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BOUNDEDNESS AND CONTINUITY OF THE FUNDAMENTAL SOLUTION OF THE TIME DEPENDENT SCHRÖDINGER EQUATION WITH SINGULAR POTENTIALS

Dedicated to Professor Kyûya Masuda on his sixtieth birthday

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Abstract. We show that the fundamental solution of the initial value problem for the time dependent Schrödinger equation is bounded and continuous for a class of non-smooth potentials. The class is large enough to accomodate Coulomb potentials if the spatial dimension is three.

1. Introduction. We consider the Cauchy problem for the time dependent Schrödinger equation

$$i\frac{\partial u}{\partial t}=-\frac{1}{2}\Delta u+V(x)u\,,\quad (t,x)\in \boldsymbol{R}^1\times \boldsymbol{R}^m\,;\quad u(0,x)=u_0(x)\,,\quad x\in\boldsymbol{R}^m \tag{1.1}$$

in the Hilbert space $L^2(\boldsymbol{R}^m)$. We assume that the potential $V(x)$ is real-valued and the operator $-(1/2)\Delta+V$ on $C_0^\infty(\boldsymbol{R}^m)$ defines a unique selfadjoint extension H in $L^2(\boldsymbol{R}^m)$. Then, the equation (1.1) has a unique solution $u(t)=e^{-itH}u_0$. The distribution kernel $E(t,x,y)$ of the propagator e^{-itH} is called the fundamental solution (FDS for short) of (1.1):

$$u(t,x)=e^{-itH}u_0(x)=\int E(t,x,y)u_0(y)dy\,.$$

The FDS $E(t,x,y)$ is a solution of (1.1) with the initial data $E(0,x,y)=\delta(x-y)$. In this paper, we show that $E(t,x,y)$ is continuous and bounded, $|E(t,x,y)|\le C_T|t|^{-m/2}$ for $0<|t|\le T<\infty$, for a class of potentials $V(x)$ which can be as singular as $|x|^{-(m-\varepsilon)/(m-1)}$ and decay at infinity as slowly as $V(x)=o(1)$. The class is wide enough to accommodate Coulomb potentials $V(x)=\sum_{j=1}^N Z_j/|x-R_j|$ in dimension three.

When $V(x)$ is C^∞, it was recently shown that the smoothness property of the FDS is determined mainly by the growth rate of $V(x)$ at infinity: The FDS is smooth and bounded for $t\neq 0$ if V is subquadratic, viz., $|V(x)|=o(|x|^2)$ roughly speaking ([20], see also [21], [11], [3]); whereas $E(t,x,y)$ is nowhere C^1 if V is superquadratic in dimension one, viz., $V(x)\ge C|x|^{2+\varepsilon}$, $\varepsilon>0$ near infinity ([20]); and at the borderline case $|V(x)|\sim$

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$C|x|^2$, $E(t, x, y)$ is smooth and bounded for at least small time ([5]) but it can in general become singular at certain later times ([21], [12]). These properties of the FDS may be explained, at least at a heuristic level, as the result of the propagation of singularities: The singularities of the initial data propagate along the limit set as energy tends to infinity of the classical trajectories $\dot{x}(t)=p(t)$, $\dot{p}(t)=-\nabla V(x)$ ([20]).

On the other hand, if V is not smooth, e.g., if V is the Coulomb potential in dimension three, the singularities of V create those of the FDS and $E(t, x, y)$ is not smooth everywhere. However, the strong dissipation property of the free propagator e^{-itH_0} moderates the singularities and we expect that $E(t, x, y)$ is bounded and continuous for $t \neq 0$ if V is bounded at infinity in a suitable norm and is not too singular locally (see Simon [14] who conjectures that this is true if V is of Kato class). Indeed it has been long known that $E(t, x, y)$ is bounded and continuous on $[\varepsilon, T] \times \boldsymbol{R}^2$ for any $0<\varepsilon<T<\infty$, if the spatial dimension $m=1$ and $V \in L^1(\boldsymbol{R})$ ([14]). In higher dimensions, the same is known under various conditions. For example, in dimension $m=3$, the FDS is bounded and continuous on $[\varepsilon, T] \times \boldsymbol{R}^6$ if V satisfies either of the following conditions:

(a) For some $\varepsilon>0$, $\|V\|_{3/2+\varepsilon}+\|V\|_{3/2-\varepsilon}$ is sufficiently small ([13]).

(b) For some $\varepsilon>0$, $\|V\|_{L^2(\{y:|x-y|\leq 1\})} \leq C\langle x\rangle^{-(11/2+\varepsilon)}$ ([18], [19]. We remark that the spectral conditions on H in [18] and [19] are not necessary when applied to the finite time problem).

(c) V is the Fourier transform of $\hat{V} \in L^1(\boldsymbol{R}^3)$. If $\hat{V}$ is a measure of bounded variation, $E(t, x, y)$ is bounded for $0<\varepsilon \leq t \leq T<\infty$ ([7], [4], [8], [10]).

Unfortunately, however, none of these results apply to physically important Coulomb potentials: (a) and (b), though they allow singularities as strong as of $L^{3/2-\varepsilon}$ functions, require V to be either small or decay rapidly at infinity; (c) requires V to be continuous.

The purpose of this paper is to prove that the FDS of (1.1) is bounded and continuous for another class of potentials which is large enough to accommodate Coulomb potentials in dimension three. We write $m_*=(m-1)/(m-2)$.

THEOREM 1.1. *Assume $m \geq 3$. Let $V=V(x)$ be real-valued. Suppose that, for any $\varepsilon>0$, V can be decomposed as $V=V_{1,\varepsilon}+V_{2,\varepsilon}$ so that $V_{1,\varepsilon}$ satisfies for some $\sigma>1/m_*$ and $\gamma>0$*

$$\|\mathscr{F}(\langle x\rangle^{1+\sigma}\langle D\rangle^{\gamma} V_{1,\varepsilon})\|_{L^{m_*}(\boldsymbol{R}^m)}<\varepsilon \tag{1.2}$$

and the Fourier transform $\hat{V}_{2,\varepsilon}$ of $V_{2,\varepsilon}$ is a signed measure of bounded variation. Then, the FDS *$E(t, x, y)$ of* (1.1) *is bounded with respect to (x, y) for $t \neq 0$ and, for any $T>0$, there exists a constant C_T such that for $0<|t| \leq T$*

$$|E(t, x, y)| \leq C_T|t|^{-m/2}, \qquad 0<|t| \leq T, \quad (x, y) \in \boldsymbol{R}^{2m}. \tag{1.3}$$

Moreover, if $\hat{V}_{2,\varepsilon} \in L^1(\boldsymbol{R}^m)$, $E(t, x, y)$ is continuous with respect to (t, x, y) for $t \neq 0$.

REMARK 1. When $m=3$, the condition (1.2) reads $\|\langle x\rangle^{\delta}\langle D\rangle^{\gamma} V_{1,\varepsilon}\|_{L^2(\boldsymbol{R}^3)}<\varepsilon$ for some $\delta>3/2$ and $\gamma>0$ and the sum of Coulomb potentials $V(x)=\sum_{j=1}^{N} Z_j/|x-R_j|$ obviously satisfies the condition of Theorem 1.1. When $m\geq 4$, the potentials with local singularities of type $\sum C_j|x-a_j|^{-m/(m-1)+\varepsilon}$ satisfy the condition of the theorem.

Since the propagator e^{-itH} is unitary in $L^2(\boldsymbol{R}^m)$, the estimate (1.3) and the interpolation theorem imply the following L^p-L^q estimate.

COROLLARY 1.2. *Let V satisfy the conditions of Theorem* 1.1 *and* $1\leq q\leq 2$, $1/p+1/q=1$. *Then, for any $T>0$, there exists a constant C_T such that for* $0<|t|\leq T$,

$$(1.4)\qquad \|e^{-itH}f\|_{L^p(\boldsymbol{R}^m)}\leq C_T|t|^{-m(1/2-1/p)}\|f\|_{L^q(\boldsymbol{R}^m)}, \qquad f\in L^q(\boldsymbol{R}^m)\cap L^2(\boldsymbol{R}^m).$$

The rest of the paper is devoted to the proof of Theorem 1.1. The basic idea is to combine the method in [18] with the one used for the case $\hat{V}\in L^1(\boldsymbol{R}^m)$. We explain it here more precisely. We denote by $X=(X_1,\ldots,X_m)$ (resp. $D=(D_1,\ldots,D_m)$) the vector whose components are the multiplication operators X_j with the variable x_j (resp. the differential operators $D_j=-i\partial/\partial x_j$), $1\leq j\leq m$. We define the Fourier transform by

$$\hat{V}(\xi)=\frac{1}{(2\pi)^m}\int e^{-ix\xi}V(x)dx\,.$$

Here and hereafter, the integrals should be taken over the whole space if no domains of integration are specified. For Banach spaces X, Y, $B(X,Y)$ stands for the space of bounded operators from X to Y. $B(X)=B(X,X)$.

If V satisfies the conditions of Theorem 1.1, $H=-(1/2)\Delta+V$ is selfadjoint with the domain $H^2(\boldsymbol{R}^m)$ and $C_0^\infty(\boldsymbol{R}^m)$ is a core. The solution $u(t)=\exp(-itH)u_0$ of (1.1) satisfies the integral equation ([15]):

$$(1.5)\qquad u(t)=e^{-itH_0}u_0-i\int_0^t e^{-i(t-s)H_0}Vu(s)ds\,,$$

where $H_0=-(1/2)\Delta$. We consider in the interaction picture and set

$$\Gamma(s)=e^{isH_0}V(X)e^{-isH_0}\,.$$

By iterating the integral equation (1.5) repeatedly, we have at least formally

$$(1.6)\qquad e^{-itH}u_0=e^{-itH_0}\sum_{n=0}^{\infty}(-i)^n G_n(t)u_0=e^{-itH_0}G_\infty(t)\,,$$

where $G_0(t)=I$, and for $n=1,2,\ldots$,

$$(1.7)\qquad G_n(t)=\int_{0\leq t_1\leq\cdots\leq t_n\leq t}\Gamma(t_n)\cdots\Gamma(t_1)dt_1\cdots dt_n\,.$$

We recall the argument that proves Theorem 1.1 if $V_{1,\varepsilon}=0$, that is, if $\hat{V}$ is a measure $d\mu$ of bounded variation. The following lemma is well known ([16], [10], [8]).

LEMMA 1.3. *Let $\hat{V}(\xi)d\xi = d\mu(\xi)$ be a (signed) measure of bounded variation. Then*

$$\Gamma(t)f(x) = \int e^{i(x\xi + t\xi^2/2)} f(x+t\xi)d\mu(\xi) \tag{1.8}$$

and, for any $1 \le p \le \infty$, $\Gamma(t)$ is bounded in $L^p(\boldsymbol{R}^m)$:

$$\|\Gamma(t)f\|_{L^p} \le |\mu| \|f\|_{L^p} \tag{1.9}$$

where $|\mu|$ is the total variation of the measure $d\mu$. Moreover, $\Gamma(t)$ is strongly continuous in $L^p(\boldsymbol{R}^m)$, $1 \le p < \infty$.

PROOF. The operators e^{-itH_0} and $e^{-iX^2/2t}$ are unitary in $L^2(\boldsymbol{R}^m)$ and as selfadjoint operators $e^{itH_0}Xe^{-itH_0} = X + tD = e^{-iX^2/2t}(tD)e^{iX^2/2t}$. Functional calculus then shows

$$\Gamma(t) = e^{itH_0}V(X)e^{-itH_0} = V(X+tD) = e^{-iX^2/2t}V(tD)e^{iX^2/2t} \tag{1.10}$$

as bounded operators in $L^2(\boldsymbol{R}^m)$. By using the Fourier transform, we have

$$V(tD)f(x) = \int e^{ix\xi}V(t\xi)\hat{f}(\xi)d\xi = \int f(x+t\xi)d\mu(\xi)\,. \tag{1.11}$$

Inserting (1.11) into (1.10), we obtain (1.8). It follows from Minkowski's inequality applied to (1.8) that $\Gamma(t)$ extends to a bounded operator in $L^p(\boldsymbol{R}^m)$ for any $1 \le p \le \infty$ and that the estimate (1.9) is satisfied. The strong continuity of $\Gamma(t)$ follows immediately from (1.8). ∎

In virtue of Lemma 1.3 and (1.7), $G_n(t)$ is bounded in $L^p(\boldsymbol{R}^m)$ and $\|G_n(t)\|_{B(L^p)} \le |\mu|^n |t|^n/n!$. Hence the series in (1.6) converges in the norm of $B(L^p(\boldsymbol{R}^m))$ for any $1 \le p \le \infty$ and $\|G_\infty(t)\|_{B(L^p)} \le \exp(|\mu||t|)$. Since e^{-itH_0} maps $L^1(\boldsymbol{R}^m)$ to $L^\infty(\boldsymbol{R}^m)$ with the operator norm bounded by $(2\pi|t|)^{-m/2}$, we see that

$$\|e^{-itH}\|_{B(L^1,L^\infty)} \le \|e^{-itH_0}\|_{B(L^1,L^\infty)} \|G_\infty(t)\|_{B(L^1)} \le (2\pi|t|)^{-m/2}\exp(|\mu||t|)\,. \tag{1.12}$$

Thus, if $\hat{V}$ is a measure of bounded variation, $E(t, x, y)$ is a bounded function of (x, y) if $t \ne 0$ and $|E(t, x, y)| \le (2\pi|t|)^{-m/2}\exp(|\mu||t|)$.

If $\hat{V} \in L^1(\boldsymbol{R}^m)$, we may approximate V by $V_j \in \mathscr{S}(\boldsymbol{R}^m)$ so that $\|\hat{V}_j - \hat{V}\|_{L^1} \to 0$ as $j \to \infty$. It is well known the FDS $E_j(t, x, y)$ for $H_j = -(1/2)\Delta + V_j$ is C^∞ for $t \ne 0$ ([21], [20]) and (1.6) and (1.9) imply that e^{-itH_j} converges to e^{-itH} in the topology of operator norm of $B(L^1, L^\infty)$ uniformly with respect to $0 < \varepsilon \le |t| \le T < \infty$. Hence $E_j(t, x, y)$ converges to $E(t, x, y)$ uniformly on $[\varepsilon, T] \times \boldsymbol{R}^m \times \boldsymbol{R}^m$ and $E(t, x, y)$ is jointly continuous in (t, x, y).

When V satisfies the conditions of Theorem 1.1, we again show that $G_\infty(t)$ is a bounded operator in $L^p(\boldsymbol{R}^m)$ for any $1 \le p \le \infty$ with $\|G_\infty(t)\|_{B(L^p)}$ bounded on every compact interval. The argument above does not apply because $\hat{V}$ decays slowly at infinity and is not integrable any more. For controlling the slow decay of $\hat{V}$ or the singularities of V, we utilize the oscillation property of e^{-itH_0} with respect to (t, x, y) as in our previous papers ([18], [19]). This will be done by integration of $\Gamma(t)$ with respect to t which transforms the oscillation into the decay property. Thus we insert into (1.7) the expression (1.8) of $\Gamma(t)$ in the operator form:

$$\Gamma(t)=\int e^{iX\xi}e^{it(D\xi+t\xi^2/2)}\hat{V}(\xi)d\xi \tag{1.13}$$

and integrate the resulting formula with respect to the variables $(t_1,\dots,t_n)$ first.

However, as we shall see in the text the estimation of the resulting expressions requires certain smoothness of $\hat{V}$ or decay of V at infinity. Recall that the argument above for the case where $\hat{V}$ is a measure did not require V to decay at infinity, though it did require $\hat{V}$ to decay instead. Thus we decompose $V=V_1+V_2$ into the singular but decaying part V_1 and the bounded but continuous part V_2 as in Theorem 1.1 and combine those two methods as follows. In what follows, we omit the subscript $\varepsilon>0$ of $V_{1,\varepsilon}$ and $V_{2,\varepsilon}$.

Denote by $\Gamma_1(t)$ and $\Gamma_2(t)$ the operator $\Gamma(t)$ corresponding to V_1 and V_2 respectively. Insert $\Gamma(t)=\Gamma_1(t)+\Gamma_2(t)$ into (1.7) and expand $G_n(t)$ into 2^n summands. If we denote, for a subset $A: R_1<\cdots<R_l$ of $\{1,2,\dots,n\}$, by $G_n^A(t)$ the summand which has the factors $\Gamma_2(t_j)$ at $j=R_1,\dots,R_l$-th places and $\Gamma_1(t_j)$ elsewhere, then, $G_n(t)$ is a sum of $G_n^A(t)$ over all subsets A of $\{1,2,\dots,n\}$:

$$G_n(t)=\sum_A G_n^A(t)\,. \tag{1.14}$$

Combining the consecutive factors $\Gamma_1(t_j)$ together, we write $G_n^A(t)$ in the form

$$\begin{aligned} G_n^A(t)=\int_{0\le t_{R_1}\le\cdots\le t_{R_l}\le t} & F(t,t_{R_l})\Gamma_2(t_{R_l})F(t_{R_l},t_{R_l-1})\cdots \\ & \cdots\Gamma_2(t_{R_2})F(t_{R_2},t_{R_1})\Gamma_2(t_{R_1})F(t_{R_1},0)dt_{R_1}\cdots dt_{R_l}\,, \end{aligned} \tag{1.15}$$

where the first factor $F(t,t_{R_l})$ and the last $F(t_{R_1},0)$ should be understood as the identity operators if $R_l=n$ or $R_1=1$, and otherwise,

$$F(t_{R_{j+1}},t_{R_j})=\int_{t_{R_j}\le t_{B_j}\le\cdots\le t_{E_j}\le t_{R_{j+1}}}\Gamma_1(t_{E_j})\cdots\Gamma_1(t_{B_j})dt_{B_j}\cdots dt_{E_j}\,. \tag{1.16}$$

Here we set $R_0=0$, $R_{l+1}=n+1$, $t_{R_0}=0$ and $t_{R_{l+1}}=t$ and

$$R_j<B_j<\cdots<E_j<R_{j+1} \text{ is a sequence of consecutive integers}\,. \tag{1.17}$$

For estimating the operator norm in $L^p(\boldsymbol{R}^m)$ of $F(t_{R_{j+1}},t_{R_j})$, we insert the expression (1.13) for V_1 into (1.16) and integrate the result with respect to $(t_{B_j},\dots,t_{E_j})$ first. In Section 2, we perform this integration and rewrite the result in a form convenient in proving the main estimate

$$\|F(t_{R_{j+1}},t_{R_j})\|_{B(L^p)}\le(C\|\mathscr{F}(\langle x\rangle^{\sigma+1}\langle D\rangle^{\gamma}V_1)\|_{L^{m_*}})^{R_{j+1}-R_j-1} \tag{1.18}$$

for $0 \leq t_{R_j} \leq t_{R_{j+1}} \leq T$. The estimate (1.18) will be proved in Section 3. Combining (1.18) with the estimate $\|\Gamma_2(t)\|_{B(L^p)} \leq C|\mu|$ obtained in Lemma 1.3, we have

$$\|G_n^A(t)\|_{B(L^p)} \leq \frac{(|\mu||t|)^l}{l!}(C\|\mathscr{F}(\langle x\rangle^{\sigma+1}\langle D\rangle^{\gamma}V_1)\|_{L^{m_*}})^{n-l}. \tag{1.19}$$

By summing (1.19) up with respect to A and n, we see that, if $\|\mathscr{F}(\langle x\rangle^{\sigma+1}\langle D\rangle^{\gamma}V_1)\|_{L^{m_*}}$ is small enough, then $G_\infty(t)$ is bounded in $L^p(\boldsymbol{R}^m)$ for any $1 \leq p \leq \infty$ and $\|G_\infty(t)\|_{B(L^p)}$ is uniformly bounded on compact intervals. This implies that the FDS is bounded on $[\varepsilon, T] \times \boldsymbol{R}^{2m}$ as explained above. An approximation argument necessary for performing the estimation and for proving the continuity of the FDS will be given in Section 4 thereby the proof of Theorem 1.1 will be completed.

We adopt the following convention. When precise values are not important, various constants are denoted by the same letter C. L^p norms are denoted by $\|\cdot\|_p$ as well as by $\|\cdot\|_{L^p}$.

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2. Preliminaries. In this section, we rewrite the operator $F(t_{R_{j+1}}, t_{R_j})$ defined by (1.16) in a form suitable for the estimation to be done in the following section. In Sections 2 and 3, we assume $V_1 \in C_0^\infty(\boldsymbol{R}^m)$. As a prototype we deal with

$$F(t,s) = \int_{s \leq t_1 \leq \cdots \leq t_n \leq t} \Gamma_1(t_n) \cdots \Gamma_1(t_1)dt_1 \ldots dt_n, \qquad 0 \leq s \leq t \leq T. \tag{2.1}$$

To make the following computation legitimate we insert the damping factor $e^{-\varepsilon\sum_{j=1}^n t_j}$ and write as $\Gamma_{1,\varepsilon}(t) = \Gamma_1(t)e^{-\varepsilon t}$. We have, in the topology of operator norm in $L^2(\boldsymbol{R}^m)$,

$$F(t,s) = \lim_{\varepsilon \to +0} F_\varepsilon(t,s) \equiv \lim_{\varepsilon \to +0} \int_{s \leq t_1 \leq \cdots \leq t_n \leq t} \Gamma_{1,\varepsilon}(t_n) \cdots \Gamma_{1,\varepsilon}(t_1)dt_1 \cdots dt_n \tag{2.2}$$

uniformly with respect to $0 \leq s \leq t \leq T$. We insert into (2.2) the expression (1.13) for $\Gamma_1(t_j)$, $1 \leq j \leq n$. Writing $\Xi = (\xi_1, \ldots, \xi_n)$ and $d\Xi = d\xi_1 \cdots d\xi_n$, we have

$$F_\varepsilon(t,s) = \int_{\boldsymbol{R}^{mn}} \hat{V}_1(\xi_n) \cdots \hat{V}_1(\xi_1)L_\varepsilon(t,s,\Xi)d\Xi, \tag{2.3}$$

where we define the strongly continuous operator-valued function $L_\varepsilon(t, s, \Xi)$ by

(2.4) $$L_\varepsilon(t, s, \Xi)=\int_{s\le t_1\le\cdots\le t_n\le t} e^{iX\xi_n}e^{it_n(D\xi_n+\xi_n^2/2+i\varepsilon)}\cdots e^{iX\xi_1}e^{it_1(D\xi_1+\xi_1^2/2+i\varepsilon)}dt_1\cdots dt_n\,.$$

We set, for $\xi\neq 0$,

$$[\xi]_\varepsilon=-i(D\xi+\xi^2/2+i\varepsilon)^{-1}\,,\quad [\xi, t]_\varepsilon^{**}=e^{iX\xi}e^{it(\xi D+\xi^2/2)}[\xi]_\varepsilon\,,\quad [\xi, t]_\varepsilon^{*}=-[\xi, t]_\varepsilon^{**}\,.$$

(2.5) $$i(D\xi+\xi^2/2+i\varepsilon)^{-1}f(x)=\int_0^\infty e^{it(\xi^2/2+i\varepsilon)}f(x+t\xi)dt\,.$$

Define for $\Xi=(\xi_1,\dots,\xi_n)\in \boldsymbol{R}^{nm}$ and a (decreasing) consecutive sequence $C=\{b,\dots,a\}\subset\{1,\dots,n\}$:

(2.6) $$[C,\Xi]_\varepsilon=[\xi_b+\cdots+\xi_a, s]^{*}_{(b-a+1)\varepsilon}[\xi_{b-1}+\cdots+\xi_a]_{(b-a)\varepsilon}\cdots[\xi_a]_\varepsilon\,,$$

(2.7) $$[C,\Xi]_{1,\varepsilon}=[\xi_b+\cdots+\xi_a, t]^{**}_{(b-a+1)\varepsilon}[\xi_{b-1}+\cdots+\xi_a]_{(b-a)\varepsilon}\cdots[\xi_a]_\varepsilon\,.$$

LEMMA 2.1. *Let $\mathscr{D}=\{D\}$ be the set of decompositions $D=\{C_1,\dots,C_l\}$ of $\{1,\dots,n\}$ into subsets of consecutive integers*:

$$C_l=\{n,\dots,n_{l-1}+1\}\,,\ \dots,\ C_2=\{n_2,\dots,n_1+1\}\,,\ C_1=\{n_1,\dots,1\}\,,$$

where $n_0=0$. Then $L_\varepsilon(t, s, \Xi)$ of (2.4) *is equal to the sum of*

$$[D,\Xi]_\varepsilon\equiv([C_l,\Xi]_{1,\varepsilon}+[C_l,\Xi]_\varepsilon)[C_{l-1},\Xi]_\varepsilon\cdots[C_1,\Xi]_\varepsilon$$

over all $D=(C_1,\dots,C_l)\in\mathscr{D}$.

In what follows we omit the variables Ξ in $[C,\Xi]_\varepsilon$, etc.

PROOF. We prove the lemma by induction on n. By integration

$$\int_s^t e^{iX\xi_1}e^{it_1(D\xi_1+\xi_1^2/2+i\varepsilon)}dt_1=e^{iX\xi_1}\cdot\frac{e^{it(D\xi_1+\xi_1^2/2+i\varepsilon)}-e^{is(D\xi_1+\xi_1^2/2+i\varepsilon)}}{i(D\xi_1+\xi_1^2/2+i\varepsilon)}=[\xi_1, t]_\varepsilon^{**}+[\xi_1, s]_\varepsilon^{*}\,,$$

which proves the lemma when $n=1$. We now suppose that the lemma is already proved for $n=1,\dots,k-1$ and prove it for $n=k$. By the induction hypothesis, we may write

$$\begin{aligned}L_\varepsilon(t, s, \Xi)&=\int_s^t dt_k\int_{s\le t_1\le\cdots\le t_{k-1}\le t_k} e^{iX\xi_k}e^{it_k(D\xi_k+\xi_k^2/2+i\varepsilon)}\cdots e^{iX\xi_1}e^{it_1(D\xi_1+\xi_1^2/2+i\varepsilon)}dt_1\cdots dt_{k-1}\\&=\sum_D\int_s^t e^{iX\xi_k}e^{it_k(D\xi_k+\xi_k^2/2+i\varepsilon)}([C_l]_{1,\varepsilon}+[C_l]_\varepsilon)[C_{l-1}]_\varepsilon\cdots[C_1]_\varepsilon dt_k\,,\end{aligned}$$

where D runs over all the decompositions $D=\{C_l,\dots,C_1\}$ of $\{1,\dots,k-1\}$ into subsets of consecutive integers and $[C_l]_\varepsilon$ is defined by (2.7) with t_k in place of t. Since $[C_l]_\varepsilon[C_{l-1}]_\varepsilon\cdots[C_1]_\varepsilon$ is independent of t_k, we may compute as in the case $n=1$ and obtain

$$(2.8)\quad \int_s^t e^{iX\xi_k}e^{it_k(D\xi_k+\xi_k^2/2+i\varepsilon)}[C_l]_\varepsilon\cdots[C_1]_\varepsilon dt_k=([\{k\}]_{1,\varepsilon}+[\{k\}]_\varepsilon)[C_l]_\varepsilon\cdots[C_1]_\varepsilon\,.$$

On the other hand, using the equation

$$e^{iX\xi}e^{it(D\xi+\xi^2/2+i\varepsilon)}e^{iX\eta}e^{it(D\eta+\eta^2/2+ij\varepsilon)}=e^{iX(\xi+\eta)}e^{it\{D(\xi+\eta)+(\xi+\eta)^2/2+i(j+1)\varepsilon\}}$$

we may compute

$$\int_s^t e^{iX\xi_k}e^{it_k(D\xi_k+\xi_k^2/2+i\varepsilon)}[\eta,t_k]_{j\varepsilon}^{**}dt_k=[\xi_k+\eta,t]_{(j+1)\varepsilon}^{**}[\eta]_{j\varepsilon}+[\xi_k+\eta,s]_{(j+1)\varepsilon}^{*}[\eta]_{j\varepsilon}\,.$$

Hence, recalling (2.6) and (2.7), we obtain

$$(2.9)\quad \int_s^t e^{iX\xi_k}e^{it_k(D\xi_k+\xi_k^2/2+i\varepsilon)}[C_l]_{1,\varepsilon}[C_{l-1}]_\varepsilon\cdots[C_1]_\varepsilon dt_k$$
$$=([C_l\cup\{k\}]_{1,\varepsilon}+[C_l\cup\{k\}]_\varepsilon)[C_{l-1}]_\varepsilon\cdots[C_1]_\varepsilon\,.$$

Since the decompositions of $\{1,\dots,k\}$ into subsets of consecutive integers may be uniquely obtained from a decomposition of $\{1,\dots,k-1\}$ either by adding the singleton $\{k\}$ or adjoining the element k to the subset containing $k-1$, (2.8) and (2.9) show that the lemma holds for $n=k$ as well. This completes the proof. ∎

It follows from Lemma 2.1 that $F_\varepsilon(t_{R_j},t_{R_{j+1}})$ decomposes into the sum over all decompositions into consecutive numbers $\{C_k,\dots,C_1\}$ of $\{B_j,\dots,E_j\}$ (recall (1.17) for the terminology):

$$(2.10)\quad F_\varepsilon(t_{R_j},t_{R_{j+1}})=\sum_{\{C_k,\dots,C_1\}}\int([C_k]_{1,\varepsilon}+[C_k]_\varepsilon)[C_{k-1}]_\varepsilon\cdots[C_1]_\varepsilon\prod_{m=B_j}^{E_j}\hat V_1(\xi_m)d\xi_m\,.$$

Since the variables ξ_j's contained in $[C_j]_\varepsilon$ and $[C_k]_\varepsilon$ are different among themselves if $j\neq k$, the integral (2.10) breaks up into factors and is equal to the product of

$$(2.11)\quad \int([C_k]_\varepsilon+[C_k]_{1,\varepsilon})\prod_{m\in C_k}\hat V_1(\xi_m)\prod_{m\in C_k}d\xi_m$$

and

$$(2.12)\quad \int[C_j]_\varepsilon\prod_{m\in C_j}\hat V_1(\xi_m)\prod_{m\in C_j}d\xi_m\,,\qquad 1\le j\le k-1\,.$$

We deal with the operators (2.11) and (2.12). Note that all factors $[C_k]_\varepsilon$, $[C_{k-1}]_\varepsilon,\dots,[C_1]_\varepsilon$ have similar form and $[C_k]_{1,\varepsilon}$ and $[C_k]_\varepsilon$ differ only by sign and parameters t_{R_j} and $t_{R_{j+1}}$. Hence, we have only to study the integral containing the factor $[C_k]_\varepsilon$ as a prototype. We assume, for notational simplicity, $C_k=\{n,\dots,1\}$ and $t_{R_k}=s$ and $t_{R_{k+1}}=t$. Then, making the change of variables $(\xi_1,\xi_2,\dots,\xi_n)\mapsto(\xi_1,\xi_2-\xi_1,\dots,\xi_n-\xi_{n-1})$, we obtain

(2.13) $$\begin{aligned} W_\varepsilon &\equiv (-1)^{n-1} \int [\{n, n-1, \dots, 1\}]_\varepsilon \prod_{j=1}^n \hat{V}_1(\xi_j) d\Xi \\ &= (-1)^{n-1} \int [\xi_n + \cdots + \xi_1, s]^*_{n\varepsilon} [\xi_{n-1} + \cdots + \xi_1]_{(n-1)\varepsilon} \cdots [\xi_1]_\varepsilon \prod_{j=1}^n \hat{V}_1(\xi_j) d\Xi \\ &= (-1)^{n-1} \int [\xi_n, s]^*_{n\varepsilon} [\xi_{n-1}]_{(n-1)\varepsilon} \cdots [\xi_1]_\varepsilon \prod_{j=1}^n \hat{V}_1(\xi_j - \xi_{j-1}) d\Xi \end{aligned}$$

where $\xi_0 = 0$. Recalling (2.5), we have

$$[\xi, s]^*_\varepsilon = ie^{iX\xi} e^{is(D\xi + \xi^2/2 + i\varepsilon)} (D\xi + \xi^2/2 + i\varepsilon)^{-1} = e^{iX\xi} \int_s^\infty e^{it(D\xi + \xi^2/2 + i\varepsilon)} dt .$$

Hence, we have, writing $dS = ds_1 \cdots ds_n$,

(2.14) $$\begin{aligned} &[\xi_n, s]^*_{n\varepsilon} [\xi_{n-1}]_{(n-1)\varepsilon} \cdots [\xi_1]_\varepsilon f(x) \\ &\quad = e^{ix\xi_n} \int_s^\infty \int_{[0,\infty)^{n-1}} e^{i\sum_{j=1}^n s_j(\xi_j^2/2 + ij\varepsilon)} f\left(x + \sum_{j=1}^n s_j \xi_j\right) dS . \end{aligned}$$

Combination of (2.13) with (2.14) yields

$$W_\varepsilon f(x) = \int \left\{ e^{ix\xi_n} \int_s^\infty \int_{[0,\infty)^{n-1}} e^{i\sum_{j=1}^n s_j(\xi_j^2/2 + ij\varepsilon)} \prod_{j=1}^n \hat{V}_1(\xi_j - \xi_{j-1}) f\left(x + \sum_{j=1}^n s_j \xi_j\right) dS \right\} d\Xi .$$

We introduce polar coordinates $\xi_j = r_j \omega_j$, $1 \le j \le n$, $0 < r_j < \infty$ and $\omega_j \in \Sigma$, Σ being the unit sphere of $\boldsymbol{R}^m$ and make the change of variables $s_j \mapsto s_j / r_j$, $1 \le j \le n$. Write $R = (R', r_n) = (r_1, \dots, r_n)$, $S = (S', s_n) = (s_1, \dots, s_n)$, $\Omega = (\omega_1, \dots, \omega_n)$, $dR = dR' dr_n = dr_1 \cdots dr_n$, etc. Then

$$\begin{aligned} &W_\varepsilon f(x) \\ &= \int_{[0,\infty)^n \times \Sigma^n} \left\{ e^{ixr_n\omega_n} \int_{sr_n}^\infty \int_{[0,\infty)^{n-1}} e^{i\sum_{j=1}^n s_j(r_j/2 + ij\varepsilon/r_j)} K_n(R, \Omega) f(x+\rho) dS' ds_n \right\} dR d\Omega , \end{aligned}$$

where $\rho = \sum_{j=1}^n s_j \omega_j$ and

(2.15) $$K_n(R, \Omega) = (r_1 \cdots r_n)^{m-2} \prod_{j=1}^n \hat{V}_1(r_j \omega_j - r_{j-1} \omega_{j-1}) , \qquad r_0 \omega_0 = 0 .$$

We then change the order of integrations with respect to R and S to obtain

(2.16) $$W_\varepsilon f(x) = \int \left\{ \int_0^{s_n/s} \int_{[0,\infty)^{n-1}} e^{i\sum_{j=1}^n s_j(r_j/2 + ij\varepsilon/r_j) + ixr_n\omega_n} K_n(R, \Omega) dR' dr_n \right\} f(x+\rho) dS d\Omega ,$$

where the first integration is taken over $[0, \infty)^n \times \Sigma^n$ and it should be understood that $s_0 \omega_0 = 0$ and that $s_n / s = \infty$ if $s = 0$. We denote $Wf = W_0 f$:

$$\text{(2.17)} \qquad Wf(x)=\int\left\{\int_0^{s_n/s}\int_{[0,\infty)^{n-1}} e^{i\sum_{j=1}^{n} s_j r_j/2+ixr_n\omega_n} K_n(R,\Omega)dR'dr_n\right\} f(x+\rho)dSd\Omega\,.$$

LEMMA 2.2. *Let* $1\le p\le\infty$ *and* $f\in L^p(\boldsymbol{R}^m)$. *Then as* $\varepsilon\to 0$, $\|W_\varepsilon f-Wf\|_p\to 0$.

PROOF. We apply integration by parts to (2.16) with respect to the variables $r_1,\dots,r_n$, twice each, using the identities

$$\frac{e^{-is_j(r_j/2+ij\varepsilon/r_j)}}{i(s_j/2)(1-2ij\varepsilon/r_j^2)}\frac{\partial}{\partial r_j}e^{is_j(r_j/2+ij\varepsilon/r_j)}=1\,,\qquad j=1,\dots,n-1$$

$$\frac{e^{-is_n(r_n/2+in\varepsilon/r_n)-ir_nx\omega_n}}{i(s_n/2+x\omega_n)(1-in\varepsilon'/r_n^2)}\frac{\partial}{\partial r_n}e^{is_n(r_n/2+in\varepsilon/r_n)+ir_nx\omega_n}=1\,,$$

where $\varepsilon'=s_n\varepsilon/(s_n/2+x\omega_n)$. Then, the lemma follows by applying the argument of the proof of Lemma 2.3 of [18] to the resulting integrals. The boundary terms which appear from the end point s_n/s of r_n-integration are easier to handle. We omit the details. ∎

It follows by applying Lemma 2.2 to (2.2) and (2.10) that the operators $F(t_{R_j},t_{R_{j+1}})$ are the product of operators of the form (2.17) with different n's and different s's. Introducing the notation

$$\text{(2.18)} \qquad G_n(r_n,S',\Omega)=\int_{[0,\infty)^{n-1}} e^{i\sum_{j=1}^{n-1} r_j s_j/2}K_n(R,\Omega)dR'$$

and

$$\text{(2.19)} \qquad \hat{K}_n(\alpha,\tau,S',\Omega)=\int_0^\alpha e^{i\tau r_n}G_n(r_n,S',\Omega)dr_n\,,$$

we write $Wf(x)$ in the form

$$\text{(2.20)} \qquad Wf(x)=\int_{[0,\infty)^n\times\Sigma^n}\hat{K}_n(s_n/s,s_n/2+x\omega_n,S',\Omega)f(x+\rho)dSd\Omega\,.$$

We recall $\rho=\sum_{j=1}^n s_j\omega_j$.

3. Estimates. In this section, we show that the operator W defined by (2.20) is bounded in $L^p(\boldsymbol{R}^m)$ for any $1\le p\le\infty$ and estimate $\|W\|_{B(L^p)}$ in terms of a certain norm of V_1, assuming $V_1\in C_0^\infty(\boldsymbol{R}^m)$. When $s=0$, W is nothing but the operator W_n studied in [18] and [19] and the estimates in Proposition 3.2 below are known. Thus we assume $s>0$ in what follows. The following argument is a modification of what is given in [18] and [19] to the case $s\ne 0$.

LEMMA 3.1. *Let W be the operator defined by* (2.20). *Then*:

$$(3.1)\qquad \|Wf\|_{L^1}\le\|f\|_{L^1}\sup_{x\in\mathbf{R}^m}\int_{[0,\infty)^n\times\Sigma^n}|\hat{K}_n(s_n/s,\,s_n/2+(x-\rho)\omega_n,\,S',\,\Omega)|dSd\Omega\,.$$

$$(3.2)\qquad \|Wf\|_{L^\infty}\le\|f\|_{L^\infty}\sup_{x\in\mathbf{R}^m}\int_{[0,\infty)^n\times\Sigma^n}|\hat{K}_n(s_n/s,\,s_n/2+x\omega_n,\,S',\,\Omega)|dSd\Omega\,.$$

PROOF. The estimate (3.2) is obvious. We prove (3.1). Integrating the modulus of both sides of (2.20) with respect to x, we have

$$\|Wf\|_{L^1}\le\int_{[0,\infty)^n\times\Sigma^n\times\mathbf{R}^m}|\hat{K}_n(s_n/s,\,s_n/2+x\omega_n,\,S',\,\Omega)f(x+\rho)|dSd\Omega dx\,.$$

Change the variables $(S,\Omega,x)\mapsto(S,\Omega,x-\rho)$ and integrate the result with respect to (S,Ω) first. The estimate (3.1) follows immediately. ∎

The integrals appearing on the right of (3.1) and (3.2) are estimated as follows.

PROPOSITION 3.2. *Let $1/m_*<\sigma<1$ and $\gamma>0$. Then there exists a constant $C>0$ independent of V and s such that both integrals*

$$(3.3)\qquad \sup_{x\in\mathbf{R}^m}\int_{[0,\infty)^n\times\Sigma^n}|\hat{K}_n(s_n/s,\,s_n/2+(x-\rho)\omega_n,\,S',\,\Omega)|dSd\Omega$$

and

$$(3.4)\qquad \sup_{x\in\mathbf{R}^m}\int_{[0,\infty)^n\times\Sigma^n}|\hat{K}_n(s_n/s,\,s_n/2+x\omega_n,\,S',\,\Omega)|dSd\Omega$$

are bounded by $(1+|s|^{1/m_*})(C\|\mathscr{F}(\langle x\rangle^{\sigma+1}\langle D\rangle^{\gamma}V)\|_{L^{m_*}})^n$.

PROOF. We prove the proposition for (3.3) only. The proof for (3.4) is similar. Using the identity

$$e^{i\tau r}=\frac{1}{1+\tau^2}\left(1-i\tau\frac{\partial}{\partial r}\right)e^{i\tau r}$$

and applying integration by parts to (2.19), we obtain

$$\hat{K}_n(\alpha,\tau,S',\Omega)=\mathrm{I}_1(\alpha,\tau,S',\Omega)+\mathrm{I}_2(\alpha,\tau,S',\Omega)+\mathrm{I}_3(\alpha,\tau,S',\Omega)\,,$$

where

$$\mathrm{I}_1(\alpha,\tau,S',\Omega)=\frac{-i\tau}{1+\tau^2}e^{i\tau\alpha}G_n(\alpha,S',\Omega)\,,$$

$$\mathrm{I}_2(\alpha,\tau,S',\Omega)=\frac{1}{1+\tau^2}\int_0^{\alpha}e^{i\tau r}G_n(r,S',\Omega)dr\,,$$

and

$$\mathrm{I}_3(\alpha, \tau, S', \Omega) = \frac{i\tau}{1+\tau^2} \int_0^\alpha e^{i\tau r} \frac{\partial}{\partial r} G_n(r, S', \Omega) dr .$$

Note that the boundary term does not appear from the zero end point since $m \geq 3$.

We first estimate the contribution of I_1 to the integral (3.3). Write $g(\tau) = -i\tau/(1+\tau^2)$ and $\langle S' \rangle_{\text{sep}} = \langle s_1 \rangle \cdots \langle s_{n-1} \rangle$. When $m \geq 4$, take δ and λ so that $\delta + \lambda = \sigma$, $\lambda > 1/(m-1)$ and $\delta > (m-3)/(m-1)$; and when $m = 3$ set $\lambda = \sigma$ and $\delta = 0$. Using Hölder's inequality with $(m-3)/(m-1) + 1/(m-1) + 1/(m-1) = 1$, we have

$$\begin{aligned}
&\int_{[0,\infty)^{n-1}} |\mathrm{I}_1(s_n/s, s_n/2 + (x-\rho)\omega_n, S', \Omega)| dS' \\
&\leq \int_{[0,\infty)^{n-1}} \langle S' \rangle_{\text{sep}}^{-\delta} \cdot |g(s_n/2 + (x-\rho)\omega_n)| \langle S' \rangle_{\text{sep}}^{-\lambda} \cdot \langle S' \rangle_{\text{sep}}^{\sigma} |G(s_n/s, S', \Omega)| dS' \\
&\leq \| \langle S' \rangle_{\text{sep}}^{-\delta} \|_{L^{(m-1)/(m-3)}([0,\infty)^{n-1})} \left(\int_{[0,\infty)^{n-1}} (|g(s_n/2 + (x-\rho)\omega_n)| \langle S' \rangle_{\text{sep}}^{-\lambda})^{m-1} dS' \right)^{1/(m-1)} \\
&\quad \times \left(\int_{[0,\infty)^{n-1}} (|G(s_n/s, S', \Omega)| \langle S' \rangle_{\text{sep}}^{\sigma})^{m-1} dS' \right)^{1/(m-1)} .
\end{aligned}$$

The first factor on the right is clearly bounded by C^n. We integrate both sides with respect to s_n and estimate the right hand side by using Hölder's inequality with $1/(m-1) + 1/m_* = 1$. Then,

$$\begin{aligned}
(3.5) \quad &\int_{[0,\infty)^n} |\mathrm{I}_1(s_n/s, s_n/2 + (x-\rho)\omega_n, S', \Omega)| dS \\
&\leq C^n \left(\int_{[0,\infty)^n} |g(s_n/2 + (x-\rho)\omega_n)|^{m-1} \langle S' \rangle_{\text{sep}}^{-(m-1)\lambda} dS \right)^{1/(m-1)} \\
&\quad \times \left\{ \int_0^\infty \left(\int_{[0,\infty)^{n-1}} (|G(s_n/s, S', \Omega)| \langle S' \rangle_{\text{sep}}^{\sigma})^{m-1} dS' \right)^{m_*/(m-1)} ds_n \right\}^{1/m_*} .
\end{aligned}$$

It is easy to see by changing the variables $(S', s_n) \mapsto (S', 2(s_n - (x-\rho)\omega_n))$ that the first integral on the right is bounded by C^n with a possibly different constant C. We then further integrate both sides with respect to Ω. Changing the variable $s_n \mapsto s s_n$ and using Hölder's inequality, we obtain

$$\begin{aligned}
(3.6) \quad &\int_{[0,\infty)^n \times \Sigma^n} |\mathrm{I}_1(s_n/s, s_n/2 + (x-\rho)\omega_n, S', \Omega)| dS d\Omega \\
&\leq C^n |\Sigma|^{n/(m-1)} |s|^{1/m_*} \left(\int_0^\infty \| G(s_n, S', \Omega) \langle S' \rangle_{\text{sep}}^{\sigma} \|_{L^{m-1}([0,\infty)_{S'}^{n-1})}^{m_*} ds_n d\Omega \right)^{1/m_*} ,
\end{aligned}$$

where $|\Sigma|$ is the measure of Σ. We estimate the last integral by interpolating the estimates for the case $\sigma = 0$ and for $\sigma = 1$. Remembering (2.18) that $G(r_n, \cdot, \Omega)$ is the Fourier transform of $K(\cdot, r_n, \Omega)$, we apply Hausdorff-Young's inequality to the S' integral. Then,

$$\begin{aligned}(3.7)\quad &\left(\int_{[0,\infty)\times\Sigma^n} \|G(s_n, S', \Omega)\|^{m_*}_{L^{m-1}([0,\infty)^{n-1}_{S'})} ds_n d\Omega\right)^{1/m_*} \\ &\le C^n\left(\int_{[0,\infty)\times\Sigma^n} \|K(R', s_n, \Omega)\|^{m_*}_{L^{m_*}([0,\infty)^{n-1}_{R'})} ds_n d\Omega\right)^{m_*} \\ &\le C^n\left(\int_{[0,\infty)\times\Sigma^n} (r_1 \dots r_n)^{m-1} \prod_{j=1}^{n} |\hat{V}_1(r_j\omega_j - r_{j-1}\omega_{j-1})|^{m_*} dR d\Omega\right)^{1/m_*} \\ &= C^n \|\hat{V}_1\|^n_{m_*}.\end{aligned}$$

Here we used the relation $m_*(m-2)=m-1$ and $(r_1\cdots r_n)^{m-1}dRd\Omega = d\xi_1\cdots d\xi_n$. To estimate the case $\sigma=1$, we let $\alpha=(\alpha_1,\dots,\alpha_{n-1})$ be a multi-index whose entries are either 0 or 1. The integration by parts shows that

$$S'^{\alpha}G_n(r_n, S', \Omega) = \int_{[0,\infty)^{n-1}} e^{i\sum_{j=1}^{n-1} r_j s_j/2}(-2D_{R'})^{\alpha} K_n(R, \Omega) dR'.$$

Here we used the assumption that $m\ge 3$ and, hence the boundary term from the zero end point did not appear. The argument which was used in the estimate (3.7) produces

$$\begin{aligned}(3.8)\quad &\left(\int_{[0,\infty)\times\Sigma^n} \|S'^{\alpha}G(s_n, S', \Omega)\|^{m_*}_{L^{m-1}((0,\infty)^{n-1}_{S'})} ds_n d\Omega\right)^{1/m_*} \\ &\le C^n\left(\int_{[0,\infty)\times\Sigma^n} \left|(-D_{R'})^{\alpha}\{(r_1\cdots r_n)^{m-2}\prod_{j=1}^{n}\hat{V}_1(r_j\omega_j - r_{j-1}\omega_{j-1})\}\right|^{m_*} dR d\Omega\right)^{1/m_*}.\end{aligned}$$

The effect of the r_j derivative is either to decrease the power of r_j^{m-2} by one or to differentiate $\hat{V}_1(r_j\omega_j - r_{j-1}\omega_{j-1})$ or $\hat{V}_1(r_{j+1}\omega_j - r_j\omega_j)$. Hence, applying Hardy's inequality, we obtain (cf. [18, p. 569]) that

$$(3.9)\quad \left(\int_{[0,\infty)\times\Sigma^n} \|S'^{\alpha}G(s_n, S', \Omega)\|^{m_*}_{L^{m-1}((0,\infty)^{n-1}_{S'})} ds_n d\Omega\right)^{1/m_*} \le C^n \|\langle D\rangle^2 \hat{V}_1\|^n_{m_*}.$$

Interpolating (3.7) and (3.9) by the multi-linear complex interpolation theorem, we conclude

$$(3.10)\quad \left(\int_{[0,\infty)\times\Sigma^n} \|\langle\cdot\rangle^{\sigma}_{\text{sep}} G(s_n, \cdot, \Omega)\|^{m_*}_{m-1} ds_n d\Omega\right)^{1/m_*} \le C^n \|\langle D\rangle^{2\sigma}\hat{V}_1\|^n_{m_*}$$

and, hence we obtain the desired estimate

$$\begin{aligned}(3.11)\quad &\sup_{x\in\boldsymbol{R}^m}\int_{[0,\infty)^n\times\Sigma^n} |I_1(s_n/s, s_n/2 + (x-\rho)\omega_n, S', \Omega)| dS d\Omega \\ &\le |s|^{1/m_*}(C_1|\Sigma|^{1/(m-1)}\|\mathscr{F}(\langle x\rangle^{2\sigma}V_1)\|_{L^{m_*}})^n.\end{aligned}$$

To estimate the contributions of I_2 and I_3 to the integral (3.3), we use the following lemma.

LEMMA 3.3. *Let* $1<p<\infty$. *Suppose* $g\in L^{p/(p-1)}(\boldsymbol{R})\cap L^{p/(p-1)+\varepsilon}(\boldsymbol{R})$ *for some* $0<\varepsilon$. *Let* $A(u, v)$ *be defined by*

$$A(u, v)=g(v)\int_0^u e^{ivx}f(x)dx\,.$$

Then there exists a constant $C>0$ *depending only on* g *and* p *such that*

$$\sup_{x\in\boldsymbol{R}}\int_{-\infty}^{\infty}|A(t/s, x\pm t/2)|dt\le C\left(\|\hat{f}\|_p+\int_{-1}^{1}\frac{1}{|h|}\|\hat{f}(\cdot+h)-\hat{f}\|_p dh\right) \tag{3.12}$$

where the Fourier transform $\hat{f}$ *is taken after setting* $f(x)=0$ *for* $x<0$.

PROOF. We set $f(x)=0$ for $x<0$. Writing the Heaviside function by $\theta(x)$: $\theta(x)=1$ for $x\ge 0$ and $\theta(x)=0$ *for* $x<0$, we compute

$$\begin{aligned}\int_0^u e^{ivx}f(x)dx&=\int_{-\infty}^{\infty}e^{ivx}\theta(u-x)f(x)dx=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{iyu}}{y-i0}\hat{f}(y-v)dy\\&=\frac{1}{2}\hat{f}(-v)+\frac{1}{2\pi i}\,\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{iyu}}{y}\hat{f}(y-v)dy\end{aligned}$$

where p.v. means Cauchy's principal value. We decompose the singular integral as follows:

$$\begin{aligned}\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{iyu}}{y}\hat{f}(y-v)dy&=\int_{|y|>1}\frac{e^{iyu}}{y}\hat{f}(y-v)dy\\&\quad+\text{p.v}\int_{-1}^{1}\frac{e^{iyu}}{y}\hat{f}(-v)dy+\int_{-1}^{1}\frac{e^{iyu}}{y}(\hat{f}(y-v)-\hat{f}(-v))dy\,.\end{aligned}$$

Correspondingly, we decompose

$$A(u, v)=\mathrm{II}_1(u, v)+\mathrm{II}_2(u, v)+\mathrm{II}_3(u, v)+\mathrm{II}_4(u, v)$$

and estimate the contribution of each summand to the integral (3.12) separately: By Hölder's inequality we have for any $1\le p<\infty$

$$\int_{-\infty}^{\infty}|\mathrm{II}_1(t/s, x\pm t/2)|dt=\int_{-\infty}^{\infty}|g(x\pm t/2)||\hat{f}(x\pm t/2)|dt\le 2\|g\|_{p/(p-1)}\|\hat{f}\|_p\,. \tag{3.13}$$

We estimate as

$$|\mathrm{II}_2(u, v)|\le|g(v)|\cdot\frac{1}{2\pi}\int_{|y|>1}\frac{|\hat{f}(v-y)|}{|y|}dy$$

and use Hölder's inequality and Young's inequality for the convolution to obtain

$$\text{(3.14)} \qquad \int_{-\infty}^{\infty} |\mathrm{II}_2(t/s, x \pm t/2)|dt \le \pi^{-1} \|g\|_q \|\hat{f}\|_p \|\chi_{\{|y| \ge 1\}} |y|^{-1}\|_r ,$$

where $q > 1$ and $r > 1$ should be taken so that $1/p + 1/q + 1/r = 2$. Since

$$\text{p.v.} \int_{-1}^{1} \frac{e^{iyu}}{y} dy = 2i \int_{0}^{1} \frac{\sin yu}{y} dy$$

is uniformly bounded with respect to u, we have as in (3.13)

$$\text{(3.15)} \qquad \int_{-\infty}^{\infty} |\mathrm{II}_3(t/s, x \pm t/2)|dt \le C \|g\|_{p/(p-1)} \|\hat{f}\|_p .$$

Finally we estimate

$$|\mathrm{II}_4(u, v)| \le |g(v)| \cdot \frac{1}{2\pi} \int_{-1}^{1} \frac{|\hat{f}(y-v) - \hat{f}(-v)|}{|y|} dy$$

and use Hölder's inequality and Minkowski's inequality to obtain

$$\text{(3.16)} \qquad \int_{-\infty}^{\infty} |\mathrm{II}_4(t/s, x \pm t/2)|dt \le \pi^{-1} \|g\|_{p/(p-1)} \int_{-1}^{1} \frac{\|\hat{f}(y + \cdot) - \hat{f}(\cdot)\|_p}{|y|} dy .$$

The combination of the estimates (3.13), (3.14), (3.15) and (3.16) yields (3.12). ∎

We now estimate the contribution of I_2 to the integral (3.3). Noting that $s_n/2 - \rho\omega_n = -s_n/2 + \rho'\omega_n$ and $\rho' = \sum_{j=1}^{n-1} s_j\omega_j$ is independent of s_n, we apply Lemma 3.3 to $f(r) = G_n(r, S', \Omega)$ with $g(v) = 1/(1+v^2)$ and $p = m-1$. It follows that

$$\text{(3.17)} \quad \sup_{x \in \boldsymbol{R}^m} \int_{0}^{\infty} |\mathrm{I}_2(s_n/s, s_n/2 + (x-\rho)\omega_n, S', \Omega)|ds_n$$
$$\le C\left(\|\hat{G}_n(\cdot, S', \Omega)\|_{m-1} + \int_{-1}^{1} \frac{1}{|h|} \|\hat{G}_n(\cdot + h, S', \Omega) - \hat{G}_n(\cdot, S', \Omega)\|_{m-1} dh \right),$$

where $\hat{G}_n(\cdot, S', \Omega)$ is the Fourier transform of $G_n(s, S', \Omega)$ with respect to the first variable s after setting $G_n(s, S', \Omega) = 0$ for $s < 0$. We integrate both sides of (3.17) with respect to S' and Ω. Using Hölder's inequality, we estimate the integral with respect to S' of first summand on the right as

$$\text{(3.18)} \quad \int_{[0,\infty)^{n-1}} \|\hat{G}_n(\cdot, S', \Omega)\|_{m-1} dS' \le C^n \left(\int_{[0,\infty)^n} |\langle S' \rangle_{\text{sep}}^{\sigma} \hat{G}_n(s, S', \Omega)|^{m-1} dS \right)^{1/(m-1)}$$

Note that $\hat{G}_n(s, S', \Omega)$ is nothing but the Fourier transform of $K(R, \Omega)$ with respect to the all radial variables $R = (r_1, \ldots, r_n)$. Hence the argument similar to the one used for obtaining the estimate (3.10) implies the following for the $d\Omega$ integral:

$$(3.19)\qquad \int_{[0,\infty)^{n-1}\times\Sigma^n} \|\hat{G}_n(\cdot, S', \Omega)\|_{m-1} dS' d\Omega \le (C_2|\Sigma|^{1/(m-1)}\|\mathscr{F}(\langle x\rangle^{2\sigma}V_1)\|_{m_*})^n .$$

In virtue of the modulus of continuity estimate for the functions in the Sobolev spaces (cf. [1]) we have for arbitrarily small $\gamma>0$.

$$(3.2)\qquad \int_{[0,\infty)^{n-1}\times\Sigma^n}\left(\int_{-1}^{1}\frac{1}{|h|}\|\hat{G}_n(\cdot+h, S', \Omega)-\hat{G}_n(\cdot, S', \Omega)\|_{m-1}dh\right)dS'd\Omega$$
$$\le C_\gamma\int_{[0,\infty)^{n-1}\times\Sigma^n}\|\langle D_1\rangle^\gamma\hat{G}_n(\cdot, S', \Omega)\|_{m-1}dS'd\Omega .$$

The integral on the right may be estimated entirely similarly as in (3.19) and is bounded by

$$(C_2|\Sigma|^{1/(m-1)}\|\mathscr{F}(\langle x\rangle^{2\sigma}\langle D\rangle^\gamma V)\|_{m_*})^n .$$

Thus we have for arbitrary small $\gamma>0$:

$$(3.21)\qquad \sup_{x\in\mathbf{R}^m}\int_{[0,\infty)^n\times\Sigma^n}|\mathrm{I}_2(s_n/s,\ s_n/2+(x-\rho)\omega_n,\ S',\ \Omega)|dSd\Omega$$
$$\le\int_{[0,\infty)^{n-1}\times\Sigma^n}\left(\sup_{x\in\mathbf{R}^m}\int_0^\infty|\mathrm{I}_2(s_n/s,\ -s_n/2+(x-\rho')\omega_n,\ S',\ \Omega)|ds_n\right)dS'd\Omega$$
$$\le(C_2|\Sigma|^{1/(m-1)}\|\mathscr{F}(\langle x\rangle^{2\sigma}\langle D\rangle^\gamma V_1)\|_{L^{m_*}})^n .$$

The contribution of I_3 may be estimated entirely similarly as above by replacing $G(r, S', \Omega)$ by $(\partial/\partial r)G(r, S', \Omega)$, and we obtain

$$(3.22)\qquad \sup_{x\in\mathbf{R}^m}\int_{[0,\infty)^n\times\Sigma^n}|\mathrm{I}_3(s_n/s,\ s_n/2+(x-\rho)\omega_n,\ S',\ \Omega)|dSd\Omega$$
$$\le(C_2|\Sigma|^{1/(m-1)}\|\mathscr{F}(\langle x\rangle^{\sigma+1}\langle D\rangle^\gamma V_1)\|_{L^{m_*}})^n .$$

Here we used the condition $\sigma\le 1$ and the fact that the variable r_n appears only once on the right side of (2.15). Combining the estimates (3.11), (3.21) and (3.22), we have proven the required estimate for (3.3) and have completed the proof of Proposition. ∎

The right hand sides of (3.1) and (3.2) of Lemma 3.1 are both bounded by the same quantity as in Proposition 3.2. Thus the complex interpolation theorem implies the following:

COROLLARY 3.4. *For $\gamma>0$ there exists $C_\gamma>0$ such that for any $1\le p\le\infty$*

$$(3.23)\qquad \|Wf\|_p\le\|f\|_p(1+|s|^{1/m_*})(C_\gamma\|\mathscr{F}(\langle x\rangle^{\sigma+1}\langle D\rangle^\gamma V_1)\|_{L^{m_*}})^n .$$

We now fix $T>0$ and assume that all relevant time variables are in the interval

$[0, T]$. We apply Corollary 3.4 to the factors (2.11) and (2.12) of $F(t_{R_j}, t_{R_{j+1}})$ in (2.10) and obtain the bound on the operator norm in $L^p(\boldsymbol{R}^m)$:

$$\|F(t_{R_j}, t_{R_{j+1}})\|_{B(L^p)} \le (2C_\gamma \|\mathscr{F}(\langle x\rangle^{\sigma+1}\langle D\rangle^\gamma V_1)\|_{L^{m_*}})^{R_{j+1}-R_j+1}\,. \tag{3.24}$$

In virtue of Lemma 1.3, it is obvious that

$$\|\Gamma_2(t)f\|_{L^p(\boldsymbol{R}^m)} \le \mathrm{Var}(\hat{V}_2)\|f\|_{L^p(\boldsymbol{R}^m)} \tag{3.25}$$

where $\mathrm{Var}(\mu)$ is the total variation of the (signed) measure μ. Applying the estimates (3.24) and (3.25) to (1.15), we see that, for all $|t|\le T$, the operator norm $\|G_n^A(t)\|$ in $B(L^p(\boldsymbol{R}^m))$ is bounded as follows:

$$\|G_n^A(t)\| \le (\mathrm{Var}(\hat{V}_2)^l T^l/l!)(C_\gamma\|\mathscr{F}(\langle x\rangle^{\sigma+1}\langle D\rangle^\gamma V_1)\|_{L^{m_*}})^{n-l}\,. \tag{3.26}$$

Now we suppose

$$C_\gamma\|\mathscr{F}(\langle x\rangle^{\sigma+1}\langle D\rangle^\gamma V_1)\|_{L^{m_*}} = \kappa < 1 \tag{3.27}$$

and write $\mathrm{Var}(\hat{V}_2) = C_\kappa$. Then

$$\|G_n(t)\|_{B(L^p(\boldsymbol{R}^m))} \le \sum_A \|G_n^A(t)\| \le \sum_{l=0}^n \binom{n}{l}\frac{(C_\kappa T)^l}{l!}\kappa^{n-l}$$

and

$$\begin{aligned}\sum_{n=0}^\infty \|G_n(t)\| &\le \sum_{n=0}^\infty\sum_{l=0}^n \frac{n(n-1)\cdots(n-l+1)}{(l!)^2}(C_\kappa T)^l\kappa^{n-l}\\ &= \sum_{l=0}^\infty \frac{(C_\kappa T)^l}{(l!)^2}\sum_{n=l}^\infty\left(\frac{d}{d\kappa}\right)^l\kappa^n = \sum_{l=0}^\infty \frac{(C_\kappa T)^l}{(l!)^2}\left(\frac{d}{d\kappa}\right)^l\sum_{n=0}^\infty\kappa^n\\ &= \sum_{l=0}^\infty\frac{(C_\kappa T)^l}{l!}\left(\frac{1}{1-\kappa}\right)^{l+1} = \frac{1}{1-\kappa}\exp\left(\frac{C_\kappa T}{1-\kappa}\right).\end{aligned} \tag{3.28}$$

4. Proof of the Theorem.

We take and fix $T>0$ arbitrarily and choose $\varepsilon>0$ small enough so that $C_{\gamma\varepsilon} = \kappa < 1$ in (3.27) is satisfied when we decompose $V = V_1 + V_2$ as in the Theorem. We then take a sequence of smooth functions $V_1^{(j)} \in C_0^\infty(\boldsymbol{R}^m)$ such that

$$\|\mathscr{F}(\langle x\rangle^{\sigma+1}\langle D\rangle^\gamma(V_1^{(j)} - V_1))\|_{L^{m_*}} \to 0\,, \qquad j\to\infty\,. \tag{4.1}$$

Denote $H_j = -(1/2)\Delta + V_1^{(j)} + V_2$. In virtue of (4.1) and Young's inequality, we have

$$\|V_1^{(j)} - V_1\|_{L^{m-1}} \le \|\langle x\rangle^{\sigma+1}\langle D\rangle^\gamma(V_1^{(j)} - V_1)\|_{L^{m-1}} \to 0\,, \qquad j\to\infty$$

and $(H_j - z)^{-1} \to (H-z)^{-1}$ in $B(L^2(\boldsymbol{R}^m))$ for all $z\in \boldsymbol{C}\setminus\boldsymbol{R}$. It follows that e^{-itH_j} converges to e^{-itH} strongly in $L^2(\boldsymbol{R}^m)$.

Denote the $G_\infty(t)$ corresponding to $V_1^{(j)} + V_2$ by $G_\infty(t)^{(j)}$. Note that $G_n^A(t)$ is multilinear in V_1. It follows by the argument used for proving (3.26) and (3.28) that $G_\infty(t)^{(j)}$ converges uniformly for $|t|\le T$ in the topology of operator norm in $L^p(\boldsymbol{R}^m)$ for any

$1 \leq p \leq \infty$, the limit of which is denoted by $G_\infty(t) = \lim_{j \to \infty} G_\infty(t)^{(j)}$. It follows by taking the limit $j \to \infty$ in

$$e^{-itH_j} = e^{-itH_0} G_\infty(t)^{(j)},$$

that, as operators in $L^2(\boldsymbol{R}^m)$,

$$e^{-itH} = e^{-itH_0} G_\infty(t)$$

and that e^{-itH} extends to a bounded operator from $L^q(\boldsymbol{R}^m)$ to $L^p(\boldsymbol{R}^m)$ for any $1 \leq q \leq 2$ and $p = q/(q-1)$ with norm

$$\|e^{-itH}\|_{B(L^q, L^p)} \leq C_T |t|^{-m(1/2-1/p)}$$

for $|t| \leq T$. Moreover, as the operators from $L^q(\boldsymbol{R}^m)$ to $L^p(\boldsymbol{R}^m)$ thus extended, the convergence of e^{-itH_j} to e^{-itH} is uniform with respect to $0 < \delta \leq t \leq T$ in the topology of operator norm.

It follows that the FDS $E(t, x, y)$ of (1.1) is bounded by $C_T |t|^{-m/2}$ and, if $\hat{V}_2$ is of $L^1(\boldsymbol{R}^m)$, $E(t, x, y)$ is continuous since the FDS $E_j(t, x, y)$ is continuous and converges uniformly to $E(t, x, y)$ with respect to $(t, x, y) \in [\delta, T] \times \boldsymbol{R}^m \times \boldsymbol{R}^m$. This completes the proof.

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