

CUT LOCUS OF A SEPARATING FRACTAL SET IN A RIEMANNIAN MANIFOLD

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(Received January 22, 1997, revised June 17, 1998)

Abstract. We study the geometry of the cut locus of a separating fractal set A in a Riemannian manifold. In particular, we prove that every point of A is a limit point of the cut locus $C(A)$ of A , and the Hausdorff dimension of $C(A)$ is greater than or equal to that of A . Furthermore, we study the cut locus of the well-known Koch snowflake, and show the Hausdorff dimension of its cut locus is $\log 6/\log 3$ which is greater than the Hausdorff dimension, $\log 4/\log 3$, of the Koch snowflake itself. We also give another example for which the Hausdorff dimension of the cut locus stays the same. These two new examples are new fractal objects which are of interest on their own right.

1. Introduction. In Riemannian geometry, the concept of the cut locus plays a very important role in investigating the global structure of a Riemannian manifold M . The cut locus $C(p)$ of a point p is defined to be the set of points $x \in M$ such that there is a geodesic γ from p to x which is minimizing up to x , but stops being minimizing beyond x . It is one of the basic tools in Riemannian geometry. It was used, for example, by Meyers [7] in studying the topological classification of Riemannian manifolds, and more recently, major advances in Riemannian geometry such as the sphere theorem, the soul theorem, etc. are achieved by tools partly employing the concept of the cut locus. One can similarly generalize it to the cut locus $C(S)$ of a set S in a Riemannian manifold. Namely, $C(S)$ can be defined to be the set of point x such that there is a geodesic, from a point p in S to x where any point z on γ between p and x minimizes the distance between the S and z , but γ stops doing so beyond x .

However, as far as we know, the cut locus of a set has not been studied as much as that of a point. People seem to be more interested in the focal locus of a submanifold as can be found in [12]. The focal locus of a submanifold corresponds rather to the conjugate locus of a point, and these concepts are more local. As for the cut locus of a set, Hartman [5] and Shiohama and Tanaka [10] have studied it. Also Wolter [9] [13] [14] [15] has studied it from various angles.

The study of the cut locus has a more practical side, too. In 1908, Voronoi

Key words and phrases: cut locus, Hausdorff dimension, fractal, dull point, sharp point, Koch snowflake, Bifurcated tree.

1991 *Mathematics Subject Classification.* Primary 28A80; Secondary 53C20.

This work was supported in part by Global Analysis Research Center, Basic Science Research Institute, Research Institute of Mathematics of Seoul National University and Korea Science and Engineering Foundation-981-0104-020-2.

studied the cut locus of a discrete set of points in the plane, which is now commonly called the *Voronoi diagram* [11]. It has numerous applications in geometry and other fields of mathematics as well as in computer science and engineering. More recently, Blum [1] proposed to study the cut locus of a plane domain as a way of compactly representing the shape of the domain. He called it the *medial axis*. The medial axis, which is also called the Voronoi diagram, has many applications in computer vision, NC (numerical control) tool path generation, and the automatic mesh generation in FEM (finite element method). Its careful mathematical study has been done by the first author and others [2], and a new algorithm to find the medial axis was obtained in [3].

It is well-known that the cut locus is a very unstable object with respect to the small boundary perturbation. In fact, none of the algorithms including that in [3] are completely stable, and the stability analysis has been one of the main issues in the study of the cut locus in the applications community. One can easily observe that if the boundary has more and more wiggles, the cut locus branches out in a more complicated fashion. The limiting case of boundary wiggling is the fractal boundary. Even if the boundary may not be a fractal set in practice, it may look like one in the resolution scale in question, in which case the cut locus also looks fractal-like.

In this paper, we investigate this limiting fractal case. It is interesting for the stability reasons as explained above as well as being itself an interesting object of study in mathematics.

In Section 2, we define the separating fractal set and study its local fractal geometry using the concept of the upper convex density; we proved that the so-called dull or sharp points are dense. (See below for definitions.) We prove that the Hausdorff dimension of the cut locus of the separating fractal set cannot decrease. An interesting question arises whether it is possible for the fractal dimension to increase. In Section 3, we give an example, the cut locus of the Koch snowflake, in which the Hausdorff dimension strictly increases. We also give another example, Bifurcated tree, in which case the Hausdorff dimension remains the same. It is worth mentioning that these cut loci are new examples of fractal sets, and it is our opinion that they are interesting objects on their own right.

Finally, we would like to thank F.-E. Wolter for his interest in our works and for the lively discussions on this subject matter.

2. Separating fractal set and its cut locus. In this section, we study fractal sets in a smooth complete Riemannian manifold M . What we are interested in is a fractal set which looks like a piece of boundary of a domain. As we can localize our argument, a fractal set need not bound a domain, which leads us to the concept of a *separating* fractal set. Below, we give definitions of *dull* and *sharp* points. Utilizing the notion of upper convex density in fractal geometry, we prove that the dull as well as sharp points are dense. Furthermore, we also prove that the Hausdorff dimension of the cut locus

of a separating fractal set is greater than or equal to that of the given separating fractal set. These results show that the cut locus of a separating fractal set is more complicated than the given one.

DEFINITION 2.1. Let A be a closed set in M , and let x be a point not in A . A geodesic segment γ from A to x is said to be in the class $\Gamma(A, x)$, if the length of γ is equal to the distance between x and A . A point $p \in M$ is called a *cut point* of the closed set A , if there exists a geodesic γ from A to p such that $\gamma(0) \in A$, $\gamma(l) = p$, and $\gamma|_{[0,l]} \in \Gamma(A, p)$, but $\gamma|_{[0,l+\varepsilon]} \notin \Gamma(A, \gamma(l+\varepsilon))$ for any $\varepsilon > 0$. The set of all cut points of A is called the cut locus of A , and is denoted by $C(A)$. (Note that some people take the closure of $C(A)$ to be the cut locus of A ; see, for example [13].)

DEFINITION 2.2. Let $A \subset M$. A is called *separating at a point* $x \in A$ if there exists $\varepsilon > 0$ such that for any $0 < \delta < \varepsilon$, A separates $B_\delta(x)$ into two nonempty disjoint parts. A set A is separating if A is separating at every interior point of A in the relative topology of A .

Before defining fractal sets, we give several definitions about the Hausdorff dimension. We will follow the basic definitions and notation in [4].

DEFINITION 2.3. Let $|U| = \text{diam}(U)$ for $U \subset M$. For a subset $A \subset M$, define $H_\delta^s(A) = \inf \sum_{i=1}^\infty |U_i|^s$, where $\{U_i\}$ is a cover of A satisfying $|U_i| < \delta$ for each i . Define the s -dimensional Hausdorff measure by

$$H^s(A) = \lim_{\delta \rightarrow 0} H_\delta^s(A).$$

Define the Hausdorff dimension $\dim_H(A)$ by $\dim_H(A) = \inf\{t \geq 0 \mid H^t(A) = 0\}$. When $\dim_H(A) = s$, define the s -set of A to be the subset of A such that for each point x in the subset of A and for any sufficiently small $\varepsilon > 0$, $\dim_H(B_\varepsilon(x) \cap A) = s$. (This definition of s -set is different from that of [4]. Note that if A is compact, then the s -set of A is not empty.) Define the upper convex density of x in the s -set of A by

$$\bar{D}_c^s(A, x) = \lim_{r \rightarrow 0} \left\{ \sup \frac{H^s(A \cap U)}{|U|^s} \right\},$$

where the supremum is over all convex sets U with $x \in U$ and $0 < |U| \leq r$. Define the upper spherical density of x in the s -set of A by

$$\bar{D}^s(A, x) = \limsup_{r \rightarrow 0} \frac{H^s(A \cap B_r(x))}{(2r)^s}.$$

Here we note that since $B_r(x)$ is convex, and since $U \subset B_r(x)$ for each $x \in A$ where $r = |U|$, we have the relations

$$(*) \quad C_s \bar{D}_c^s(A, x) \leq \bar{D}^s(A, x) \leq \bar{D}_c^s(A, x),$$

where C_s is a positive constant depending only on s and M .

Now let us define a *fractal set*. The definition could vary, but we choose the most general.

DEFINITION 2.4. A set in a Riemannian manifold is *fractal* if its Hausdorff dimension is not an integer.

DEFINITION 2.5. Suppose A is separating at x . Choose $\varepsilon > 0$ so that $B_\varepsilon(x)$ is separated by A into two disjoint subsets. Then taking the closure, we can get two closed subsets $A_1(\varepsilon, x)$ and $A_2(\varepsilon, x)$ whose union is $B_\varepsilon(x)$ and each of A_1 and A_2 contains $A \cap B_\varepsilon(x)$. (When there is no danger of confusion, we abbreviate them as A_1 and A_2 , respectively.) $x \in A \cap B_\varepsilon(x)$ is called a *dull point* of A_1 (resp. A_2) if there exists a point $q \in \text{int}(A_1)$ (resp. A_2) and $\delta > 0$ such that $x \in \overline{B_\delta(q)} \subset A_1$ (resp. A_2). x is called a *sharp point* of A_1 (resp. A_2) if x is not a dull point of A_1 (resp. A_2).

LEMMA 2.1. *Suppose A is itself an s -set and a separating fractal set. Then for each $x \in A$, it cannot be a dull point of both A_1 and A_2 .*

PROOF. We use 2.3 in p. 24 in [4] which says that if A is an s -set in \mathbf{R}^n , then $\overline{D}_c^s(A, x) = 1$ at $x \in A$. (Here, we note that our definition of an s -set is different from that of [4], so that ‘ H^s -almost all’ in [4] can be dropped.) This result also applies to a Riemannian manifold. Now suppose that a point $x \in A$ is a dull point of both A_1 and A_2 . Then there exist two balls $B_r(p) \subset A_1$ and $B_r(q) \subset A_2$ which contact each other at x . Then for sufficiently small $\delta > 0$, $B_\delta(x) \cap A \subset B_\delta(x) \cap B_r(p)^c \cap B_r(q)^c$. But this means that there exists a tangent hyperplane touching at x . Then using the same calculation as in Lemma 4.5 and Corollary 4.6 in [4], we can get $\overline{D}^s(A, x) = 0$. This is a contradiction to (*). □

LEMMA 2.2. *Suppose A is itself an s -set and a separating fractal set. Then the set of sharp points of A_1 is dense in A and the set of dull points of A_1 is dense in A . The same holds for A_2 .*

PROOF. First, we note by Lemma 2.1 that x is a sharp point of A_1 if x is a dull point of A_2 . Suppose that the set of dull points of A_1 is not dense in A . Then there exists a point $x \in A$ such that for some $\varepsilon > 0$, all points in $B_\varepsilon(x) \cap A$ are sharp points of A_1 . Since A is separating at x , we may assume that $B_\varepsilon(x)$ contains both the interior and the exterior of A_1 . Choose a point $p \in \text{int}(A_1) \cap B_{\varepsilon/2}(x)$. Then there exists a point $x_0 \in A$ such that $d(p, A) = d(p, x_0) < \varepsilon/2$. Then $B_{d(p, x_0)}(p)$ is contained in A_1 , that is, x_0 is a dull point of A_1 . But x_0 is in $B_\varepsilon(x) \cap A$. This contradicts the assumption. Hence the set of the dull points of A_1 must be dense in A . This also proves that the set of sharp points of A_1 is dense in A , if we reverse the role of A_1 and A_2 and use Lemma 2.1. □

Here we remark that if x is a sharp point of A_1 , then x can be either a sharp point or a dull point of A_2 in the proof of Lemma 2.2.

LEMMA 2.3. *Let Ω be any domain in M such that $\partial\Omega$ is separating. Then the centers of the maximal balls contained in Ω are contained in $C(\partial\Omega)$, the cut locus of $\partial\Omega$.*

PROOF. Let $B_r(p)$ be a maximal ball contained in Ω . It touches $\partial\Omega$ at least at one point. To show $p \in C(\partial\Omega)$, we will show that for each point $q \in \partial\Omega \cap B_r(p)$, any minimal geodesic $\gamma \in \Gamma(\partial\Omega, p)$ from q to p has the property that $r + \alpha > d(\partial\Omega, \gamma(r + \alpha))$ for any $\alpha > 0$. If the center p is already a cut point of q , then $r + \alpha > d(q, \gamma(r + \alpha)) \geq d(\partial\Omega, \gamma(r + \alpha))$. If the center p is not a cut point of q , then $d(q, \gamma(r + \alpha)) = r + \alpha$ for sufficiently small $\alpha > 0$. Let p' be the point $\gamma(r + \alpha)$. Consider a geodesic ball $B_{r+\alpha}(p')$. It contains $B_r(p)$. Since $B_r(p)$ is a maximal ball contained in Ω , we have $B_{r+\alpha}(p') \not\subset \Omega$. So there exists a point $y \in B_{r+\alpha}(p') - \bar{\Omega}$. The distance minimizing geodesic from p' to y must intersect $\partial\Omega$. So we have $d(p', \partial\Omega) < r + \alpha$. □

Now our main theorem is the following.

THEOREM 2.1. *Let A be a separating fractal set in M . Then:*

- (1) *The Hausdorff dimension $\dim_H(A)$ satisfies $n - 1 < \dim_H(A) < n$.*
- (2) *Every point of A is a limit point of the cut locus of A .*
- (3) *$\dim_H(C(A)) \geq \dim_H(A)$, where $\dim_H(C(A))$ is the Hausdorff dimension of the cut locus $C(A)$.*

PROOF. (1) Choose an n -dimensional box B in a neighborhood of x in $B_\epsilon(x)$, and give a local coordinate (x_1, x_2, \dots, x_n) to B where the n -th coordinate can be chosen so that each point in $B \cap A$ has positive n -th coordinate values. This is possible because A is separating at x so B is also separated by A into two disjoint subsets. Consider the projection map P from B to the $(n - 1)$ -dimensional space defined by $P(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{n-1})$. Then the image $P(B)$ is an $(n - 1)$ -dimensional box. Each n -dimensional δ -cover $\{U_i\}$ of B is projected to an $(n - 1)$ -dimensional δ -cover $\{P(U_i)\}$ of $P(B)$. So for any $t > 0$, $\sum_i |P(U_i)|^t \leq C \sum_i |U_i|^t$ for some fixed constant C . Hence $H'_\delta(P(B)) \leq CH'_\delta(B)$. As $\delta \rightarrow 0$, we get $H'(P(B)) \leq CH'(B)$. If $H'(B) = 0$, then $H'(P(B)) = 0$. Hence $\dim_H(B) = \inf\{t \geq 0 \mid H'(B) = 0\} \geq \dim_H(P(B)) = n - 1$. Since the Hausdorff dimension of a fractal set cannot be an integer, we get $n - 1 < \dim_H(B) = \dim_H(A \cap B_\epsilon(x)) < n$.

(2) Let the Hausdorff dimension of A be s . To calculate the Hausdorff dimension of $C(A)$, we only consider the set $A \cap B_\epsilon(x)$ for a point x in the s -set of A . (Because A is compact, the s -set of A is not empty.) From now on, denote $A \cap B_\epsilon(x)$ by A for convenience. Consider $C(A) \cap A_1$. We will show that for every sharp point $x_0 \in A$ of A_1 and for every $\delta > 0$, $B_\delta(x_0)$ contains a point of $C(A) \cap A_1$. Choose a sequence of points $\{x_i\} \subset A_1$ which converges to x_0 . Choose a ball $B_{r_i}(p_i)$ containing x_i which is a maximal ball in A_1 . Suppose that for some $\delta > 0$, $B_\delta(x_0)$ does not contain any point of $C(A) \cap A_1$. Then by Lemma 2.3, each center p_i is outside of $B_\delta(x_0)$. So for sufficiently large i , r_i is bigger than $\delta/2$. Choose $\delta/4$ -balls $B_{\delta/4}(q_i)$ in A_1 such that $x_i \in \partial B_{\delta/4}(q_i)$ and $B_{\delta/4}(q_i) \subset B_{r_i}(p_i)$. Then the limit ball of these balls is contained in A_1 because $\{\bar{B}_{\delta/4}(q_i)\}$ is a sequence of compact sets contained in the closed set A_1 . Since x_i goes to x_0 , the

limit ball touches x_0 at the boundary. But this means that x_0 is a dull point of A_1 . This is a contradiction. So we proved that each sharp point of A_i is a limit point of the interior cut locus $C(A)$. The proof is complete since the sharp points are dense.

(3) Now compute the Hausdorff dimension of $C(A)$. When we compute it, every neighborhood of each sharp point, which is dense in A by Lemma 2.2, contributes to the calculation of the Hausdorff measure. Thus we get $\dim_{\text{H}}(C(A)) \geq \dim_{\text{H}}(A)$. \square

REMARK. Theorem 2.1 can be used to construct new fractal sets. Suppose A is a given separating fractal set. Then Theorem 2.1 says that the Hausdorff dimension $\dim_{\text{H}}(C(A))$ of the cut locus $C(A)$ of A is greater than or equal to $\dim_{\text{H}}(A)$. If one can prove that $\dim_{\text{H}}(C(A))$ is not an integer, $C(A)$ is a newly found fractal set.

3. Examples. Theorem 2.1 shows that the cut locus of a separating fractal set is at least as complicated as the original one. The natural question that comes up immediately is if the cut locus of a separating fractal set is more complicated.

We give two examples here to address this question. The first one is the cut locus of the *Koch snowflake*. The Koch snowflake is a well-known fractal object whose Hausdorff dimension is $\log 4/\log 3$, and we show that the part of the cut locus which lies inside the Koch snowflake is a fractal object whose Hausdorff dimension is $\log 6/\log 3$. So this example shows that the Hausdorff dimension of the cut locus of a separating fractal set can be greater than that of the separating fractal set itself. The second example is the *Bifurcated tree* whose Hausdorff dimension is the same as that of the interior cut locus of its boundary.

Another equivalent way of approaching the cut locus of a domain in \mathbf{R}^n is via the so-called *medial axis*. The medial axis $\text{MA}(D)$ of a domain D in \mathbf{R}^n is defined to be the set of centers of maximal inscribed balls in D . The proof of this equivalence is not hard, and it can be found in [13]. It should be noted that in applying the argument in [13], one only needs the fact that ∂D is closed.

3.1. The Koch snowflake. The *Koch snowflake* is a well-known fractal object, and was invented by Koch in 1906 who wanted a nowhere differentiable curve.

The Koch snowflake is iteratively constructed as follows. We first start with an equilateral triangle. We partition each side into three equal parts, and then attach at the middle third of each side an equilateral triangle of one-third the size of the original one, and take away both segments situated at the middle third. This is the basic construction step. Let KS_1 be the domain obtained after one iterative step. Thus KS_1 resembles the star of David whose boundary has twelve sides. A reduction of this figure, made of twelve line segments, will be reused in the next step, partitioned into three equal parts, and so on. The 6-th step object in the construction of the Koch snowflake is shown in Figure 1. We have some immediate facts from its construction.

FACT 3.1. (i) *Let KS_n be the domain bounded by the closed curve constructed*

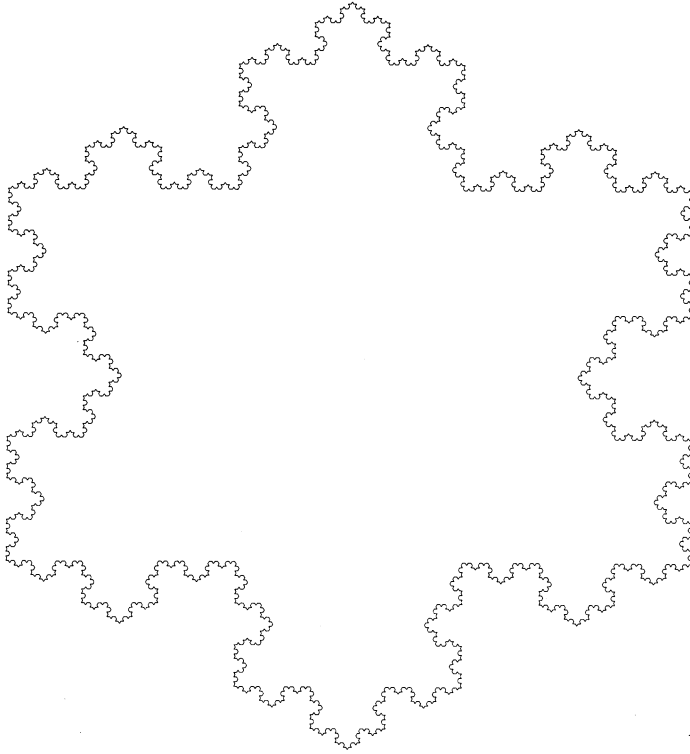


FIGURE 1. The sixth iteration object of the Koch snowflake.

in the n -th step above and let KS be the closure of the domain enclosed by the Koch snowflake. Then

$$\text{KS}_m \subset \text{KS}_n \quad \text{for } m < n, \quad \text{and} \quad \text{KS}_n \subset \text{KS}.$$

(ii) The boundary points of KS_n are not contained in the boundary of KS except the vertices of KS_n .

In order to decide $\text{MA}(\text{KS})$, we need the following result which depends heavily on the two-dimensionality [2].

THEOREM 3.1 (2-dimensional domain decomposition). *For a given domain Ω in \mathbf{R}^2 and any maximal ball $B(p)$ in Ω , suppose A_1, A_2, \dots are the connected components of $\Omega - B(p)$. Denote $\Omega_i = A_i \cup B(p)$ for $i = 1, 2, \dots$. Then*

$$\text{MA}(\Omega) = \bigcup_{i=1}^{\infty} \text{MA}(\Omega_i).$$

Moreover, we have

$$MA(\Omega_i) \cap MA(\Omega_j) = \{p\},$$

for every distinct i and j .

PROOF. Suppose q is the center of a maximal ball $B_r(q)$ in Ω_i for some i . We will show that $B_r(q)$ is also a maximal ball in Ω . Suppose there is another ball $B_{r'}(q') \subset \Omega$ such that $B_r(q) \subset B_{r'}(q')$. If $B_{r'}(q') \subset \Omega_i$, then by the maximality of $B_r(q)$ in Ω_i , we get $B_r(q) = B_{r'}(q')$. If $B_{r'}(q') \not\subset \Omega_i$, then there exists $v \in B_{r'}(q') - B_r(q)$ such that $v \in A_j$ for some $j \neq i$ and $d(q', v) = r'$. Define $T(p)$ to be the union of all segments from p to contact points of $B(p)$ with $\partial\Omega$. By $T(p)$, $B(p)$ is divided into separate regions B_i 's each of which is contained in Ω_i . So Ω is divided into $A_i \cup B_i$. Furthermore, since $B_{d(q',v)}(q') \subset \Omega$ and $q' \in A_i$ and $v \in A_j$, the line $\overline{q'v}$ meets $T(p)$. Let w be one of them. Also, let w' be a point in $T(p) \cap \partial\Omega$ such that $d(w, w') = d(w, \partial\Omega)$. Then $r' = d(q', \partial\Omega) \leq d(q', w') \leq d(q', w) + d(w, w')$. Since $v \notin B(p)$, we have $d(w, w') < d(w, v)$. So $r' = d(q', \partial\Omega) \leq d(q', w') \leq d(q', w) + d(w, w') < d(q', w) + d(w, v) = d(q', v) = r'$, which is a contradiction. This proves $\bigcup_{i=1}^{\infty} MA(\Omega_i) \subset MA(\Omega)$.

On the other hand, suppose q is the center of maximal disk $B_r(q)$ in Ω . Then there exists a unique i such that $B_r(q) \subset \Omega_i$. If $B_r(q) \cap A_j \neq \emptyset$ for some $j \neq i$, then by an argument similar to that above we can derive a contradiction. Since $B_r(q)$ is maximal in Ω , $B_r(q)$ is also maximal in Ω_i . Thus we proved $MA(\Omega) \subset \bigcup_{i=1}^{\infty} MA(\Omega_i)$. \square

We need to compute $MA(KS)$, which is equivalent to computing the cut locus of KS . But, as KS is iteratively defined as a limit object and mere inclusion is of no help, it cannot be done directly. However, our specific construction suggests that it can be done iteratively also. To do so, we need to decompose the domain into simpler pieces whose medial axis is easier to handle due to Theorem 3.1.

THEOREM 3.2. *The closure of the medial axis of KS is the same as the closure of the union of that of KS_n 's at each iteration step, i.e.,*

$$\overline{MA(KS)} = \overline{\bigcup_{n=1}^{\infty} MA(KS_n)}.$$

PROOF. Starting from the center p of KS_1 , we get the maximal circle passing through all six dull corners of KS_1 whose center is p . By this maximal circle, the object is decomposed into six equal subdomains as described in Theorem 3.1. The medial axis of this object is also decomposed into six parts as also described in Theorem 3.1. The line segments from p to the six sharp corners of KS_1 is contained in the medial axis of KS . This is proved by simple examination as follows. Let L be one of the six line segments, and choose a point on L . Enlarge the circle from that point regarding it as a center of that circle until touching the boundary of the Koch snowflake. But then we must get pairs of points touching the boundary by the symmetry of the Koch snowflake, so it is contained in the medial axis of the Koch snowflake. Thus any point

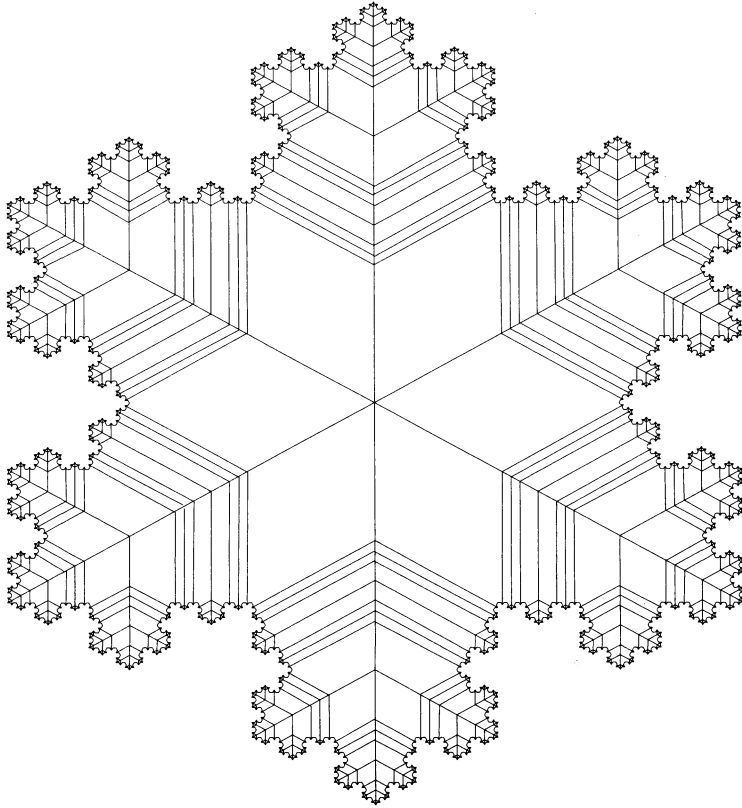


FIGURE 2. The interior cut locus of the sixth iteration object of the Koch snowflake.

in L has at least two legs to its boundary so L itself is contained in the medial axis of the Koch snowflake.

Now observe the object KS_2 of the second construction. We can choose six maximal circles passing through each of six sets each of which consists of four vertices created in the second iteration. Then by each of these maximal circles, each subdomain of KS_1 described above is further decomposed into three equal parts and one large base part. Each of the three equal parts is then scaled by one-third of the previous one. As in the first construction, each of these three equal parts has a segment from the center of the maximal circle to the sharp corner of the second iteration. These line segments constitute $MA(KS_2)$. Similarly, we can show inductively that $MA(KS_n)$ belong to $MA(KS)$. (cf. $MA(KS_6)$ is drawn in Figure 2.) Thus we get $\bigcup_{n=1}^{\infty} MA(KS_n) \subset MA(KS)$.

Let us now prove that $\overline{MA(KS)} \subset \bigcup_{n=1}^{\infty} \overline{MA(KS_n)}$. Here we note that $\bigcup_{n=1}^{\infty} MA(KS_n)$ consists of the symmetric line segments which start from the sharp corners of KS_n for each n , bisecting the sharp corners. So it is enough to prove that the point

which is not in symmetric line segments cannot be a center of the maximal disk of KS .

Suppose that there exists a point $p \in \overline{MA(KS)} - \bigcup_{n=1}^{\infty} \overline{MA(KS_n)}$. There exists m such that $p \in \text{int}(KS_m)$ (Here $\text{int}(KS_m)$ means the interior of KS_m). Consider the point $q \in \partial KS$ satisfying $d(p, q) = d(p, \partial KS) = r > 0$. If $q \in \partial KS_l$ for some l , then the ball $B_r(p)$ is in KS_l and is not maximal. This is a contradiction to the maximality of $B_r(p)$ in KS . So we can assume that $q \notin \partial KS_n$ for all n . Now for each $n \geq m$, take a sequence $\{q_n\}$, where $q_n \in \partial KS_n$ satisfies that $d(p, q_n) = d(p, \partial KS_n)$. Since $B_{d(p, q_n)}(p)$ is not maximal in KS_n , we can find a maximal ball $B_{r_n}(c_n)$ in KS_n satisfying $B_{d(p, q_n)}(p) \subset B_{r_n}(c_n)$ for each $n \geq m$. In a bounded domain KS , we have a sequence of balls $B_{r_n}(c_n)$, so there exists a converging subsequence. Denote the limit of this subsequence by $B_R(c)$. Then $c \in \bigcup_{n=1}^{\infty} \overline{MA(KS_n)}$ and $B_r(p) \subset B_R(c)$. This is a contradiction to the choice of p . \square

Here we note that the Hausdorff dimension of a set is the same as that of the closure of the set. By this observation, we get the following result.

COROLLARY 3.1. *The Hausdorff dimension of the medial axis of KS is the same as that of the union of the medial axis of KS_n 's at each iteration step, i.e.,*

$$\dim_H(MA(KS)) = \dim_H\left(\bigcup_{n=1}^{\infty} MA(KS_n)\right).$$

The dimension of the boundary of the Koch snowflake is $\log 4 / \log 3$, since at each iteration, the boundary is divided into three equal parts and pasting an equilateral triangle gives four equal line segments at each side (cf. [6]). Now let us introduce the method to compute the dimension of the medial axis of the Koch snowflake. Observe that the medial axis consists of line segments. To look at the increasing procedure of the line segments, we cut off some region from the Koch snowflake as follows. KS_1 has six sharp corners and six dull corners. Draw three line segments from the center p to one of the sharp corners and two dull corners in the side of the chosen sharp corner, cut the object by the line to get two congruent triangles, turn over one of them, and paste them by the line to get a square-like object. The resulting figure has one medial line segment which divides the object into two parts. In the second construction, two medial line segments which divide the square-like object is added. In the third step, four line segments across the object is added, and so on. Here we observe that the medial axis of this square-like object look like the product of the Cantor ternary set with an interval. By this observation, we can compute the Hausdorff dimension of the medial axis of the Koch snowflake by the product dimension of the Cantor ternary set with an interval. It is also well-known that the dimension of the Cantor ternary set is $\log 2 / \log 3$. So we get

$$\dim_H(MA(KS)) = \frac{\log 6}{\log 3}.$$

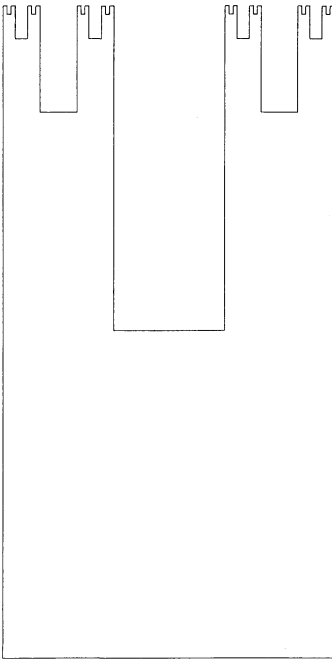


FIGURE 3. The fourth iteration object of the bifurcated tree.

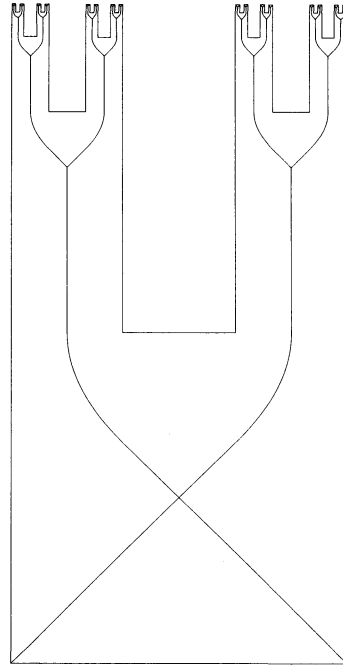


FIGURE 4. The interior cut locus of the fourth iteration object of the bifurcated tree.

We also observe that the Hausdorff dimension of the exterior cut locus of the Koch snowflake is the same as that of the interior one.

3.2. The bifurcated tree. Here we present an example, which we call the bifurcated tree, whose Hausdorff dimension is equal to that of the cut locus of it.

The construction of the bifurcated tree is as follows: It starts from an equilateral square whose length of each side is 1. Attach two rectangles, whose length of bases are $1/3$ and whose length of heights are $2/3$, to each of two ends of the upper side of the first equilateral square, and remove the two intersected bases of the rectangles. Then, for each rectangles, attach two rectangles, whose length of base is $1/3^2$ and whose length of height is $2^2/3^2$, and remove the intersected part. And so on. The resulting domain is shown in Figure 3, and its interior cut locus is shown in Figure 4, which consists of line segments and pieces of parabolas.

Now, we shall compute the dimension of them. First of all, we can find that the dimension of the bifurcated tree is the same as that of its interior cut locus: Roughly speaking, the number of δ -balls which cover the bifurcated tree, is the same as the number of the balls covering its interior cut locus, regardless of some constant multiple. So by the definition of the Hausdorff dimension (Def. 2.3), the dimensions of the two objects are the same.

Then let us compute the dimension of the bifurcated tree. The length of the boundary of the n -th iteration object of the bifurcated tree is as follows:

$$4 + \left(\frac{2}{3}\right) \cdot 2^2 + \left(\frac{2}{3}\right)^2 \cdot 2^3 + \cdots + \left(\frac{2}{3}\right)^n \cdot 2^n.$$

If we cover the ball of its diameter, $\delta = 3^{-n}$, the total number N of the balls which cover the bifurcated tree is as follows:

$$2 \cdot 4^n < N < 10 \cdot 4^n.$$

So by Def 2.3, the Hausdorff measure H is as follows:

$$\frac{2 \cdot 4^n}{3^{ns}} \leq H \leq \frac{10 \cdot 4^n}{3^{ns}}.$$

Thus as $n \rightarrow \infty$, s is given to $\log 4 / \log 3$.

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