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THE RIGIDITY FOR REAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

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Abstract. We prove a rigidity theorem for real hypersurfaces in a complex projective space of complex dimension $n \geq 4$. As an application of this rigidity theorem, we classify all intrinsically homogeneous real hypersurfaces in the complex projective space.

Introduction. Let $P_n(C)$ be an n-dimensional complex projective space. It is an open question whether a real hypersurface in $P_n(C)$ has rigidity or not. More precisely, *if M is a $(2n-1)$-dimensional Riemannian manifold and ι, $\hat{\iota}$ are two isometric immersions of M into $P_n(C)$, then are ι and $\hat{\iota}$ congruent*?

To this problem, many authors including the present ones gave some partial solutions (see [1], [3], [4] and [5]). Recall that an almost contact structure (ϕ, ξ, η) is naturally induced on a real hypersurface in $P_n(C)$ from the complex structure of $P_n(C)$, and ξ is called the *structure vector field*. The rank of the second fundamental tensor or the shape operator of a real hypersurface in $P_n(C)$ is said to be the *type number*. As one of the above-mentioned solutions, the following is known.

THEOREM A ([1]). *Let M be a $(2n-1)$-dimensional connected Riemannian manifold, and ι and $\hat{\iota}$ be two isometric immersions of M into $P_n(C)$ $(n \geq 3)$. If the two structure vector fields coincide up to sign on M and the type number of (M, ι) or $(M, \hat{\iota})$ is not equal to 2 at every point of M, then ι and $\hat{\iota}$ are rigid, that is, there exists an isometry φ of $P_n(C)$ such that $\varphi \circ \iota = \hat{\iota}$.*

The purpose of this paper is to give a solution of the rigidity problem using Theorem A. Namely, first of all we shall prove:

THEOREM 1. *Let M be a $(2n-1)$-dimensional Riemannian manifold, and ι and $\hat{\iota}$ be two isometric immersions of M into $P_n(C)$ $(n \geq 4)$. Then the two structure vector fields coincide up to sign on M.*

The following is immediate from Theorems A and 1.

THEOREM 2. *Let M be a $(2n-1)$-dimensional connected Riemannian manifold, and ι and $\hat{\iota}$ be two isometric immersions of M into $P_n(C)$ $(n \geq 4)$. If the type number of (M, ι) or $(M, \hat{\iota})$ is not equal to 2 at every point of M, then ι and $\hat{\iota}$ are rigid, that is, there exists*

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an isometry φ *of* $P_n(\boldsymbol{C})$ *such that* $\varphi \circ \imath = \hat{\imath}$.

There are two concepts of homogeneous real hypersurfaces in $P_n(\boldsymbol{C})$. A real hypersurface M in $P_n(\boldsymbol{C})$ is said to be *intrinsically homogeneous* if for any points p and q in M there exists an isometry σ of M such that $\sigma(p)=q$, and *extrinsically homogeneous* if for any points p and q in M there exists an isometry φ of $P_n(\boldsymbol{C})$ such that $\varphi(p)=q$ and $\varphi(M)=M$. It is clear that an extrinsically homogeneous real hypersurface in $P_n(\boldsymbol{C})$ is intrinsically homogeneous. The first author classified all extrinsically homogeneous real hypersurfaces in $P_n(\boldsymbol{C})$, which consist of the so-called six model spaces of types A_1, A_2, B, C, D and E ([6], [7]).

As an application of Theorem 2, we shall classify all intrinsically homogeneous real hypersurfaces in $P_n(\boldsymbol{C})$ ($n \geq 4$). Namely, we can state:

THEOREM 3. *Let* M *be a* $(2n-1)$*-dimensional connected homogeneous Riemannian manifold. If* M *admits an isometric immersion* $\imath$ *into* $P_n(\boldsymbol{C})$ ($n \geq 4$), *then* $\imath(M)$ *is extrinsically homogeneous, that is, congruent to one of the model spaces of six types.*

1. Preliminaries. We denote by $P_n(\boldsymbol{C})$ a complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $4c$ and M a $(2n-1)$-dimensional Riemannian manifold. Let $\imath$ be an isometric immersion of M into $P_n(\boldsymbol{C})$. For a local orthonormal frame field $\{e_1, \dots, e_{2n-1}\}$ of M, we denote its dual 1-forms by θ_i, where and in the sequel the indices $i, j, k, l, \dots$ run over the range $\{1, 2, \dots, 2n-1\}$ unless otherwise stated. Then the connection forms θ_{ij} and the curvature forms Θ_{ij} of M are defined by

$$d\theta_i + \sum \theta_{ij} \wedge \theta_j = 0\,, \qquad \theta_{ij} + \theta_{ji} = 0\,,$$
$$\Theta_{ij} = d\theta_{ij} + \sum \theta_{ik} \wedge \theta_{kj}$$

respectively. We denote the components of the shape operator or the second fundamental tensor A of $(M, \imath)$ by A_{ij}, and put $\psi_i = \sum A_{ij}\theta_j$. Then we have the equations of Gauss and Codazzi

$$\Theta_{ij} = \psi_i \wedge \psi_j + c\theta_i \wedge \theta_j + c\sum (\phi_{ik}\phi_{jl} + \phi_{ij}\phi_{kl})\theta_k \wedge \theta_l\,,$$
$$d\psi_i + \sum \psi_j \wedge \theta_{ji} = c\sum (\xi_j \phi_{ik} + \xi_i \phi_{jk})\theta_j \wedge \theta_k \tag{1.1}$$

respectively, where the triplet ($\phi = (\phi_{ij})$, $\xi = \sum \xi_i e_i$, $\eta = \sum \xi_i \theta_i$) is the almost contact structure on M. The tensor fields ϕ and ξ satisfy

$$\sum \phi_{ik}\phi_{kj} = \xi_i \xi_j - \delta_{ij}\,, \quad \sum \xi_j \phi_{ji} = 0\,, \quad \sum \xi_i^2 = 1\,,$$
$$d\phi_{ij} = \sum (\phi_{ik}\theta_{kj} - \phi_{jk}\theta_{ki}) - \xi_i \psi_j + \xi_j \psi_i\,, \tag{1.2}$$
$$d\xi_i = \sum (\xi_j \theta_{ji} - \phi_{ji}\psi_j)\,.$$

For another isometric immersion $\hat{\imath}$ of M into $P_n(\boldsymbol{C})$, we shall denote the differential forms and tensor fields of $(M, \hat{\imath})$ by the same symbol as the ones in $(M, \imath)$ but with a hat. Then, since $\theta_i=\hat{\theta}_i$ and $\Theta_{ij}=\hat{\Theta}_{ij}$, from (1.1) we have

$$A_{ik}A_{jl}-A_{il}A_{jk}+c(\phi_{ik}\phi_{jl}-\phi_{il}\phi_{jk}+2\phi_{ij}\phi_{kl}) = \hat{A}_{ik}\hat{A}_{jl}-\hat{A}_{il}\hat{A}_{jk}+c(\hat{\phi}_{ik}\hat{\phi}_{jl}-\hat{\phi}_{il}\hat{\phi}_{jk}+2\hat{\phi}_{ij}\hat{\phi}_{kl}) . \tag{1.3}$$

2. Proof of the theorems. In this section we shall prove Theorems 1, 2 and 3.

PROOF OF THEOREM 1. We choose a local orthonormal frame field $\{e_1, e_2, \ldots, e_{2n-2}, e_0\}$ in such a way that

$$\hat{\xi}_1=0, \ldots, \hat{\xi}_{2n-2}=0 \quad \text{and} \quad \hat{\xi}_0=1 , \tag{2.1}$$

where 0 denotes the last index $2n-1$. Then it follows from (1.2) that

$$\hat{\phi}_{i0}=0 . \tag{2.2}$$

If we put $l=0$ in (1.3) and make use of (2.2), then we have

$$A_{ik}A_{j0}-A_{i0}A_{jk}+c(\phi_{ik}\phi_{j0}-\phi_{i0}\phi_{jk}+2\phi_{ij}\phi_{k0})=\hat{A}_{ik}\hat{A}_{j0}-\hat{A}_{i0}\hat{A}_{jk} . \tag{2.3}$$

Here we consider a local vector field

$$u_1=\sum_{j=1}^{2n-2}\hat{A}_{0j}e_j .$$

This vector field is independent of a choice of a local orthonormal frame field $\{e_1, \ldots, e_{2n-2}, e_0\}$ satisfying (2.1), up to sign. In particular, we can define a subset N_1 of M by

$$N_1=\{p\in M \mid u_1(p)\neq 0\} .$$

Assume that $N_1\neq\emptyset$. Then we can take another local orthonormal frame field $\{f_1, \ldots, f_{2n-2}, e_0\}$ on N_1 in such a way that the unit vector field f_{2n-2} is parallel to u_1. If we denote the components of tensor fields with respect to this new orthonormal frame field by the same symbols as those to the old frame field, then we have

$$\hat{A}e_0=\sum_{i=1}^{2n-1}\hat{A}_{0i}e_i=\left\|\sum_{j=1}^{2n-2}\hat{A}_{0j}e_j\right\|f_{2n-2}+\hat{A}_{00}e_0 = \sum_{q=1}^{2n-2}\hat{A}_{0q}f_q+\hat{A}_{00}e_0 ,$$

where $\| \ \|$ indicates the length of a vector field. It implies that

$$\hat{A}_{p0}=0 \qquad \text{for} \quad p=1, 2, \ldots, 2n-3 . \tag{2.4}$$

If $M\setminus\bar{N}_1\neq\emptyset$, then it is obvious that (2.4) holds on $M\setminus\bar{N}_1$ for any local orthonormal frame field $\{e_1, \ldots, e_{2n-2}, e_0\}$ satisfying (2.1). Thus we may assume that (2.4) holds on $M \setminus \partial N_1$. For a while we consider all forms on $M \setminus \partial N_1$. In terms of this new local frame field, it follows from (2.3) and (2.4) that

$$A_{pi}A_{q0} - A_{qi}A_{p0} + c(\phi_{pi}\phi_{q0} - \phi_{p0}\phi_{qi} + 2\phi_{pq}\phi_{i0}) = 0 , \tag{2.5}$$

where $p, q = 1, 2, \dots, 2n-3$, and $i = 1, 2, \dots, 2n-1$.

Here we consider another local vector field

$$u_2 = \sum_{p=1}^{2n-3} A_{p0} f_p .$$

Then, by the same method as in the above, we can define a subset N_2 of M by

$$N_2 = \{ p \in M \setminus \partial N_1 \mid u_2(p) \neq 0 \}$$

and can assume that

$$A_{a0} = 0 \quad \text{on} \quad (M \setminus \partial N_1) \cap (M \setminus \partial N_2) , \tag{2.6}$$

where and in the sequel the indices $a, b, c, \dots$ run over the range $\{1, 2, \dots, 2n-4\}$. Putting $p = a$ and $q = b$ in (2.5) and making use of (2.6), we have

$$\phi_{ai}\phi_{b0} - \phi_{a0}\phi_{bi} + 2\phi_{ab}\phi_{i0} = 0 . \tag{2.7}$$

If we put $i = a$ in (2.7), then we get

$$\phi_{ab}\phi_{a0} = 0 . \tag{2.8}$$

Multiplying (2.7) by ϕ_{ab} and making use of (2.8), we have

$$\phi_{ab}\phi_{i0} = 0 , \tag{2.9}$$

and hence it follows from (2.7) and (2.9) that

$$\phi_{ai}\phi_{b0} = \phi_{a0}\phi_{bi} . \tag{2.10}$$

Let v_1, v_2, v_3 be vectors in the $(2n-4)$-dimensional vector space $\boldsymbol{R}^{2n-4}$ given by

$$v_1 = (\phi_{1\,2n-3}, \phi_{2\,2n-3}, \dots, \phi_{2n-4\,2n-3}) ,$$

$$v_2 = (\phi_{1\,2n-2}, \phi_{2\,2n-2}, \dots, \phi_{2n-4\,2n-2}) ,$$

$$v_3 = (\phi_{10}, \phi_{20}, \dots, \phi_{2n-4\,0}) .$$

Then (2.10) shows that $\{v_1, v_2, v_3\}$ is a linearly dependent subset of $\boldsymbol{R}^{2n-4}$.

Finally, we assert that $\phi_{ab} \neq 0$ for some indices a and b. Indeed, if (ϕ_{ab}) is the zero matrix, then the matrix ϕ is given by

$$\phi=\left(\begin{array}{cc|ccc} 0 & & {}^t v_1 & {}^t v_2 & {}^t v_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -v_1 & & & & \\ -v_2 & & & * & \\ -v_3 & & & & \end{array}\right).$$

Since v_1, v_2 and v_3 are linearly dependent, the rank of ϕ is not greater than 4, a contradiction, because the rank of ϕ is equal to $2n-2$ and $n \geq 4$. Therefore the matrix (ϕ_{ab}) is not zero.

Since there exists a non-zero entry of (ϕ_{ab}), it follows from (2.9) that $\phi_{i0}=0$. It is easily seen from (1.2) that $\xi_0=\pm 1$, and hence the two structure vector fields ξ and $\hat{\xi}$ coincide up to sign on $(M \setminus \partial N_1) \cap (M \setminus \partial N_2)$, and hence on the whole M. This completes the proof of Theorem 1.

PROOF OF THEOREM 2. It is immediate from Theorems 1 and A.

PROOF OF THEOREM 3. Since M is homogeneous, both M and $\iota(M)$ are complete. We denote by $t(p)$ the type number of ι at a point p of M, and define a subset U of M by

$$U=\{p \in M \mid t(p) \geq 3\}.$$

Then obviously U is open. Moreover, by a theorem in [2] or [5], there exists a point p in M such that $t(p) \geq 3$. Therefore the set U is non-empty.

For any points p and q in U, there exists an isometry σ of M such that $\sigma(p)=q$. Then, by Theorem 2, the two isometric immersions $\iota \upharpoonright_U$ and $(\iota \circ \sigma) \upharpoonright_U$ are rigid, that is, $\iota(U)$ is congruent to $\iota(\sigma(U))$. Thus the principal curvatures at p coincide with those at q. This implies that the principal curvatures of M are constant on U. Hence U is closed.

Since M is connected, we have $U=M$ and hence the two isometric immersions ι and $\iota \circ \sigma$ are rigid, that is, there exists an isometry φ of $P_n(\boldsymbol{C})$ such that $\varphi \circ \iota = \iota \circ \sigma$. Therefore $\iota(M)$ is an extrinsically homogeneous real hypersurface in $P_n(\boldsymbol{C})$. As we have already seen in the Introduction, $\iota(M)$ is congruent to one of the model spaces of six types A_1, A_2, B, C, D and E.

REMARK. Theorem 2 is valid for connected real hypersurfaces in a hyperbolic complex space form $H_n(\boldsymbol{C})$ of the same complex dimensions as $P_n(\boldsymbol{C})$ because we can replace $P_n(\boldsymbol{C})$ by $H_n(\boldsymbol{C})$ in the proofs of both Theorem A (see [1]) and Theorem 1.

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