

A CORRESPONDENCE BETWEEN THE MODULI SPACES OF VECTOR BUNDLES OVER A CURVE

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Abstract. After establishing a correspondence between a smooth moduli space of vector bundles on a curve and a self-product of the Jacobian, the nontriviality of the Griffiths group of the moduli space for a general curve is proved.

1. Introduction. Let \mathcal{N} denote the moduli space of isomorphism classes of stable vector bundles, over a compact Riemann surface X , of rank r and of a fixed determinant of degree d , with r and d being mutually coprime. Let $P(\mathcal{E})$ be the universal projective bundle over the Cartesian product $X \times \mathcal{N}$. The characteristic classes $a_k \in H^{2k}(X \times \mathcal{N}, \mathcal{Q})$, where $k = 2, 3, \dots, r$, of $P(\mathcal{E})$, give rise to homomorphisms, from $H_1(X, \mathcal{Q})$ to $H^{2k-1}(\mathcal{N}, \mathcal{Q})$, by the “slant product” operation. Since the homology algebra, $H_*(J(X), \mathcal{Q})$, of the Jacobian $J(X)$ of X is the exterior algebra $\bigwedge H_1(X, \mathcal{Q})$, combining all these homomorphisms given by the characteristic classes of $P(\mathcal{E})$, we get an algebra homomorphism from $H_*(J^{r-1}, \mathcal{Q})$ (the homology algebra of the $(r-1)$ -fold self-product of J) to the cohomology algebra $H^*(\mathcal{N}, \mathcal{Q})$. The details of this construction are given in Section 2.

We show that the above formally constructed homomorphism has a geometric origin. More precisely, there is a correspondence cycle on the product $\mathcal{N} \times J^{r-1}$, which is canonical as an element of the Chow ring (cycles modulo rational equivalence) of $\mathcal{N} \times J^{r-1}$, such that the above homomorphism is induced by this cycle (Theorem 2.5). As an application of this result, we construct nonzero elements of the Griffiths group of the moduli space \mathcal{N} for a general Riemann surface (Theorem 4.8). These elements are constructed from the nonzero elements of the Jacobian of a general curve discovered by Ceresa [C].

The above results extend to the more general context of smooth moduli spaces of parabolic bundles.

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2. Map between Hodge cycles. Let X be a compact Riemann surface, or equivalently, a connected smooth projective curve over \mathbb{C} . Assume that the genus $g := \text{genus}(X) \geq$

2. Choose and fix a point $x_0 \in X$. Fix a pair of mutually coprime integers r and d with $r > 1$. Let \mathcal{N} denote the moduli space of isomorphism classes of stable vector bundles E on X of rank r and with the top exterior product $\bigwedge^r E$ isomorphic to $\mathcal{O}_X(dx_0)$. The space \mathcal{N} has a structure of a connected smooth projective variety over \mathbf{C} of dimension $(r^2 - 1)(g - 1)$.

Let $\mathcal{E} \rightarrow X \times \mathcal{N}$ be a universal vector bundle; which means that for any $z \in \mathcal{N}$, the vector bundle on X , given by the restriction $\mathcal{E}|_{X \times z}$, is in the isomorphism class represented by the point z . If \mathcal{E}' is another universal bundles on $X \times \mathcal{N}$, then from the projection formula we have the equality $\mathcal{E}' = \mathcal{E} \otimes p_2^* \mathcal{L}$, where \mathcal{L} is a line bundle on \mathcal{N} and $p_2 : X \times \mathcal{N} \rightarrow \mathcal{N}$ is the obvious projection. Thus the projective bundle $P(\mathcal{E})$ is independent of the choice of the universal bundle.

Let $a_k \in H^{2k}(X \times \mathcal{N}, \mathcal{Q})$, $2 \leq k \leq r$, be the characteristic classes of the principal $PGL(r)$ bundle $P(\mathcal{E}) \rightarrow X \times \mathcal{N}$ given by the space of lines in \mathcal{E} . We will recall a description of a_k in terms of the Chern classes of \mathcal{E} . Consider the k -th degree polynomial, P_k , of r variables $\{x_1, \dots, x_r\}$, defined by the identity

$$\det(\lambda + D_{\underline{x}}) = \sum_{j=0}^r \lambda^{r-j} P_j,$$

where $D_{\underline{x}}$ is the $r \times r$ diagonal matrix with x_1, x_2, \dots, x_r as the diagonal entries. So the i -th Chern class $c_i(\mathcal{E}) = P_i(\alpha_1, \dots, \alpha_r)$, where $\{\alpha_1, \dots, \alpha_r\}$ are the Chern roots of \mathcal{E} . Define

$$\beta_i := r\alpha_i - \sum_{j=1}^r \alpha_j.$$

With this notation we have

$$(2.1) \quad a_i := P_i(\beta_1, \beta_2, \dots, \beta_r).$$

Note that $a_1 = 0$.

For a cohomology class $c \in H^j(X \times \mathcal{N}, \mathcal{Q})$, using the Künneth decomposition of the \mathcal{Q} -vector space $H^j(X \times \mathcal{N}, \mathcal{Q})$, as well as the obvious duality between $H^k(X, \mathcal{Q})$ and $H_k(X, \mathcal{Q})$, we have a homomorphism

$$(2.2) \quad S(c, k) : H_k(X, \mathcal{Q}) \rightarrow H^{j-k}(\mathcal{N}, \mathcal{Q}),$$

which is usually called the “slant product”. Taking c to be the class a_i , we get

$$S(a_i, k) : H_k(X, \mathcal{Q}) \rightarrow H^{2i-k}(\mathcal{N}, \mathcal{Q}).$$

Let the direct sum $H_1(X, \mathcal{Q})^{\oplus(r-1)}$ be denoted by W . Taking the direct sum of the above maps, we have a \mathcal{Q} -vector space homomorphism

$$\sum_{i=2}^r S(a_i, 1) : W = H_1(X, \mathcal{Q})^{\oplus(r-1)} \rightarrow H^{\text{odd}}(\mathcal{N}, \mathcal{Q}),$$

which, in turn, induces an algebra homomorphism

$$(2.3) \quad F : \bigwedge W \rightarrow H^*(\mathcal{N}, \mathcal{Q}),$$

where $\bigwedge W$ is the exterior algebra for W . Consider the complex vector space $W_C := W \otimes_{\mathcal{Q}} \mathcal{C}$. Tensoring (2.3) with \mathcal{C} , we get a \mathcal{C} -algebra homomorphism

$$F_C : \bigwedge W_C \rightarrow H^*(\mathcal{N}, \mathcal{C}).$$

Let $J := \text{Pic}^0(X)$ be the Jacobian of X . The homology algebra $H_*(J, \mathcal{Q})$ (the algebra structure is given by the cap product) is canonically isomorphic to the exterior algebra $\bigwedge H_1(X, \mathcal{Q})$. Indeed, consider the map $X \rightarrow J$, defined by $x \mapsto \mathcal{O}_X(x - x_0)$. The induced map of the first homology is extended as an algebra homomorphism from the exterior algebra $\bigwedge H_1(X, \mathcal{Q})$ to $H_*(J, \mathcal{Q})$. It is easy to see that this homomorphism is actually an isomorphism, and that it does not depend upon the choice of the base point x_0 . The canonical isomorphism between $H_*(J, \mathcal{Q})$ and $\bigwedge H_1(X, \mathcal{Q})$ induces an isomorphism between the homology algebra

$$H_*(J^{r-1}, \mathcal{Q}) = \bigwedge H_1(J^{r-1}, \mathcal{Q}) = \bigwedge W.$$

Let \mathcal{J} denote the $(r - 1)$ -fold self-product, namely J^{r-1} , of J with itself. Let

$$\gamma : H_*(\mathcal{J}, \mathcal{Q}) \rightarrow \bigwedge W$$

be the isomorphism obtained above. Combining this with the homomorphism F in (2.3), we have

$$(2.4) \quad \bar{F} : H_*(\mathcal{J}, \mathcal{Q}) \rightarrow H^*(\mathcal{N}, \mathcal{Q})$$

defined by $c \mapsto F(\gamma(c))$. On account of the obvious duality, namely $H_i(\mathcal{J}, \mathcal{Q})^* = H^i(\mathcal{J}, \mathcal{Q})$, the homomorphism \bar{F} gives an element in $H^*(\mathcal{N}, \mathcal{Q}) \otimes H^*(\mathcal{J}, \mathcal{Q})$. The cohomology algebra $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$ of $\mathcal{N} \times \mathcal{J}$ is canonically isomorphic to the graded tensor product $H^*(\mathcal{N}, \mathcal{Q}) \otimes H^*(\mathcal{J}, \mathcal{Q})$. Thus, \bar{F} gives an element in $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$; this element will be denoted by Φ .

For a smooth projective variety Y over \mathcal{C} , an element of the \mathcal{Q} -vector space

$$\bigoplus_p (H^{p,p}(Y) \cap H^*(Y, \mathcal{Q}))$$

will be called, following [D], a *Hodge cycle* on Y . There is a natural homomorphism, known as the *cycle class map*, from the space of cycles on X to the space of Hodge cycles Y . The cycle class of a cycle of codimension p on Y is in $H^{p,p}(Y) \cap H^{2p}(Y, \mathcal{Q})$. Let $\text{CH}^*(Y)$ denote the Chow group of cycles on Y modulo rational equivalence. The image of $\text{CH}^*(Y) \otimes_{\mathcal{Z}} \mathcal{Q}$ in the space of Hodge cycles (by the cycle class map) is called the space of *algebraic cohomology classes* on Y .

From the definition of the cohomology class Φ it is clear that Φ is actually an element of the image of $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Z})$ in $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$ by the obvious map. Indeed, this is a consequence of the fact that the homomorphism \bar{F} in (2.4) maps $H_*(\mathcal{J}, \mathcal{Z})$ into $H^*(\mathcal{N}, \mathcal{Z})$.

THEOREM 2.5. *The cohomology class Φ is algebraic, i.e., it is the cycle class of an algebraic cycle on $\mathcal{N} \times \mathcal{J}$.*

PROOF. Let $G_{\mathcal{Z}}$ denote the group of all automorphisms of $H_1(X, \mathcal{Z})$ preserving the cap product. Choosing a symplectic basis of $H_1(X, \mathcal{Z})$, this group $G_{\mathcal{Z}}$ can be identified with $Sp(g, \mathcal{Z})$, the group of $2g \times 2g$ symplectic matrices with entries in \mathcal{Z} . The proof of Theorem 2.5

consists of three steps. First, the group G_Z acts on the cohomology algebra $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$. Second, any invariant class in $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$, for the action of G_Z , is algebraic. Third, the class Φ is an invariant for the action of G_Z on $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$.

We will first describe the action of G_Z on $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$. Let \bar{W} denote the direct sum $H_0(X, \mathcal{Q}) \oplus H_2(X, \mathcal{Q})$. Setting the class c to be a_i in (2.2), we have

$$(2.6) \quad S(a_i) := S(a_i, 0) \oplus S(a_i, 2) : \bar{W} \rightarrow H^{\text{even}}(\mathcal{N}, \mathcal{Q}).$$

These homomorphisms combine together to induce a homomorphism from the symmetric algebra

$$S : S(\bar{W}^{\oplus(r-1)}) \rightarrow H^{\text{even}}(\mathcal{N}, \mathcal{Q}).$$

Consider the tensor product of the homomorphism S above with F defined in (2.3):

$$(2.7) \quad S \otimes F : S(\bar{W}^{\oplus(r-1)}) \otimes \bigwedge W \rightarrow H^*(\mathcal{N}, \mathcal{Q})$$

The group G_Z acts on the direct sum W by the diagonal action corresponding to the natural action on $H_1(X, \mathcal{Q})$. Let G_Z act on \bar{W} as the trivial action. In [BR] and [B1] it is proved that the homomorphism $S \otimes F$ in (2.7) is surjective and the kernel is left invariant by the induced action of G_Z on $S(\bar{W}^{\oplus(r-1)}) \otimes \bigwedge W$. Thus we have an action of G_Z on $H^*(\mathcal{N}, \mathcal{Q})$ induced by the action of G_Z on $S(\bar{W}^{\oplus(r-1)}) \otimes \bigwedge W$.

We will recall a theorem proved in [B1]. Take a smooth family of one pointed curves parametrized by T . Consider the corresponding family of moduli spaces of vector bundles parametrized by T . The holonomy of the Gauss-Manin connection for this family gives rise to a representation of the fundamental group $\pi_1(T)$ into the cohomology algebra of the typical fiber. This representation factors through the homomorphism of $\pi_1(T)$ into G_Z , given by the holonomy of the Gauss-Manin connection for the first homology of the family of curves. The action of G_Z on the cohomology of the typical fiber of the family of moduli spaces is precisely the action of G_Z on $H^*(\mathcal{N}, \mathcal{Q})$ obtained above.

Since $H^*(J, \mathcal{Q}) = \bigwedge H^1(X, \mathcal{Q}) = \bigwedge H_1(X, \mathcal{Q})^*$ (Poincaré duality on X), we have the equality $H^*(\mathcal{J}, \mathcal{Q}) = \bigwedge W^*$. Thus the cohomology algebra $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$ is the graded tensor product $H^*(\mathcal{N}, \mathcal{Q}) \otimes \bigwedge W^*$. The group G_Z acts on $H_1(X, \mathcal{Z})^*$ by the adjoint action and on W^* by the diagonal action. Consider the induced action of G_Z on $\bigwedge W^*$. The actions of G_Z on $H^*(\mathcal{N}, \mathcal{Q})$ (obtained above) and on $\bigwedge W^*$ combine together to induce an action of G_Z on $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$.

As the next step we want to show that any invariant class in $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$ for the action of G_Z is an algebraic class.

The action of G_Z on $H_1(X, \mathcal{Z})$ extends naturally to an action of its complexification G_C on $H_1(X, \mathcal{C})$. The Borel density theorem asserts that the subgroup G_Z is Zariski dense in G_C . This implies that the space of invariants in $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$ for the action of G_Z is precisely the image of $S(\bar{W}^{\oplus(r-1)}) \otimes (\bigwedge W \otimes \bigwedge W^*)^{G_Z}$ (in $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$) by the map $S \otimes \bar{F} \otimes \text{Id}$, where $(\bigwedge W \otimes \bigwedge W^*)^{G_Z}$ denotes the space of all G_Z invariants in $\bigwedge W \otimes \bigwedge W^* = \text{End}(\bigwedge W)$. (Note that the action of G_Z on $S(\bar{W}^{\oplus(r-1)})$ is the trivial action.) From Theorem 2 of [Ho] we know that the space of G_C invariants in $\text{End}(\bigwedge W_C)$ is generated by a subclass of degree

two elements (when considered as elements of $\bigwedge(W \oplus W)$ using the symplectic form). The subclass in question is described as follows: identify $H_1(X, \mathcal{Q})$ with $H^1(X, \mathcal{Q})$ using the Poincaré duality on X . Thus

$$\bigwedge W \otimes \bigwedge W^* = \bigwedge(W \oplus W) = \bigwedge(H_1(X, \mathcal{Q})^{\oplus 2(r-1)}).$$

For $1 \leq i \leq j \leq 2(r-1)$, let $f_{i,j}$ denote the diagonal inclusion of $H_1(X, \mathcal{Q})$ in $H_1(X, \mathcal{Q})^{\oplus 2(r-1)}$ along the (i, j) -th coordinates, and let $\bigwedge^2 f_{i,j}$ denote the induced homomorphism of the second exterior products. Let

$$\tau \in \bigwedge^2 H_1(X, \mathcal{Q})$$

be the element corresponding to the Poincaré duality pairing, namely

$$\omega_1 \otimes \omega_2 \mapsto \int_X \omega_1 \wedge \omega_2,$$

where $\omega_1, \omega_2 \in H^1(X, \mathcal{Q})$. Now define

$$\tau_{i,j} := (\bigwedge^2 f_{i,j})(\tau).$$

The space of all $G_{\mathbb{Z}}$ invariants in $\bigwedge W \otimes \bigwedge W$ is generated, as an algebra, by the collection of elements $\{\tau_{i,j}\}$.

Since the class $a_k \in H^{2k}(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$ is algebraic (any Chern class of an algebraic vector bundle is algebraic), the image of the homomorphism S is contained in the algebraic classes in $H^*(\mathcal{N}, \mathcal{Q})$. Indeed, $S(a_k, 0)(1)$ is the restriction of a_k to \mathcal{N} (1 is the canonical generator of $H_0(X, \mathcal{Q})$); and, $S(a_k, 2)([X])$ is the image of a_k by the Gysin map for the projection of $X \times \mathcal{N}$ onto \mathcal{N} ($[X]$ is the oriented generator of $H_2(X, \mathbb{Z})$). We note that the proof of the assertion that any $G_{\mathbb{Z}}$ invariant class in $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$ is algebraic will be completed once we are able to show that any $\bar{F} \otimes \text{Id}(\tau_{i,j})$ in $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$ is actually algebraic.

For notational simplicity we will also use $\tau_{i,j}$ to denote $\bar{F} \otimes \text{Id}(\tau_{i,j})$. From the context it will be clear which one is being used.

If $1 \leq i, j \leq r-1$, then $\tau_{i,j}$ is the pullback of a cohomology class on \mathcal{N} (by the projection of $\mathcal{N} \times \mathcal{J}$ onto \mathcal{N}); we will denote this class in \mathcal{N} by t . Moreover, t is an invariant for the action of $G_{\mathbb{Z}}$ (since $\tau_{i,j}$ is, by definition, an invariant). Hence from Proposition 2.4 of [BN] it follows that t is algebraic. Thus $\tau_{i,j}$ is algebraic.

If $r \leq i < j \leq 2(r-1)$, then $\tau_{i,j}$ is the pullback of a cohomology class in $J \times J$, denoted by t , using the projection of $\mathcal{N} \times \mathcal{J}$ along the i -th and j -th factors in \mathcal{J} . If $i = j$, then it is the pullback of a cohomology class on J , denoted by t , using the projection onto the i -th factor of \mathcal{J} . Since $\tau_{i,j}$ is invariant under the action of $G_{\mathbb{Z}}$, the cohomology class t , in either case, is also invariant under the action of $G_{\mathbb{Z}}$. If $i = j$, then t is the cycle class of the theta divisor in J . (The theta divisor on J is the pullback of the theta divisor on $\text{Pic}^{g-1}(X)$ using the map defined by $L \mapsto L \otimes \mathcal{O}_X((g-1)x_0)$.) Let Θ denote this theta divisor on J . Let

$$A : J \times J \rightarrow J$$

be the addition map defined by $(L, L') \mapsto L \otimes L'$. If $i \neq j$, then t is the cycle class of the divisor

$$A^*\Theta - p_1^*\Theta - p_2^*\Theta$$

on $J \times J$, where p_1 (respectively, p_2) is the projection of $J \times J$ to the first (respectively, second) factor. It is a straightforward calculation to check the above statement. Thus the cohomology class $\tau_{i,j}$ is algebraic for $r \leq i \leq j \leq 2(r - 1)$.

Finally, consider a cohomology class $\tau_{i,j}$ with $i \leq r - 1$ and $j \geq r$. Let f denote the projection of $\mathcal{N} \times \mathcal{J}$ onto $\mathcal{N} \times J$ which maps any

$$(n, L_1, \dots, L_{r-1}) \in \mathcal{N} \times J^{r-1} = \mathcal{N} \times \mathcal{J}$$

to $(n, L_j) \in \mathcal{N} \times J$. There is a unique cohomology class on $\mathcal{N} \times J$, say t , such that $f^*t = \tau_{i,j}$.

Consider the characteristic class $a_k \in H^{2k}(X \times \mathcal{N}, \mathcal{Q})$ defined above. Let

$$(2.8) \quad a_{i+1} = a \otimes 1 + b + c \otimes [X]$$

be the Künneth decomposition of the cohomology class a_{i+1} , where the cohomology class $[X]$ is the oriented generator of $H^2(X, \mathbf{Z})$. The classes $a, c \in H^*(\mathcal{N}, \mathcal{Q})$ are algebraic. Indeed, a is the restriction of a_{i+1} to $x_0 \times \mathcal{N}$, and c is the push-forward of a_{i+1} by the projection of $X \times \mathcal{N}$ onto \mathcal{N} . Thus the cohomology class b in (2.8) is also algebraic.

Let \mathcal{L} be a Poincaré line bundle over $X \times J$. Let θ be the component of $c_1(\mathcal{L})$ in $H^1(X, \mathcal{Q}) \otimes H^1(J, \mathcal{Q})$ for the Künneth decomposition of $H^2(X \times J, \mathcal{Q})$. Note that, since any two choices of the Poincaré line bundle differ only by the pullback of a line bundle on J , this Künneth component of $c_1(\mathcal{L})$ does not depend upon the choice of the Poincaré bundle. Let q_1 and q_2 denote the obvious projections of $X \times \mathcal{N} \times J$ onto $X \times \mathcal{N}$ and $X \times J$, respectively. The class θ is algebraic for the same reason as for b . Consider

$$\omega := q_1^*b \cup q_2^*\theta \in H^{2i+4}(X \times \mathcal{N} \times J, \mathcal{Q}).$$

Now, since both b and θ are algebraic, so is the class ω . Let

$$p : X \times \mathcal{N} \times J \rightarrow \mathcal{N} \times J$$

be the obvious projection, and p_* the corresponding Gysin map of cohomologies. It is a straightforward calculation (the details are in the proof of the Proposition 2.4 of [BN]) to check that $t = p_*\omega$. In other words, t is the cycle class of the push-forward of a cycle whose cycle class is ω . Thus the cohomology class $\tau_{i,j}$ must be algebraic. This completes the proof of the assertion that any invariant class is algebraic.

To show that Φ is an invariant for the action of $G_{\mathbf{Z}}$, first note that, since the classes a_i are canonical in the sense that they do not depend upon the choice of the universal bundle, these classes are invariant under the monodromy action on $H^*(X \times \mathcal{N}, \mathcal{Q})$ for a family of curves. This, in turn, implies that the class Φ is an invariant for the monodromy action on $H^*(\mathcal{N} \times \mathcal{J}, \mathcal{Q})$. The monodromy action factors through $G_{\mathbf{Z}}$ [B1]. Now, since the mapping class group projects onto $G_{\mathbf{Z}}$, the class Φ must be an invariant for the action of $G_{\mathbf{Z}}$. This completes the proof of the theorem. □

Theorem 2.5 implies the following: by virtue of the natural identification of $H^*(\mathcal{J}, \mathcal{Q})$ with $H_*(\mathcal{J}, \mathcal{Q})$ using the Poincaré duality on X , the homomorphism \bar{F} in (2.4) maps any algebraic class in $H^*(\mathcal{J}, \mathcal{Q})$ into an algebraic class in $H^*(\mathcal{N}, \mathcal{Q})$. In the case where $r = 2$, this is in fact a consequence of Proposition 6.1 of [BKN].

Fix, once and for all, a polynomial $P(\{\tau_{i,j}\})$ of $\binom{2r-2}{2}$ variables (same as the number of $\tau_{i,j}$) and with coefficients in \mathcal{Q} such that

$$P(\{\tau_{i,j}\}) = \Phi.$$

We will show that there is a natural element in the Chow group of $\mathcal{N} \times \mathcal{J}$ whose cycle class is Φ . Before proceeding, we note that the Chern classes of an algebraic vector bundle can be defined in the Chow group [F]. Let $\hat{a}_i \in \text{CH}^i(X \times \mathcal{N})$ be the element of the Chow group given by (2.1), with any β_j being considered as an element of $\text{CH}^j(X \times \mathcal{N})$. So the cycle class of \hat{a}_i is a_i .

There is a natural choice of an element in the Chow group of $\mathcal{N} \times \mathcal{J}$, namely

$$E_{i,j} \in \text{CH}^*(\mathcal{N} \times \mathcal{J})$$

such that the cycle class of $E_{i,j}$ is the cohomology class $\tau_{i,j}$. Indeed, for the two cases $r \leq i, j \leq 2(r-1)$ and $i \leq r-1 < j$, in the proof of Theorem 2.5, we have produced an explicit element of the Chow group of $\mathcal{N} \times \mathcal{J}$ (depending only upon the point x_0) whose cycle class is $\tau_{i,j}$. We consider the remaining case, namely $1 \leq i \leq j \leq r-1$. As in the proof of Theorem 2.5, let t denote the cohomology class on \mathcal{N} which pulls back to $\tau_{i,j}$ by the projection of $\mathcal{N} \times \mathcal{J}$ onto \mathcal{N} . Let a' denote the cycle on \mathcal{N} obtained by taking the intersection of the cycle \hat{a}_{j+1} with $x_0 \times \mathcal{N}$, and let c' be the cycle on \mathcal{N} given by the push-forward of \hat{a}_{j+1} using the projection of $X \times \mathcal{N}$ onto \mathcal{N} . Imitating the Künneth decomposition (2.8), define the cycle b_j on \mathcal{N} by

$$b_j := \hat{a}_{j+1} - a' \otimes 1 - c' \otimes [X].$$

Similarly, construct b_i from \hat{a}_{i+1} . Now consider the image of the intersection cycle $b_i \cap b_j$ ($\in \text{CH}^*(X \times \mathcal{N})$) in $\text{CH}^*(\mathcal{N})$ by the push-forward map for the projection of $X \times \mathcal{N}$ onto \mathcal{N} . The cycle class of this cycle on \mathcal{N} is the cohomology class t . This is a straightforward computation, which is already done in the proof of Proposition 2.4 of [BN]. Thus the cycle class of the pullback of this cycle to $\mathcal{N} \times \mathcal{J}$, which we will denote by $E_{i,j}$, is actually $\tau_{i,j}$.

Define

$$(2.9) \quad \Gamma := P(\{E_{i,j}\}) \in \text{CH}^*(\mathcal{N} \times \mathcal{J}),$$

where $E_{i,j}$ and the polynomial P are defined as above. Thus we have that the cycle class of Γ is Φ .

3. Map between cycles. Using the cycle Γ defined in Section 2, we get a map (correspondence) from the Chow group $\text{CH}^*(\mathcal{J})$ of \mathcal{J} to $\text{CH}^*(\mathcal{N})$ [F, Definition 16.1.2], which we will now describe. Let q_1 (respectively, q_2) be the projection of $\mathcal{N} \times \mathcal{J}$ onto \mathcal{N} (respectively, \mathcal{J}). For a cycle c on \mathcal{J} consider the intersection cycle, $q_2^*c \cap \Gamma$, on $\mathcal{N} \times \mathcal{J}$. Let $q_{1*}(q_2^*c \cap \Gamma)$

be the cycle on \mathcal{N} obtained by taking the push-forward of $q_2^*c \cap \Gamma$ by the projection q_1 [F, §1.4]. Now let

$$(3.1) \quad \Pi : \text{CH}^*(\mathcal{J}) \rightarrow \text{CH}^*(\mathcal{N})$$

be the homomorphism defined by $c \mapsto q_{1*}(q_2^*c \cap \Gamma)$.

With a slight abuse of notation, the Gysin map $H^*(\mathcal{N} \times \mathcal{J}, \mathbf{C}) \rightarrow H^*(\mathcal{N}, \mathbf{C})$, for the projection q_1 , will also be denoted by q_{1*} . Consider the homomorphism

$$(3.2) \quad [\Phi] : H^*(\mathcal{J}, \mathbf{C}) \rightarrow H^*(\mathcal{N}, \mathbf{C})$$

defined by $a \mapsto q_{1*}((q_2^*a) \cup \Phi)$. Note that $[\Phi]$ maps $H^*(\mathcal{J}, \mathbf{Q})$ into $H^*(\mathcal{N}, \mathbf{Q})$. Moreover, $[\Phi]$ maps the space of Hodge cycles on \mathcal{J} into that on \mathcal{N} . The cycle class $[\Pi(c)] \in H^*(\mathcal{N}, \mathbf{Q})$ is actually $[\Phi]([c])$.

Using the Poincaré duality on \mathcal{J} , we have an isomorphism

$$(3.3) \quad \rho : H^*(\mathcal{J}, \mathbf{Q}) \rightarrow H_*(\mathcal{J}, \mathbf{Q}).$$

It is a routine calculation to check that for any $\omega \in H^*(\mathcal{J}, \mathbf{Q})$, the equality

$$(3.4) \quad [\Phi](\omega) = \bar{F}(\rho(\omega))$$

holds. (The homomorphism \bar{F} was defined in (2.4).) From (3.4) it follows that for a cycle c on \mathcal{J} ,

$$[\Pi(c)] = (\bar{F} \circ \rho)([c]).$$

We give an application of the correspondence using Γ .

It is easy to construct a curve X of genus g such that the rank of the Neron-Severi group of its Jacobian, J , is greater than one. (Take a curve X with a nontrivial automorphism, f , such that the genus of the quotient curve X/f is at least one. Then pullback the theta line bundle on the Jacobian $J(X/f)$ to J using the map defined by $L \mapsto \det(\pi_*L)$, where π is the quotient map. The first Chern class of this pullback bundle and that of the theta line bundle on J are linearly independent in $\text{NS}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$.) Let $a \in H^2(J, \mathbf{Q})$ be an element of the Neron-Severi group of J which is linearly independent with the theta bundle on J . Since $H^1(X, \mathbf{Q}) = H_1(X, \mathbf{Q})$ by the Poincaré duality on X , we get that

$$H^2(J, \mathbf{Q}) = \bigwedge^2 H^1(X, \mathbf{Q}) = \bigwedge^2 H_1(X, \mathbf{Q}) = H_2(J, \mathbf{Q}).$$

Let $\bar{a} \in H_2(J, \mathbf{Q})$ be the element corresponding to a . Define

$$b_k := 1 \otimes \cdots \otimes \bar{a} \otimes 1 \otimes \cdots \otimes 1 \in H_2(\mathcal{J}, \mathbf{Q})$$

to be the element of the Künneth decomposition of $H_2(\mathcal{J}, \mathbf{Q})$, where \bar{a} is at the k -th position.

It is easy to check that $\rho^{-1}(b_k)$ is a Hodge cycle on \mathcal{J} . Since any Hodge cycle of degree $2d - 2$ on a projective manifold of complex dimension d is actually algebraic, the cohomology class $\rho^{-1}(b_k)$ must be algebraic.

Consider $\bar{F}(b_k) \in H^{4k+2}(\mathcal{N}, \mathbf{Q})$, which is a Hodge cycle on \mathcal{N} . The class $\bar{F}(b_k)$ is not zero, for example, when $r = 2$ and $k = 1$ ([N], [KN]). Let C be a cycle on \mathcal{J} whose cycle class is $\rho^{-1}(b_k)$. The equality (3.4) implies that the cohomology class $\bar{F}(b_k)$ is the cycle class of $\Pi(C)$. Thus the Hodge cycle $\bar{F}(b_k)$ must be algebraic.

The cohomology class $\rho^{-1}(b_k) \in H^*(\mathcal{J}, \mathcal{Q})$ is not an invariant for the action of $G_{\mathcal{Z}}$. Indeed, the invariants in $H^2(J, \mathcal{Q})$ are spanned by the class of the theta line bundle. Thus the cohomology class a , and hence b_k , is not an invariant for the action of $G_{\mathcal{Z}}$. If $\bar{F}(b_k)$ is not zero, then it is not an invariant for the action of $G_{\mathcal{Z}}$, since the homomorphism \bar{F} is equivariant for the action of $G_{\mathcal{Z}}$. In [BN] it was proved that the space of Hodge cycles on \mathcal{N} for a general Riemann surface is exactly the space of invariants for the action of $G_{\mathcal{Z}}$. The above construction shows that the ‘‘generality’’ is essential. In other words, for special \mathcal{N} there are more Hodge cycles (and also algebraic classes) than those simply being the invariants.

4. Map between Deligne-Beilinson cohomologies. Let Y be a connected smooth projective variety over \mathbb{C} . For $p \geq 0$, let $\mathbf{Z}(p)$ denote the constant sheaf $\mathbf{Z} \cdot (2\pi\sqrt{-1})^p$ on Y . The following complex

$$\mathcal{D}(p)Y : 0 \rightarrow \mathbf{Z}(p) \rightarrow \mathcal{O} \xrightarrow{d} \Omega_Y^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_Y^{p-1} \rightarrow 0,$$

where the sheaf $\mathbf{Z}(p)$ is at the 0-th position, is called the p -th *Deligne complex*. Since we are working over the field of complex numbers, the distinction between $\mathbf{Z}(p)$ and \mathbf{Z} is not important for us. The j -th hypercohomology group of $\mathcal{D}(p)Y$ is called the j -th *Deligne-Beilinson cohomology group*, and it is denoted by $H_{\mathcal{D}}^j(Y, p)$ [EV, 1.1].

Consider the following complex, $\mathcal{C}(p)$, of \mathcal{O}_Y -modules

$$\mathcal{C}(p) : 0 \rightarrow \mathcal{O} \xrightarrow{d} \Omega_Y^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_Y^{p-1} \rightarrow 0.$$

The following exact sequences of complexes, that is,

$$0 \rightarrow \mathcal{C}(p)[1] \rightarrow \mathcal{D}(p) \rightarrow \mathbf{Z}(p) \rightarrow 0,$$

where $\mathcal{C}(p)[1]$ denotes the one step right translation of $\mathcal{C}(p)$ (i.e., \mathcal{O} is in the first position), induces the following exact sequence [EV, 7.9]

$$(4.1) \quad 0 \rightarrow J^p(Y) \rightarrow H_{\mathcal{D}}^{2p}(Y, p) \xrightarrow{\gamma} \text{Hg}^p(Y) \rightarrow 0,$$

where $J^p(Y)$ is the p -th Griffiths intermediate Jacobian, and $\text{Hg}^p(Y)$ is the subgroup of $H^{2p}(Y, \mathbf{Z}(p))$ consisting of those elements in $H^{2p}(Y, \mathbf{Z}(p))$ which are of type (p, p) .

Let $H_{\mathcal{D}}^*(Y)$ denote the direct sum $\bigoplus_{p \geq 0} H_{\mathcal{D}}^{2p}(Y, p)$. The cycle map in the Deligne-Beilinson cohomology

$$(4.2) \quad \psi : \text{CH}^*(Y) \rightarrow H_{\mathcal{D}}^*(Y)$$

is a ring homomorphism [EV, Corollary 7.7]. For $c \in \text{CH}^p(Y)$, the class

$$(\gamma \circ \psi)(c) \in \text{Hg}^p(Y)$$

(γ is defined in (4.1)) is $(2\pi\sqrt{-1})^p$ times the integral class corresponding to c . If c is homologically equivalent to zero, then $\psi(c)$, as an element of $J^p(Y)$, coincides with the image of c in $J^p(Y)$ under the Abel-Jacobi map [EV, Theorem 7.11].

The direct sum $\bigoplus_p H_{\mathcal{D}}^{2p}(Y, p)$ will be denoted by $H_{\mathcal{D}}^*(Y)$.

There is the pullback homomorphism $q_2^* : H_{\mathcal{D}}^*(\mathcal{J}) \rightarrow H_{\mathcal{D}}^*(\mathcal{N} \times \mathcal{J})$ such that for any cycle c on \mathcal{J}

$$\psi(q_2^*(c)) = q_2^*(\psi(c)),$$

where ψ is the homomorphism defined in (4.2) [EV, Proposition 7.5].

We know that for a cycle d on $\mathcal{N} \times \mathcal{J}$, the following equality holds:

$$\psi(d \cdot \Gamma) = \psi(d) \cup \psi(\Gamma).$$

The push-forward map, which corresponds to the Gysin map of cohomology,

$$q_{1*} : H_{\mathcal{D}}^*(\mathcal{N} \times \mathcal{J}) \rightarrow H^*(\mathcal{N})$$

satisfies the condition that for any cycle d on $\mathcal{N} \times \mathcal{J}$ the following equality holds:

$$\psi(q_{1*}(d)) = q_{1*}(\psi(d)),$$

where $q_{1*}(d)$ is the push-forward of d .

The map Π , defined in (3.1), is the composite of a pullback, the intersection with Γ , and a push-forward. Thus, from the above observations it follows that for any cycle c on \mathcal{J} ,

$$\psi(\Pi(c)) = [\Phi](\psi(c))$$

($[\Phi]$ was defined in (3.2)).

We want to describe how one can construct, using the map Π , nonzero elements of Griffiths group of \mathcal{N} . But before that we need to set up some notation.

For any integer $l \geq 0$, let

$$(4.3) \quad \Gamma_l \subset \Gamma$$

be the union of components of Γ of codimension l . In other words, $\Gamma_l \in \text{CH}^l(\mathcal{J} \times \mathcal{N})$. Let

$$(4.4) \quad [\Phi_l] : H^*(\mathcal{J}, \mathbf{C}) \rightarrow H^*(\mathcal{N}, \mathbf{C})$$

be the induced map of cohomologies obtained by taking Γ_l as the correspondence cycle. Note that $[\Phi_l]$ maps $H^*(\mathcal{J}, \mathcal{Q})$ to $H^*(\mathcal{N}, \mathcal{Q})$.

Let p_i , where $1 \leq i \leq r - 1$, denote the projection of \mathcal{J} onto the i -th factor. It is easy to check that for any $\omega \in H^k(J, \mathbf{C})$, the cohomology class $[\Phi](p_i^*\omega) \in H^*(\mathcal{N}, \mathbf{C})$ is actually of degree

$$\sum_{j=2}^r 2g(2j - 1) - k(2i + 1) = 2g(r^2 - 1) - k(2i + 1).$$

On the other hand, we have that

$$(4.5) \quad [\Phi_l](p_i^*\omega) \in H^{2l+k-2g(r-1)}(\mathcal{N}, \mathbf{C}).$$

Since Γ_l is the union of components of codimension l of Γ , when $l = g(r^2 + r - 2) - k(i + 1)$, the equality

$$(4.6) \quad [\Phi](p_i^*\omega) = [\Phi_l](p_i^*\omega)$$

is valid for any $\omega \in H^k(J, \mathbf{C})$.

For any $k \in \mathbb{N}$, let $f : X^k \rightarrow J$ be the morphism defined by $(x_1, \dots, x_k) \mapsto \sum_j (x_j - x_0)$, where x_0 is the base point of the pointed curve. Let W_k denote the cycle on J given by the image of f . Define $C_k := W_{g-k} - W_{g-k}^-$ to be the cycle of codimension k , where W_{g-k}^- denotes the image of W_{g-k} under the involution $j \mapsto -j$ of J . Clearly, C_k is homologically equivalent to zero. Ceresa in [C] proved that if $g \geq 3$ and $1 \leq k \leq g-1$, then for a generic curve X the cycle C_k is not algebraically equivalent to zero, i.e., it represents a nonzero element of the Griffiths group of J .

Define $C_{k,i}$ to be the following pullback cycle on \mathcal{J} :

$$C_{k,i} := p_i^* C_k,$$

where p_i is the projection of \mathcal{J} onto the i -th factor. The cycle $C_{k,i}$ is homologically equivalent to zero, since C_k is homologically equivalent to zero.

Let Π_l denote the correspondence map between Chow groups obtained by taking Γ_l as the correspondence cycle (as in (3.1)). In other words, for any $c \in \text{CH}^*(\mathcal{J})$, we have

$$\Pi_l(c) := q_{1*}(q_2^* c \cap \Gamma_l) \in \text{CH}^*(\mathcal{N}).$$

Since $C_{k,i}$ is homologically equivalent to zero, so is the cycle $\Pi_l(C_{k,i})$.

Let $L \in H^2(J, \mathbf{Z})$ be the polarization on the Jacobian J given by the Poincaré dual of the theta divisor (which is an ample divisor on J). Let $P_k \subset H^k(J, \mathbf{C})$ denote the space of primitive cohomology classes. Consider the Lefschetz decomposition

$$(4.7) \quad H^{2p-1}(J, \mathbf{Q}) = \bigoplus_{1 \leq k \leq p} L^{p-k} \wedge P_{2k-1}^{\mathbf{Q}},$$

where $P_j^{\mathbf{Q}}$ is the space of all rational primitive cohomology classes of degree j , i.e., $P_j^{\mathbf{Q}} = P_j \cap H^j(J, \mathbf{Q})$. Recall that $G_{\mathbf{Z}}$ denotes the group of all automorphisms of $H_1(X, \mathbf{Z})$ preserving the cap product. The subspaces $P_j^{\mathbf{Q}}$ are all irreducible $G_{\mathbf{Z}}$ modules, and (4.7) is the decomposition into irreducible $G_{\mathbf{Z}}$ modules. Indeed, this is a consequence of the fact that the complexification of the decomposition (4.7) is the irreducible decomposition of the $G_{\mathbf{C}}$ module $H^{2p-1}(J, \mathbf{C})$, where $G_{\mathbf{C}}$ is the complexification of $G_{\mathbf{Z}}$, i.e., the group of all automorphisms of $H_1(X, \mathbf{C})$ preserving the cap product. As a $G_{\mathbf{Z}}$ module, the irreducibility of any $P_{2k-1}^{\mathbf{Q}}$ is a consequence of the Borel density theorem which asserts that $G_{\mathbf{Z}}$ is Zariski dense in $G_{\mathbf{C}}$.

Let \mathcal{M}_g^1 denote the moduli space of one pointed Riemann surfaces of genus g . In other words, it is the moduli space of pairs of the form (X, p) , where p is a point of a Riemann surface X of genus g .

THEOREM 4.8. *Let g and p be integers such that $g-1 \geq p \geq 1$. Assume that the image $[\Phi_l](p_i^*(L^{p-2} P_3^{\mathbf{Q}}))$ is a nontrivial subspace of $H^*(\mathcal{N}, \mathbf{Q})$. Then there is a countable union of proper subvarieties, say $\bigcup_k S_k$, of the moduli space \mathcal{M}_g^1 with the following property: For any $(X, x_0) \in \mathcal{M}_g^1 - \bigcup_k S_k$, the cycle $\Pi_l(C_{p,i})$ on \mathcal{N} is not algebraically equivalent to zero, and it is of infinite order as an element of the Griffiths group of \mathcal{N} .*

Before proving the theorem we remark some observations.

REMARK 4.9. (1) The condition that $[\Phi_l](p_i^*(L^{p-2}P_3^{\mathcal{Q}}))$ is a nontrivial subspace is independent of the choice of the curve X . For a smooth family of pointed curves, the subspace $p_i^*(L^{p-2}P_3^{\mathcal{Q}}) \subset H^{2p-1}(\mathcal{J}, \mathcal{Q})$ gives rise to a sub-local system of the local system on the parameters space given by $H^{2p-1}(\mathcal{J}, \mathcal{Q})$. The homomorphism Φ_l actually gives a homomorphism of local systems. Thus, for a smooth family of curves parametrized by a connected space, if $[\Phi_l](p_i^*(L^{p-2}P_3^{\mathcal{Q}}))$ is zero for one curve, then it is zero for all curves in the family.

(2) Set $r = 2$ and $l = 6$. So the equality (4.6) is valid with $k = 2g - 3$. Now using the equality (3.4), the image $[\Phi_l](L^{p-2}P_3^{\mathcal{Q}})$ is identified with \bar{F} on $\wedge^3 H_1(X, \mathbb{C})$. If $g \geq 4$, then \bar{F} maps $\wedge^3 H_1(X, \mathbb{C})$ injectively into $H^9(\mathcal{N}, \mathcal{Q})$ ([N], [KN, Proposition 2.1]). So in this situation Theorem 4.8 implies that, for a general one pointed curve (X, x_0) , the cycle $\Pi_l(C_{g-1})$ is not algebraically equivalent to zero.

(3) The cycle $\Pi(C_{p,i})$ is not algebraically equivalent to zero if the component $\Pi_l(C_{p,i})$ is so. However we consider $\Pi_l(C_{p,i})$ to get a cycle of pure dimension.

PROOF OF THEOREM 4.8. The image $[\Phi_l](p_i^*(H^{2p-1}(J, \mathbb{C})))$ is contained in $H^m(\mathcal{N}, \mathcal{Q})$, where m is some odd integer which depends on l, i, p and r . Consider the Griffiths intermediate Jacobian

$$J^{(m+1)/2}(\mathcal{N}) := \frac{H^m(\mathcal{N}, \mathbb{C})}{F^{(m+1)/2} + H^m(\mathcal{N}, \mathbb{Z}((m+1)/2))}$$

of \mathcal{N} , where $F^{(m+1)/2} = \bigoplus_{j \geq 0} H^{(m+1)/2+j, (m-1)/2-j}(\mathcal{N})$. (Note that m is odd.) Since both $[\Phi_l]$ and the pullback operation on cohomology, namely p_i^* , are given by correspondence cycles, we have a homomorphism

$$\nu : J^p(J) := \frac{H^{2p-1}(J, \mathbb{C})}{F^p + H^{2p-1}(J, \mathbb{Z}(p))} \rightarrow J^{(m+1)/2}(\mathcal{N}),$$

where $F^p = \bigoplus_{j \geq 0} H^{p+j, p-1-j}(J)$.

Since the Abel-Jacobi map ψ , defined in (4.2), is compatible with correspondence, the following equality is valid:

$$(4.10) \quad \nu(\psi(C_{p,i})) = \psi(\Pi_l(C_{p,i})).$$

If $\Pi_l(C_{p,i})$ is algebraically equivalent to zero, then there is a closed algebraic subgroup of the complex torus $J^{(m+1)/2}(\mathcal{N})$ which contains $\psi(\Pi_l(C_{p,i}))$ [H1, Proposition 10.1]. Assuming that $\Pi_l(C_{p,i})$ is algebraically equivalent to zero, denote this algebraic subgroup of $J^{(m+1)/2}(\mathcal{N})$ by A .

Consider the following complex torus:

$$(4.11) \quad J_3 := \frac{L^{p-2}P_3}{(F^p + H^{2p-1}(J, \mathbb{Z}(p))) \cap L^{p-2}P_3},$$

where $L^{p-2}P_3$ is a factor of the Lefschetz decomposition in (4.7). Using the decomposition (4.7), a finite (unramified) covering of the torus $J^p(J)$, say \hat{J} , is identified with the Cartesian product $J_3 \times B$, where B is a complex torus. Since the decomposition (4.7) is not necessarily

over \mathbf{Z} , we need to go to a finite covering of $J^p(J)$. Let $\hat{\nu}$ be the homomorphism from \hat{J} to $J^{(m+1)/2}(\mathcal{N})$ given by ν . Let f (respectively g) denote the projection (respectively inclusion) homomorphism between \hat{J} and J_3 .

In the statement of Theorem 4.8 it is assumed that the image $[\Phi_l](p_i^*(L^{p-2}P_3^{\mathcal{Q}}))$ is a nonzero subspace of $H^m(\mathcal{N}, \mathcal{Q})$. Now, the homomorphism $[\Phi_l]$ in (4.4) is equivariant for the action of $G_{\mathbf{Z}}$ described in Section 2. Indeed, this is a consequence of the fact that the homomorphism \bar{F} in (2.4) is equivariant for the action of $G_{\mathbf{Z}}$ ([B1], [BN]). Since $P_3^{\mathcal{Q}}$ is an irreducible $G_{\mathbf{Z}}$ module, $[\Phi_l](p_i^*(L^{p-2}P_3^{\mathcal{Q}}))$ must be isomorphic to $P_3^{\mathcal{Q}}$ as a $G_{\mathbf{Z}}$ module. This implies that $\hat{\nu} \circ g$ is an embedding of J_3 in $J^{(m+1)/2}(\mathcal{N})$. (The homomorphism $\hat{\nu} \circ g$ coincides with the restriction of ν to the image of J_3 in $J^p(J)$ by the obvious inclusion.) Moreover, since (4.7) is an isotypical decomposition, i.e., all the $P_{2k-1}^{\mathcal{Q}}$ are distinct $G_{\mathbf{Z}}$ modules, we conclude that for any $k \neq 2$, the two subspaces of $H^m(\mathcal{N}, \mathcal{Q})$, namely $[\Phi_l](p_i^*(L^{p-2}P_3^{\mathcal{Q}}))$ and $[\Phi_l](p_i^*(L^{p-k}P_{2k-1}^{\mathcal{Q}}))$, belong to two distinct components of the isotypical decomposition of the $G_{\mathbf{Z}}$ module $H^m(\mathcal{N}, \mathcal{Q})$.

Consider the natural projection of $G_{\mathbf{Z}}$ module $H^m(\mathcal{N}, \mathcal{Q})$ onto its isotypical component corresponding to the $G_{\mathbf{Z}}$ module $P_3^{\mathcal{Q}}$. This projection induces a projection, denoted by h , of the torus $J^{(m+1)/2}(\mathcal{N})$ to another complex torus, say T . Evidently, $h \circ \hat{\nu} \circ g$ is a homomorphism from J_3 with finite kernel (recall that $\hat{\nu} \circ g$ is an embedding). Given this, since $f(\psi(C_{p,i}))$ in J_3 is of infinite order (a consequence of Proposition 8.8 of [H2]), the equality (4.10) implies that the element $h(\psi(\Pi_l(C_{p,i})))$ is of infinite order in T . Since A is an abelian variety, so is $h(A)$. Since $h(A)$ contains an element of infinite order, namely $h(\psi(\Pi_l(C_{p,i})))$, it must be of strictly positive dimension.

Thus $(h \circ \hat{\nu} \circ g)^{-1}(h(A))$ is an algebraic subgroup of J_3 of strictly positive dimension. That the dimension of $(h \circ \hat{\nu} \circ g)^{-1}(h(A))$ is strictly positive is a consequence of the fact that the order of the element $f(\psi(C_{p,i}))$ of $(h \circ \hat{\nu} \circ g)^{-1}(h(A))$ is infinite.

We will now complete the proof of Theorem 4.8 imitating the argument given in the proof of Theorem 10.3 of [H1]. The Lemma 10.2 in [H1, page 125] asserts the following: let S be the set of all curves, X , such that the complex torus J_3 (defined in 4.11) corresponding to X contains a closed algebraic subgroup (abelian variety) of strictly positive dimension. Then the subset of \mathcal{M}_g^1 defined by S is actually contained in a countable union of proper closed subvarieties of \mathcal{M}_g^1 . In view of this lemma, the proof of Theorem 4.8 is completed. \square

REMARK 4.12. The product $\mathcal{N} \times J$ is an étale cover of $M(r, d)$, where $M(r, d)$ is the moduli of stable bundle of rank r and degree d . The covering map is given by $(E, L) \mapsto E \times L$. Thus the image in $M(r, d)$, of any cycle on \mathcal{N} (or J), that represents a nontrivial element of the Griffiths group of \mathcal{N} (or J), would represent a nontrivial element of the Griffiths group of $M(r, d)$.

REMARK 4.13. Deligne proved that any Hodge cycle on an abelian variety is an absolute Hodge cycle [D, Main Theorem 2.11]. So, in particular, any Hodge cycle on any self-product (arbitrary number of times) of a Jacobian is an absolute Hodge cycle. Using this

and the existence of the correspondence cycle Γ , it is easy to check that any Hodge cycle on \mathcal{N} is an absolute Hodge cycle. In the case of rank two, this was proved in [B2].

Both Theorem 2.5 and Theorem 4.8 generalize to any (smooth) moduli space of parabolic bundles. Indeed, the results of [BR] and [BN] used here are actually proved in that generality. If \mathcal{N} denotes a smooth moduli space of parabolic bundles of rank r and fixed determinant, then the homomorphism \bar{F} in (2.4) remains valid. The monodromy action of the mapping class group, for Riemann surfaces with marked points, on the cohomology of a smooth moduli space of parabolic bundles factors through the symplectic group. Since invariant rational cohomology classes continue to be algebraic, the argument for Theorems 2.4 and 4.8 remains valid.

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