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# HOMOGENEOUS FRACTIONAL INTEGRALS ON HARDY SPACES

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**Abstract.** Mapping properties for the homogeneous fractional integral operator $T_{\Omega,\alpha}$ on the Hardy spaces $H^p(\boldsymbol{R}^n)$ are studied. Our results give the extension of Stein-Weiss and Taibleson-Weiss's results for the boundedness of the Riesz potential operator $I_\alpha$ on the Hardy spaces $H^p(\boldsymbol{R}^n)$.

**1. Introduction and results.** Let $S^{n-1}$ denote the unit sphere in Euclidean $n$-space $\boldsymbol{R}^n$. Suppose $0 < \alpha < n$, and let $\Omega \in L^r(S^{n-1})$ with $r \geq 1$ be homogeneous of degree zero on $\boldsymbol{R}^n$. Then we define the homogeneous fractional integral operator $T_{\Omega,\alpha}$ by

$$(T_{\Omega,\alpha}f)(x) = \int_{\boldsymbol{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy\,.$$

When $\alpha = 0$, we denote $T_{\Omega,0}$ by $T_\Omega$, and the integration is taken by the Cauchy principal value. Since the operator $T_{\Omega,\alpha}$ is closely connected with the singular integral operator $T_\Omega$, $T_{\Omega,\alpha}$ plays an important roles in the study for homogeneous operator $T_\Omega$. For example, recently the authors applied several results on $T_{\Omega,\alpha}$ to a study of a mapping property for a class of multilinear singular integral operator with homogeneous kernel [4]. As an application of this mapping property, in [4] we obtained the $L^p$ boundedness of the commutator $[T_\Omega, b]$ formed by the homogeneous singular integral operator $T_\Omega$ with a function $b$ in *BMO*.

In 1971, Muckenhoupt and Wheeden [7] proved the weighted $(L^p, L^q)$ boundedness of $T_{\Omega,\alpha}$ for power weight when $1 < p < n/\alpha$. In 1998, we obtained the weighted $(L^p, L^q)$ boundedness of $T_{\Omega,\alpha}$ for $A(p,q)$ weight [2]. Moreover, when $p = 1$, the $(L^1, L^{n/(n-\alpha),\infty})$ boundedness of $T_{\Omega,\alpha}$ can also be found in [1] (unweighted) and in [5] (with power weights). For $p = n/\alpha$, an exponential integral inequality of $T_{\Omega,\alpha}$ was proved in [3].

On the other hand, the Hardy-Littlewood-Sobolev theorem showed that the Riesz potential operator $I_\alpha$ is bounded from $L^p$ to $L^q$. In 1960, Stein and Weiss [10] used the theory of harmonic functions of several variables to prove that $I_\alpha$ is bounded from $H^1$ to $L^{n/(n-\alpha)}$. In 1980, using the molecular characterization of the real Hardy spaces, Taibleson and Weiss [11] proved that $I_\alpha$ is also bounded from $H^p$ to $L^q$ or $H^q$, where $0 < p < 1$ and $1/q = 1/p - \alpha/n$.

Since the Riesz potential operator $I_\alpha$ is essentially the homogeneous fractional integral operators $T_{\Omega,\alpha}$ when $\Omega \equiv 1$, by comparing mapping properties of $I_\alpha$ and $T_{\Omega,\alpha}$, it is natural

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to ask whether the homogeneous fractional integral operator $T_{\Omega,\alpha}$ has the same mapping properties on $H^p$ as those of the Riesz potential operator $I_\alpha$.

The purpose of this paper is to answer this question. Using the atomic-molecular decomposition of $H^p$, we prove that if $\Omega$ satisfies a class of $L^r$-Dini conditions on $S^{n-1}$, then $T_{\Omega,\alpha}$ is bounded from $H^p$ to $L^q$ or $H^q$ for some $p \le 1$. Thus, we verify that Stein-Weiss's conclusion (for $p = 1$) and Taibleson-Weiss's conclusion (for some $p < 1$) hold also for $T_{\Omega,\alpha}$.

Before stating our results, let us recall the definition of the $L^r$-Dini condition.

We say that $\Omega$ satisfies the $L^r$-Dini condition if $\Omega \in L^r(S^{n-1})$ with $r \ge 1$ is homogeneous of degree zero on $\boldsymbol{R}^n$, and

$$\int_0^1 \frac{\omega_r(\delta)}{\delta} d\delta < \infty,$$

where $\omega_r(\delta)$ denotes the integral modulus of continuity of order $r$ of $\Omega$ defined by

$$\omega_r(\delta) = \sup_{|\rho|<\delta} \left( \int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^r dx' \right)^{1/r}$$

and $\rho$ is a rotation in $\boldsymbol{R}^n$ and $|\rho| = \|\rho - I\|$.

Now, let us formulate our results as follows.

THEOREM 1. *Let $0 < \alpha < n$, and let $\Omega \in L^r(S^{n-1})$ for $r > n/(n-\alpha)$ be homogeneous of degree zero on $\boldsymbol{R}^n$. If $\Omega$ satisfies the $L^r$-Dini condition, then there is a $C > 0$ such that $\|T_{\Omega,\alpha} f\|_{L^{n/(n-\alpha)}} \le C\|f\|_{H^1}$.*

THEOREM 2. *Let $0 < \alpha < 1$, $n/(n+\alpha) \le p < 1$, $1/q = 1/p - \alpha/n$ and $\Omega \in L^r(S^{n-1})$ with $r > n/(n-\alpha)$ be homogeneous of degree zero on $\boldsymbol{R}^n$. If the integral modulus of continuity $\omega_r(\delta)$ of order $r$ of $\Omega$ satisfies*

$$\int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\alpha}} d\delta < \infty, \tag{1.1}$$

*then there is a $C > 0$ such that $\|T_{\Omega,\alpha} f\|_{L^q} \le C\|f\|_{H^p}$.*

Theorems 1 and 2 give the $(H^p, L^q)$ boundedness of $T_{\Omega,\alpha}$. The following theorem will give the $(H^p, H^q)$ boundedness of $T_{\Omega,\alpha}$.

THEOREM 3. *Let $0 < \alpha < 1/2$, $1/q = 1/p - \alpha/n$ and let $\Omega \in L^r(S^{n-1})$ with $r > 1/(1-2\alpha)$ be homogeneous of degree zero on $\boldsymbol{R}^n$. If for $\alpha < \beta \le 1$ the integral modulus of continuity $\omega_r(\delta)$ of order $r$ of $\Omega$ satisfies*

$$\int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\beta}} d\delta < \infty, \tag{1.2}$$

*then for $n/(n+\beta) < p \le n/(n+\alpha)$, there is a $C > 0$ such that $\|T_{\Omega,\alpha} f\|_{H^q} \le C\|f\|_{H^p}$.*

In the following the letter $C$ will denote a constant which varies at each occurrence.

**2. Some preliminary facts.** In this section we give the $(L^p, L^q)$ boundedness of the fractional integral operator with rough kernel $T_{\Omega,\alpha}$, which will be used in the proof of Theorems 1 through 3.

PROPOSITION 1. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and let $\Omega \in L^r(S^{n-1})$ with $r > n/(n-\alpha)$ be homogeneous of degree zero on $\boldsymbol{R}^n$. Then $T_{\Omega,\alpha}$ is an operator of type $(p, q)$.*

To prove Proposition 1, let us first give some lemmas.

LEMMA 1 (see [1]). *Let $0 < \alpha < n$, $r \geq n/(n-\alpha)$ and let $\Omega \in L^r(S^{n-1})$ be homogeneous of degree zero on $\boldsymbol{R}^n$. Then for any $\lambda > 0$ and any $f \in L^1$,*

$$\left|\left\{x \in \boldsymbol{R}^n : |(T_{\Omega,\alpha} f)(x)| > \lambda\right\}\right| \leq C\left(\frac{1}{\lambda}\|f\|_{L^1}\right)^{n/(n-\alpha)}$$

*where $C$ is independent of $\lambda$ and $f$.*

LEMMA 2 (see [5]). *Suppose that $0 < \alpha < n$, and $\Omega \in L^r(S^{n-1})$ with $r \geq 1$. Then there is a $C > 0$ dependent only on $n$ and $\alpha$ such that $(M_{\Omega,\alpha} f)(x) \leq C(T_{|\Omega|,\alpha})(|f|)(x)$, where $M_{\Omega,\alpha}$ denotes the homogeneous fractional maximal operator defined by*

$$(M_{\Omega,\alpha} f)(x) = \sup_{t>0} \frac{1}{t^{n-\alpha}} \int_{|x-y|<t} |\Omega(x-y) f(y)| dy\,.$$

LEMMA 3 (see [2]). *Suppose that $0 < \alpha < n$, and $\Omega \in L^r(S^{n-1})$ with $r \geq 1$. Then for any $0 < \varepsilon < \min\{\alpha, n-\alpha\}$, there is a $C = C(n, \alpha, \varepsilon)$ such that*

$$|(T_{\Omega,\alpha} f)(x)| \leq C[(M_{\Omega,\alpha+\varepsilon} f)(x)]^{1/2}[(M_{\Omega,\alpha-\varepsilon} f)(x)]^{1/2}\,.$$

*Proof of Proposition* 1.

It is easy to see that under the conditions of Proposition 1, $M_{\Omega,\alpha}$ is of weak-type $(1, n/(n-\alpha))$ (by Lemmas 1 and 2) and of type $(n/\alpha, \infty)$. Thus, by the Marcinkiewicz interpolation theorem, we get the $(L^p, L^q)$-boundedness $(1 < p < n/\alpha)$ of $M_{\Omega,\alpha}$ under the conditions of Proposition 1, i.e.,

$$\|M_{\Omega,\alpha} f\|_{L^q} \leq C\|f\|_{L^p}\,. \tag{2.1}$$

Since $r > n/(n-\alpha)$, we can choose $\varepsilon > 0$ with $\varepsilon < \min\{\alpha, n-\alpha\}$ so small that $1/q - \varepsilon/n > 0$, $1/q + \varepsilon/n < 1$ and $r > n/[n-(\alpha+\varepsilon)]$. By letting $1/q_1 = 1/q - \varepsilon/n$ and $1/q_2 = 1/q + \varepsilon/n$, we have

$$1/q_1 = 1/p - (\alpha+\varepsilon)/n \quad \text{and} \quad 1/q_2 = 1/p - (\alpha-\varepsilon)/n\,.$$

Noting that $r > n/[n-(\alpha+\varepsilon)] > n/(n-\alpha) > n/[n-(\alpha-\varepsilon)]$, by (2.1) it follows that

$$\|M_{\Omega,\alpha+\varepsilon} f\|_{L^{q_1}} \leq C\|f\|_{L^p} \quad \text{and} \quad \|M_{\Omega,\alpha-\varepsilon} f\|_{L^{q_2}} \leq C\|f\|_{L^p}\,. \tag{2.2}$$

Hence, if we denote $l_1 = 2q_1/q$ and $l_2 = 2q_2/q$, then $l_1, l_2 > 1$ and $1/l_1 + 1/l_2 = 1$. By Lemma 3, Hölder's inequality for $l_1, l_2$ and (2.2), we get

$$\begin{aligned}
\|(T_{\Omega,\alpha} f)(x)\|_{L^q} &\le C\left(\int_{\mathbf{R}^n} [(M_{\Omega,\alpha+\varepsilon} f)(x)]^{q/2}[(M_{\Omega,\alpha-\varepsilon} f)(x)]^{q/2} dx\right)^{1/q} \\
&\le C\left(\int_{\mathbf{R}^n} [(M_{\Omega,\alpha+\varepsilon} f)(x)]^{l_1 q/2} dx\right)^{1/ql_1}\left(\int_{\mathbf{R}^n} [(M_{\Omega,\alpha-\varepsilon} f)(x)]^{l_2 q/2} dx\right)^{1/ql_2} \\
&= C\left(\int_{\mathbf{R}^n} [(M_{\Omega,\alpha+\varepsilon} f)(x)]^{q_1} dx\right)^{1/2q_1}\left(\int_{\mathbf{R}^n} [(M_{\Omega,\alpha-\varepsilon} f)(x)]^{q_2} dx\right)^{1/2q_2} \\
&\le C\|f\|_{L^p}.
\end{aligned}$$

This proves Proposition 1.

In the proof of Theorem 1 through 3, we need the following fact.

LEMMA 4. *Suppose that* $0 < \alpha < n$, $r > 1$, *and* $\Omega$ *satisfies the* $L^r$*-Dini condition. If there is a constant* $a_0 > 0$ *such that* $|y| < a_0 R$, *then*

$$\begin{aligned}
&\left(\int_{R<|x|<2R}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}}\right|^r dx\right)^{1/r} \\
&\qquad \le C R^{n/r-(n-\alpha)}\left\{\frac{|y|}{R} + \int_{|y|/2R<\delta<|y|/R} \frac{\omega_r(\delta)}{\delta} d\delta\right\}.
\end{aligned}$$

Using a method similar to that in the proof of Lemma 5 in [7], we can prove Lemma 4. We omit the detail here.

**3. Proofs of Theorems 1 and 2.** Let us first give the proof of Theorem 1. By the atomic decomposition theory of Hardy spaces [9], it is sufficient to prove that there is a constant $C$ such that for any $(1, l, 0)$-atom $a(x)$, the inequality

$$\|(T_{\Omega,\alpha} a)(x)\|_{L^q} \le C \tag{3.1}$$

holds, where $l > 1$ and $q = n/(n-\alpha)$. To do so, we take $1 < l_1 < l_2 < \infty$, such that $1/l_1 - 1/l_2 = \alpha/n$. Without loss of generality, we may assume that $a(x)$ is $(1, l_1, 0)$-atom supported in a ball $B = B(0, d)$ with center at zero and radius $d$. That means

$$\text{(i)}\quad \operatorname{supp}(a) \subset B\,; \qquad \text{(ii)}\quad \|a\|_{L^{l_1}} \le |B|^{1/l_1 - 1}\,; \qquad \text{(iii)}\quad \int a(x)dx = 0\,.$$

We have

$$\|T_{\Omega,\alpha} a\|_{L^q} \le \left(\int_{2B} |(T_{\Omega,\alpha} a)(x)|^q dx\right)^{1/q} + \left(\int_{(2B)^C} |(T_{\Omega,\alpha} a)(x)|^q dx\right)^{1/q} := I_1 + I_2\,.$$

Applying Hölder's inequality and Proposition 1, we get

$$I_1 \le C\|T_{\Omega,\alpha} a\|_{L^{l_2}} |B|^{1/q-1/l_2} \le C\|a\|_{L^{l_1}} |B|^{1/q-1/l_2} \le C\,.$$

For $I_2$, by the vanishing condition (iii) of $a(x)$, we have

$$I_2 \leq \int_B |a(y)| \sum_{j=1}^{\infty} \left( \int_{2^j d \leq |x| < 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}} \right|^q dx \right)^{1/q} dy . \tag{3.2}$$

Noting that $r > n/(n-\alpha) = q$, therefore

$$\begin{aligned} &\left( \int_{2^j d \leq |x| < 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}} \right|^q dx \right)^{1/q} \\ &\quad \leq C(2^j d)^{n(1/q-1/r)} \left( \int_{2^j d \leq |x| < 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}} \right|^r dx \right)^{1/r} . \end{aligned} \tag{3.3}$$

Applying Lemma 4, we get

$$\begin{aligned} &\left( \int_{2^j d \leq |x| < 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}} \right|^r dy \right)^{1/r} \\ &\quad \leq C(2^j d)^{n/r-(n-\alpha)} \left\{ \frac{1}{2^j} + \int_{|y|/2^{j+1} d}^{|y|/2^j d} \frac{\omega_r(\delta)}{\delta} d\delta \right\} . \end{aligned} \tag{3.4}$$

By (3.3), (3.4) and the $L^r$-Dini condition, we get

$$\begin{aligned} &\sum_{j=1}^{\infty} \left( \int_{2^j d \leq |x| < 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}} \right|^q dx \right)^{1/q} \\ &\quad \leq C \sum_{j=1}^{\infty} (2^j d)^{n(1/q-1/r)} \cdot (2^j d)^{n/r-(n-\alpha)} \left\{ \frac{1}{2^j} + \int_{|y|/2^{j+1} d}^{|y|/2^j d} \frac{\omega_r(\delta)}{\delta} d\delta \right\} \\ &\quad \leq C \sum_{j=1}^{\infty} \left\{ \frac{1}{2^j} + \int_{|y|/2^{j+1} d}^{|y|/2^j d} \frac{\omega_r(\delta)}{\delta} d\delta \right\} \\ &\quad \leq C \left( 1 + \int_0^1 \frac{\omega_r(\delta)}{\delta} d\delta \right) < \infty . \end{aligned} \tag{3.5}$$

Thus, by (3.2) and (3.5)

$$I_2 \leq C \int_B |a(y)| dy \leq C \|a\|_{L^{l_1}} |B|^{1/l_1'} \leq C .$$

Hence we complete the proof of Theorem 1.

The proof of Theorem 2 is similar to that of Theorem 1. Here we only give the main steps of the proof. Taking $1 < l_1 < l_2 < \infty$ such that $1/l_1 - 1/l_2 = 1/p - 1/q = \alpha/n$. Let $a(x)$ be $(p, l_1, 0)$-atom supported in the ball $B(0, d)$. We need to prove (3.1) for the atom $a(x)$. As in the proof of Theorem 1, we give the estimates for $I_1$ and $I_2$, respectively. Using

Hölder's inequality and Proposition 1, we can easily obtain the estimates of $I_1$. On the other hand, by (3.3) and (3.4), we get

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left(\int_{2^j d\le|x|<2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(x)}{|x|^{n-\alpha}}\right|^q dx\right)^{1/q}\\
&\le C\sum_{j=1}^{\infty}(2^j d)^{n(1/p-1)}\left\{\frac{1}{2^j}+\int_{|y|/2^{j+1}d}^{|y|/2^j d}\frac{\omega_r(\delta)}{\delta}d\delta\right\} \qquad (3.6)\\
&\le C|B|^{1/p-1}\sum_{j=1}^{\infty}\left\{2^{j[n(1/p-1)-1]}+2^{j[n(1/p-1)-\alpha]}\int_{|y|/2^{j+1}d}^{|y|/2^j d}\frac{\omega_r(\delta)}{\delta^{1+\alpha}}d\delta\right\}.
\end{aligned}$$

Using the conditions of Theorem 2, we get $n(1/p-1)-1 < n(1/p-1)-\alpha \le 0$. If $p > n/(n+\alpha)$, then $n(1/p-1)-\alpha < 0$. In this case, by (3.6) and (1.1) we have

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left(\int_{2^j d\le|x|<2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(x)}{|x|^{n-\alpha}}\right|^q dx\right)^{1/q}\\
&\le C|B|^{1/p-1}\sum_{j=1}^{\infty}\left\{2^{j[n(1/p-1)-1]}+2^{j[n(1/p-1)-\alpha]}\int_0^1\frac{\omega_r(\delta)}{\delta^{1+\alpha}}d\delta\right\}\\
&\le C|B|^{1/p-1}\left(1+\int_0^1\frac{\omega_r(\delta)}{\delta^{1+\alpha}}d\delta\right)\le C|B|^{1/p-1}.
\end{aligned}$$

If $p = n/(n+\alpha)$, then $n(1/p-1)-\alpha = 0$. In this case, by (3.6) and (1.1) we have

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left(\int_{2^j d\le|x|<2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(x)}{|x|^{n-\alpha}}\right|^q dx\right)^{1/q}\\
&\le C|B|^{1/p-1}\sum_{j=1}^{\infty}\left\{2^{j[n(1/p-1)-1]}+\int_{|y|/2^{j+1}d}^{|y|/2^j d}\frac{\omega_r(\delta)}{\delta^{1+\alpha}}d\delta\right\}\\
&\le C|B|^{1/p-1}\left(1+\int_0^1\frac{\omega_r(\delta)}{\delta^{1+\alpha}}d\delta\right)\le C|B|^{1/p-1}.
\end{aligned}$$

Finally, from the discussion above and (3.2) we have

$$I_2 \le C|B|^{1/p-1}\int_B|a(y)|dy \le C|B|^{1/p-1}\|a\|_{L^{l_1}}|B|^{1/l_1'} \le C.$$

The conclusion of Theorem 2 is proved.

**4. Proof of Theorem 3.** Since $r > n/(n-\alpha)$, we can choose $1 < l_1 < l_2$ so that $1/l_1 - 1/l_2 = 1/p - 1/q = \alpha/n$ and $n/(n-\alpha) < l_2 < r$. Take $\varepsilon$ so that $1/q - 1 < \varepsilon < (\beta-\alpha)/n \le (1-\alpha)/n$. Denote $a_0 = 1 - 1/q + \varepsilon$, $b_0 = 1 - 1/l_2 + \varepsilon$ and let $a(x)$ be a $(p, l_1, 0)$-atom supported in the ball $B(0,d)$. By the atomic-molecular decomposition theory of real Hardy spaces [9], it suffices to show that $T_{\Omega,\alpha}a$ is a $(q, l_2, 0, \varepsilon)$-molecule for proving Theorem 3. This means that we need to verify that $(T_{\Omega,\alpha}a)(x)$ satisfies the following conditions:

(i) $|x|^{nb_0}(T_{\Omega,\alpha}a)(x) \in L^{l_2}$;

(ii) $\mathcal{N}_{l_2}(T_{\Omega,\alpha}a) := \|T_{\Omega,\alpha}a\|_{L^{l_2}}^{a_0/b_0} \||\cdot|^{nb_0}(T_{\Omega,\alpha}a)(\cdot)\|_{L^{l_2}}^{1-a_0/b_0} < \infty$;

(iii) $\int (T_{\Omega,\alpha}a)(x)dx = 0$.

Moreover, we also need to show that there is a constant $C > 0$, independent of $a(x)$, such that

$$\mathcal{N}_{l_2}(T_{\Omega,\alpha}a) \leq C\,.$$

Let us begin with proving (i). Write

$$\begin{aligned}\||\cdot|^{nb_0}(T_{\Omega,\alpha}a)(\cdot)\|_{L^{l_2}} &\leq \||\cdot|^{nb_0}(T_{\Omega,\alpha}a)(\cdot)\chi_{2B}(\cdot)\|_{L^{l_2}} + \||\cdot|^{nb_0}(T_{\Omega,\alpha}a)(\cdot)\chi_{(2B)^c}(\cdot)\|_{L^{l_2}} \\ &:= J_1 + J_2\,.\end{aligned}$$

Noting that $n/(n-\alpha) < l_2 < r$ and $1/l_1 - 1/l_2 = \alpha/n$, by Proposition 1 we have

$$J_1 \leq C|B|^{b_0}\|T_{\Omega,\alpha}a\|_{L^{l_2}} \leq C|B|^{b_0}\|a\|_{L^{l_1}}\,. \tag{4.1}$$

On $J_2$, by the vanishing condition of $a(x)$ we get

$$J_2 \leq \int_B |a(y)| \sum_{j=1}^{\infty} \left( \int_{2^j d \leq |x| < 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}} \right|^{l_2} |x|^{nb_0 l_2} dx \right)^{1/l_2} dy\,. \tag{4.2}$$

Applying Hölder's inequality and (3.4), we get

$$\begin{aligned}&\left( \int_{2^j d \leq |x| < 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}} \right|^{l_2} |x|^{nb_0 l_2} dx \right)^{1/l_2} \\ &\quad\leq \left( \int_{2^j d \leq |x| < 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}} \right|^{r} dx \right)^{1/r} \\ &\quad\quad\times \left( \int_{2^j d \leq |x| < 2^{j+1} d} |x|^{nb_0 l_2 \cdot (r/l_2)'} dx \right)^{1/[l_2 (r/l_2)']} \\ &\quad\leq C(2^j d)^{n/r-(n-\alpha)} \left\{ \frac{1}{2^j} + \int_{|y|/2^{j+1}d}^{|y|/2^j d} \frac{\omega_r(\delta)}{\delta} d\delta \right\} \cdot (2^j d)^{nb_0} (2^j d)^{n(1/l_2 - 1/r)} \\ &\quad\leq C(2^j d)^{n\varepsilon+\alpha} \left\{ \frac{1}{2^j} + \int_{|y|/2^{j+1}d}^{|y|/2^j d} \frac{\omega_r(\delta)}{\delta} d\delta \right\} \\ &\quad\leq C|B|^{\varepsilon+\alpha/n} \left\{ 2^{j(n\varepsilon+\alpha-1)} + 2^{j(n\varepsilon+\alpha-\beta)} \int_{|y|/2^{j+1}d}^{|y|/2^j d} \frac{\omega_r(\delta)}{\delta^{1+\beta}} d\delta \right\}\,.\end{aligned}$$

Since $\varepsilon < (\beta - \alpha)/n \le (1-\alpha)/n$, we have $n\varepsilon + \alpha - 1 \le n\varepsilon + \alpha - \beta < 0$. Thus, by the inequality above, (4.2) and (1.2) we have

$$\begin{aligned} J_2 &\le C|B|^{\varepsilon+\alpha/n} \sum_{j=1}^{\infty} \left\{ 2^{j(n\varepsilon+\alpha-1)} + 2^{j(n\varepsilon+\alpha-\beta)} \int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\beta}} d\delta \right\} \int_B |a(y)| dy \\ &\le C|B|^{\varepsilon+\alpha/n} \left( 1 + \int_0^1 \frac{\omega_r(\delta)}{\delta^{1+\beta}} d\delta \right) \|a\|_{L^{l_1}} |B|^{1/l_1'} \qquad (4.3) \\ &= C|B|^{b_0} \|a\|_{L^{l_1}} . \end{aligned}$$

By (4.1) and (4.3) we know that (i) holds and

$$\begin{aligned} \mathcal{N}_{l_2}(T_{\Omega,\alpha}a) &= \|T_{\Omega,\alpha}a\|_{L^{l_2}}^{a_0/b_0} \| | \cdot |^{nb_0} (T_{\Omega,\alpha}a)(\cdot) \|_{L^{l_2}}^{1-a_0/b_0} \\ &\le C\|a\|_{L^{l_1}}^{a_0/b_0} \cdot |B|^{b_0(1-a_0/b_0)} \|a\|_{L^{l_1}}^{1-a_0/b_0} \\ &\le C\|a\|_{L^{l_1}} |B|^{b_0-a_0} \le C . \end{aligned}$$

From the process of the proof above, it is easy to check that the constant $C$ is independent of the atom $a(x)$. Hence, it remains to verify (iii) to complete the proof of Theorem 3.

To this end, we first show that $(T_{\Omega,\alpha}a)(x) \in L^1(\boldsymbol{R}^n)$. Write

$$\int_{\boldsymbol{R}^n} |(T_{\Omega,\alpha}a)(x)| dx = \int_{|x|<1} |(T_{\Omega,\alpha}a)(x)| dx + \int_{|x|\ge 1} |(T_{\Omega,\alpha}a)(x)| dx := E_1 + E_2 .$$

Clearly, $E_1 \le C$ by $T_{\Omega,\alpha}a(x) \in L^{l_2}$. On the other hand, by $b_0 - 1/l_2' = \varepsilon > 0$ and $|x|^{nb_0}(T_{\Omega,\alpha}a)(x) \in L^{l_2}$, we have

$$E_2 \le \| | \cdot |^{nb_0} (T_{\Omega,\alpha}a)(\cdot) \|_{L^{l_2}} \cdot \left( \int_{|x|\ge 1} |x|^{(-nb_0)l_2'} dx \right)^{1/l_2'} < \infty .$$

Therefore, $(T_{\Omega,\alpha}a)\hat{}(\xi) \in C(\boldsymbol{R}^n)$. In order to verify

$$\int (T_{\Omega,\alpha}a)(x) dx = (T_{\Omega,\alpha}a)\hat{}(0) = 0 ,$$

it is sufficient to prove

$$\lim_{|\xi|\to 0} (T_{\Omega,\alpha}a)\hat{}(\xi) = 0 . \qquad (4.4)$$

We know that $(T_{\Omega,\alpha}a)\hat{}(\xi) = \hat{a}(\xi) \cdot (\Omega(\cdot)/|\cdot|^{n-\alpha})\hat{}(\xi)$, and

$$\left( \frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}} \right)^{\widehat{}} (\xi) = \int_{|x|<1} \frac{\Omega(x)}{|x|^{n-\alpha}} e^{-2\pi i \xi \cdot x} dx + \sum_{j=1}^{\infty} \int_{2^{j-1}\le|x|<2^j} \frac{\Omega(x)}{|x|^{n-\alpha}} e^{-2\pi i \xi\cdot x} dx .$$

Thus,

$$\left| \left( \frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}} \right)^{\widehat{}} (\xi) \right| \le C + \sum_{j=1}^{\infty} |\widehat{K_j}(\xi)| , \quad \text{where} \quad K_j(x) = \frac{\Omega(x)}{|x|^{n-\alpha}} \chi_{[2^{j-1},2^j)}(|x|) .$$

Below we give the estimate of $|\widehat{K_j}(\xi)|$ for any $j \ge 1$.

LEMMA 5. *Suppose that* $0 < \alpha < 1/2$, *and* $\Omega \in L^r(S^{n-1})$ *with* $r > 1/(1-2\alpha)$ *is homogeneous of degree zero on* $\boldsymbol{R}^n$. *Then there are* $C$ *and* $\sigma > 0$, *such that* $2\alpha < \sigma < 1/r' \leq 1$ *and for* $j \geq 1$

$$|\widehat{K_j}(\xi)| \leq C2^{(\alpha-\sigma/2)j}|\xi|^{-\sigma/2}.$$

PROOF. Denote

$$|\widehat{K_j}(\xi)| = \left|\int_{2^{j-1}\leq|x|<2^j} \frac{\Omega(x)}{|x|^{n-\alpha}} e^{-2\pi i\xi\cdot x}dx\right| = \left|\int_{2^{j-1}}^{2^j} t^\alpha I_t(\xi)\frac{dt}{t}\right|, \tag{4.5}$$

where

$$I_t(\xi) = \int_{S^{n-1}} e^{-2\pi i\xi\cdot t\theta}\Omega(\theta)d\theta.$$

From [6], we have

$$\left|\int_{2^{j-1}}^{2^j} e^{-2\pi it\xi\cdot(\theta-\phi)}\frac{dt}{t}\right| \leq C\min\{1, |2^j\xi\cdot(\theta-\phi)|^{-1}\}. \tag{4.6}$$

On the other hand, by the conditions of Lemma 5, we get $2\alpha < 1/r'$. Hence we can choose $\sigma > 0$ so that $2\alpha < \sigma < 1/r' \leq 1$. Using the interpolation method, from (4.6) we obtain

$$\left|\int_{2^{j-1}}^{2^j} e^{-2\pi it\xi\cdot(\theta-\phi)}\frac{dt}{t}\right| \leq C|2^j\xi\cdot(\theta-\phi)|^{-\sigma} = C|2^j\xi|^{-\sigma}\cdot|\xi'(\theta-\phi)|^{-\sigma}, \tag{4.7}$$

where $\xi' = \xi/|\xi|$. Thus, by (4.5) we get

$$|\widehat{K_j}(\xi)|^2 \leq \left(\int_{2^{j-1}}^{2^j} t^{2\alpha}\frac{dt}{t}\right)\left(\int_{2^{j-1}}^{2^j} |I_t(\xi)|^2\frac{dt}{t}\right) \leq C2^{2\alpha j}\left(\int_{2^{j-1}}^{2^j} |I_t(\xi)|^2\frac{dt}{t}\right).$$

Since

$$|I_t(\xi)|^2 = \iint_{S^{n-1}\times S^{n-1}} \Omega(\theta)\overline{\Omega(\phi)}e^{-2\pi it\xi\cdot(\theta-\phi)}d\theta d\phi,$$

by (4.7) and the conclusion in [6] (noting that $\Omega \in L^r(S^{n-1})$ and $\sigma r' < 1$), we get

$$\begin{aligned}|\widehat{K_j}(\xi)| &\leq C2^{\alpha j}\left(\iint_{S^{n-1}\times S^{n-1}} |\Omega(\theta)\overline{\Omega(\phi)}|\,|\xi'(\theta-\phi)|^{-\alpha}d\theta d\phi\right)^{1/2}|2^j\xi|^{-\sigma/2}\\ &\leq C2^{\alpha j}|2^j\xi|^{-\sigma/2}.\end{aligned}$$

Thus, we complete the proof of Lemma 5.

Now let us return to the proof of Theorem 3. Applying the conclusion of Lemma 5, we have

$$\left|\left(\frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}}\right)^{\widehat{}}(\xi)\right| \leq C+\sum_{j=1}^\infty|\widehat{K_j}(\xi)| \leq C+C\sum_{j=1}^\infty 2^{(\alpha-\sigma/2)j}|\xi|^{-\sigma/2} \leq C(1+|\xi|^{-\sigma/2}). \tag{4.8}$$

On the other hand, for $\hat{a}(\xi)$ we have

$$\left|\int_{\boldsymbol{R}^n} a(x)e^{-2\pi i\xi\cdot x}dx\right| = \left|\int_B a(x)[e^{-2\pi i\xi\cdot x}-1]dx\right| \leq C\int_B |a(x)|\,|\xi|\,|x|dx \leq C|\xi|. \tag{4.9}$$

Thus, by (4.8) and (4.9) we get

$$|(T_{\Omega,\alpha}a)\hat{}(\xi)| \leq |\hat{a}(\xi)| \cdot \left| \left( \frac{\Omega(\cdot)}{|\cdot|^{n-\alpha}} \right)^{\widehat{}} (\xi) \right| \leq C(|\xi| + |\xi|^{1-\sigma/2}). \tag{4.10}$$

By the choice of $\sigma$ we know that $1 - \sigma/2 > 0$. Thus, (4.4) holds by (4.10). Hence $(T_{\Omega,\alpha}a)(x)$ satisfies the condition (iii) and Theorem 3 follows.

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