $|F'_\varepsilon(\rho)|$ is non decreasing. Hence we have

$$\begin{aligned}|F'_\varepsilon(\rho)| &= \max_{0\leq t\leq\rho} |F'_\varepsilon(t)| \\ &\geq \max_{0\leq t\leq\rho}\left|\sum_{j=2}^{\gamma(\Sigma)} jb_j(z, \omega, \varepsilon)t^{j-1}\right| - \max_{0\leq t\leq\rho}\left|\sum_{j=\gamma(\Sigma)+1}^{\infty} jb_j(z, \omega, \varepsilon)t^{j-1}\right|\end{aligned}$$

$$\geq \max_{0\leq t\leq 1}\left|\sum_{j=2}^{\gamma(\Sigma)} jb_j(z,\omega,\varepsilon)\rho^{j-1}t^{j-1}\right| - C_1 \max_{0\leq t\leq \rho}|t\pi(t)|$$

$$\geq (C-C_1\rho)\pi(\rho)\,.$$

Here we have used the compatibility of the norms $\max_{0\leq t\leq 1}|\sum_{j=1}^{\gamma(\Sigma)} c_j t^{j-1}|$ and $\sum_{j=1}^{\gamma(\Sigma)}|c_j|$ on $\boldsymbol{C}^{\gamma(\Sigma)}$, and the estimate (7) with $\nu=1$. Thus we have obtained the estimate (5) for sufficiently small ρ. This completes the proof of Lemma 10, and hence that of Theorem 3.

4. Proof of Theorem 2. We shall prove Theorem 2 in this section. First we note that $p(2\pi\xi)\leq\lambda$ if and only if

$$R(\lambda)^2 f_-(2\pi R(\lambda)^{-1}\xi_n)\leq|2\pi\xi'|^2\leq R(\lambda)^2 f_+(2\pi R(\lambda)^{-1}\xi_n)\,,$$

where $R(\lambda)=\lambda^{1/(4l)}$, $\xi=(\xi',\xi_n)$, $\xi'=(\xi_1,\xi_2,\dots,\xi_{n-1})$, and

$$f_+(t)=\begin{cases} t^2+(1-t^{4l})^{1/(2l)} & |t|\leq 1\\ 0 & \text{otherwise}\,,\end{cases}$$

$$f_-(t)=\begin{cases} t^2-(1-t^{4l})^{1/(2l)} & A=(1/2)^{1/(4l)}\leq|t|\leq 1\\ 0 & \text{otherwise}\,.\end{cases}$$

Then, by Lemma 1, we have

$$N(\lambda)=\sum_{\nu\in\boldsymbol{Z}} N^{n-1}(R(\lambda)^2 f_+(2\pi R(\lambda)^{-1}\nu))-\sum_{\nu\in\boldsymbol{Z}} N^{n-1}(R(\lambda)^2 f_-(2\pi R(\lambda)^{-1}\nu))$$
$$+\sum_{\nu\in\boldsymbol{Z}} dN^{n-1}(R(\lambda)^2 f_-(2\pi R(\lambda)^{-1}\nu))\,,$$

where $N^n(\lambda)$ and $dN^n(\lambda)$ denote the numbers of $k\in\boldsymbol{Z}^n$ which satisfy the inequality $|2\pi k|^2\leq\lambda$ and the equality $|2\pi k|^2=\lambda$, respectively. By the result in Example 2 and Lemma 1 again, we have

$$N^n(\lambda)=(2\pi)^{-n}|B^n|\lambda^{n/2}+O(\lambda^{(n-1)/2-(n-1)/\{2(n+1)\}})\,,$$

where $B^n=\{\xi\in\boldsymbol{R}^n;\,|\xi|\leq 1\}$. Since $dN^n(\lambda)\leq N^n(\lambda+1)-N^n(\lambda-1)$, we have

$$dN^n(\lambda)=O(\lambda^{(n-1)/2-(n-1)/\{2(n+1)\}})$$

as well. Hence we have

$$(8)\qquad N(\lambda)=\sum_{\substack{\nu\in\boldsymbol{Z}\\ |\nu|\leq R(\lambda)/(2\pi)}}\left\{(2\pi)^{-(n-1)}|B^{n-1}|R(\lambda)^{n-1}f\left(\frac{2\pi\nu}{R(\lambda)}\right)+O(R(\lambda)^{n-2-(n-2)/n})\right\}$$

$$=(2\pi)^{-n}|B^{n-1}|R(\lambda)^n\sum_{\nu=-L}^{L}\frac{1}{L}f\left(\frac{\nu}{L}\right)+O(R(\lambda)^{n-1-(n-2)/n})$$

$$=(2\pi)^{-n}|B^{n-1}|R(\lambda)^n\left\{2\sum_{\nu=0}^{L-1}\frac{1}{L}f\left(\frac{\nu}{L}\right)-\frac{1}{L}f(0)\right\}+O(R(\lambda)^{n-1-(n-2)/n}),$$

where

$$f(t)=f_+(t)^{(n-1)/2}-f_-(t)^{(n-1)/2}$$

and $L=R(\lambda)/(2\pi)$. We have taken here $\lambda=\lambda_L$ so that $L\in\boldsymbol{N}$, that is,

$$\lambda_L=(2\pi L)^{4l}\,;\qquad L=1,2,\dots.$$

We remark here that $f(t)\in C^0(\boldsymbol{R})\cap C^2(\boldsymbol{R}\setminus\{\pm A,\ \pm 1\})$ and $f(t)$ is convex in a neighborhood of $t=\pm A$. In fact, we have $\lim_{t\to A-0}f'(t)\geq\lim_{A+0}f'(t)$ and $\lim_{t\to A\pm 0}f''(t)<0$. We also remark that $\sup_{t\neq\pm A,\pm 1}f''(t)$ is finite, since $\lim_{t\to 1-0}(1-t^{4l})^{2-1/(2l)}f''(t)<0$.

The following *trapezoidal rule* is the key to the proof.

LEMMA 12. (i) *Suppose* $F(t)\in C^2([a,a+h])$. *Then*

$$\int_a^{a+h}F(t)dt=hF(a)+\frac{h}{2}(F(a+h)-F(a))-\frac{1}{12}F''(a+\theta h)h^3$$

for some $0<\theta<1$.

(ii) *Suppose that* $F(t)$ *is continuous and convex on the interval* $[a,a+h]$. *Then*

$$\int_a^{a+h}F(t)dt\geq hF(a)+\frac{h}{2}(F(a+h)-F(a)).$$

PROOF OF LEMMA 12. (i) Apply Taylor's formula

$$\varphi(h)=\varphi(0)+\varphi'(0)h+h^2\int_0^1(1-\theta)\varphi''(\theta h)d\theta$$

to

$$\varphi(h)=\int_a^{a+h}F(t)dt-\frac{h}{2}(F(a+h)+F(a))$$

and use the mean value theorem. (ii) is trivial. ∎

By Lemma 12, we have for $\nu=0,1,\dots,L-2$,

$$\frac{1}{L}f\left(\frac{\nu}{L}\right)\leq\int_{\nu/L}^{(\nu+1)/L}f(t)dt-\frac{1}{2L}\left(f\left(\frac{\nu+1}{L}\right)-f\left(\frac{\nu}{L}\right)\right)+O\left(\left(\frac{1}{L}\right)^3\right).\tag{9}$$

On the other hand, since

$$f(1-t)=(1+(4lt)^{1/(2l)}+O(t))^{(n-1)/2}-(1-(4lt)^{1/(2l)}+O(t))^{(n-1)/2}$$
$$=(n-1)(4lt)^{1/(2l)}+O(t^{1/l})$$

for small t, we have for $v=L-1$

$$(10)\quad \frac{1}{L}f\left(\frac{L-1}{L}\right)=\int_{(L-1)/L}^{1} f(t)dt+\frac{1}{2L}f\left(\frac{L-1}{L}\right)-\int_{0}^{1/L} f(1-t)dt+\frac{1}{2L}f\left(1-\frac{1}{L}\right)$$
$$=\int_{(L-1)/L}^{1} f(t)dt+\frac{1}{2L}f\left(\frac{L-1}{L}\right)-C\left(\frac{1}{L}\right)^{1+1/(2l)}+O\left(\left(\frac{1}{L}\right)^{1+1/l}\right),$$

where C is a positive constant which is independent of L. From (9) and (10), we obtain

$$2\sum_{v=0}^{L-1}\frac{1}{L}f\left(\frac{v}{L}\right)-\frac{1}{L}f(0)\leq 2\int_{0}^{1} f(t)dt-2C\left(\frac{1}{L}\right)^{1+1/(2l)}+O\left(\left(\frac{1}{L}\right)^{1+1/l}\right),$$

and hence from (8)

$$N(\lambda_L)\leq(2\pi)^{-n}|\Omega|R(\lambda_L)^n-2(2\pi)^{-n}|B^{n-1}|CR(\lambda_L)^{n-1-1/(2l)}$$
$$+O(R(\lambda_L)^{n-1-1/l})+O(R(\lambda_L)^{n-1-(n-2)/n}).$$

Consequently, we have

$$\limsup_{L\to\infty}\frac{N(\lambda_L)-(2\pi)^{-n}|\Omega|\lambda_L^{n/(4l)}}{\lambda_L^{(n-1)/(4l)-1/(4l)\cdot 1/(2l)}}<0$$

if $l>1/2+1/(n-2)$. This inequality and the formula (1) imply $d\leq 1/(4l)\cdot 1/(2l)=1/(m\gamma_0(\Sigma))$. We have thus completed the proof of Theorem 2.

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