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GALOIS QUANTUM GROUPS OF II_1-SUBFACTORS

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Abstract. We give a correspondence between a class of quantum groups (face algebras) and a class of AFD II_1-subfactors, which contains both all of those of index less than 4 and all of those of principal graph $D_n^{(1)}$ or $E_n^{(1)}$. Ocneanu's flat connection and a variant of Woronowicz's compact quantum group theory play central roles.

Introduction. It is widely expected that Jones' index theory is deeply connected with quantum groups. One of the evidence is an apparent similarity between Ocneanu's Galois invariants (flat biunitary connections) of II_1-subfactors and Boltzmann weights of solvable lattice models (SLM). In fact, quantum groups originated from the so-called L-operators of SLM of vertex type.

Investigating the algebraic structure of L-operators of SLM of face type, the author found the notion of **face algebras**, which is an unexpected generalization of bialgebras. Although the definition of face algebras is more complicated than that of bialgebras, many important concepts in the bialgebra theory—such as antipodes, Haar functionals and universal R-matrixes—have natural generalization in the theory of face algebras. In particular, the category $\mathcal{C}$ of (co-)modules of a face algebra still has a binary operation $\bar{\otimes}$ which makes $\mathcal{C}$ a monoidal category. In a previous paper [H2], the author used face algebras in order to prove certain technical lemmas arising from the classification problem of II_1-subfactors of index less than 4.

In this paper, we establish a new relation between II_1-subfactors and face algebras. More precisely, we give a correspondence between a class of irreducible AFD II_1-subfactors and a class of face algebras with specified comodules. The correspondence covers all AFD II_1-subfactors $N \subset M$ of index less than 4, and gives a "group-theoretic" interpretation of these, which is just like the construction, due to Goodman, de la Harpe and Jones [G-H-J], of II_1-subfactors of index 4 via subgroups of $SU(2)$.

In consequence of our construction, we obtain new examples of quantum groups $\mathfrak{G}$ which have rich representation theory. We classify their irreducible comodules, and compute their dimensions and fusion (branching) rules with respect to $\bar{\otimes}$. When $N \subset M$ is of type A_{l+1}, $\mathfrak{G}$ has fusion rules which coincide with those of $SU(2)_l$-WZNW models in conformal field theory. In a forthcoming paper [H6], we construct face algebras whose fusion rules are the same as those of $SU(N)_l$-WZNW models, using the results of this paper. We will also give applications of these to quantum invariants of 3-manifolds.

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Our construction of II_1-subfactors (Theorem 2.8) is a generalization of that of [G-H-J] and Wassermann [Wa]. However, the proof is more involved. In fact, it deeply depends on abstract harmonic analysis of face algebras, which is a variant of Woronowicz's theory of compact quantum groups (cf. [Wo]). Category-theoretic properties of comodules of face algebras also play important roles.

The construction of face algebras $\mathfrak{S}$ is inspired by Schur's reciprocity theorem between $GL(N, \boldsymbol{C})$ and the symmetric group $\mathfrak{S}_m$. The algebras $\mathfrak{S}$ are defined so as to be satisfied a reciprocity theorem between $\mathfrak{S}$ and the string algebras (cf. Proposition 4.4(1)). Ocneanu's notion of flatness plays a crucial role.

In Section 1, we recall basic properties of face algebras and their comodules. In particular, we recall the notion of hollowless compact Hopf face algebras $\mathfrak{H}$ and functionals $\boldsymbol{Q}$ on them, which we call the **Woronowicz functionals**.

In Subsection 2.1, we define the $\boldsymbol{Q}$-dimension $\dim_{\boldsymbol{Q}}(V)$ and the $\boldsymbol{Q}$-trace $\mathrm{Tr}_{\boldsymbol{Q}}(f)$ for each $\mathfrak{H}$-comodule V and its endomorphism $f \in \mathrm{End}_{\mathfrak{H}}(V)$. In Subsections 2.2 and 2.3, we construct a commuting square for each of three $\mathfrak{H}$-comodules. In Sections 2.4 and 2.5, we construct a II_1-subfactor of index $\dim_{\boldsymbol{Q}}(V)^2$ for each irreducible $\mathfrak{H}$-comodule V, provided that $\mathfrak{H}$ is finite dimensional.

In Section 3, we begin to study **flat face models** (V, w) which are variants of Ocneanu's flat biunitary connections. They also contain Wenzl's Hecke algebra representations at a root of unity in some sense. For each (V, w), we define its string algebra $\mathrm{Str}^m(V)$ and construct an action of $\mathrm{Str}^m(V)$ on the "full" path space. Using these, we define a face algebra $\mathrm{Cost}(V)$ which is called the **costring algebra**.

In Section 4, we define a quotient $\mathfrak{S}(V)$ of $\mathrm{Cost}(V)$ for each flat biunitary connection such that its principal graph $\mathcal{G}$ is finite and coincides with the dual principal graph. We prove that $\mathfrak{S}(V)$ is a finite-dimensional hollowless compact Hopf face algebra and that its irreducible comodules are labeled by vertexes of $\mathcal{G}$. Using a result of Ocneanu and S. Popa, we verify that $\mathfrak{S}(V)$ has enough information to reconstruct the original II_1-subfactor.

The author would like to thank Professor M. Izumi for explaining Ocneanu's notion of flatness.

We refer the reader to [G-H-J] for basic facts on Jones' index theory.

NOTATIONS AND TERMINOLOGIES. Throughout this paper, $\Delta : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}$ (resp. $\varepsilon : \mathfrak{H} \to \boldsymbol{K}$) denotes the coproduct (resp. counit) of a coalgebra $\mathfrak{H}$ over a field $\boldsymbol{K}$, and $\rho = \rho_V : V \to V \otimes \mathfrak{H}$ denotes the structure map of a right $\mathfrak{H}$-comodule V. We also use Sweedler's "sigma" notation: $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$, $(\Delta \otimes \mathrm{id}) \circ \Delta(x) = (\mathrm{id} \otimes \Delta) \circ \Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$, $\rho_V(u) = \sum_{(u)} u_{(0)} \otimes u_{(1)}$ $(x \in \mathfrak{H}, u \in V)$, etc. (cf. [S]).

1. Preliminaries. We summarize facts on face algebras and their comodules (see [H4] and [H5]).

1.1. Face algebras. Let $\mathfrak{H}$ be an algebra over a field $\boldsymbol{K}$, which also has a coalgebra structure $(\mathfrak{H}, \Delta, \varepsilon)$. Let $\mathcal{V}$ be a finite non-empty set and $\{e_i, \mathring{e}_j \mid i, j \in \mathcal{V}\}$ elements of $\mathfrak{H}$. We say that $\mathfrak{H} = (\mathfrak{H}, \{e_i, \overset{\circ}{e}_j\})$ is a $\mathcal{V}$-**face algebra** if the following axioms are satisfied:

$$\Delta(ab) = \Delta(a)\Delta(b)\,, \tag{1.1}$$

$$e_i e_j = \delta_{ij} e_i\,, \quad \overset{\circ}{e}_i \overset{\circ}{e}_j = \delta_{ij} \overset{\circ}{e}_i\,, \quad e_i \overset{\circ}{e}_j = \overset{\circ}{e}_j e_i\,, \qquad \sum_{k\in\mathcal{V}} e_k = \sum_{k\in\mathcal{V}} \overset{\circ}{e}_k = 1\,, \tag{1.2}$$

$$\Delta(\overset{\circ}{e}_i e_j) = \sum_{k\in\mathcal{V}} \overset{\circ}{e}_i e_k \otimes \overset{\circ}{e}_k e_j\,, \quad \varepsilon(\overset{\circ}{e}_i e_j) = \delta_{ij}\,, \tag{1.3}$$

$$\sum_{k\in\mathcal{V}} \varepsilon(a e_k)\varepsilon(\overset{\circ}{e}_k b) = \varepsilon(ab) \tag{1.4}$$

for each $a, b \in \mathfrak{H}$ and $i, j \in \mathcal{V}$. If, in addition, $\{\overset{\circ}{e}_i e_j \mid i, j \in \mathcal{V}\}$ are linearly independent, then $\mathfrak{H}$ is called **hollowless**. A subspace $\mathfrak{I}$ of a $\mathcal{V}$-face algebra $\mathfrak{H}$ is called a **biideal** if it is both an ideal and a coideal. In this case, the quotient $\mathfrak{H}/\mathfrak{I}$ naturally becomes a $\mathcal{V}$-face algebra. A $\mathcal{V}$-face algebra becomes a bialgebra if and only if $\sharp(\mathcal{V}) = 1$.

EXAMPLE 1.1. Let $\mathcal{G}$ be a finite oriented graph. We denote by $\mathcal{V} = \mathcal{G}^0$ the set of vertexes of $\mathcal{G}$ and by $\mathcal{G}^1$ the set of edges of $\mathcal{G}$. We denote the source (start) and the range (end) of an edge $\boldsymbol{p}$ of $\mathcal{G}$ by $\mathfrak{s}(\boldsymbol{p})$ and $\mathfrak{r}(\boldsymbol{p})$, respectively. For each $m > 0$, let $\mathcal{G}^m = \coprod_{i,j\in\mathcal{V}} \mathcal{G}^m_{ij}$ be the set of **paths** on $\mathcal{G}$ of length m. That is, $\boldsymbol{p} \in \mathcal{G}^m_{ij}$ if $\boldsymbol{p}$ is a sequence $(\boldsymbol{p}_1, \dots, \boldsymbol{p}_m)$ of edges of $\mathcal{G}$ such that $\mathfrak{s}(\boldsymbol{p}) := \mathfrak{s}(\boldsymbol{p}_1) = i$, $\mathfrak{r}(\boldsymbol{p}_1) = \mathfrak{s}(\boldsymbol{p}_2), \dots, \mathfrak{r}(\boldsymbol{p}_{m-1}) = \mathfrak{s}(\boldsymbol{p}_m)$, $\mathfrak{r}(\boldsymbol{p}) := \mathfrak{r}(\boldsymbol{p}_m) = j$. We also set $\mathcal{G}^0 = \coprod_{i,j\in\mathcal{V}} \mathcal{G}^0_{ij}$, $\mathcal{G}^0_{ii} = \{i\}$ $(i \in \mathcal{V})$, $\mathcal{G}^0_{ij} = \emptyset$ $(i \neq j)$ and $\mathcal{G}^m_{i,-} = \coprod_j \mathcal{G}^m_{ij}$, $\mathcal{G}^m_{-,j} = \coprod_i \mathcal{G}^m_{ij}$. Let $\mathfrak{H}(\mathcal{G})$ be the linear span of the symbols

$$\left\{ e\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix} \,\middle|\, \boldsymbol{p}, \boldsymbol{q} \in \mathcal{G}^m,\ m \geq 0 \right\}.$$

Then, $\mathfrak{H}(\mathcal{G})$ becomes a $\mathcal{V}$-face algebra by setting

$$\begin{gathered} \overset{\circ}{e}_i = \sum_{j\in\mathcal{V}} e\begin{pmatrix} i \\ j \end{pmatrix}, \qquad e_j = \sum_{i\in\mathcal{V}} e\begin{pmatrix} i \\ j \end{pmatrix}. \\ e\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix} e\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \delta_{\mathfrak{r}(\boldsymbol{p}),\mathfrak{s}(\boldsymbol{a})}\delta_{\mathfrak{r}(\boldsymbol{q}),\mathfrak{s}(\boldsymbol{b})} e\begin{pmatrix} \boldsymbol{p}\cdot\boldsymbol{a} \\ \boldsymbol{q}\cdot\boldsymbol{b} \end{pmatrix}, \\ \Delta\left(e\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix}\right) = \sum_{\boldsymbol{t}\in\mathcal{G}^m} e\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{t} \end{pmatrix} \otimes e\begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{q} \end{pmatrix}, \\ \varepsilon\left(e\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix}\right) = \delta_{\boldsymbol{pq}} \quad (\boldsymbol{p}, \boldsymbol{q} \in \mathcal{G}^m,\ \boldsymbol{a}, \boldsymbol{b} \in \mathcal{G}^n). \end{gathered} \tag{1.5}$$

Here, for paths $\boldsymbol{p} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_m)$ and $\boldsymbol{a} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$, we set $\boldsymbol{p}\cdot\boldsymbol{a} := (\boldsymbol{p}_1, \dots, \boldsymbol{p}_m, \boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ if $\mathfrak{r}(\boldsymbol{p})$ coincides with $\mathfrak{s}(\boldsymbol{a})$. Also, we set $i \cdot \boldsymbol{p} = \boldsymbol{p} \cdot j = \boldsymbol{p}$ for each $i, j \in \mathcal{G}^0$ and $\boldsymbol{p} \in \mathcal{G}^m_{ij}$. It is known that each finitely generated face algebra is isomorphic to $\mathfrak{H}(\mathcal{G})/\mathfrak{I}$ for some $\mathcal{G}$ and a biideal $\mathfrak{I} \subset \mathfrak{H}(\mathcal{G})$ (cf. [H7]).

Throughout this paper, we frequently use the notations for graphs defined in the example above.

Let S be a linear endomorphism on a $\mathcal{V}$-face algebra $\mathfrak{H}$. We say that S is an **antipode** if it satisfies:

$$\sum_{(a)} S(a_{(1)})a_{(2)} = \sum_{i\in\mathcal{V}} \varepsilon(ae_i)e_i\,, \tag{1.6}$$

$$\sum_{(a)} a_{(1)}S(a_{(2)}) = \sum_{i\in\mathcal{V}} \varepsilon(e_i a)\mathring{e}_i\,, \tag{1.7}$$

$$\sum_{(a)} S(a_{(1)})a_{(2)}S(a_{(3)}) = S(a) \tag{1.8}$$

for each $a \in \mathfrak{H}$. A $\mathcal{V}$-face algebra is called a $\mathcal{V}$-**Hopf face algebra** if it has an antipode. When $\sharp(\mathcal{V}) = 1$, this definition coincides with the usual one. The antipode is unique if it exists, and it is both an anti-algebra and an anti-coalgebra endomorphism of $\mathfrak{H}$ such that

$$S(\mathring{e}_i e_j) = \mathring{e}_j e_i \quad (i, j \in \mathcal{V})\,. \tag{1.9}$$

LEMMA 1.2. *For a $\mathcal{V}$-face algebra $\mathfrak{H}$, $i, j, i', j' \in \mathcal{V}$ and $a \in \mathfrak{H}$, we have the following formulas*:

$$\varepsilon(ae_i) = \varepsilon(a\mathring{e}_i)\,, \quad \varepsilon(e_i a) = \varepsilon(\mathring{e}_i a)\,, \tag{1.10}$$

$$\sum_{(a)} a_{(1)}\varepsilon(e_i a_{(2)} e_j) = e_i a e_j\,, \tag{1.11}$$

$$\sum_{(a)} \varepsilon(e_i a_{(1)} e_j)a_{(2)} = \mathring{e}_i a \mathring{e}_j\,, \tag{1.12}$$

$$\sum_{(a)} e_i a_{(1)} e_j \otimes a_{(2)} = \sum_{(a)} a_{(1)} \otimes \mathring{e}_i a_{(2)} \mathring{e}_j\,, \tag{1.13}$$

$$\Delta(\mathring{e}_i e_j a \mathring{e}_{i'} e_{j'}) = \sum_{(a)} \mathring{e}_i a_{(1)} \mathring{e}_{i'} \otimes e_j a_{(2)} e_{j'}\,. \tag{1.14}$$

See [H4] for a proof of these formulas.

1.2. Comodules. For a $\mathcal{V}$-face algebra $\mathfrak{H}$, we define linear functionals ε_i, $\mathring{\varepsilon}_i \in \mathfrak{H}^*$ $(i \in \mathcal{V})$ by

$$\varepsilon_i(a) = \varepsilon(ae_i)\,, \quad \mathring{\varepsilon}_i(a) = \varepsilon(e_i a) \quad (a \in \mathfrak{H})\,. \tag{1.15}$$

As elements of the dual algebra $\mathfrak{H}^*$, they satisfy the following relations:

$$\varepsilon_i\varepsilon_j = \delta_{ij}\varepsilon_i\,, \quad \mathring{\varepsilon}_i\mathring{\varepsilon}_j = \delta_{ij}\mathring{\varepsilon}_i\,, \quad \mathring{\varepsilon}_i\varepsilon_j = \varepsilon_j\mathring{\varepsilon}_i\,, \tag{1.16}$$

$$\sum_{i\in\mathcal{V}} \varepsilon_i = 1 = \sum_{i\in\mathcal{V}} \mathring{\varepsilon}_i\,. \tag{1.17}$$

Hence each right $\mathfrak{H}$-comodule V has a direct sum decomposition given by

$$V = \bigoplus_{i,j\in\mathcal{V}} V(i,j)\,, \quad V(i,j) := \pi_V(\overset{\circ}{\varepsilon}_i \varepsilon_j)(V)\,, \tag{1.18}$$

where the representation $\pi_V : \mathfrak{H}^* \to \mathrm{End}(V)$ is given by

$$\pi_V(X)(u) = \sum_{(u)} u_{(0)}\langle X, u_{(1)}\rangle \quad (u \in V,\ X \in \mathfrak{H}^*)\,. \tag{1.19}$$

We call (1.18) the **face space decomposition** of V. When V is finite dimensional, we define a graph $\mathcal{G}$ by $\mathcal{G}^0 = \mathcal{V}$ and $\sharp(\mathcal{G}^1_{ij}) = \dim(V(i,j))$ and call it the **dimension graph** of V. Let $\{u_{\boldsymbol{q}} \mid \boldsymbol{q} \in \mathcal{G}^1_{ij}\}$ be a basis of $V(i,j)$. We define a matrix $[x^{\boldsymbol{p}}_{\boldsymbol{q}}] \in \mathrm{Mat}(\mathcal{G}^1, \mathfrak{H})$ by

$$\rho_V(u_{\boldsymbol{q}}) = \sum_{\boldsymbol{p}} u_{\boldsymbol{p}} \otimes x^{\boldsymbol{p}}_{\boldsymbol{q}}$$

and call it the **matrix corepresentation** of $(V, \{u_{\boldsymbol{q}}\})$. The following lemma easily follows from (1.11) and (1.12).

LEMMA 1.3. *Let* $[x^{\boldsymbol{p}}_{\boldsymbol{q}}]$ *be as above. Then we have*

$$\overset{\circ}{e}_i e_j x^{\boldsymbol{p}}_{\boldsymbol{q}} \overset{\circ}{e}_{i'} e_{j'} = \delta_{i\mathfrak{s}(\boldsymbol{p})}\delta_{j\mathfrak{s}(\boldsymbol{q})}\delta_{i'\mathfrak{r}(\boldsymbol{p})}\delta_{j'\mathfrak{r}(\boldsymbol{q})} x^{\boldsymbol{p}}_{\boldsymbol{q}}$$

for each $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{G}^1$ *and* $i, i', j, j' \in \mathcal{V}$.

Let W be another $\mathfrak{H}$-comodule. We define an $\mathfrak{H}$-comodule $V \bar{\otimes} W$ by

$$V \bar{\otimes} W = \bigoplus_{i,j,k\in\mathcal{V}} V(i,k) \otimes W(k,j)\,,$$

$$\rho_{V\bar{\otimes}W}(u \otimes v) = \sum_{(u)}\sum_{(v)} (u_{(0)} \otimes v_{(0)}) \otimes u_{(1)}v_{(1)} \quad (u \in V(i,k),\ v \in W(k,j))$$

and call it the **truncated tensor product** of V and W. For $\mathfrak{H}$-comodule maps $f : V \to V'$ and $g : W \to W'$, $f \bar{\otimes} g := (f \otimes g)\big|_{V\bar{\otimes}W}$ gives an $\mathfrak{H}$-comodule map from $V \bar{\otimes} W$ into $V' \bar{\otimes} W'$.

Let g be an element of $\mathfrak{H}$. We say that g is **group-like** if the following three relations are satisfied:

$$\Delta(g) = \sum_{k\in\mathcal{V}} g e_k \otimes g \overset{\circ}{e}_k\,,$$

$$\overset{\circ}{e}_i e_j g = g \overset{\circ}{e}_i e_j\,, \quad \varepsilon(g \overset{\circ}{e}_i e_j) = \delta_{ij} \quad (i, j \in \mathcal{V})\,.$$

By (1.3) and (1.2), the unit of a face algebra is group-like. For a group-like element g, let Rg denote the linear span of the symbols $\{e_j g \mid j \in \mathcal{V}\}$ equipped with an $\mathfrak{H}$-comodule structure given by

$$\rho_{Rg}(e_j g) = \sum_{i\in\mathcal{V}} e_i g \otimes g \overset{\circ}{e}_i e_j$$

Then $R := R1$ satisfies $R \bar{\otimes} V \simeq V \simeq V \bar{\otimes} R$ for each $\mathfrak{H}$-comodule V. We call R the **unit comodule** of $\mathfrak{H}$. Explicitly, the isomorphisms are given by

$$V \xrightarrow{\gamma} R \bar{\otimes} V\,; \quad u \mapsto e_i \otimes u\,, \quad V \xrightarrow{\delta} V \bar{\otimes} R\,; \quad u \mapsto u \otimes e_j \quad (u \in V(i,j))\,.$$

Note that R is irreducible if $\mathfrak{H}$ is hollowless.

Next, suppose that $\mathfrak{H}$ has the antipode S and that V is finite dimensional. Then the dual space V^* of V has a unique structure of a right $\mathfrak{H}$-comodule such that

$$\sum_{(u)}\langle v, u_{(0)}\rangle S(u_{(1)}) = \sum_{(v)}\langle v_{(0)}, u\rangle v_{(1)} \quad (u \in V,\ v \in V^*)\,.$$

We denote this comodule by $V^\vee$ and call it the **left dual comodule** of V. This terminology is compatible with that of monoidal category theory. That is, there exist comodule maps $\% : R \to V \bar{\otimes} V^\vee$ and $\$: V^\vee \bar{\otimes} V \to R$ such that both of the following two composite maps are identities (see e.g. [D2]):

$$V \xrightarrow{\gamma} R \bar{\otimes} V \xrightarrow{\% \bar{\otimes} \mathrm{id}} V \bar{\otimes} V^\vee \bar{\otimes} V \xrightarrow{\mathrm{id} \bar{\otimes} \$} V \bar{\otimes} R \xrightarrow{\delta^{-1}} V\,,$$

$$V^\vee \xrightarrow{\delta} V^\vee \bar{\otimes} R \xrightarrow{\mathrm{id} \bar{\otimes} \%} V^\vee \bar{\otimes} V \bar{\otimes} V^\vee \xrightarrow{\$ \bar{\otimes} \mathrm{id}} R \bar{\otimes} V^\vee \xrightarrow{\gamma^{-1}} V^\vee\,.$$

Explicitly, these maps are given by

$$\%(e_i) = \sum_{\nu} \pi(\mathring{\varepsilon}_i) u_\nu \otimes v^\nu \quad (i \in \mathcal{V})\,,$$

$$\$(v \otimes u) = \langle v, u\rangle e_i \quad (v \in V^\vee(i,k),\ u \in V(k,j),\ i,j,k \in \mathcal{V})\,,$$

where $\{u_\nu\}$ denotes a basis of V and $\{v^\nu\}$ denotes its dual basis.

Let W be another finite-dimensional $\mathfrak{H}$-comodule. Then, there exists an $\mathfrak{H}$-comodule isomorphism $(V \bar{\otimes} W)^\vee \simeq W^\vee \bar{\otimes} V^\vee$, which is compatible with the usual linear isomorphism $(V \otimes W)^* \simeq W^* \otimes V^*$. We identify the vector space $V \otimes V^*$ with $\mathrm{End}(V)$ in the obvious way. Then, the subspace $V \bar{\otimes} V^\vee$ is identified with

(1.20) $$E_V := \{f \in \mathrm{End}(V) \mid f\pi_V(\varepsilon_i) = \pi_V(\varepsilon_i) f \ (i \in \mathcal{V})\}\,.$$

We regard $E = E_V$ as an $\mathfrak{H}$-comodule via this identification. Then we have

(1.21) $$\mathrm{End}_{\mathfrak{H}}(V) = \left\{ f \in E \,\middle|\, \rho_E(f) = \sum_{i \in \mathcal{V}} \pi_E(\varepsilon_i)(f) \otimes \mathring{e}_i \right\}.$$

1.3. Compact face algebras. Let $\mathfrak{H}$ be a $\mathcal{V}$-face algebra over the complex number field $\boldsymbol{C}$ and $\times : \mathfrak{H} \to \mathfrak{H}$ an antilinear map such that $(a^\times)^\times = a$ for each $a \in \mathfrak{H}$. We say that $\times$ is a **costar structure** of $\mathfrak{H}$ (or $\mathfrak{H}$ is a **costar face algebra**) if the following relations are satisfied:

(1.22) $$e_i^\times = \mathring{e}_i \quad (i \in \mathcal{V})$$

(1.23) $$(ab)^\times = a^\times b^\times\,, \quad \Delta(a^\times) = \sum_{(a)} a_{(2)}^\times \otimes a_{(1)}^\times \quad (a, b \in \mathfrak{H})\,.$$

PROPOSITION 1.4. *Let $\times$ be a costar structure of $\mathfrak{H}$. Then the following hold.*

(i) $\varepsilon(a^\times) = \overline{\varepsilon(a)}$ $(a \in \mathfrak{H})$.

(ii) *The dual algebra $\mathfrak{H}^*$ has a unique $*$-algebra structure such that $\langle X^*, a\rangle = \overline{\langle X, a^\times\rangle}$ $(X \in \mathfrak{H}^*, a \in \mathfrak{H})$. Moreover, we have $\varepsilon_i^* = \varepsilon_i$ and $\mathring{\varepsilon}_i^* = \mathring{\varepsilon}_i$.*

(iii) *If $\mathfrak{H}$ is a Hopf face algebra, then its antipode is bijective.*

Let V be a finite-dimensional right $\mathfrak{H}$-comodule equipped with a Hilbert space structure $(\,|\,)$. We say that $V = (V, (\,|\,))$ is **unitary** if

$$\sum_{(u)}(u_{(0)}|v)u_{(1)} = \sum_{(v)}(u|v_{(0)})v_{(1)}^{\times} \quad (u, v \in V).$$

For a unitary comodule V, (1.19) gives a $*$-representation π_V of $\mathfrak{H}^*$ on V. We say that $\mathfrak{H}$ is **compact** if each finite-dimensional right $\mathfrak{H}$-comodule is isomorphic to a unitary comodule.

PROPOSITION 1.5. *Let $\mathfrak{H}$ be a compact $\mathcal{V}$-Hopf face algebra and let V and W be unitary $\mathfrak{H}$-comodules. Then the following hold.*

(i) *The face space decomposition of V is orthogonal.*

(ii) *The comodule $V \bar{\otimes} W$ is unitary with respect to the following Hermitian inner product*:

$$(u \otimes v|u' \otimes v') = (u|u')(v|v')$$
$$(u \in V(i, j),\ v \in V(j, k),\ u' \in W(i', j'),\ v' \in W(j', k')).$$

PROOF. Part (i) follows from Proposition 1.4(ii). Part (ii) is straightforward. □

For a compact $\mathcal{V}$-Hopf face algebra $\mathfrak{H}$, there exists the unique linear functional $\boldsymbol{Q}$ on $\mathfrak{H}$ which satisfies the following two conditions:

(i) For each unitary comodule V, $\pi_V(\boldsymbol{Q})$ is a positive invertible element of $\mathrm{End}(V)$, which satisfies $\mathrm{Tr}(\pi_V(\boldsymbol{Q})) = \mathrm{Tr}(\pi_V(\boldsymbol{Q})^{-1})$.

(ii) For each $a, b \in \mathfrak{H}$ and $i, j \in \mathcal{V}$, the following relations are satisfied:

(1.24) $$S^2(a) = \sum_{(a)}\langle \boldsymbol{Q}, a_{(1)}\rangle a_{(2)}\langle \boldsymbol{Q}^{-1}, a_{(3)}\rangle,$$

(1.25) $$\langle \boldsymbol{Q}, ab\rangle = \sum_{k\in\mathcal{V}}\langle \boldsymbol{Q}\varepsilon_k, a\rangle\langle \boldsymbol{Q}\mathring{\varepsilon}_k, b\rangle,$$

(1.26) $$S^*(\boldsymbol{Q}) = \boldsymbol{Q}^{-1},$$

(1.27) $$\boldsymbol{Q}\varepsilon_i = \varepsilon_i\boldsymbol{Q}, \quad \boldsymbol{Q}\mathring{\varepsilon}_i = \mathring{\varepsilon}_i\boldsymbol{Q},$$

(1.28) $$\langle \boldsymbol{Q}, \mathring{e}_i e_j\rangle = \delta_{ij}.$$

We call $\boldsymbol{Q}$ the **Woronowicz functional** of $\mathfrak{H}$ (cf. [H5], [Wo], [Ko]).

1.4. Fusion rules. Let $\mathfrak{H}$ be a face algebra which has a coalgebra isomorphism $\mathfrak{H} \simeq \bigoplus_{\lambda\in\Lambda}\mathrm{End}(L_\lambda)^*$ for some $\mathfrak{H}$-comodules $\{L_\lambda \mid \lambda \in \Lambda\}$. We define nonnegative integers $N_{\lambda\mu}^{\nu}$ $(\lambda, \mu, \nu \in \Lambda)$ via the irreducible decomposition

$$L_\lambda \bar{\otimes} L_\mu \simeq \bigoplus_{\nu\in\Lambda} N_{\lambda\mu}^{\nu} L_\nu,$$

and call them the **fusion rules** (or the **branching rules**) of $\mathfrak{H}$.

Next, let $\mathfrak{H}$ be a compact Hopf face algebra. Since each unitary comodule is completely reducible, $\mathfrak{H}$ satisfies the condition stated above. We define a bijection $\check{}\, : \Lambda \xrightarrow{\sim} \Lambda;\ \lambda \mapsto \lambda\check{}$

by $(L_\lambda)^\vee \simeq L_{\lambda^\vee}$. Since $\pi_{L_\lambda}(\boldsymbol{Q}) : L_\lambda \xrightarrow{\sim} (L_\lambda)^{\vee\vee}$ by (1.24), we have $\lambda^{\vee\vee} = \lambda$ $(\lambda \in \Lambda)$. It follows from $(L_{\lambda^\vee} \bar{\otimes} L_{\mu^\vee})^\vee \simeq L_\mu \bar{\otimes} L_\lambda$ that

$$N^{\nu^\vee}_{\lambda^\vee \mu^\vee} = N^\nu_{\mu\lambda}\,. \tag{1.29}$$

Moreover, we have

$$N^\nu_{\lambda\mu^\vee} = N^\lambda_{\nu\mu}\,, \quad N^\nu_{\lambda^\vee\mu} = N^\mu_{\lambda\nu} \tag{1.30}$$

(see e.g., [H4], (4.22)). If $\mathfrak{H}$ is hollowless, then there exists the unique element $* \in \Lambda$ such that $L_* \simeq R$.

2. Construction of II_1-subfactors. Throughout this section, $\mathfrak{H}$ denotes a fixed hollowless compact $\mathcal{V}$-Hopf face algebra, and U, V and W denote finite-dimensional unitary right $\mathfrak{H}$-comodules. In Subsection 2.5, we also assume that $\mathfrak{H}$ is finite dimensional.

2.1. $\boldsymbol{Q}$-traces. For each $f \in \mathrm{End}_{\mathfrak{H}}(V)$, we set $\mathrm{tr}_{\boldsymbol{Q}}(f) = \sharp(\mathcal{V})^{-1}\mathrm{Tr}(\pi_V(\boldsymbol{Q})f)$, and call it the **$\boldsymbol{Q}$-trace** of f. We also set $\dim_{\boldsymbol{Q}}(V) = \mathrm{tr}_{\boldsymbol{Q}}(1)$, and call it the **$\boldsymbol{Q}$-dimension** of V.

LEMMA 2.1. *For each $f \in \mathrm{End}_{\mathfrak{H}}(V)$ and $i \in \mathcal{V}$, we have*

$$\mathrm{Tr}(\pi_V(\mathring{\varepsilon}_i\boldsymbol{Q})f) = \mathrm{tr}_{\boldsymbol{Q}}(f)\,.$$

PROOF. Let %, $\{u_\nu\}$, etc. be as in Subsection 1.2. Since the unit comodule R is irreducible, the map

$$R \xrightarrow{\%} V \bar{\otimes} V^\vee \xrightarrow{f\,\bar{\otimes}\,\mathrm{id}} V \bar{\otimes} V^\vee \xrightarrow{\pi_V(\boldsymbol{Q})\,\bar{\otimes}\,\mathrm{id}} V^{\vee\vee} \bar{\otimes} V^\vee \xrightarrow{\$} R$$

is a scalar multiple, say $c \cdot \mathrm{id}_R$. Comparing the image of $\sum_i e_i \in R$, we see that

$$c \sum_{i \in \mathcal{V}} e_i = \$\left(\sum_\nu (\pi_V(\boldsymbol{Q}) \circ f)(u_\nu) \otimes v^\nu\right) = \sum_{i \in \mathcal{V}} \mathrm{Tr}(\pi_V(\mathring{\varepsilon}_i\boldsymbol{Q})f)e_i\,.$$

Comparing the coefficient of e_i, we get $c = \mathrm{Tr}(\pi_V(\mathring{\varepsilon}_i\boldsymbol{Q})f)$. Summing over all $i \in \mathcal{V}$, we find $c \cdot \sharp(\mathcal{V}) = \mathrm{Tr}(\pi_V(\boldsymbol{Q})f)$. The last two formulas complete the proof of the lemma. □

As usual, we identify $\mathrm{End}(V \bar{\otimes} W)$ with $q\mathrm{End}(V \otimes W)q$, where the projection $q = q_{VW}$ is defined by $q = \sum_i \pi_V(\varepsilon_i) \otimes \pi_W(\mathring{\varepsilon}_i)$. In particular, for $f \in \mathrm{End}_{\mathfrak{H}}(V)$ and $g \in \mathrm{End}_{\mathfrak{H}}(W)$, we identify $f \bar{\otimes} g$ with $q(f \otimes g)$.

PROPOSITION 2.2. (i) *For each $f \in \mathrm{End}_{\mathfrak{H}}(V)$ and $g \in \mathrm{End}_{\mathfrak{H}}(W)$, we have*

$$\mathrm{tr}_{\boldsymbol{Q}}(f \bar{\otimes} g) = \mathrm{tr}_{\boldsymbol{Q}}(f)\mathrm{tr}_{\boldsymbol{Q}}(g)\,. \tag{2.1}$$

(ii) *We have the following formulas*:

$$\dim_{\boldsymbol{Q}}(V \oplus W) = \dim_{\boldsymbol{Q}}(V) + \dim_{\boldsymbol{Q}}(W)\,, \tag{2.2}$$

$$\dim_{\boldsymbol{Q}}(V \bar{\otimes} W) = \dim_{\boldsymbol{Q}}(V)\dim_{\boldsymbol{Q}}(W)\,, \tag{2.3}$$

$$\dim_{\boldsymbol{Q}}(V^\vee) = \dim_{\boldsymbol{Q}}(V)\,, \tag{2.4}$$

$$\dim_{\boldsymbol{Q}}(R) = 1\,. \tag{2.5}$$

PROOF. By (1.25), we have $\pi_{V\bar{\otimes}W}(Q) = \sum_i \pi_V(Q\varepsilon_i) \otimes \pi_W(Q\mathring{\varepsilon}_i)$. Hence

$$\mathrm{tr}_Q(f \bar{\otimes} g) = \frac{1}{\sharp(\mathcal{V})} \sum_{i\in\mathcal{V}} \mathrm{Tr}(\pi_V(\varepsilon_i Q) f)\mathrm{Tr}(\pi_W(\mathring{\varepsilon}_i Q) g),$$

from which together with Lemma 2.1 the relation (2.1) follows. The relations (2.3), (2.4) and (2.5) follow from (2.1), (1.26) and (1.28), respectively.

PROPOSITION 2.3. *If* $V \neq 0$, *then* $\dim_Q(V) \geq 1$. *Moreover,* $\dim_Q(V) = 1$ *if and only if* $V \bar{\otimes} V^{\vee} \simeq V^{\vee} \bar{\otimes} V \simeq R$.

PROOF. By (2.2)–(2.5), we have $\dim_Q(V)^2 = 1 + \dim_Q(X)$, where X denotes an $\mathfrak{H}$-comodule such that $V \bar{\otimes} V^{\vee} \simeq R \oplus X$. Since $\dim_Q(X) > 0$ if $X \neq 0$, we get the proposition. □

2.2. $*$-structure of $\mathrm{End}_{\mathfrak{H}}(V)$. Since the category of finite-dimensional right $\mathfrak{H}$-comodules is equivalent to that of finite-dimensional left $\mathfrak{H}^*$-modules, we have $\mathrm{End}_{\mathfrak{H}}(V) = \mathrm{End}_{\mathfrak{H}^*}(V)$. Since π_V is a $*$-representation, $\mathrm{End}_{\mathfrak{H}}(V)$ is a $*$-subalgebra of $\mathrm{End}(V)$. Moreover, $\tau_V := \dim_Q(V)^{-1}\mathrm{tr}_Q$ is a faithful tracial state on $\mathrm{End}_{\mathfrak{H}}(V)$.

LEMMA 2.4. *The following map is a $*$-algebra inclusion*:

(2.6) $$\mathrm{End}_{\mathfrak{H}}(V) \otimes \mathrm{End}_{\mathfrak{H}}(W) \hookrightarrow \mathrm{End}_{\mathfrak{H}}(V \bar{\otimes} W)\,; \quad f \otimes g \mapsto f \bar{\otimes} g\,.$$

In particular, $V \bar{\otimes} W \neq 0$ *if* $V, W \neq 0$.

PROOF. We define Hermitian inner products on $\mathrm{End}_{\mathfrak{H}}(V)\otimes\mathrm{End}_{\mathfrak{H}}(W)$ and $\mathrm{End}_{\mathfrak{H}}(V\bar{\otimes}W)$ via $\tau_V \otimes \tau_W$ and $\tau_{V\bar{\otimes}W}$, respectively. Using (2.1), we see that (2.6) gives an isometry. □

2.3. Commuting squares. By Lemma 2.4, we obtain the following $*$-algebra inclusions:

(2.7) $$\mathrm{End}_{\mathfrak{H}}(V) \hookrightarrow \mathrm{End}_{\mathfrak{H}}(V \bar{\otimes} W)\,; \quad f \mapsto f \bar{\otimes} \mathrm{id}_W\,,$$

(2.8) $$\mathrm{End}_{\mathfrak{H}}(W) \hookrightarrow \mathrm{End}_{\mathfrak{H}}(V \bar{\otimes} W)\,; \quad g \mapsto \mathrm{id}_V \bar{\otimes} g\,.$$

LEMMA 2.5. *Let $\mathcal{E}$ be the conditional expectation of the inclusion* (2.7) *with respect to* $\tau_{V\bar{\otimes}W}$. *Then, for each element* $h = \sum_\mu f_\mu \otimes g_\mu$ *of* $\mathrm{End}_{\mathfrak{H}}(V\bar{\otimes}W) \subset q\mathrm{End}(V \otimes W)q$, *we have*

(2.9) $$\mathcal{E}(h) = \frac{1}{\dim_Q(W)} \sum_\mu \mathrm{Tr}(\pi_W(Q) g_\mu) f_\mu\,.$$

PROOF. We define an $\mathfrak{H}$-comodule map $\tilde{\mathcal{E}}$ as follows:

$$E_{V\bar{\otimes}W} \xrightarrow{\sim} V \bar{\otimes} W \bar{\otimes} W^{\vee} \bar{\otimes} V^{\vee} \xrightarrow{\mathrm{id}\bar{\otimes}\pi(Q)\bar{\otimes}\mathrm{id}\bar{\otimes}\mathrm{id}} V \bar{\otimes} W^{\vee\vee} \bar{\otimes} W^{\vee} \bar{\otimes} V^{\vee}$$
$$\xrightarrow{\mathrm{id}\bar{\otimes}\$\bar{\otimes}\mathrm{id}} V \bar{\otimes} R \bar{\otimes} V^{\vee} \xrightarrow{\mathrm{id}\bar{\otimes}\gamma^{-1}} V \bar{\otimes} V^{\vee} \xrightarrow{\sim} E_V\,.$$

By a direct computation, we see that $\tilde{\mathcal{E}}(h)$ formally coincides with the right-hand side of (2.9) up to the constant factor $\dim_Q(W)$, where $h = \sum_\mu f_\mu \otimes g_\mu$ is an arbitrary element of $E_{V\bar{\otimes}W}$. On the other hand, using (1.21), we obtain $\tilde{\mathcal{E}}(\mathrm{End}_{\mathfrak{H}}(V \bar{\otimes} W)) \subset \mathrm{End}_{\mathfrak{H}}(V)$. Hence

(2.9) gives a well-defined map $\mathcal{E}'$ into $\mathrm{End}_{\mathfrak{H}}(V)$. Obviously, $\mathcal{E}'$ is an $\mathrm{End}_{\mathfrak{H}}(V)$-bimodule map. The relation $\mathcal{E}'(1) = 1$ follows from Lemma 2.1, while $\tau_V(\mathcal{E}'(h)) = \tau_{V\bar{\otimes}W}(h)$ follows from (1.25) and (2.3). Thus $\mathcal{E}'$ is the conditional expectation with respect to $\tau_{V\bar{\otimes}W}$. □

Using the lemma above, we obtain the following proposition.

PROPOSITION 2.6. *The diagram*

$$\begin{array}{ccc} \mathrm{End}_{\mathfrak{H}}(U\bar{\otimes}V) & \subset & \mathrm{End}_{\mathfrak{H}}(U\bar{\otimes}V\bar{\otimes}W) \\ \cup & & \cup \\ \mathrm{End}_{\mathfrak{H}}(V) & \subset & \mathrm{End}_{\mathfrak{H}}(V\bar{\otimes}W) \end{array}$$

is a commuting square with respect to $\tau_{U\bar{\otimes}V\bar{\otimes}W}$, where the horizontal and the vertical inclusions are given by $f \mapsto f\bar{\otimes}\mathrm{id}_W$ and $g \mapsto \mathrm{id}_U\bar{\otimes}g$, respectively.

2.4. Representations of $\mathrm{End}_{\mathfrak{H}}(V)$. Let L_λ, Λ, etc. be as in Subsection 1.4. We define a subset $\Lambda(V)$ of Λ by

$$\Lambda(V) = \{\lambda \in \Lambda \mid \mathrm{Hom}_{\mathfrak{H}}(L_\lambda, V) \neq 0\}.$$

For $\lambda \in \Lambda(V)$, we define $e_\lambda \in \pi_V(\mathfrak{H}^*)$ to be the unique minimal central idempotent such that $e_\lambda V$ is isomorphic to a direct sum of copies of L_λ. By Proposition 2.2.3 of [G-H-J], $\{e_\lambda \mid \lambda \in \Lambda(V)\}$ is the set of all minimal central idempotents of both $\pi_V(\mathfrak{H}^*)$ and $\mathrm{End}_{\mathfrak{H}}(V)$. For $\lambda \in \Lambda(V)$, we set $K_\lambda(V) = \mathrm{Hom}_{\mathfrak{H}}(L_\lambda, V)$ and regard it as an $\mathrm{End}_{\mathfrak{H}}(V)$-module via $(af)(u) = af(u)$ ($a \in \mathrm{End}_{\mathfrak{H}}(V)$, $f \in \mathrm{Hom}_{\mathfrak{H}}(L_\lambda, V)$, $u \in L_\lambda$). Then $K_\lambda(V)$ is irreducible and $e_\lambda V$ is isomorphic to $L_\lambda \otimes K_\lambda(V)$ as a $\pi_V(\mathfrak{H}^*) \otimes \mathrm{End}_{\mathfrak{H}}(V)$-module.

PROPOSITION 2.7. *We regard $K_\nu(V\bar{\otimes}W)$ ($\nu \in \Lambda(V\bar{\otimes}W)$) as an $\mathrm{End}_{\mathfrak{H}}(V)\otimes\mathrm{End}_{\mathfrak{H}}(W)$-module via* (2.6). *Then, we have*:

$$K_\nu(V\bar{\otimes}W) \simeq \bigoplus_{\lambda\in\Lambda(V)} \bigoplus_{\mu\in\Lambda(W)} N^\nu_{\lambda\mu} K_\lambda(V) \otimes K_\mu(W).$$

PROOF. For algebras $A \subset B$, we set $C_B(A) = \{b \in B \mid ab = ba \ (a \in A)\}$. Applying Proposition 2.2.5 of [G-H-J] to $q = q_{VW}$, $F = \mathrm{End}(V \otimes W)$ and $M = \pi_V(\mathfrak{H}^*) \otimes \pi_W(\mathfrak{H}^*)$, we get

$$C_{qFq}(qMq) = qC_F(M)q = q(\mathrm{End}_{\mathfrak{H}}(V) \otimes \mathrm{End}_{\mathfrak{H}}(W))q.$$

Hence the inclusion matrix for $q(\mathrm{End}_{\mathfrak{H}}(V)\otimes\mathrm{End}_{\mathfrak{H}}(W))q \subset \mathrm{End}_{\mathfrak{H}}(V\bar{\otimes}W)$ is the transpose of the inclusion matrix for $\pi_{V\bar{\otimes}W}(\mathfrak{H}^*) \subset qMq$ (cf. [G-H-J, Proposition 2.3.5] and [G, Theorem 6.2]). This proves the proposition. □

2.5. II_1-subfactors associated with comodules. Let $\mathfrak{H}$ be a finite-dimensional holllowless compact $\mathcal{V}$-Hopf face algebra. Let $V = L_\square$ ($\square \in \Lambda$) be an irreducible unitary $\mathfrak{H}$-comodule such that $\dim_Q(V) > 1$. For $m \geq 1$, we set $B_m = \mathrm{End}_{\mathfrak{H}}(V_m)$ and $C_m = \mathrm{End}_{\mathfrak{H}}(W_m)$, where V_m and W_m denote $\mathfrak{H}$-comodules defined by

$$V_1 = V, \quad W_1 = V^\vee,$$
$$V_{2m+1} = V_{2m}\bar{\otimes}V, \quad V_{2m+2} = V_{2m+1}\bar{\otimes}V^\vee,$$
$$W_{2m+1} = W_{2m}\bar{\otimes}V^\vee, \quad W_{2m+2} = W_{2m+1}\bar{\otimes}V.$$

By Proposition 2.6, we have the following ladder of commuting squares:

$$\begin{array}{ccccccccc} \cdots & \subset & C_m & \subset & C_{m+1} & \subset & C_{m+2} & \subset & \cdots \\ & & \cap & & \cap & & \cap & & \\ \cdots & \subset & B_{m+1} & \subset & B_{m+2} & \subset & B_{m+3} & \subset & \cdots . \end{array} \tag{2.10}$$

Here the horizontal and the vertical inclusions are given respectively by $f \mapsto f \bar{\otimes} \mathrm{id}$ and $f \mapsto \mathrm{id} \bar{\otimes} f$, and the trace on B_m is τ_{V_m}.

Let M and N denote respectively the weak closures of $\pi_\tau\left(\varinjlim B_m\right)$ and $\pi_\tau\left(\varinjlim C_m\right)$, where π_τ denote the GNS constructions with respect to the traces induced by τ_{V_m}. We call $N \subset M$ the **pair of the von Neumann algebras** associated with V.

THEOREM 2.8. *Let $\mathfrak{H}$ be a finite-dimensional hollowless compact $\mathcal{V}$-Hopf face algebra. Let $N \subset M$ be the pair of the von Neumann algebras associated with an irreducible unitary $\mathfrak{H}$-comodule V such that* $\dim_Q(V) > 1$. *Then $N \subset M$ is an irreducible* II_1*-subfactor of Jones' index* $[M : N] = \dim_Q(V)^2$.

We will prove this theorem by using Wenzl's index formula and his estimate of the relative commutant (cf. [We]). Since $\% : R \hookrightarrow V \bar{\otimes} V^{\check{}}$ and $\% : R \hookrightarrow V^{\check{}} \bar{\otimes} V^{\check{}\check{}} \simeq V^{\check{}} \bar{\otimes} V$, we have $\Lambda(V_m) \subset \Lambda(V_{m+2})$ and $\Lambda(W_m) \subset \Lambda(W_{m+2})$ for each $m \geq 0$. Since $\sharp(\Lambda) < \infty$, there exists a positive integer m_0 such that $\Lambda(V_{m+2}) = \Lambda(V_m)$ and $\Lambda(W_{m+2}) = \Lambda(W_m)$ for each $m \geq m_0$. Let n be an even integer such that $n \geq m_0$. For a pair $C \subset B$ of multimatrix algebras, let $\mathrm{Inc}(C \subset B)$ denote its inclusion matrix. Using Proposition 2.7 and (1.30), we then obtain

$$\begin{aligned} \mathrm{Inc}(B_n \subset B_{n+1}) &= Y = {}^t\mathrm{Inc}(B_{n+1} \subset B_{n+2}) , \\ {}^t\mathrm{Inc}(C_n \subset C_{n+1}) &= Y = \mathrm{Inc}(C_{n+1} \subset C_{n+2}) , \\ \mathrm{Inc}(C_n \subset B_{n+1}) &= Z = \mathrm{Inc}(C_{n+1} \subset B_{n+2}) , \end{aligned} \tag{2.11}$$

where $Y = [N^{\lambda}_{\mu\Box}]_{\lambda\mu}$ and $Z = [N^{\lambda}_{\Box\mu}]_{\lambda\mu}$.

LEMMA 2.9. *All of the matrixes ${}^tYY$, $Y\,{}^tY$, ${}^tZZ$ and $Z\,{}^tZ$ are primitive and irreducible* (cf. [G-H-J, §1]).

PROOF. Let μ be an element of $\Lambda(V_n)$. Since $\dim_Q(L_\mu \bar{\otimes} V) > 0$, there exists an element $\lambda \in \Lambda(V_{n+1})$ such that $N^{\lambda}_{\mu\Box} > 0$. Hence each column of Y is never 0. Considering similarly, we see that both Y and Z are irredundant. Hence by Lemma 1.3.2 of [G-H-J], it suffices to show that neither Y nor Z is decomposable. We define a bipartite graph $\mathcal{H}$ as follows:

$$\begin{aligned} &\mathcal{H}^0 = \mathcal{W} = \mathcal{W}_{\mathrm{odd}} \coprod \mathcal{W}_{\mathrm{even}} , \\ &\mathcal{W}_{\mathrm{odd}} = \Lambda(V_{n+1}) , \quad \mathcal{W}_{\mathrm{even}} = \Lambda(V_n) , \\ &\sharp(\mathcal{H}^1_{\lambda\mu}) = Y_{\lambda\mu} \quad (\lambda \in \mathcal{W}_{\mathrm{odd}}, \mu \in \mathcal{W}_{\mathrm{even}}) . \end{aligned} \tag{2.12}$$

By induction on $m > 0$, we obtain $\Lambda(V_m) = \{\lambda \in \mathcal{W} \mid \mathcal{H}^m_{*\lambda} \neq \emptyset\}$. Therefore $\mathcal{H}$ is connected and Y is not decomposable. The proof of the indecomposability of Z is similar. □

By computing the $\boldsymbol{Q}$-dimensions of $V^{\vee}\bar{\otimes}V\bar{\otimes}L_{\mu}$ ($\mu \in \Lambda(W_n)$) in two ways, we see that the Perron-Frobenius eigenvalue of ${}^tZZ$ is $\dim_{\boldsymbol{Q}}(V)^2$. Thus, we get $\|Z\| = \dim_{\boldsymbol{Q}}(V)$, and similarly, we obtain $\|Y\| = \dim_{\boldsymbol{Q}}(V)$. In particular, both of the sequences $\{B_m \mid m \geq m_0\}$ and $\{C_m \mid m \geq m_0\}$ are strictly increasing. Thus we complete to check all hypotheses of Wenzl's index formula.

Finally, we show that the resulting II_1-subfactor $N \subset M$ is irreducible. By Theorem 1.6 of [We], it suffices to show that there exists a projection $p \in C_n$ such that $\dim(C_{B_{n+1}}(C_n)p) = 1$. Let $\{e_\lambda \mid \lambda \in \Lambda(V_{n+1})\}$ and $\{f_\mu \mid \mu \in \Lambda(W_n)\}$ denote the sets of minimal central projections of B_{n+1} and C_n, respectively. It is easy to see that $C_{B_{n+1}}(C_n)$ is the direct product of the simple subalgebras of the form $C_{B_{n+1}}(C_n)h = C_{hB_{n+1}h}(hC_nh)$, $h = e_\lambda f_\mu \neq 0$ and that $\dim(C_{B_{n+1}}(C_n)h) = (Z_{\lambda\mu})^2$ (cf. [G-H-J, p. 43]). The element $*$ belongs to $\Lambda(W_n)$, so that we obtain $\dim(C_{B_{n+1}}(C_n)p) = \sum_\lambda (N_{\square *}^{\lambda})^2 = 1$ for $p = f_*$. We have completed the proof of Theorem 2.8.

REMARK. If $\mathfrak{H}$ is a Hopf algebra (i.e., $\sharp(\mathcal{V}) = 1$), its Woronowicz functional is given by $\boldsymbol{Q}(a) = \varepsilon(a)$ (cf. [H5, §5]). Hence $\dim_{\boldsymbol{Q}}(V) = \dim(V)$ for each $\mathfrak{H}$-comodule V. Therefore, in this case, Theorem 2.8 gives only II_1-subfactors with square-integer indices.

3. Flat face models and face algebras.

3.1. Face models. Let V be a finite-dimensional vector space which is the direct sum of the subspaces $V(i, j)$ indexed by two elements i and j of a finite set $\mathcal{V}$. We call such a vector space a $\mathcal{V}$-**face premodel**. For each $i \in \mathcal{V}$, we set

$$V(i, -) = \bigoplus_{j\in\mathcal{V}} V(i, j)\,, \quad V(-, i) = \bigoplus_{j\in\mathcal{V}} V(j, i)\,.$$

For each $m \geq 0$, we define a $\mathcal{V}$-face premodel V^m as follows:

$$V^0(i, j) = \begin{cases} \boldsymbol{K}e_i & (i = j) \\ 0 & (i \neq j)\,, \end{cases}$$
$$V^1 = V\,, \quad V^{m+1}(i, j) = \bigoplus_{k\in\mathcal{V}} V^m(i, k) \otimes V^1(k, j)\,.$$

Here e_i denotes a non-zero vector indexed by $i \in \mathcal{V}$. By definition, we may regard V^m ($m \geq 1$) as a subspace of $V^{\otimes m}$. Let w be a linear automorphism of V^2. We say that a pair (V, w) is a $\mathcal{V}$-**face model** if $w(V^2(i, j)) \subset V^2(i, j)$ for each $i, j \in \mathcal{V}$. For a $\mathcal{V}$-face model (V, w), we define $w_i \in \mathrm{End}(V^m)$ ($1 \leq i \leq m-1$) and $w_{mn} \in \mathrm{End}(V^{m+n})$ ($m, n \geq 1$) as follows:

$$\begin{aligned} w_i &= (\mathrm{id}_V^{\otimes i-1} \otimes w \otimes \mathrm{id}_V^{\otimes m-i-1})|_{V^m}\,, \\ w_{mn} &= (w_n w_{n+1} \cdots w_{m+n-1})(w_{n-1} w_n \cdots w_{m+n-2}) \cdots (w_1 w_2 \cdots w_m)\,. \end{aligned} \tag{3.1}$$

It is sometimes convenient to describe a $\mathcal{V}$-face model via a fixed basis. For a $\mathcal{V}$-face premodel V, we define an oriented graph $\mathcal{G}$ by $\mathcal{G}^0 = \mathcal{V}$, and $\sharp(\mathcal{G}^1_{ij}) = \dim(V(i, j))$, which is called the **dimension graph** of V. Let $\{u_{\boldsymbol{p}} \mid \boldsymbol{p} \in \mathcal{G}^1_{ij}\}$ be a basis of $V(i, j)$. Then, we obtain a basis

$\{u_{\boldsymbol{p}} \,|\, \boldsymbol{p} \in \mathcal{G}^m\}$ of V^m by setting $u_{(p_1,\dots,p_m)} := u_{p_1} \otimes \cdots \otimes u_{p_m}$, which we call a **path basis** of V^m. We say that a quadraple

$$\boldsymbol{B} = \left(\boldsymbol{c} \, {\boldsymbol{a} \atop \boldsymbol{d}} \, \boldsymbol{b} \right)$$

or a diagram

$$\boldsymbol{B} = \begin{array}{ccc} i & \xrightarrow{\boldsymbol{a}} & j \\ {\scriptstyle \boldsymbol{c}}\big\downarrow & & \big\downarrow{\scriptstyle \boldsymbol{b}} \\ k & \xrightarrow[\boldsymbol{d}]{} & l \end{array}$$

is a **boundary condition** on $\mathcal{G}$ of size $m \times n$ if $\boldsymbol{a}, \boldsymbol{d} \in \mathcal{G}^n$, $\boldsymbol{b}, \boldsymbol{c} \in \mathcal{G}^m$ and $\mathfrak{s}(\boldsymbol{a}) = i = \mathfrak{s}(\boldsymbol{c})$, $\mathfrak{r}(\boldsymbol{a}) = j = \mathfrak{s}(\boldsymbol{b})$, $\mathfrak{r}(\boldsymbol{c}) = k = \mathfrak{s}(\boldsymbol{d})$, $\mathfrak{r}(\boldsymbol{b}) = l = \mathfrak{r}(\boldsymbol{d})$. For each boundary condition $\boldsymbol{B}$ of size $m \times n$, we define a scalar $w(\boldsymbol{B})$ by

$$w_{nm}(u_{\boldsymbol{a}} \otimes u_{\boldsymbol{b}}) = \sum_{\boldsymbol{c},\boldsymbol{d}} w \left(\boldsymbol{c} \, {\boldsymbol{a} \atop \boldsymbol{d}} \, \boldsymbol{b} \right) u_{\boldsymbol{c}} \otimes u_{\boldsymbol{d}}\,, \quad (\boldsymbol{a} \in \mathcal{G}^n_{-,j},\ \boldsymbol{b} \in \mathcal{G}^m_{j,-},\ j \in \mathcal{V}) \tag{3.2}$$

and call it the **partition function** of $(V, \{u_{\boldsymbol{p}}\})$, where the summation is taken over all $\boldsymbol{c} \in \mathcal{G}^m_{\mathfrak{s}(\boldsymbol{a}),-}$ and $\boldsymbol{d} \in \mathcal{G}^n_{-,\mathfrak{r}(\boldsymbol{b})}$ such that $\mathfrak{r}(\boldsymbol{c}) = \mathfrak{s}(\boldsymbol{d})$. For convenience, we set $w(\boldsymbol{B}) = 0$ for each quadraple $\boldsymbol{B}$ of paths, which is not a boundary condition. For example, the above summation may be taken over all $\boldsymbol{c} \in \mathcal{G}^m$ and $\boldsymbol{d} \in \mathcal{G}^n$.

Let $V = (V, w, *)$ be a $\mathcal{V}$-face model with a fixed vertex $* \in \mathcal{V}$. We assume that V satisfies the following two conditions:

(A) For each $i \in \mathcal{V}$, there exists $m \geq 0$ such that $\mathcal{G}^m_{*i} \neq \emptyset$.

(B) For each $m \geq 0$, there exists $i \in \mathcal{V}$ such that $\mathcal{G}^m_{*i} \neq \emptyset$.

We define sets $\Lambda_V = \coprod_{m \geq 0} \Lambda^m_V$, $\mathcal{V}(m)$ and an algebra $\mathrm{Str}^m(V)$ $(m \geq 0)$ by

$$\begin{aligned} \Lambda^m_V &= \{(i, m) \in \mathcal{V} \times \mathbf{Z}_{\geq 0} \,|\, V^m(*, i) \neq 0\}\,, \\ \mathcal{V}(m) &= \{i \in \mathcal{V} \,|\, (i, m) \in \Lambda_V\}\,, \\ \mathrm{Str}^m(V) &= \bigoplus_{i \in \mathcal{V}(m)} \mathrm{End}(V^m(*, i))\,. \end{aligned}$$

We call $\mathrm{Str}^m(V)$ the **string algebra** of V. For each $m, n > 0$, we define an algebra map $\iota = \iota_{mn} : \mathrm{Str}^m(V) \to \mathrm{Str}^{m+n}(V)$ by

$$\iota_{mn}(x)(u_{\boldsymbol{p}} \otimes u_{\boldsymbol{q}}) = x u_{\boldsymbol{p}} \otimes u_{\boldsymbol{q}} \quad (x \in \mathrm{Str}^m(V),\ \boldsymbol{p} \in \mathcal{G}^m_{*i},\ \boldsymbol{q} \in \mathcal{G}^n_{ij})\,. \tag{3.3}$$

We say that $V = (V, w, *)$ is a **flat $\mathcal{V}$-face model** if the relation

$$\iota(x) \overset{*}{w}_{nm} \iota(y) \overset{*}{w}{}^{-1}_{nm} = \overset{*}{w}_{nm} \iota(y) \overset{*}{w}{}^{-1}_{nm} \iota(x) \tag{3.4}$$

holds in $\mathrm{Str}^{m+n}(V)$ for each $m, n \geq 0$, $x \in \mathrm{Str}^m(V)$ and $y \in \mathrm{Str}^n(V)$, where $\overset{*}{w}_{nm}$ denotes the restriction of w_{nm} on $V^{m+n}(*, -)$.

Let $E_{\boldsymbol{pq}} \in \mathrm{End}(V^m)$ $(\boldsymbol{p}, \boldsymbol{q} \in \mathcal{G}^m)$ be a matrix unit which corresponds to a path basis $\{u_{\boldsymbol{p}} \,|\, \boldsymbol{p} \in \mathcal{G}^m\}$ of V^m, that is, $E_{\boldsymbol{pq}} u_{\boldsymbol{r}} = \delta_{\boldsymbol{qr}} u_{\boldsymbol{p}}$. Substituting $x = E_{\boldsymbol{ef}}$ $(\boldsymbol{e}, \boldsymbol{f} \in \mathcal{G}^m_{*i})$ and $y = E_{\boldsymbol{ab}}$

($\boldsymbol{a}, \boldsymbol{b} \in \mathcal{G}^n_{*j}$) into (3.4), we obtain

$$\delta_{\boldsymbol{qe}} \sum_{\boldsymbol{t} \in \mathcal{G}^m} w^{-1}\left(\boldsymbol{b}\ \genfrac{}{}{0pt}{}{\boldsymbol{p}}{\boldsymbol{t}}\ \boldsymbol{p}'\right) w\left(\boldsymbol{f}\ \genfrac{}{}{0pt}{}{\boldsymbol{a}}{\boldsymbol{q}'}\ \boldsymbol{t}\right) = \delta_{\boldsymbol{pf}} \sum_{\boldsymbol{t} \in \mathcal{G}^m} w^{-1}\left(\boldsymbol{b}\ \genfrac{}{}{0pt}{}{\boldsymbol{e}}{\boldsymbol{t}}\ \boldsymbol{p}'\right) w\left(\boldsymbol{q}\ \genfrac{}{}{0pt}{}{\boldsymbol{a}}{\boldsymbol{q}'}\ \boldsymbol{t}\right)$$

for each $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{G}^m$ and $\boldsymbol{p}', \boldsymbol{q}' \in \mathcal{G}^n$. Hence $(V, w, *)$ is flat if and only if there exists a function $\gamma : \coprod_{n \geq 0} (\mathcal{G}^n_{*,-})^2 \times (\mathcal{G}^n)^2 \to \boldsymbol{K}$ such that

$$\sum_{\boldsymbol{t} \in \mathcal{G}^m} w^{-1}\left(\boldsymbol{b}\ \genfrac{}{}{0pt}{}{\boldsymbol{u}}{\boldsymbol{t}}\ \boldsymbol{c}\right) w\left(\boldsymbol{v}\ \genfrac{}{}{0pt}{}{\boldsymbol{a}}{\boldsymbol{d}}\ \boldsymbol{t}\right) = \delta_{\boldsymbol{uv}} \gamma(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}) \tag{3.5}$$

for each $m, n \geq 0$, $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{G}^n_{*,-}$, $\boldsymbol{c}, \boldsymbol{d} \in \mathcal{G}^n$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{G}^m_{*,-}$ such that $\mathfrak{r}(\boldsymbol{u}) = \mathfrak{s}(\boldsymbol{c})$ and $\mathfrak{r}(\boldsymbol{v}) = \mathfrak{s}(\boldsymbol{d})$.

Next, we will construct an action of $\mathrm{Str}^m(V)$ on the "full" path space V^m for a flat $\mathcal{V}$-face model $V = (V, w, *)$. For each $j \in \mathcal{V}$, $n \geq 0$ and $(i, m) \in \Lambda_V$, we define a linear map

$$\Phi_m : V^n(i, j) \to \mathrm{Hom}_{\mathrm{Str}^m(V)}(V^m(*, i), V^{m+n}(*, j)) \tag{3.6}$$

by $\Phi_m(\xi)(\eta) = \eta \otimes \xi$ ($\xi \in V^n(i, j)$, $\eta \in V^m(*, i)$), where the action of $\mathrm{Str}^m(V)$ on $V^{m+n}(*, j)$ is given by ι_{mn}. Comparing dimensions, we see that Φ_m is an isomorphism. By (3.4), the right-hand side of (3.6) becomes a $\mathrm{Str}^n(V)$-module via $x \otimes f \mapsto \overset{*}{w}_{nm} \iota(x) \overset{*}{w}{}^{-1}_{nm} f$ ($x \in \mathrm{Str}^n(V)$, $f \in \mathrm{Im}(\Phi_m)$). Hence, $V^n(i, j)$ also becomes a $\mathrm{Str}^n(V)$-module. Explicitly, corresponding representation Γ is given by

$$\Gamma(E_{\boldsymbol{ab}}) u_{\boldsymbol{c}} = \sum_{\boldsymbol{d} \in \mathcal{G}^n_{ij}} \gamma(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{d}) u_{\boldsymbol{d}} \tag{3.7}$$

for each $h, i, j \in \mathcal{V}$, $\boldsymbol{c} \in \mathcal{G}^n_{ij}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{G}^n_{*,h}$, where γ is as in (3.5). In particular, the action does not depend on the choice of m. We have thus obtained an action Γ of $\mathrm{Str}^n(V)$ on V^n.

3.2. Costring algebras. For $x \in \mathrm{Str}^m(V)$ and $y \in \mathrm{Str}^n(V)$, we denote the left-hand side of (3.4) by $\nabla_{mn}(x \otimes y)$. Then, ∇_{mn} gives an algebra map from $\mathrm{Str}^m(V) \otimes \mathrm{Str}^n(V)$ into $\mathrm{Str}^{m+n}(V)$. By definition, we have

$$\nabla_{mn}(1 \otimes y)(\eta \otimes \xi) = \eta \otimes \Gamma(y)\xi\,, \tag{3.8}$$

$$\nabla_{mn}(x \otimes 1) = \iota_{mn}(x) \tag{3.9}$$

for each $i \in \mathcal{V}$, $\eta \in V^m(*, i)$, $\xi \in V^n(i, -)$, $x \in \mathrm{Str}^m(V)$ and $y \in \mathrm{Str}^n(V)$. We also define an algebra map ∇^0_{mn} from $\mathrm{End}(V^m) \otimes \mathrm{End}(V^n)$ into $\mathrm{End}(V^{m+n})$ by $\nabla^0_{mn}(f \otimes g) = \mu_{mn} \circ (f \otimes g) \circ \delta_{mn}$, where $\delta_{mn} : V^{m+n} \to V^m \otimes V^n$ is the natural inclusion and $\mu_{mn} : V^m \otimes V^n \to V^{m+n}$ is given by $\mu_{mn}(u_{\boldsymbol{p}} \otimes u_{\boldsymbol{q}}) = \delta_{\mathfrak{r}(\boldsymbol{p})\mathfrak{s}(\boldsymbol{q})} u_{\boldsymbol{p}} \otimes u_{\boldsymbol{q}}$.

LEMMA 3.1. *Let V be a flat face model. Then the following hold.*

(i) *The family of maps* $\{\nabla_{mn} \mid m, n \geq 0\}$ *is associative, that is,* $\nabla_{l+m,n}(\nabla_{lm}(x \otimes y) \otimes z) = \nabla_{l,m+n}(x \otimes \nabla_{mn}(y \otimes z))$ *for each* $l, m, n \geq 0$ *and* $x \in \mathrm{Str}^l(V)$, $y \in \mathrm{Str}^m(V)$, $z \in \mathrm{Str}^n(V)$.

(ii) *We have the following commutative diagram.*

$$\begin{array}{ccc} \mathrm{Str}^m(V)\otimes\mathrm{Str}^n(V) & \xrightarrow{\nabla_{mn}} & \mathrm{Str}^{m+n}(V) \\ {\scriptstyle\Gamma\otimes\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle\Gamma} \\ \mathrm{End}(V^m)\otimes\mathrm{End}(V^n) & \xrightarrow{\nabla^0_{mn}} & \mathrm{End}(V^{m+n})\,. \end{array}$$

PROOF. Using the fact that $w_i w_j = w_j w_i$ for $|i-j| \geq 2$, we obtain

$$\begin{aligned} w_{mn} &= (w_n w_{n-1})(w_{n+1}\cdots w_{m+n-1})(w_n\cdots w_{m+n-2}) \\ &\qquad \cdots(w_{n-2}\cdots w_{m+n-3})\cdots(w_1\cdots w_m) \\ &= \quad \cdots \\ &= (w_n\cdots w_1)(\mathrm{id}_V\otimes w_{m-1,n})\big|_{V^{m+n}} \\ &= (w_n\cdots w_1)(w_{n+1}\cdots w_2)\cdots(w_{m+n-1}\cdots w_m)\,. \end{aligned} \tag{3.10}$$

Combining (3.1) with this formula, we obtain

$$w_{l+m,n} = (w_{ln}\otimes\mathrm{id}_{V^m})\circ(\mathrm{id}_{V^l}\otimes w_{mn})\big|_{V^{l+m+n}}\,, \tag{3.11}$$

$$w_{l,m+n} = (\mathrm{id}_{V^m}\otimes w_{ln})\circ(w_{lm}\otimes\mathrm{id}_{V^n})\big|_{V^{l+m+n}}\,. \tag{3.12}$$

Using these two formulas and the fact that $\iota(y)$ commutes with $(\mathrm{id}_{V^m}\otimes w_{nl})\big|_{V^{l+m+n}(*,-)}$, we obtain

$$\begin{aligned} \overset{*}{w}{}^{-1}_{n,m+l}\,\iota(\overset{*}{w}_{ml})^{-1} &= \iota(\overset{*}{w}_{nm})^{-1}\,\overset{*}{w}{}^{-1}_{m+n,l}\,, \\ \iota(\overset{*}{w}_{ml})\iota(y)\overset{*}{w}_{n,m+l} &= \overset{*}{w}_{m+n,l}\,\iota(y)\iota(\overset{*}{w}_{nm})\,. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \nabla_{l+m,n}(\nabla_{lm}(x\otimes y)\otimes z) &= \iota(x)\iota(\overset{*}{w}_{ml})\iota(y)\iota(\overset{*}{w}_{ml})^{-1}\,\overset{*}{w}_{n,m+l}\,\iota(z)\,\overset{*}{w}{}^{-1}_{n,m+l} \\ &= \iota(x)\iota(\overset{*}{w}_{ml})\iota(y)\,\overset{*}{w}_{n,m+l}\,\iota(z)\,\overset{*}{w}{}^{-1}_{n,m+l}\,\iota(\overset{*}{w}_{ml})^{-1}\,, \end{aligned}$$

where the second equality follows from (3.4). Applying the above two formulas to the right-hand side of this equality, we obtain (i).

Using (3.8), we obtain

$$\begin{aligned} &\Phi_l(\nabla^0_{mn}(\Gamma(x)\otimes\Gamma(y))(\eta\otimes\xi))(\zeta) = \iota(\nabla_{lm}(1\otimes x))\nabla_{l+m,n}(1\otimes y)(\zeta\otimes\eta\otimes\xi)\,, \\ &\Phi_l(\Gamma(\nabla_{mn}(x\otimes y))(\eta\otimes\xi))(\zeta) = \nabla_{l,m+n}(1\otimes\nabla_{mn}(x\otimes y))(\zeta\otimes\eta\otimes\xi) \end{aligned}$$

for each $i, j, k \in \mathcal{V}$, $\zeta \in V^l(*, i)$, $\eta \in V^m(i, j)$, $\xi \in V^n(j, k)$, $x \in \mathrm{Str}^m(V)$ and $y \in \mathrm{Str}^n(V)$. Hence, (ii) follows easily from (i) and (3.9). □

We define a linear map $\Delta_{mn} : \mathrm{End}(V^{m+n}) \to \mathrm{End}(V^m)\otimes\mathrm{End}(V^n)$ by $\Delta_{mn}(f) = \delta_{mn}\circ f\circ\mu_{mn}$. It is easy to verify that the coalgebra $\mathfrak{H}(V) := \bigoplus_{m\geq 0}\mathrm{End}(V^m)^*$ becomes a

$\mathcal{V}$-face algebra via the product $\bigoplus_{m,n\geq 0} \Delta^*_{mn}$ and that it is isomorphic to $\mathfrak{H}(\mathcal{G})$ (cf. Example 1.1). We define a coalgebra $\mathrm{Cost}(V)$ by

$$\mathrm{Cost}(V) = \bigoplus_{n\geq 0} \mathrm{Cost}^n(V)\,, \quad \mathrm{Cost}^n(V) = \mathrm{End}_{\mathrm{Str}^n(V)}(V^n)^*\,,$$

and call it the **costring algebra** of V.

PROPOSITION 3.2. *For a flat $\mathcal{V}$-face model V,* $\mathrm{Cost}(V)$ *becomes a quotient $\mathcal{V}$-face algebra of $\mathfrak{H}(V)$.*

PROOF. Let x and y be elements of $\mathrm{Str}^m(V)$ and $\mathrm{Str}^n(V)$, respectively. Since $\Gamma(x)\otimes\Gamma(y)$ preserves V^{m+n}, it commutes with $\Delta_{mn}(1)$. Using this fact, we calculate

$$\begin{aligned}(\Gamma(x)\otimes\Gamma(y))\circ\Delta_{mn}(f) &= \Delta_{mn}(1)\circ(\Gamma(x)\otimes\Gamma(y))\circ\Delta_{mn}(f)\\ &= \delta_{mn}\circ(\Gamma(\nabla_{mn}(x\otimes y)))\circ f\circ\mu_{mn} \quad (f\in\mathrm{End}(V^{m+n}))\,,\end{aligned}$$

where the second equality follows from the lemma above and the definition of Δ_{mn} and ∇^0. Computing $\Delta_{mn}(f)\circ(\Gamma(x)\otimes\Gamma(y))$ similarly, we see that $\Gamma(x)\otimes\Gamma(y)$ commutes with $\Delta_{mn}(f)$ for each $f\in\mathrm{End}_{\mathrm{Str}^{m+n}(V)}(V^{m+n})$, or equivalently,

$$\Delta_{mn}(\mathrm{End}_{\mathrm{Str}^{m+n}(V)}(V^{m+n}))\subset\mathrm{End}_{\mathrm{Str}^m(V)}(V^m)\otimes\mathrm{End}_{\mathrm{Str}^n(V)}(V^n)\,.$$

This proves the proposition. □

As $\mathrm{Cost}(V)$-comodules, V^m $(m>0)$ and V^0 are isomorphic to $V^{\bar{\otimes}m}$ and the unit comodule R, respectively. Moreover, the definition of $V^m(i,j)$ is consistent with (1.18).

LEMMA 3.3. *Let V be a flat face model with dimension graph $\mathcal{G}$. Let $\mathfrak{H}(\mathcal{G})$ and $e\binom{\boldsymbol{p}}{\boldsymbol{q}}$ be as in Example* 1.1, *and $\mathfrak{I}$ the linear span of the following elements of $\mathfrak{H}(\mathcal{G})$:*

$$\sum_{t\in\mathcal{G}^m}\gamma(\boldsymbol{p},\boldsymbol{q};\boldsymbol{r},\boldsymbol{t})e\binom{\boldsymbol{s}}{\boldsymbol{t}}-\sum_{t\in\mathcal{G}^m}\gamma(\boldsymbol{p},\boldsymbol{q};\boldsymbol{t},\boldsymbol{s})e\binom{\boldsymbol{t}}{\boldsymbol{r}}$$
$$(m\geq 0,\ i\in\mathcal{V},\ \boldsymbol{p},\boldsymbol{q}\in\mathcal{G}^m_{*i},\ \boldsymbol{r},\boldsymbol{s}\in\mathcal{G}^m)\,.$$

Then $\mathfrak{I}$ is a biideal of $\mathfrak{H}(\mathcal{G})$ and $\mathrm{Cost}(V)\simeq\mathfrak{H}(\mathcal{G})/\mathfrak{I}$.

PROOF. The assertion easily follows from $\mathrm{Cost}^m(V)\simeq\mathrm{End}(V^m)^*/C^\perp$, where $C^\perp=\{X\in\mathrm{End}(V^m)^*\mid\langle X,C\rangle=0\}$ and $C=\mathrm{End}_{\mathrm{Str}^m(V)}(V^m)$. □

3.3. Representations of $\mathrm{Cost}(V)$. For each $\lambda=(i,m)\in\Lambda^m_V$, we define an $\mathrm{Str}^m(V)$-module V_λ and a space L_λ as follows:

$$V_\lambda=V^m(*,i)\,,\quad L_\lambda=\mathrm{Hom}_{\mathrm{Str}^m(V)}(V_\lambda,V^m)\,.$$

Since L_λ naturally becomes an irreducible $\mathrm{End}_{\mathrm{Str}^m(V)}(V^m)$-module, it also becomes an irreducible $\mathrm{Cost}^m(V)$-comodule (cf. Subsection 2.4). Moreover, $\{L_\lambda\mid\lambda\in\Lambda^m_V\}$ gives a set of complete representatives of irreducible comodules of $\mathrm{Cost}^m(V)$.

For each $i, j \in \mathcal{V}$ and $(k, m) \in \Lambda_V$, we define a non-negative integer $N_{ik}^j(m)$ by the following irreducible decomposition of a $\mathrm{Str}^m(V)$-module:

$$V^m(i, j) \simeq \bigoplus_{k \in \mathcal{V}(m)} N_{ik}^j(m) V_{(k,m)} . \tag{3.13}$$

We call the integers $N_{ik}^j(m)$ the **fusion rules** of V. By definition, we have

$$N_{*k}^j(m) = \delta_{jk} \tag{3.14}$$

for each $j \in \mathcal{V}$ and $(k, m) \in \Lambda_V$.

PROPOSITION 3.4. *For a flat $\mathcal{V}$-face model V, the following hold.*

(i) *For each $i, j \in \mathcal{V}$ and $(k, m) \in \Lambda_V$, we have*

$$\dim(L_{(k,m)}(i, j)) = N_{ik}^j(m) .$$

(ii) *For each finite-dimensional* $\mathrm{Cost}^m(V)$*-comodule M, we have*

$$M \simeq \bigoplus_{i \in \mathcal{V}(m)} \dim(M(*, i)) L_{(i,m)} .$$

In particular, we have $M \simeq L_{(i,m)}$ if $\dim(M(*, i)) = 1$.

(iii) *For each $(i, m), (j, n) \in \Lambda_V$, we have*

$$L_{(i,m)} \bar{\otimes} L_{(j,n)} \simeq \bigoplus_{k \in \mathcal{V}(m+n)} N_{ij}^k(n) L_{(k,m+n)} .$$

PROOF. Part (i) follows from

$$L_{(k,m)}(i, j) = \mathrm{Hom}_{\mathrm{Str}^m(V)}(V_{(k,m)}, V^m(i, j)) .$$

For $i \in \mathcal{V}(m)$, let μ_i denote the multiplicity of $L_{(i,m)}$ in M. Then, using (i) and (3.14), we obtain

$$\dim(M(*, i)) = \sum_{j \in \mathcal{V}(m)} \mu_j N_{*j}^i(m) = \mu_i ,$$

$$\dim((L_{(i,m)} \otimes L_{(j,n)})(*, k)) = \sum_{l \in \mathcal{V}} N_{*i}^l(m) N_{lj}^k(n) = N_{ij}^k(n) .$$

Part (ii) follows from the first formula, while (iii) follows from (ii) and the second formula. □

We say that $\square \in \mathcal{V}$ is the **generating vertex** of a flat $\mathcal{V}$-face model V if $\sharp(\mathcal{G}_{*i}^1) = \delta_{i\square}$ for each $i \in \mathcal{V}$.

LEMMA 3.5. *If a flat face model V has the generating vertex $\square$, then V is isomorphic to $L_{(\square,1)}$ as* $\mathrm{Cost}(V)$*-comodules. Moreover, we have*:

$$N_{i\square}^j(1) = \sharp(\mathcal{G}_{ij}^1) \quad (i, j \in \mathcal{V}) . \tag{3.15}$$

PROOF. The first assertion is obvious and the second assertion follows from (3.13). □

3.4. Unitary flat face models. Let $V = (V, w, *)$ be a flat face model over the complex number field $\mathbf{C}$, and $(\ |\ )$ a Hilbert space structure on V such that $(V(i, j) \mid V(i', j')) = 0$

unless $(i,j)=(i',j')$. We define a Hermitian inner product on V^m by $(u_p\,|\,u_q)=\delta_{pq}$ $(p,q\in\mathcal{G}^m)$, where $\{u_p\,|\,p\in\mathcal{G}^m\}$ $(m\geq 0)$ denotes a path basis of V^m such that $\{u_p\,|\,p\in\mathcal{G}^1_{ij}\}$ is a orthonormal basis of $V(i,j)$. We call such $\{u_p\}$ an **orthonormal path basis** of V^m. We say that $V=(V,(\,|\,))$ is **unitary** if w is unitary with respect to $(\,|\,)$. The partition function and the function γ with respect to $\{u_p\}$ satisfy the following relations:

$$\overline{w^{\pm 1}\begin{pmatrix} c \begin{smallmatrix} a \\ d \end{smallmatrix} b \end{pmatrix}} = w^{\mp 1}\begin{pmatrix} a \begin{smallmatrix} c \\ b \end{smallmatrix} d \end{pmatrix}, \tag{3.16}$$

$$\overline{\gamma(a,b;c,d)}=\gamma(b,a;d,c)\,. \tag{3.17}$$

Let $\left[e\begin{pmatrix}p\\q\end{pmatrix}\middle|\,p,q\in\mathcal{G}^m\right]$ be the matrix corepresentation of the $\mathrm{Cost}(V)$-comodule $(V^m,\{u_p\,|\,p\in\mathcal{G}^m\})$. By (3.17), Lemma 3.3 and Lemma 2.1(5) of [H5], $\mathrm{Cost}(V)$ becomes a compact face algebra via

$$e\begin{pmatrix}p\\q\end{pmatrix}^{\times}=e\begin{pmatrix}q\\p\end{pmatrix}\quad (p,q\in\mathcal{G}^m,\ m\geq 0)\,. \tag{3.18}$$

It is easy to verify that the costar structure $\times$ does not depend on the choice of $\{u_p\}$.

4. Galois face algebras.

4.1. The face algebra $\mathfrak{G}(V)$. Let $\mathcal{G}$ be a finite connected non-oriented graph. We identify $\mathcal{G}$ with an oriented graph equipped with a bijection $^\sim:\mathcal{G}^1\xrightarrow{\sim}\mathcal{G}^1;\,p\mapsto p^\sim$ such that $(p^\sim)^\sim=p$ and $p^\sim\in\mathcal{G}^1_{ji}$ for each $p\in\mathcal{G}^1_{ij}$ and $i,j\in\mathcal{V}=\mathcal{G}^0$. Let $V=(V,w,*)$ be a unitary flat face model whose dimension graph is $\mathcal{G}$, and $\{u_p|\in\mathcal{G}^m\}$ an orthonormal path basis of V^m. We say that $(V,\{u_p\})$ is of **connection type** if its partition function with respect to $\{u_p\}$ satisfies the following **renormalization rule**:

$$\begin{aligned} w\begin{pmatrix} i & \xrightarrow{p} & j \\ r\big\downarrow & & \big\downarrow s \\ k & \xrightarrow{q} & l \end{pmatrix} &= \left(\frac{\mu(j)\mu(k)}{\mu(i)\mu(l)}\right)^{1/2} \overline{w\begin{pmatrix} j & \xrightarrow{p^\sim} & i \\ s\big\downarrow & & \big\downarrow r \\ l & \xrightarrow{q^\sim} & k \end{pmatrix}} \\ &= \left(\frac{\mu(j)\mu(k)}{\mu(i)\mu(l)}\right)^{1/2} \overline{w\begin{pmatrix} k & \xrightarrow{q} & l \\ r^\sim\big\downarrow & & \big\downarrow s^\sim \\ i & \xrightarrow{p} & j \end{pmatrix}}\,. \end{aligned} \tag{4.1}$$

Here, we denote by $[\mu(i)]_{i\in\mathcal{V}}$ the Perron-Frobenius eigenvector of $[\sharp(\mathcal{G}^1_{ij})]_{i,j\in\mathcal{V}}$ such that $\mu(*)=1$, and by β its eigenvalue. We call $[\mu(i)]$ the **normalized Perron-Frobenius eigenvector** of V. For a flat face model V of connection type, we define operators e_J and $b_J=b_J(\varepsilon)$ on V^2 by

$$e_J=\beta^{-1}\sum_{i\in\mathcal{V}}\sum_{p,q\in\mathcal{G}^1_{i,-}}\frac{\sqrt{\mu(\mathfrak{r}(p))\mu(\mathfrak{r}(q))}}{\mu(i)}E_{p\cdot p^\sim,q\cdot q^\sim}\,, \tag{4.2}$$

$$b_J = \varepsilon \cdot \mathrm{id} + \beta\varepsilon^{-1} e_J\,, \tag{4.3}$$

and call e_J the **Jones projection** of V, where ε denotes a fixed solution of the equation $\varepsilon^2 + \varepsilon^{-2} + \beta = 0$. It is known that e_J actually is a projection and that (V, b_J) is a face model which satisfies the braid relation: $(b_J)_1(b_J)_2(b_J)_1 = (b_J)_2(b_J)_1(b_J)_2$ in the algebra $\mathrm{End}(V^3)$ (cf. [G-H-J]). Moreover, using the unitarity and the renormalization rule, we find that b_J satisfies

$$w_1 w_2 (b_J)_1 = (b_J)_2 w_1 w_2 \tag{4.4}$$

in $\mathrm{End}(V^3)$ (see e.g., [Ka, p. 70]).

Let $N \subset M$ be an irreducible AFD II_1-subfactor of finite index with finite principal graph $\mathcal{G}$. Then, by Popa's classification theory of II_1-subfactors, $N \subset M$ is completely determined by its standard invariant (see [P] and also [O1]). Moreover, by Ocnecanu's theory, the standard invariant is described by the flat biunitary connection W (cf. [O1], [O2], [Ka]). When $\mathcal{G}$ coincides with the dual principal graph, W is a function which assigns a complex number $W(\boldsymbol{B})$ to each boundary condition $\boldsymbol{B}$ on $\mathcal{G}$ of size 1×1. Set $V = \mathrm{span}\{u_{\boldsymbol{p}} \,|\, \boldsymbol{p} \in \mathcal{G}^1\}$ and define $w = w_{11}$ by (3.2). Then, $V = (V, w)$ becomes a flat face model of connection type with generating vertex $\square$. We call V the **flat face model associated with** $N \subset M$. Let $N \subset M$ be either an AFD II_1-subfactor of index < 4, or an irreducible AFD II_1-subfactor of index $= 4$ with finite principal graph. Then, $N \subset M$ satisfies the conditions stated above and its principal graph is either A_n $(n \geq 2)$, D_{2n} $(n \geq 2)$, E_6, E_8, $D_n^{(1)}$ $(n \geq 4)$ or $E_n^{(1)}$ $(n = 6, 7, 8)$. Except for the case of $D_n^{(1)}$, the corresponding face model is given by $w = b_J(\varepsilon)$ for some ε. When $\mathcal{G}$ is of type $D_n^{(1)}$, there are $n-2$ subfactors and corresponding face models. The explicit formulas of these are given in [I-K].

LEMMA 4.1. *Let $V = (V, w, *)$ be a flat $\mathcal{V}$-face model. If b is a linear operator on V^2 such that (V, b) is a $\mathcal{V}$-face model and that $w_1 w_2 b_1 = b_2 w_1 w_2$ on V^3, then the element* $\overset{*}{b}_i = b_i\big|_{V^n(*,-)}$ $(1 \leq i \leq n-1)$ *of* $\mathrm{Str}^n(V)$ *satisfies* $\Gamma(\overset{*}{b}_i) = b_i$.

PROOF. The assertion easily follows from $w_{nm} b_i = b_{i+m} w_{nm}$ $(1 \leq i \leq n-1)$ and the definition of Γ. □

Applying the lemma above to $b = b_J$, we see that b_J commutes with the coaction of $\mathrm{Cost}(V)$ on V^2. Computing $\rho(b_J u_{\boldsymbol{p}}) = (b_J \otimes \mathrm{id})(\rho(u_{\boldsymbol{p}}))$ $(\boldsymbol{p} \in \mathcal{G}^2)$, we find that the following "L-operator" relation is satisfied in $\mathrm{Cost}(V)$:

$$\sum_{\boldsymbol{r}\cdot\boldsymbol{s}\in\mathcal{G}^2} b_J\begin{pmatrix} & \boldsymbol{p} & \\ \boldsymbol{r} & & \boldsymbol{q} \\ & \boldsymbol{s} & \end{pmatrix} \cdot e\begin{pmatrix} \boldsymbol{a}\cdot\boldsymbol{b} \\ \boldsymbol{r}\cdot\boldsymbol{s} \end{pmatrix} = \sum_{\boldsymbol{c}\cdot\boldsymbol{d}\in\mathcal{G}^2} b_J\begin{pmatrix} & \boldsymbol{c} & \\ \boldsymbol{a} & & \boldsymbol{d} \\ & \boldsymbol{b} & \end{pmatrix} \cdot e\begin{pmatrix} \boldsymbol{c}\cdot\boldsymbol{d} \\ \boldsymbol{p}\cdot\boldsymbol{q} \end{pmatrix} \tag{4.5}$$

$$(\boldsymbol{a}\cdot\boldsymbol{b},\ \boldsymbol{p}\cdot\boldsymbol{q} \in \mathcal{G}^2)\,,$$

where $\left[e\begin{pmatrix}p\\q\end{pmatrix}\right]$ is as in Subsection 3.4 (cf. [R-T-F], [H6]). Hence, by Lemma 7.4 of [H5], Cost(V) has a central group-like element det which satisfies the following relations:

$$\det = \sum_{i,j\in\mathcal{V}} \det\begin{pmatrix}i\\j\end{pmatrix},$$

$$(4.6)\qquad \det\begin{pmatrix}i\\j\end{pmatrix} = \sum_{t\in\mathcal{G}^1_{j,-}} \left(\frac{\mu(i)\mu(\mathfrak{r}(t))}{\mu(j)\mu(\mathfrak{r}(p))}\right)^{1/2} e\begin{pmatrix}p\cdot p^{\sim}\\t\cdot t^{\sim}\end{pmatrix}$$

$$= \sum_{t\in\mathcal{G}^1_{i,-}} \left(\frac{\mu(j)\mu(\mathfrak{r}(t))}{\mu(i)\mu(\mathfrak{r}(q))}\right)^{1/2} e\begin{pmatrix}t\cdot t^{\sim}\\q\cdot q^{\sim}\end{pmatrix},$$

where p and q denote arbitrary elements of $\mathcal{G}^1_{i,-}$ and $\mathcal{G}^1_{j,-}$, respectively. Note that we have $R\det \simeq \mathrm{Im}(e_J) \simeq L_{(*,2)}$ as Cost(V)-comodules. By Lemma 7.2 and Lemma 7.4 of [H5], the quotient $\mathfrak{G}(V) := \mathrm{Cost}(V)/(\det -1)$ becomes a compact $\mathcal{V}$-Hopf face algebra via (3.18), and

$$S\left(e\begin{pmatrix}p\\q\end{pmatrix}\right) = \left(\frac{\mu(\mathfrak{s}(q))\mu(\mathfrak{r}(p))}{\mu(\mathfrak{s}(p))\mu(\mathfrak{r}(q))}\right)^{1/2} e\begin{pmatrix}q^{\sim}\\p^{\sim}\end{pmatrix} \quad (p, q \in \mathcal{G}^m,\ m \geq 0).$$

If V is a flat face model associated with a II_1-subfactor $N \subset M$, we call $\mathfrak{G}(V)$ the **Galois face algebra** of $N \subset M$.

4.2. The main results.

THEOREM 4.2. *Let $(V, w, *)$ be a flat $\mathcal{V}$-face model of connection type such that its dimension graph $\mathcal{G}$ is bipartite. Then the following hold.*

(i) *The compact Hopf face algebra $\mathfrak{G}(V)$ is hollowless and finite dimensional, and the fusion rules $N^j_{ik} := N^j_{ik}(m)$ do not depend on the choice of m.*

(ii) *For each $i \in \mathcal{V}$, there exists a $\mathfrak{G}(V)$-comodule L_i such that* $\dim(L_i(*, j)) = \delta_{ij}$ *($j \in \mathcal{V}$). The comodule L_i is irreducible and unique up to isomorphism. Moreover, we have:*

$$(4.7)\qquad \mathfrak{G}(V) \simeq \bigoplus_{i\in\mathcal{V}} \mathrm{End}(L_i)^*,$$

$$(4.8)\qquad L_{(i,m)} \simeq L_i \quad ((i, m) \in \Lambda_V),$$

$$(4.9)\qquad L_* \simeq R,$$

$$(4.10)\qquad L_i \bar{\otimes} L_j \simeq \bigoplus_{k\in\mathcal{V}} N^k_{ij} L_k \quad (i, j \in \mathcal{V}),$$

$$(4.11)\qquad \dim(L_k(i, j)) = N^j_{ik} \quad (i, j, k \in \mathcal{V}),$$

where (4.7) *stands for an isomorphism of coalgebras and* (4.8)–(4.10) *stand for isomorphisms of $\mathfrak{G}(V)$-comodules.*

We give the proof of this theorem in the next subsection.

THEOREM 4.3. *Let V be as in the theorem above. Assume also that V has the generating vertex* $\square$. *Then we have*:

$$\dim_{\boldsymbol{Q}}(L_i) = \mu(i) \quad (i \in \mathcal{V}), \tag{4.12}$$

$$\boldsymbol{Q} = \sum_{i,j\in\mathcal{V}} \frac{\mu(j)}{\mu(i)} \overset{\circ}{\varepsilon}_i \varepsilon_j, \tag{4.13}$$

where $[\mu(i)]_{i\in\mathcal{V}}$ *denotes the normalized Perron-Frobenius eigenvector of V.*

PROOF. By (4.10) and (2.3), we have

$$\dim_{\boldsymbol{Q}}(L_i)\dim_{\boldsymbol{Q}}(L_j) = \sum_{k\in\mathcal{V}} N_{ij}^k \dim_{\boldsymbol{Q}}(L_k). \tag{4.14}$$

Using (3.15), we see that $[\dim_{\boldsymbol{Q}}(L_i)]_{i\in\mathcal{V}}$ is an eigenvector of the matrix $[\sharp(\mathcal{G}_{ij}^1)]_{i,j\in\mathcal{V}}$. Hence (4.12) follows from the uniqueness of the Perron-Frobenius eigenvector. Let $\tilde{\boldsymbol{Q}}$ be the right-hand side of (4.13). Using (4.11) and (4.14), we obtain

$$\operatorname{Tr}\pi_{L_k}(\tilde{\boldsymbol{Q}}) = \sum_{i,j\in\mathcal{V}} \frac{\mu(j)}{\mu(i)} \cdot N_{ik}^j = \sharp(\mathcal{V}) \cdot \mu(k).$$

Similarly, using (1.30) and (2.4) in addition, we get $\operatorname{Tr}_{L_k}(\tilde{\boldsymbol{Q}}^{-1}) = \sharp(\mathcal{V})\cdot\mu(k)$. The verification of the relations (1.24)–(1.28) is straightforward. $\square$

PROPOSITION 4.4. *Let V and* $\square$ *be as in the theorem above.*

(i) (*A reciprocity of Schur type*) *For each* $m \geq 0$, *we have*

$$C(\pi_{V^m}(\mathfrak{G}(V)^*)) \simeq \operatorname{Str}^m(V), \quad C(\operatorname{Str}^m(V)) \simeq \pi_{V^m}(\mathfrak{G}(V)^*),$$

where C denotes the commutant in the algebra $\operatorname{End}(V^m)$.

(ii) (cf. [O1]) *Let* π_{V^m} *be the tracial state on* $\operatorname{End}_{\mathfrak{G}(V)}(V^m)$ *defined as in Subsection 2.2. Then we have*

$$\tau_{V^m}(\Gamma(E_{\boldsymbol{ab}})) = \mu(i)\mu(\square)^{-m}\delta_{\boldsymbol{ab}} \quad (m \geq 0,\ i \in \mathcal{V},\ \boldsymbol{a}, \boldsymbol{b} \in \mathcal{G}_{*i}^m). \tag{4.15}$$

PROOF. (i) As $\mathfrak{G}(V)^*$-modules, $\{L_\lambda \mid \lambda \in \Lambda_V^m\}$ are still irreducible and mutually non-isomorphic. Hence, the map $\pi_{V^m} : \mathfrak{G}(V)^* \to \operatorname{Cost}^m(V)^*$ is surjective. This proves the second isomorphism. The other isomorphism follows from the double commutant theorem.

(ii) Using (4.13), we obtain

$$\tau_{V^m}(\Gamma(E_{\boldsymbol{ab}})) = \frac{1}{\sharp(\mathcal{V})\mu(\square)^m} \sum_{j,k\in\mathcal{V}} \frac{\mu(k)}{\mu(j)} \sum_{\boldsymbol{c}\in\mathcal{G}_{jk}^m} \gamma(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{c})$$

for each $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{G}_{*i}^m$. On the other hand, using the unitarity and the renormalization rule, we obtain

$$\sum_{k\in\mathcal{V}} \mu(k) \sum_{\boldsymbol{c}\in\mathcal{G}_{jk}^m} \gamma(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}, \boldsymbol{c}) = \mu(i)\mu(j)\delta_{\boldsymbol{ab}}.$$

This proves (ii). $\square$

For a flat face model (V, w) of connection type, we define another flat face model $(V, \bar{w})$ of connection type via

$$\bar{w}(u_{a} \otimes u_{b}) = \sum_{c,d} \overline{w\left(c \begin{smallmatrix} a \\ \\ d \end{smallmatrix} b\right)}\, u_{c} \otimes u_{d} \quad (\boldsymbol{a} \cdot \boldsymbol{b} \in \mathcal{G}^{2})\,.$$

THEOREM 4.5. *Let V, $\square$ and $\mu(i)$ be as in Theorem* 4.3. *For each $i \in \mathcal{V}$ such that $\mu(i) > 1$, the Jones' index of the* II_1*-subfactor $N(i) \subset M(i)$ associated with L_i is $\mu(i)^2$, where L_i is as in Theorem* 4.2. *If $(V, \bar{w})$ is associated with a* II_1*-subfactor $N \subset M$ which satisfies the conditions stated above, then $N(\square) \subset M(\square)$ is isomorphic to $N \subset M$.*

PROOF. The first assertion is obvious. Let V_m, W_m etc. be as in Subsection 2.5. Using (1.30) and (3.15), we compute

$$\delta_{\square,\square^{\vee}} = N_{*\square^{\vee}}^{\square} = \sharp(\mathcal{G}_{\square *}^{1}) = 1\,.$$

Hence, we have $V_m \simeq W_m \simeq V^m$ as $\mathfrak{G}(V)$-comodules. Hence, by the proposition above, we have $B_m \simeq C_m \simeq \mathrm{Str}^m(V)$. Moreover, the inclusions $C_m \subset C_{m+1}$ and $B_m \subset B_{m+1}$ are identified with ι_{m1}, and the inclusion $C_m \subset B_{m+1}$ is identified with $\nabla_{1m} : \mathrm{Str}^1(V) \otimes \mathrm{Str}^m(V) \hookrightarrow \mathrm{Str}^{m+1}(V)$. Hence, by (4.15), the ladder (2.10) of the commuting squares is identified with that of Ocneanu which appeared in [O1, p. 131]. Therefore the second assertion follows from a theorem of [O1, p. 134], whose proof is given by Popa [P]. $\square$

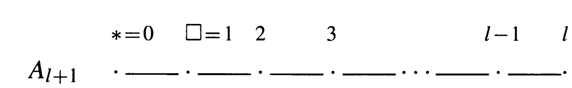
Let $N \subset M$ be an AFD II_1-subfactor of index less than 4 with principal graph $\mathcal{G}$. Let $\mathfrak{G}$ be its Galois face algebra. By (1.30), we have

$$\dim(\mathrm{Hom}_{\mathfrak{G}}(L_{*}, L_{i} \bar{\otimes} L_{j})) = \delta_{ij^{\vee}}\,. \tag{4.16}$$

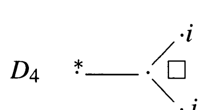
In [I], M. Izumi shows that the fusion rules of sectors corresponding to subfactors with index < 4 are computable by means of results which are analogous to (1.29) and (4.16). Hence, the fusion rules of $\mathfrak{G}$ are also computable. For example, in case $\mathcal{G} = A_{l+1}$, these are given by

$$N_{ij}^{k} = \begin{cases} 1 & (|i - j| \le k \le i + j,\ i + j + k \in 2\mathbf{Z} \text{ and } \le 2l) \\ 0 & \text{otherwise}\,, \end{cases}$$

where the labeling of the vertexes of A_{l+1} is as follows:



These numbers are well-known as fusion rules of $SU(2)_l$-Wess-Zumino-Novikov-Witten models (cf. [T-K]). In general, the fusion algebra of $\mathfrak{G}$ (i.e., the representation ring of $\mathfrak{G}^*$) is commutative, and the involution $^{\vee} : \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ is of order 2 if $\mathcal{G} = D_{4n}$ $(n \ge 1)$ and is an identity if otherwise. For the convenience of readers, we write down the fusion rules of $\mathfrak{G}$ when $\mathcal{G} = D_4$.



$$L_\square \bar{\otimes} L_\square \simeq L_* \oplus L_i \oplus L_j\,, \quad L_i \bar{\otimes} L_\square \simeq L_\square\,, \quad L_j \bar{\otimes} L_\square \simeq L_\square\,,$$
$$L_i \bar{\otimes} L_i \simeq L_j\,, \quad L_j \bar{\otimes} L_i \simeq L_*\,, \quad L_j \bar{\otimes} L_j \simeq L_i\,,$$
$$L_\square{}^\vee \simeq L_\square\,, \quad L_i{}^\vee \simeq L_j\,, \quad L_j{}^\vee \simeq L_i\,.$$

It is natural to ask the relation between Izumi's descendant sectors and the family of II_1-subfactors which is obtained by applying the theorem above to $\mathfrak{G}$.

4.3. Simply reducible group like element. Let $\mathfrak{H}$ be a $\mathcal{V}$-face algebra such that $\mathfrak{H} \simeq \bigoplus_{\lambda\in\Lambda} \mathrm{End}(L_\lambda)^*$ as coalgebras for some irreducible right comodules $\{L_\lambda \mid \lambda \in \Lambda\}$. Let g be a central group-like element of $\mathfrak{H}$. We say that g is **simply reducible** if there exist a subset $\bar{\Lambda} \subset \Lambda$ and a bijection $\varphi : \bar{\Lambda} \times \mathbf{Z}_{\geq 0} \xrightarrow{\sim} \Lambda$ such that $L_{\varphi(\lambda,n)} \simeq Rg^n \bar{\otimes} L_\lambda$ for each $\lambda \in \bar{\Lambda}$ and $n \in \mathbf{Z}_{\geq 0}$, and that $\varphi(\lambda, 0) = \lambda$.

THEOREM 4.6. *Let $\mathfrak{H}$, g etc. be as above. Then the following hold.*

(i) *The element g is not a zero divisor of $\mathfrak{H}$.*

(ii) *The quotient $\bar{\mathfrak{H}} := \mathfrak{H}/\mathfrak{H}(g-1)\mathfrak{H}$ is isomorphic to $\bigoplus_{\lambda\in\bar{\Lambda}} \mathrm{End}(L_\lambda)^*$ as coalgebras.*

(iii) *As an $\bar{\mathfrak{H}}$-comodule, $L_{\varphi(\lambda,n)}$ is irreducible and isomorphic to L_λ. In particular, $\bar{\mathfrak{H}}$ is hollowless if $\mathfrak{H}$ is hollowless.*

(iv) *The fusion rules of $\bar{\mathfrak{H}}$ are given by*

$$L_\lambda \bar{\otimes} L_\mu \simeq \bigoplus_{\nu\in\bar{\Lambda}} \left(\sum_{n\geq 0} N_{\lambda,\mu}^{\varphi(\nu,n)} \right) L_\nu\,,$$

where $N_{\lambda\mu}^{\nu}$ denote the fusion rules of $\mathfrak{H}$.

PROOF. For each $\lambda \in \bar{\Lambda}$, let $\mathcal{G}_\lambda$ denote the dimension graph of L_λ. Let $\{u_{\boldsymbol{q}} \mid \boldsymbol{q} \in (\mathcal{G}_\lambda)^1_{ij}\}$ be a basis of $L_\lambda(i,j)$, and $[x^{\boldsymbol{p}}_{\boldsymbol{q}}]$ the corresponding matrix corepresentation. Using Lemma 1.3, we see that $[g^n x^{\boldsymbol{p}}_{\boldsymbol{q}}]_{\boldsymbol{pq}}$ is a matrix corepresentation of $(Rg^n \bar{\otimes} L_\lambda,\ \{e_{\mathfrak{s}(\boldsymbol{q})} g^n \otimes u_{\boldsymbol{q}} \mid \boldsymbol{q} \in \mathcal{G}^1_\lambda\})$. Hence $\left\{ g^n x^{\boldsymbol{p}}_{\boldsymbol{q}} \;\middle|\; \lambda \in \bar{\Lambda}, \boldsymbol{p}, \boldsymbol{q} \in \mathcal{G}^1_\lambda, n \in \mathbf{Z}_{\geq 0} \right\}$ is a basis of $\mathfrak{H}$. Therefore (i) is obvious.

Since g is central, we have

$$\begin{aligned}\mathfrak{H}(g-1)\mathfrak{H} &= (g-1)\mathfrak{H} \\ &= \sum_{\lambda\in\bar{\Lambda}} \sum_{\boldsymbol{p},\boldsymbol{q}\in\mathcal{G}^1_\lambda} \sum_{n\geq 0} \boldsymbol{K}(g-1) g^n x^{\boldsymbol{p}}_{\boldsymbol{q}}\,.\end{aligned}$$

Since $\left\{ x^{\boldsymbol{p}}_{\boldsymbol{q}}, (g-1) g^n x^{\boldsymbol{p}}_{\boldsymbol{q}} \;\middle|\; \lambda \in \bar{\Lambda}, \boldsymbol{p}, \boldsymbol{q} \in \mathcal{G}^1_\lambda, n \in \mathbf{Z}_{\geq 0} \right\}$ is a basis of $\mathfrak{H}$, $\left\{ \bar{x}^{\boldsymbol{p}}_{\boldsymbol{q}} \;\middle|\; g \in \bar{\Lambda}, \boldsymbol{p}, \boldsymbol{q} \in \mathcal{G}^1_\lambda \right\}$ gives a basis of $\bar{\mathfrak{H}}$, where $\bar{x}^{\boldsymbol{p}}_{\boldsymbol{q}}$ denotes the image of $x^{\boldsymbol{p}}_{\boldsymbol{q}}$ via the projection $\mathfrak{H} \twoheadrightarrow \bar{\mathfrak{H}}$. Hence (ii) follows from $\mathrm{span}\left\{ \bar{x}^{\boldsymbol{p}}_{\boldsymbol{q}} \;\middle|\; \boldsymbol{p}, \boldsymbol{q} \in \mathcal{G}^1_\lambda \right\} \simeq \mathrm{End}(L_\lambda)^*$. The proof of the other assertions is now obvious. □

Now we are in a position to give the proof of Theorem 4.2. Since $\mathcal{G}$ is bipartite, we have $\Lambda := \Lambda_V = \coprod_{i\in\mathcal{V}} \{(i, m(i) + 2n) \mid n \geq 0\}$, where $m(i) = \min\{m \mid (i,m) \in \Lambda\}$. We define a subset $\bar{\Lambda}$ of Λ and a bijection $\varphi : \bar{\Lambda} \times \mathbf{Z}_{\geq 0} \xrightarrow{\sim} \Lambda$ by $\bar{\Lambda} = \{(i, m(i)) \mid i \in \mathcal{V}\}$ and $\varphi(i,n) = (i, m(i) + 2n)$. Using the second assertion of Proposition 3.4(ii), we see that

$g =$ det satisfies the conditions of the theorem above. Therefore Theorem 4.2 follows from Proposition 3.4.

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