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ON θ-STABLE BOREL SUBALGEBRAS OF LARGE TYPE FOR REAL REDUCTIVE GROUPS

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Abstract. Vogan-Zuckerman's standard representation X for a real reductive group $G(\boldsymbol{R})$ is constructed from a θ-stable parabolic subalgebra $\mathfrak{q}$ of the complexified Lie algebra $\mathfrak{g}$ of $G(\boldsymbol{R})$. Adams and Vogan showed that the set of $\mathfrak{g}$-principal K-orbits in the associated variety $\mathrm{Ass}(X)$ of X is in one-to-one correspondence with the set $\mathcal{B}^L_{\mathfrak{q}^-}/K$ of K-conjugacy classes of θ-stable Borel subalgebras of large type having representatives in the opposite parabolic subalgebra $\mathfrak{q}^-$ of $\mathfrak{q}$. In this paper, we give a description of $\mathcal{B}^L_{\mathfrak{q}}/K$ and show that $\mathcal{B}^L_{\mathfrak{q}}/K \neq \emptyset$ under certain condition on the positive system of imaginary roots contained in $\mathfrak{q}$. Furthermore, we construct a finite group which acts on $\mathcal{B}^L_{\mathfrak{q}}/K$ transitively.

0. Introduction. Let G be a complex connected reductive algebraic group and $\tau : G \to G$ a complex conjugation which defines a real form $G(\boldsymbol{R})$ of G. Let $\theta : G \to G$ be a (complexified) Cartan involution of G which commutes with τ. Write $K = \{g \in G; \theta(g) = g\}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ the Cartan decomposition with respect to θ. For a closed subgroup H of G, we denote its Lie algebra by the corresponding small German letter $\mathfrak{h}$ and its group of real points by $H(\boldsymbol{R})$. Let $Q = LU$ be a θ-stable parabolic subgroup of G with θ-stable and τ-stable quasisplit Levi factor L, and with unipotent radical U. Let H be a θ-stable and τ-stable maximal torus of L such that $H(\boldsymbol{R})$ is a maximally split Cartan subgroup of $L(\boldsymbol{R})$. Then for a quadruplet $(\mathfrak{q}, H(\boldsymbol{R}), \delta, \nu)$ of θ-stable data for $G(\boldsymbol{R})$ and a minimal parabolic subgroup $P_L(\boldsymbol{R})$ of $L(\boldsymbol{R})$, the standard $(\mathfrak{g}, K)$-module

$$X = X_{G(\boldsymbol{R})}(\mathfrak{q}, H(\boldsymbol{R}), \delta, \nu) = \left(\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}}\right)^{\dim(\mathfrak{u}\cap\mathfrak{k})}\left(\mathrm{Ind}^{L(\boldsymbol{R})}_{P_L(\boldsymbol{R})}(\delta \otimes \nu)\right)$$

is defined in [V], where we write $\mathrm{Ind}^{L(\boldsymbol{R})}_{P_L(\boldsymbol{R})}$ the parabolic induction by a real parabolic subgroup and $(\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}})^{\dim(\mathfrak{u}\cap\mathfrak{k})}$ the cohomological parabolic induction by a θ-stable parabolic subalgebra.

In [AV], Adams and Vogan described the set $\mathrm{Ass}(X)^{\mathfrak{g}-\mathrm{pr}}/K$ of $\mathfrak{g}$-principal K-orbits in the associated variety of the standard $(\mathfrak{g}, K)$-module X for a quasisplit group $G(\boldsymbol{R})$. $\mathrm{Ass}(X)^{\mathfrak{g}-\mathrm{pr}}/K$ is parametrized by the set $\mathcal{B}^L_{\mathfrak{q}^-}/K$ of K-conjugacy classes of θ-stable Borel subalgebras of large type (cf. Definition 1.1(ii)) which have representatives contained in $\mathfrak{q}^-$, where $\mathfrak{q}^-$ is the opposite parabolic subalgebra of $\mathfrak{q}$.

In this paper, corresponding to each $\mathfrak{l}$-principal nilpotent $L \cap K$-orbit $\mathcal{O}_{\mathfrak{l}}$ in $\mathfrak{l} \cap \mathfrak{s}$, we construct a θ-stable Borel subalgebra $\mathfrak{b}(\mathcal{O}_{\mathfrak{l}})$ of $\mathfrak{g}$ contained in $\mathfrak{q}$. If the positive system Σ of imaginary roots of $\mathfrak{h}$ contained in $\mathfrak{u}$ is of large type (cf. Definition 1.1(i)), we show that $\mathfrak{b}(\mathcal{O}_{\mathfrak{l}})$ is of large type for a suitable choice of $\mathcal{O}_{\mathfrak{l}}$. This implies the converse of a result of Adams and

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Vogan ([AV, Proposition 6.30(a)]) which claims that if Σ is of large type, Σ is special with respect to some K-conjugacy class of θ-stable Borel subalgebras of $\mathfrak{g}$ of large type. Thus we obtain that $\mathcal{B}^L_{\mathfrak{q}}/K \neq \emptyset$ if and only if Σ is of large type. Furthermore, we find a finite group $F_L(\mathfrak{q})$ which acts on $\mathcal{B}^L_{\mathfrak{q}}/K$ transitively. The construction of $\mathfrak{b}(\mathcal{O}_{\mathfrak{l}})$ also gives an algorithm to obtain $\mathcal{B}^L_{\mathfrak{q}}/K$.

Finally we mention the relation between the closure of the K-orbit of $\mathfrak{b} := \mathfrak{p}_L + \mathfrak{u}$ in the flag variety and the list $\mathcal{L} := \{\mathfrak{b}(\mathcal{O}_{\mathfrak{l}})\}$ of θ-stable Borel subalgebras of $\mathfrak{g}$. Let $\mathcal{B}_{\mathfrak{g}}$ be the set of Borel subalgebras of $\mathfrak{g}$ (flag variety) and $\pi : T^*\mathcal{B}_{\mathfrak{g}} \to \mathfrak{g}^*$ the moment map. By the identification $\mathfrak{g}^* \simeq \mathfrak{g}$ which is given by a G-invariant bilinear form on $\mathfrak{g}$, we can regard π as $\pi : T^*\mathcal{B}_{\mathfrak{g}} \to \mathfrak{g}$. Let Y_i $(i = 1, \dots, n)$ be the K-orbits contained in the closure $\overline{K \cdot \mathfrak{b}} \subset \mathcal{B}_{\mathfrak{g}}$ and Z_j $(j = 1, \dots, m)$ the K-orbits in $\mathcal{B}_{\mathfrak{g}}$ generated by the Borel subalgebras in $\mathcal{L}$. Then Z_j $(j = 1, \dots, m)$ are closed orbits contained in $\overline{K \cdot \mathfrak{b}}$ and we have

$$\pi\left(\bigcup_{i=1}^{n} T^*_{Y_i}\mathcal{B}_{\mathfrak{g}}\right) = \bigcup_{j=1}^{m} \overline{\pi(T^*_{Z_j}\mathcal{B}_{\mathfrak{g}})} = \bigcup_{\mathcal{O} \in \mathrm{Ind}^{\theta}((\mathfrak{l},\mathfrak{q})\uparrow\mathfrak{g})(\mathcal{N}^{\mathfrak{l}-\mathrm{pr}}_{\mathfrak{l}\cap\mathfrak{s}}/L\cap K)} \bar{\mathcal{O}}$$

(for the definition of $\mathrm{Ind}^{\theta}((\mathfrak{l},\mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{N}^{\mathfrak{l}-\mathrm{pr}}_{\mathfrak{l}\cap\mathfrak{s}}/L \cap K)$, see 1.4). The proof of this result will be given in a forthcoming paper.

1. θ-stable Borel subalgebras of large type. In this section, we review the basic facts on θ-stable Borel subalgebras of large type. We also recall the relation among the K-conjugacy classes of θ-stable Borel subalgebras of large type which have representatives contained in a θ-stable parabolic subalgebra $\mathfrak{q}$, $\mathfrak{g}$-principal nilpotent K-orbits in $\mathfrak{s}$ which have representatives contained in $\mathfrak{q}$ and the induction of nilpotent orbits by $\mathfrak{q}$.

1.1. Preliminaries. Let G be a complex reductive algebraic group defined over $\boldsymbol{R}$ and $\tau : G \to G$ a complex conjugation which defines the real form $G(\boldsymbol{R}) = \{g \in G; \tau(g) = g\}$ of G. Let $\theta : G \to G$ be a (complexified) Cartan involution of G which commutes with τ. Throughout this paper, we use the following notation. For a closed subgroup of G, its Lie algebra is denoted by the corresponding small German letter. The involution of $\mathfrak{g}$, which is induced from τ (resp. θ), is also denoted by τ (resp. θ). Write $K := \{g \in G; \theta(g) = g\}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ the Cartan decomposition with respect to θ. The action of G on $\mathfrak{g}$ is always understood to be the adjoint action. For a τ-stable subset A of G (resp. $\mathfrak{g}$), we write $A(\boldsymbol{R}) = \{x \in A; \tau(x) = x\}$. For a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we denote by $R(\mathfrak{g}, \mathfrak{h})$ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $\mathfrak{g}_\alpha$ the root space corresponding to a root $\alpha \in R(\mathfrak{g}, \mathfrak{h})$. For a $\mathfrak{h}$-stable subspace $V \subset \mathfrak{g}$, we write $R(V, \mathfrak{h}) := \{\alpha \in R(\mathfrak{g}, \mathfrak{h}); \mathfrak{g}_\alpha \subset V\}$. If $\mathfrak{h}$ is θ-stable, we write $R(V, \mathfrak{h})_{i\boldsymbol{R}}$ (resp. $R(V, \mathfrak{h})_{\boldsymbol{R}}$) the set of imaginary (resp. real) roots in $R(V, \mathfrak{h})$:

$$R(V, \mathfrak{h})_{i\boldsymbol{R}} = \{\alpha \in R(V, \mathfrak{h}); \theta(\alpha) = \alpha\}, \quad R(V, \mathfrak{h})_{\boldsymbol{R}} = \{\alpha \in R(V, \mathfrak{h}); \theta(\alpha) = -\alpha\}$$

The set of all nilpotent elements in $\mathfrak{g}$ (resp. $\mathfrak{s}$, $\mathfrak{g}(\boldsymbol{R})$) is denoted by $\mathcal{N}_{\mathfrak{g}}$ (resp. $\mathcal{N}_{\mathfrak{s}}$, $\mathcal{N}_{\mathfrak{g}(\boldsymbol{R})}$). The set of orbits in $\mathcal{N}_{\mathfrak{g}}$ (resp. $\mathcal{N}_{\mathfrak{s}}$, $\mathcal{N}_{\mathfrak{g}(\boldsymbol{R})}$) under the action of G (resp. K, $G(\boldsymbol{R})$) is denoted by $\mathcal{N}_{\mathfrak{g}}/G$ (resp. $\mathcal{N}_{\mathfrak{s}}/K, \mathcal{N}_{\mathfrak{g}(\boldsymbol{R})}/G(\boldsymbol{R})$).

1.2. Basic facts on θ-stable Borel subalgebras of large type. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ and Σ a positive system of $R(\mathfrak{g}, \mathfrak{h})_{i\boldsymbol{R}}$. We say that a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ belongs to $(\mathfrak{h}, \Sigma)$ if the following conditions are satisfied:

(1.1.a) $\mathfrak{h} \subset \mathfrak{q}$

(1.1.b) Write $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ the Levi decomposition of $\mathfrak{q}$ such that $\mathfrak{h} \subset \mathfrak{l}$. Then $\mathfrak{h}$ is a maximally split Cartan subalgebra of $\mathfrak{l}$.

(1.1.c) $\Sigma = R(\mathfrak{u}, \mathfrak{h})_{i\boldsymbol{R}}$ (hence $\mathfrak{l}$ is quasisplit).

DEFINITION 1.1 ([AV]). (i) Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. A positive system Σ of $R(\mathfrak{g}, \mathfrak{h})_{i\boldsymbol{R}}$ is called of large type if every simple root α of Σ is non-compact (i.e., $\mathfrak{g}_\alpha \subset \mathfrak{s}$).

(ii) A θ-stable Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is called of large type if every simple root of $R(\mathfrak{b}, \mathfrak{h})$ is complex or non-compact imaginary for any θ-stable Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. We write $\mathcal{B}^L_{\mathfrak{g}}$ the set of θ-stable Borel subalgebras of $\mathfrak{g}$ of large type.

(iii) For a K-conjugacy class $\mathcal{B} \in \mathcal{B}^L_{\mathfrak{g}}/K$, a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is called special (with respect to $\mathcal{B}$) if there exists $\mathfrak{b} \in \mathcal{B}$ such that $\mathfrak{b} \subset \mathfrak{q}$.

(iv) Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{l}$ the Levi subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ such that $R(\mathfrak{l}, \mathfrak{h}) = R(\mathfrak{g}, \mathfrak{h})_{\boldsymbol{R}}$. A positive system Σ of $R(\mathfrak{g}, \mathfrak{h})_{i\boldsymbol{R}}$ is called special (with respect to $\mathcal{B}$) if there exists a special θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ such that $\Sigma = R(\mathfrak{u}, \mathfrak{h})_{i\boldsymbol{R}}$.

REMARK 1.2. (i) For a θ-stable Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, $\mathfrak{b}$ is of large type if and only if there exists a θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{b}$ such that every simple root of $R(\mathfrak{b}, \mathfrak{h})$ is complex or non-compact imaginary, since any θ-stable Cartan subalgebras in $\mathfrak{b}$ are conjugate under the action of $B \cap K$.

(ii) In the setting of Definition 1.1(iv), we can verify that Σ is special with respect to $\mathcal{B}(\in \mathcal{B}^L_{\mathfrak{g}}/K)$ if and only if there exists a special θ-stable parabolic subalgebra $\mathfrak{q}'$ of $\mathfrak{g}$ which belongs to $(\mathfrak{h}, \Sigma)$ (cf. Proof of Proposition 2.8(ii)).

PROPOSITION 1.3. (i) ([AV, Proposition 6.30(a)]) *Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ and Σ a positive system of $R(\mathfrak{g}, \mathfrak{h})_{i\boldsymbol{R}}$. If Σ is special (with respect to some $\mathcal{B} \in \mathcal{B}^L_{\mathfrak{g}}/K$), Σ is of large type.*

(ii) ([AV, Proposition 6.25]) *Let $\mathfrak{b}$ be a θ-stable Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{t} \subset \mathfrak{b}$ a θ-stable Cartan subalgebra (such $\mathfrak{t}$'s are all conjugate under the action of $B \cap K$). Then the positive system $R(\mathfrak{b}, \mathfrak{h})_{i\boldsymbol{R}}$ is of large type if and only if $\mathfrak{b}$ is of large type.*

A nilpotent element $X \in \mathfrak{g}$ is called $\mathfrak{g}$-principal if X is regular in $\mathfrak{g}$. We write $\mathcal{N}^{\mathfrak{g}-\mathrm{pr}}_{\mathfrak{g}}$ (resp. $\mathcal{N}^{\mathfrak{g}-\mathrm{pr}}_{\mathfrak{s}}$, $\mathcal{N}^{\mathfrak{g}-\mathrm{pr}}_{\mathfrak{g}(\boldsymbol{R})}$) the set of $\mathfrak{g}$-principal elements in $\mathcal{N}_{\mathfrak{g}}$ (resp. $\mathcal{N}_{\mathfrak{s}}$, $\mathcal{N}_{\mathfrak{g}(\boldsymbol{R})}$).

PROPOSITION 1.4 ([AV, Proposition 6.24]). *The following conditions on θ are equivalent.*

(a) *$\mathfrak{g}$ is quasisplit (i.e., there exists a Borel subalgebra of $\mathfrak{g}$ defined over $\boldsymbol{R}$).*

(b) $\mathcal{N}^{\mathfrak{g}-\mathrm{pr}}_{\mathfrak{s}} \neq \emptyset$.

(c) $\mathcal{B}^L_{\mathfrak{g}} \neq \emptyset$.

(d) *For any θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $R(\mathfrak{g}, \mathfrak{h})_{iR}$ has a positive system of large type.*

(d′) *There exists a θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $R(\mathfrak{g}, \mathfrak{h})_{iR}$ has a positive system of large type.*

1.3. The correspondence $[\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K]_{\mathfrak{q}} \simeq \mathcal{B}_{\mathfrak{q}}^{L}/K$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{s} \cap [\mathfrak{g}, \mathfrak{g}]$ and define a finite group F_G by

$$F_G := \{a \in \exp(\mathfrak{a}); \mathrm{Ad}(a^2) = \mathrm{id}\} .$$

REMARK 1.5. F_G normalizes K and satisfies $\mathrm{Ad}(N_G(\mathfrak{k})) = \mathrm{Ad}(N_G(\mathfrak{s})) = \mathrm{Ad}(F_G K)$ (cf. [O2]).

The natural correspondence between $\mathcal{B}_{\mathfrak{g}}^{L}/K$ and the set $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K$ of K-orbits in $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}$ is given as follows. For $x \in \mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}$, we can take a normal S-triple (h, x, y) $(h \in \mathfrak{k}, y \in \mathfrak{s})$ (cf. Kostant and Rallis [KR]). Since h is a regular semisimple element of $\mathfrak{g}$, $\mathfrak{t} := \mathfrak{z}_{\mathfrak{g}}(h)$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$. Define a Borel subalgebra $\mathfrak{b} \supset \mathfrak{t}$ of $\mathfrak{g}$ by $R(\mathfrak{b}, \mathfrak{t}) = \{\alpha \in R(\mathfrak{g}, \mathfrak{t}); \alpha(h) > 0\}$ and write Δ the set of simple roots in $R(\mathfrak{b}, \mathfrak{t})$. Then we have $\Delta = \{\alpha \in R(\mathfrak{g}, \mathfrak{t}); \alpha(h) = 2\}$. Since $[h, x] = 2x$, x can be written as a sum

$$x = \sum\nolimits_{\alpha \in \Delta} X_\alpha$$

for some root vectors $X_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ and it holds that $\theta(X_\alpha) = -X_{\theta(\alpha)}$ $(\alpha \in \Delta)$. Hence any roots in Δ are complex or non-compact imaginary, so that $\mathfrak{b}$ is of large type. Since $x \in \mathfrak{b}$ and x is $\mathfrak{g}$-principal, $\mathfrak{b}$ is the unique Borel subalgebra containing x. Then the correspondence $x \mapsto \mathfrak{b}$ defines a map

$$\varphi : \mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K \to \mathcal{B}_{\mathfrak{g}}^{L}/K .$$

PROPOSITION 1.6 ([AV, Proposition A.7]). *The map $\varphi : \mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K \to \mathcal{B}_{\mathfrak{g}}^{L}/K$ is a bijection. Furthermore the finite group F_G acts naturally and transitively on the sets $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K$ and $\mathcal{B}_{\mathfrak{g}}^{L}/K$, where the map φ is F_G-equivariant.*

Let $Q = LU$ be a θ-stable parabolic subgroup of G with θ-stable Levi factor L. For a maximal abelian subspace $\mathfrak{a}_L$ of $[\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{s}$, we write

$$F_L = \{a \in \exp(\mathfrak{a}_L); \mathrm{Ad}(a^2)\,|_{\mathfrak{l}} = \mathrm{id}_{\mathfrak{l}}\} , \quad F_L^G = \{a \in F_L; \mathrm{Ad}(a^2) = \mathrm{id}_{\mathfrak{g}}\} .$$

We put

$$\mathcal{B}_{\mathfrak{q}}^{L} := \{\mathfrak{b} \in \mathcal{B}_{\mathfrak{g}}^{L}; k\mathfrak{b} \subset \mathfrak{q} \text{ for some } k \in K\} , \quad [\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K]_{\mathfrak{q}} := \{\mathcal{O} \in \mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K; \mathcal{O} \cap \mathfrak{q} \neq \emptyset\} .$$

Then we have the following

PROPOSITION 1.7. *The map φ in Proposition 1.6 induces a bijection*

$$[\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K]_{\mathfrak{q}} \simeq \mathcal{B}_{\mathfrak{q}}^{L}/K .$$

F_L^G acts on these sets and the bijection is F_L^G-equivariant.

1.4. Induction of nilpotent orbits by θ-stable parabolic subalgebras. Let $Q = LU$ be θ-stable parabolic subgroup of G with θ-stable Levi factor L and unipotent radical U, and write $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ its Lie algebra. We put

$$K_L := L \cap K, \quad \mathfrak{s}_L := \mathfrak{l} \cap \mathfrak{s}.$$

DEFINITION 1.8. (i) For a nilpotent K_L-orbit $\mathcal{O} \in \mathcal{N}_{\mathfrak{s}_L}/K_L$, there exists a unique nilpotent K-orbit $\tilde{\mathcal{O}}$ in $\mathcal{N}_{\mathfrak{s}}$ such that $(\mathcal{O} + \mathfrak{u} \cap \mathfrak{s}) \cap \tilde{\mathcal{O}}$ is open and dense in $\mathcal{O} + \mathfrak{u} \cap \mathfrak{s}$. We write

$$\tilde{\mathcal{O}} = \mathrm{Ind}^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{O}) \in \mathcal{N}_{\mathfrak{s}}/K.$$

(ii) For a subset $S \in 2^{\mathcal{N}_{\mathfrak{s}_L}/K_L}$ of $\mathcal{N}_{\mathfrak{s}_L}/K_L$, we write $\mathrm{Ind}^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(S)$ the set of orbits in $\{\mathrm{Ind}^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{C}); \mathcal{C} \in S\}$ which are maximal with respect to the closure relation. This defines a map

$$\mathrm{Ind}^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g}) : 2^{\mathcal{N}_{\mathfrak{s}_L}/K_L} \to 2^{\mathcal{N}_{\mathfrak{s}}/K}.$$

PROPOSITION 1.9 ([O2, Proposition 2.4]). *Let $Q = LU$ be a θ-stable parabolic subgroup of G with θ-stable Levi factor L and unipotent radical U. Suppose that L is quasi-split. Then the set* $[\mathrm{Ind}^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{N}_{\mathfrak{s}_L}^{\mathfrak{l}-\mathrm{pr}}/K_L)]^{\mathfrak{g}-\mathrm{pr}}$ *of $\mathfrak{g}$-principal K-orbits in* $\mathrm{Ind}^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{N}_{\mathfrak{s}_L}^{\mathfrak{l}-\mathrm{pr}}/K_L)$ *can be written as*

$$[\mathrm{Ind}^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{N}_{\mathfrak{s}_L}^{\mathfrak{l}-\mathrm{pr}}/K_L)]^{\mathfrak{g}-\mathrm{pr}} = [\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K]_{\mathfrak{q}}.$$

2. A construction of a θ-stable Borel subalgebra of large type. Throughout this section, we write H a τ-stable and θ-stable maximal torus of G, Σ a positive system of $R(\mathfrak{g}, \mathfrak{h})_{i\mathbf{R}}$ and $Q = LU$ a θ-stable parabolic subgroup of G with θ-stable Levi factor L and unipotent radical U of which Lie algebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ belongs to $(\mathfrak{h}, \Sigma)$. In this section, for a nilpotent orbit $\mathcal{O}_{\mathfrak{l}} \in \mathcal{N}_{\mathfrak{s}_L}^{\mathfrak{l}-\mathrm{pr}}/K_L$, we construct a θ-stable Borel subalgebra $\mathfrak{b}(\mathcal{O}_{\mathfrak{l}})$ of $\mathfrak{g}$ contained in $\mathfrak{q}$. If Σ is of large type, we will show that $\mathfrak{b}(\mathcal{O}_{\mathfrak{l}})$ is of large type for a suitable choice of $\mathcal{O}_{\mathfrak{l}}$. This implies the converse of Proposition 1.4(i) which says that if Σ is of large type, Σ is special with respect to some $\mathcal{B} \in \mathcal{B}_{\mathfrak{g}}^L/K$. Furthermore, we find a subgroup $F_L(\mathfrak{q})$ of F_L which can act on $\mathcal{B}_{\mathfrak{q}}^L/K$ transitively. We see that $F_L(\mathfrak{q}) = F_L^G$ if θ is of inner type, and hence conclude that the action of F_L^G on $\mathcal{B}_{\mathfrak{q}}^L/K \simeq [\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K]_{\mathfrak{q}}$ is transitive.

Let $(\mathfrak{q}, H(\mathbf{R}), \delta, \nu)$ be a quadruplet of θ-stable data and $X = X_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta, \nu)$ the corresponding standard $(\mathfrak{g}, K)$-module. Suppose that Σ is of large type. Then, via the identification $\mathfrak{g} \simeq \mathfrak{g}^*$ by a G-invariant bilinear form on $\mathfrak{g}$, the set $\mathrm{Ass}(X)^{\mathfrak{g}-\mathrm{pr}}/K$ of $\mathfrak{g}$-principal K-orbits contained in the associated variety $\mathrm{Ass}(X)$ of X is described in [AV] and coincides with $[\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K]_{\mathfrak{q}^-}$:

$$\mathrm{Ass}(X)^{\mathfrak{g}-\mathrm{pr}}/K = [\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K]_{\mathfrak{q}^-} \simeq \mathcal{B}_{\mathfrak{q}^-}^L/K,$$

where we write $\mathfrak{q}^-$ the opposite parabolic subalgebra of $\mathfrak{q}$([O2]). Hence the construction of $\mathfrak{b}(\mathcal{O}_{\mathfrak{l}})$ gives an algorithm to obtain $\mathrm{Ass}(X)^{\mathfrak{g}-\mathrm{pr}}/K$ and $F_L(\mathfrak{q}) = F_L(\mathfrak{q}^-)$ is a finite group which acts on $\mathrm{Ass}(X)^{\mathfrak{g}-\mathrm{pr}}/K$ transitively.

2.1. Certain Cayley transforms associated with $\mathfrak{l}$-principal K_L-orbits in $\mathfrak{s}_L$. Let us first recall the Sekiguchi correspondence.

THEOREM 2.1 (Sekiguchi [S], see also [O1]). *For a nilpotent orbit $\mathcal{O}_\theta \in \mathcal{N}_{\mathfrak{s}}/K$, there exists an S-triple (h, x, y) of $\mathfrak{g}$ satisfying the following conditions*:

(2.1.a) $h \in \mathfrak{k},\ x, y \in \mathfrak{s}$,

(2.1.b) $\tau(h) = -h,\ \tau(x) = y$,

(2.1.c) $x \in \mathcal{O}_\theta$.

(*We call an S-triple satisfying* (2.1.a–b) *a strictly normal S-triple.*) *Such an S-triple* (h, x, y) *is unique up to conjugation of* $K(\boldsymbol{R})$.

For (h, x, y), *we write*

$$h_{\boldsymbol{R}} = i(x - y), \quad x_{\boldsymbol{R}} = (x + y + ih)/2, \quad y_{\boldsymbol{R}} = (x + y - ih)/2.$$

Then $(h_{\boldsymbol{R}}, x_{\boldsymbol{R}}, y_{\boldsymbol{R}})$ *is an S-triple of* $\mathfrak{g}(\boldsymbol{R})$ *such that*

$$\theta(h_{\boldsymbol{R}}) = -h_{\boldsymbol{R}}, \quad \theta(x_{\boldsymbol{R}}) = -y_{\boldsymbol{R}}.$$

Write $\mathcal{O}_{\boldsymbol{R}} := G(\boldsymbol{R})x_{\boldsymbol{R}} \in \mathcal{N}_{\mathfrak{g}(\boldsymbol{R})}/G(\boldsymbol{R})$ *the* $G(\boldsymbol{R})$*-orbit of* $x_{\boldsymbol{R}}$. *Then the correspondence* $\mathcal{O}_\theta \mapsto \mathcal{O}_{\boldsymbol{R}}$ *defines a bijection*

$$S_G : \mathcal{N}_{\mathfrak{s}}/K \xrightarrow{\sim} \mathcal{N}_{\mathfrak{g}(\boldsymbol{R})}/G(\boldsymbol{R}),$$

which we call the Sekiguchi correspondence.

Let $P_L = HN_L$ be a τ-stable Borel subgroup of L. Write $\mathfrak{b} := \mathfrak{p}_L + \mathfrak{u}$ the Borel subalgebra of $\mathfrak{g}$. Since $[\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{h} \cap \mathfrak{s}$ is a maximal abelian subspace of $[\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{s}$, we can take

$$F_L = \left\{a \in \exp([\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{h} \cap \mathfrak{s});\ \mathrm{Ad}(a^2)\,\big|_{\mathfrak{l}} = \mathrm{id}_{\mathfrak{l}}\right\}$$

as a finite group associated to L (cf. 1.3). By [S], we can take an S-triple $(h^0_{\boldsymbol{R}}, x^0_{\boldsymbol{R}}, y^0_{\boldsymbol{R}})$ of $\mathfrak{l}(\boldsymbol{R})$ satisfying the following conditions:

(2.2.a) $h^0_{\boldsymbol{R}} \in \mathfrak{h}$,

(2.2.b) $\theta(h^0_{\boldsymbol{R}}) = -h^0_{\boldsymbol{R}}, \theta(x^0_{\boldsymbol{R}}) = -y^0_{\boldsymbol{R}}$,

(2.2.c) $x^0_{\boldsymbol{R}} \in \mathfrak{n}_L(\boldsymbol{R}) \cap \mathcal{N}^{\mathfrak{l}-\mathrm{pr}}_{\mathfrak{l}(\boldsymbol{R})}$.

Since the action of F_L on $\mathcal{N}^{\mathfrak{l}-\mathrm{pr}}_{\mathfrak{l}(\boldsymbol{R})}/L(\boldsymbol{R})$ is transitive (cf. Proposition 1.6), the subset $\mathcal{C} := F_L x^0_{\boldsymbol{R}} \subset \mathfrak{n}_L(\boldsymbol{R})$ contains representatives of $\mathcal{N}^{\mathfrak{l}-\mathrm{pr}}_{\mathfrak{l}(\boldsymbol{R})}/L(\boldsymbol{R})$. Fix an element $x_{\boldsymbol{R}} := a x^0_{\boldsymbol{R}}$ ($a \in F_L$) and put

$$(h_{\boldsymbol{R}}, x_{\boldsymbol{R}}, y_{\boldsymbol{R}}) = (a h^0_{\boldsymbol{R}}, a x^0_{\boldsymbol{R}}, a y^0_{\boldsymbol{R}}).$$

Then the S-triple $(h_{\boldsymbol{R}}, x_{\boldsymbol{R}}, y_{\boldsymbol{R}})$ also satisfies the conditions (2.2.a–c). Let us write $\Delta(\mathfrak{p}_L, \mathfrak{h})$ the set of simple roots of $R(\mathfrak{p}_L, \mathfrak{h})$. We notice that $\alpha(h_{\boldsymbol{R}}) = 2$ for $\alpha \in \Delta(\mathfrak{p}_L, \mathfrak{h})$ and that $x_{\boldsymbol{R}}$ can be written as $x_{\boldsymbol{R}} = \sum_{\alpha \in \Delta(\mathfrak{p}_L, \mathfrak{h})} X_\alpha$, where X_α ($\alpha \in \Delta(\mathfrak{p}_L, \mathfrak{h})$) is a non-zero root vector in $\mathfrak{g}_\alpha$. Corresponding to the S-triple $(h_{\boldsymbol{R}}, x_{\boldsymbol{R}}, y_{\boldsymbol{R}})$, define elements $s_{x_{\boldsymbol{R}}}$ and $\sigma_{x_{\boldsymbol{R}}}$ of L by

$$s_{x_{\boldsymbol{R}}} := \exp(\pi i(x_{\boldsymbol{R}} + y_{\boldsymbol{R}})/4), \quad \sigma_{x_{\boldsymbol{R}}} := s^2_{x_{\boldsymbol{R}}}. \tag{2.3}$$

We also write $s = s_{x_{\boldsymbol{R}}}$ and $\sigma = \sigma_{x_{\boldsymbol{R}}}$ for short. Define an S-triple $(h_\theta, x_\theta, y_\theta)$ by

$$h_\theta := s_{x_{\boldsymbol{R}}} h_{\boldsymbol{R}}, \quad x_\theta := s_{x_{\boldsymbol{R}}} x_{\boldsymbol{R}}, \quad y_\theta := s_{x_{\boldsymbol{R}}} y_{\boldsymbol{R}}. \tag{2.4}$$

Then we have the following.

LEMMA 2.2. (i) *$h_\theta, x_\theta, y_\theta$ can be written as*

$$h_\theta = -i(x_{\boldsymbol{R}} - y_{\boldsymbol{R}}), \quad x_\theta = (x_{\boldsymbol{R}} + y_{\boldsymbol{R}} - ih_{\boldsymbol{R}})/2, \quad y_\theta = (x_{\boldsymbol{R}} + y_{\boldsymbol{R}} + ih_{\boldsymbol{R}})/2.$$

$(h_\theta, x_\theta, y_\theta)$ is a strictly normal S-triple of $\mathfrak{l}$.

(ii) *It holds that $h_{\boldsymbol{R}} = i(x_\theta - y_\theta)$, $x_{\boldsymbol{R}} = (x_\theta + y_\theta + ih_\theta)/2$, $y_{\boldsymbol{R}} = (x_\theta + y_\theta - ih_\theta)/2$. Therefore the K_L-orbit $K_L x_\theta \in \mathcal{N}_{\mathfrak{s}_L}^{\mathfrak{l}-\mathrm{pr}}/K_L$ corresponds to the $L(\boldsymbol{R})$-orbit $L(\boldsymbol{R})x_{\boldsymbol{R}}$ via the Sekiguchi correspondence $S_L : \mathcal{N}_{\mathfrak{s}_L}/K_L \xrightarrow{\sim} \mathcal{N}_{\mathfrak{l}(\boldsymbol{R})}/L(\boldsymbol{R})$.*

(iii) *$s^4 = \exp(\pi i h_{\boldsymbol{R}})$.*

(iv) *$\sigma h_{\boldsymbol{R}} = -h_{\boldsymbol{R}}$, $\sigma x_{\boldsymbol{R}} = y_{\boldsymbol{R}}$, $\sigma y_{\boldsymbol{R}} = x_{\boldsymbol{R}}$.*

PROOF. Let

$$(h_{\boldsymbol{R}})_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (x_{\boldsymbol{R}})_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (y_{\boldsymbol{R}})_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the canonical basis of $\mathfrak{sl}(2, \boldsymbol{C})$. Since $SL(2, \boldsymbol{C})$ is simply connected, the homomorphism $\phi : \mathfrak{sl}(2, \boldsymbol{C}) \to \mathfrak{g}$ defined by $(h_{\boldsymbol{R}})_1 \mapsto h_{\boldsymbol{R}}$, $(x_{\boldsymbol{R}})_1 \mapsto x_{\boldsymbol{R}}$, $(y_{\boldsymbol{R}})_1 \mapsto y_{\boldsymbol{R}}$ induces a homomorphism $\Phi : SL(2, \boldsymbol{C}) \to G$ which makes the following diagram commutative:

$$\begin{array}{ccccc} & \mathfrak{sl}(2, \boldsymbol{C}) & \xrightarrow{\phi} & \mathfrak{g} & \\ \exp & \downarrow & & \downarrow & \exp \\ & SL(2, \boldsymbol{C}) & \xrightarrow{\Phi} & G & \end{array}$$

Therefore, to show (i) and (iii), it is sufficient to show that the corresponding equalities hold in $\mathfrak{sl}(2, \boldsymbol{C})$ or $SL(2, \boldsymbol{C})$. It is easily verified that

$$s_1 := \exp(\pi i\{(x_{\boldsymbol{R}})_1 + (y_{\boldsymbol{R}})_1\}/4) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

which implies (i). The equality $s_1^4 = \exp(\pi i (h_{\boldsymbol{R}})_1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in $SL(2, \boldsymbol{C})$ implies (iii). (ii) and (iv) easily follow from (i). q.e.d.

LEMMA 2.3. (i) *$\sigma\mathfrak{h} = \mathfrak{h}$. Hence σ acts on $R(\mathfrak{g}, \mathfrak{h})$ as an element of the Weyl group $W_G = N_G(H)/H$ of G. Then we have $\sigma R(\mathfrak{p}_L, \mathfrak{h}) = -R(\mathfrak{p}_L, \mathfrak{h})$. In particular, the element of $W_L = N_L(H)/H$ defined by σ is the longest element of W_L.*

(ii) *$\sigma^{-1}\theta R(\mathfrak{b}, \mathfrak{h}) = R(\mathfrak{b}, \mathfrak{h})$. Hence $\mathfrak{b}$ is an $\mathrm{Ad}(\sigma^{-1}) \circ \theta$-stable Borel subalgebra of $\mathfrak{g}$.*

(iii) *For any root $\alpha \in R(\mathfrak{g}, \mathfrak{h})$, we have*

$$\sigma^2 X_\alpha = e^{\pi i \alpha(h_{\boldsymbol{R}})} X_\alpha = \pm X_\alpha \quad (X_\alpha \in \mathfrak{g}_\alpha).$$

In particular, $\mathrm{Ad}(\sigma^2)\big|_{\mathfrak{l}} = \mathrm{id}_{\mathfrak{l}}$.

(iv) *$\theta \circ \mathrm{Ad}(s) = \mathrm{Ad}(s^{-1}) \circ \theta$ and $\theta \circ \mathrm{Ad}(s^{-1}) = \mathrm{Ad}(s) \circ \theta$.*

(v) *The action of σ on $R(\mathfrak{g}, \mathfrak{h})$ commutes with that of θ.*

(vi) *$s\mathfrak{b} \subset \mathfrak{q}$ is a θ-stable Borel subalgebra of $\mathfrak{g}$.*

PROOF. (i) Since $h_{\boldsymbol{R}}$ is a regular element of $\mathfrak{l}$, the centralizer $\mathfrak{z}_{\mathfrak{l}}(h_{\boldsymbol{R}})(\supset \mathfrak{h})$ of $h_{\boldsymbol{R}}$ in $\mathfrak{l}$ is a Cartan subalgebra and hence $\mathfrak{z}_{\mathfrak{l}}(h_{\boldsymbol{R}}) = \mathfrak{h}$. Therefore we have

$$\sigma\mathfrak{h} = \mathfrak{z}_{\mathfrak{l}}(\sigma h_{\boldsymbol{R}}) = \mathfrak{z}_{\mathfrak{l}}(-h_{\boldsymbol{R}}) = \mathfrak{h}\,.$$

The assertion $\sigma R(\mathfrak{p}_L, \mathfrak{h}) = -R(\mathfrak{p}_L, \mathfrak{h})$ follows from $R(\mathfrak{p}_L, \mathfrak{h}) = \{\alpha \in R(\mathfrak{l}, \mathfrak{h}); \alpha(h_{\boldsymbol{R}}) > 0\}$ and $\sigma h_{\boldsymbol{R}} = -h_{\boldsymbol{R}}$.

The facts $\sigma R(\mathfrak{p}_L, \mathfrak{h}) = -R(\mathfrak{p}_L, \mathfrak{h}), \theta R(\mathfrak{p}_L, \mathfrak{h}) = -R(\mathfrak{p}_L, \mathfrak{h})$, $\sigma\mathfrak{u} = \mathfrak{u}$ and $\theta(\mathfrak{u}) = \mathfrak{u}$ imply (ii).

(iii) follows from Lemma 2.2(iii), $\alpha(h_{\boldsymbol{R}}) \in \boldsymbol{Z}$ ($\alpha \in R(\mathfrak{g}, \mathfrak{h})$) and $\beta(h_{\boldsymbol{R}}) \in 2\boldsymbol{Z}$ ($\beta \in R(\mathfrak{l}, \mathfrak{h})$).

(iv) follows from $\theta(s) = s^{-1}$.

By (iii), the action of σ^2 on $R(\mathfrak{g}, \mathfrak{h})$ is trivial. Then (v) follows from $\theta(\sigma) = \sigma^{-1}$.

Since $\mathfrak{b}$ is stable under the action of $\mathrm{Ad}(\sigma^{-1}) \circ \theta = \mathrm{Ad}(s^{-1}) \circ \theta \circ \mathrm{Ad}(s)$, we have $\theta(s\mathfrak{b}) = s\mathfrak{b}$. q.e.d.

Let us write $\mathfrak{h}^c = s\mathfrak{h}$.

LEMMA 2.4. *$\mathfrak{h}^c$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing a Cartan subalgebra of $\mathfrak{k}$.*

PROOF. Since $\sigma\mathfrak{h} = \mathfrak{h}$, we have

$$\theta(\mathfrak{h}^c) = \theta(s\mathfrak{h}) = s^{-1}\mathfrak{h} = s(\sigma^{-1}\mathfrak{h}) = \mathfrak{h}^c\,.$$

Hence $\mathfrak{h}^c$ is θ-stable. Since $\mathfrak{h}^c \subset s\mathfrak{b}$ and $s\mathfrak{b}$ is a θ-stable Borel subalgebra of $\mathfrak{g}$ (Lemma 2.3(vi)), $R(\mathfrak{g}, \mathfrak{h}^c)$ have no real roots. Therefore $\mathfrak{h}^c$ contains a Cartan subalgebra of $\mathfrak{k}$. q.e.d.

LEMMA 2.5. (i) *F_L acts on the set $\{s_{x_{\boldsymbol{R}}}\mathfrak{b}; x_{\boldsymbol{R}} \in \mathcal{C} = F_L x_\theta^0\}$ transitively.*

(ii) *If $\mathcal{B}_{\mathfrak{q}}^L/K \neq \emptyset$, $\{s_{x_{\boldsymbol{R}}}\mathfrak{b}; x_{\boldsymbol{R}} \in \mathcal{C}\}$ contains a representative of $\mathcal{B}_{\mathfrak{q}}^L/K$.*

PROOF. For $x_{\boldsymbol{R}} \in \mathcal{C}$ and $a \in F_L$, since $s_{ax_{\boldsymbol{R}}} = as_{x_{\boldsymbol{R}}}a^{-1}$ and $a^{-1}\mathfrak{b} = \mathfrak{b}$, we have $s_{ax_{\boldsymbol{R}}}\mathfrak{b} = as_{x_{\boldsymbol{R}}}\mathfrak{b}$. This implies (i).

Let us write $(h_\theta^0, x_\theta^0, y_\theta^0)$ the normal S-triple corresponding to $(h_{\boldsymbol{R}}^0, x_{\boldsymbol{R}}^0, y_{\boldsymbol{R}}^0)$ by (2.4). Then we have

$$s_{ax_{\boldsymbol{R}}^0}(ax_{\boldsymbol{R}}^0) = as_{x_{\boldsymbol{R}}^0}a^{-1}(ax_{\boldsymbol{R}}^0) = a(s_{x_{\boldsymbol{R}}^0}x_{\boldsymbol{R}}^0) = ax_\theta^0 \quad (a \in F_L)$$

and hence $K_L(ax_\theta^0) \in \mathcal{N}_{\mathfrak{s}_L}/K_L$ corresponds to $L(\boldsymbol{R})(ax_{\boldsymbol{R}}^0) \in \mathcal{N}_{\mathfrak{l}(\boldsymbol{R})}/L(\boldsymbol{R})$ by the Sekiguchi correspondence. Since $\mathcal{C} = F_L x_{\boldsymbol{R}}^0$ contains representatives of $\mathcal{N}_{\mathfrak{l}(\boldsymbol{R})}^{\mathfrak{l}-\mathrm{pr}}/L(\boldsymbol{R})$, $F_L x_\theta^0$ contains those of $\mathcal{N}_{\mathfrak{s}_L}^{\mathfrak{l}-\mathrm{pr}}/K_L$.

Here we notice that $\mathcal{B}_{\mathfrak{q}}^L/K \xrightarrow{\sim} [\mathrm{Ind}^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{N}_{\mathfrak{s}_L}^{\mathfrak{l}-\mathrm{pr}}/K_L)]^{\mathfrak{g}-\mathrm{pr}}$ (cf. Proposition 1.7, Proposition 1.9). For any $\mathcal{B}_1 \in \mathcal{B}_{\mathfrak{q}}^L/K$, there exists $ax_\theta^0 \in F_L x_\theta^0$ ($a \in F_L$) such that $\mathcal{O}_1 := \mathrm{Ind}^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(K_L(ax_\theta^0))$ corresponds to $\mathcal{B}_1$ via the above correspondence. Then $\mathcal{O}_1$ is the unique $\mathfrak{g}$-principal K-orbit which meets $K_L(ax_\theta^0) + (\mathfrak{u} \cap \mathfrak{s})$. Take $Y \in \mathfrak{u} \cap \mathfrak{s}$ such that $ax_\theta^0 + Y \in \mathcal{O}_1$. Then

$$ax_\theta^0 + Y = s_{ax_{\boldsymbol{R}}^0}(ax_{\boldsymbol{R}}^0) + Y \in s_{ax_{\boldsymbol{R}}^0}(ax_{\boldsymbol{R}}^0 + \mathfrak{u}) \subset s_{ax_{\boldsymbol{R}}^0}(\mathfrak{p}_L + \mathfrak{u}) = s_{ax_{\boldsymbol{R}}^0}\mathfrak{b}$$

and hence $s_{ax_{\boldsymbol{R}}^0}\mathfrak{b} \in \mathcal{B}_1$. q.e.d.

2.2. A construction of a θ-stable Borel subalgebra of large type. Let H, Σ, $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and $\mathfrak{b}$ be as in 2.1. The main theorem of this section is the following.

THEOREM 2.6. *If Σ is of large type, then $\mathcal{B}_{\mathfrak{q}}^L/K$ is non-empty.*

To prove Theorem 2.6, it is sufficient to show the following

THEOREM 2.7. *If Σ is of large type, we can choose $x_{\boldsymbol{R}} \in \mathcal{C} = F_L x_{\boldsymbol{R}}^0$* (cf. (2.2)) *such that $s_{x_{\boldsymbol{R}}}\mathfrak{b}$ is a θ-stable Borel subalgebra of large type.*

Theorem 2.6 implies the following.

PROPOSITION 2.8. *Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ and Σ a positive system of $R(\mathfrak{g}, \mathfrak{h})_{i\boldsymbol{R}}$.*

(i) *If Σ is of large type, then Σ is special with respect to some $\mathcal{B} \in \mathcal{B}_{\mathfrak{g}}^L/K$.*

(ii) *For any θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ which belongs to $(\mathfrak{h}, \Sigma)$, Σ is of large type if and only if $\mathcal{B}_{\mathfrak{q}}^L/K \neq \emptyset$.*

PROOF. (i) Let us take a θ-stable parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}'$ which belongs to $(\mathfrak{h}, \Sigma)$ such that $R(\mathfrak{l}', \mathfrak{h}) = R(\mathfrak{g}, \mathfrak{h})_{\boldsymbol{R}}$. Then by Theorem 2.6, there exists $\mathfrak{b}_1 \in \mathcal{B}_{\mathfrak{g}}^L$ such that $\mathfrak{b}_1 \subset \mathfrak{q}'$. Hence Σ is special with respect to $K\{\mathfrak{b}_1\} \in \mathcal{B}_{\mathfrak{g}}^L/K$.

(ii) The "only if" part is just Theorem 2.6. To prove the "if" part, suppose that $\mathcal{B}_{\mathfrak{q}}^L/K \neq \emptyset$. Take $\mathfrak{b}_1 \in \mathcal{B}_{\mathfrak{q}}^L$ such that $\mathfrak{b}_1 \subset \mathfrak{q}$. Let $\mathfrak{l}'$ be the Levi subalgebra of $\mathfrak{g}$ such that $\mathfrak{l}' \supset \mathfrak{h}$ and $R(\mathfrak{l}', \mathfrak{h}) = R(\mathfrak{l}, \mathfrak{h})_{\boldsymbol{R}} = R(\mathfrak{g}, \mathfrak{h})_{\boldsymbol{R}}$. Let $\mathfrak{t} \subset \mathfrak{l}'$ be a fundamental Cartan subalgebra of $\mathfrak{g}$ (cf. Lemma 2.4). Since $\mathfrak{b}_1$ contains a fundamental Cartan subalgebra of $\mathfrak{g}$, and any fundamental Cartan subalgebras in $\mathfrak{q}$ are $Q \cap K$-conjugate, we can assume that $\mathfrak{t} \subset \mathfrak{b}_1$ by taking a $Q \cap K$-conjugate of $\mathfrak{b}_1$ instead of $\mathfrak{b}_1$. Define a parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}'$ by $R(\mathfrak{u}', \mathfrak{t}) = R(\mathfrak{b}_1, \mathfrak{t}) \setminus R(\mathfrak{l}', \mathfrak{t})$. Then $\mathfrak{u} \subset \mathfrak{u}'$ and hence $\Sigma = R(\mathfrak{u}, \mathfrak{h})_{i\boldsymbol{R}} = R(\mathfrak{u}', \mathfrak{h})_{i\boldsymbol{R}}$. Therefore $\mathfrak{q}'$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$ which belongs to $(\mathfrak{h}, \Sigma)$ and contains $\mathfrak{b}_1$. Thus Σ is special with respect to $K\{\mathfrak{b}_1\} \in \mathcal{B}_{\mathfrak{g}}^L/K$. By Proposition 1.3(i), Σ is of large type. q.e.d.

Let us define another involution θ' of $\mathfrak{g}$ by $\theta' := \mathrm{Ad}(\sigma^{-1}) \circ \theta = \mathrm{Ad}(s^{-1}) \circ \theta \circ \mathrm{Ad}(s)$. We consider the isomorphism $R(\mathfrak{g}, \mathfrak{h}) \xrightarrow{\sim} R(\mathfrak{g}, \mathfrak{h}^c)$, $(\alpha \mapsto \tilde{\alpha} := \alpha \circ \mathrm{Ad}(s^{-1}))$ of root systems. Since $\theta(\tilde{\alpha})(sA) = \theta'(\alpha)(A)$ $(\alpha \in R(\mathfrak{g}, \mathfrak{h}), A \in \mathfrak{h})$, a root $\alpha \in R(\mathfrak{g}, \mathfrak{h})$ is non-compact imaginary (compact imaginary, complex, or real) with respect to θ' if and only if, so is $\tilde{\alpha}$ with respect to θ. Therefore $s\mathfrak{b}(\supset \mathfrak{h}^c)$ is of large type with respect to θ if and only if $\mathfrak{b}(\supset \mathfrak{h})$ is of large type with respect to θ'.

Let us write $LW(\mathfrak{u}, \mathfrak{h})$ the set of lowest weights of $\mathfrak{h}$ (with respect to the positive system $R(\mathfrak{p}_L, \mathfrak{h})$) in the $\mathfrak{l}$-module $\mathfrak{u}$. For $\alpha \in LW(\mathfrak{u}, \mathfrak{h})$, we write $\mathfrak{u}(\alpha)$ the irreducible $\mathfrak{l}$-submodule of $\mathfrak{u}$ generated by $\mathfrak{g}_\alpha$ and α^h the highest weight of $\mathfrak{u}(\alpha)$.

LEMMA 2.9. *For $\alpha \in LW(\mathfrak{u}, \mathfrak{h})$, we have the following:*

(i) *$\mathfrak{u}(\alpha)$ is s-stable and hence σ-stable.*

(ii) *$\sigma\alpha = \alpha^h$.*

(iii) *If $\mathfrak{u}(\alpha)$ is θ-stable, $\theta(\alpha) = \alpha^h$.*

PROOF. (i) follows from $s \in L$.

(ii) Since $R(\mathfrak{p}_L, \mathfrak{h}) = \{\alpha \in R(\mathfrak{g}, \mathfrak{h}); \alpha(h_{\boldsymbol{R}}) > 0\}$ and $\sigma h_{\boldsymbol{R}} = -h_{\boldsymbol{R}}$, we have $\sigma \mathfrak{n}_L = \mathfrak{n}_L^-$. This implies

$$[\mathfrak{n}_L, \mathfrak{g}_{\sigma\alpha}] = [\mathfrak{n}_L, \sigma\mathfrak{g}_\alpha] = \sigma[\sigma\mathfrak{n}_L, \mathfrak{g}_\alpha] = \sigma[\mathfrak{n}_L^-, \mathfrak{g}_\alpha] = 0\,,$$

and hence $\sigma\alpha = \alpha^h$.

(iii) Since $\theta(\mathfrak{n}_L) = \mathfrak{n}_L^-$ and $\mathfrak{g}_{\theta\alpha} \subset \mathfrak{u}(\alpha)$, we can show that $\theta\alpha = \alpha^h$ similarly. q.e.d.

To know the action of $\theta' = \mathrm{Ad}(\sigma^{-1}) \circ \theta$ on root spaces $\mathfrak{g}_\alpha$ ($\alpha \in R(\mathfrak{p}_L, \mathfrak{h})$), we need the following lemmas.

LEMMA 2.10. *Let (h, x, y) be a basis of $\mathfrak{sl}(2, \boldsymbol{C})$ such that $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$. Write $\sigma := \exp(\pi i(x + y)/2) \in SL(2, \boldsymbol{C})$. Let U be the irreducible $SL(2, \boldsymbol{C})$-module of dimension $k + 1$ ($k \geq 0$). By the representation theory of $\mathfrak{sl}(2, \boldsymbol{C})$, we can choose a basis $u_{-k}, u_{-k+2}, \dots, u_{k-2}, u_k$ of U such that*

$$hu_j = ju_j\,, \quad xu_j = \frac{k-j}{2}u_{j+2}\,, \quad yu_j = \frac{k+j}{2}u_{j-2}\,. \tag{2.5}$$

Then the action of σ on U is given by

$$\sigma u_j = i^k u_{-j} \quad (j = -k, -k+2, \dots, k-2, k)\,.$$

PROOF. Let V be the two dimensional $\mathfrak{sl}(2, \boldsymbol{C})$-module with basis e, f such that

$$he = e\,, \quad xe = 0\,, \quad ye = f\,, \quad hf = -f\,, \quad xf = e\,, \quad yf = 0\,.$$

We may assume that U is the space of symmetric k-tensors of V : $U = S^k(V)$. Moreover, since $e^{(k+j)/2} f^{(k-j)/2}$ ($-k \leq j \leq k$, $j \in k+2\boldsymbol{Z}$) is a basis of U satisfying the condition (2.5), we may assume $u_j = e^{(k+j)/2} f^{(k-j)/2}$. It is easy to see that $(x + y)(e + f)^k = k(e + f)^k$, which implies $\sigma(e + f)^k = e^{k\pi i/2}(e + f)^k = i^k(e + f)^k$. We notice that $(e + f)^k = \sum_{j\in k+2\boldsymbol{Z}, |j|\leq k} \binom{k}{(k-j)/2} u_j$ and $h(\sigma u_j) = \sigma(\mathrm{Ad}(\sigma^{-1})h)u_j = \sigma(-hu_j) = -j\sigma u_j$. Then by comparing the h-weight vectors of weight $-j$ in $\sum \binom{k}{(k-j)/2}\sigma u_j = \sigma(e+f)^k = i^k(e+f)^k = \sum \binom{k}{(k-j)/2} i^k u_j$, we have $\binom{k}{(k-j)/2}\sigma u_j = \binom{k}{(k+j)/2} i^k u_{-j}$ and hence $\sigma u_j = i^k u_{-j}$. q.e.d.

LEMMA 2.11. *Suppose that $\mathfrak{u}(\alpha)$ ($\alpha \in LW(\mathfrak{u}, \mathfrak{h})$) is θ-stable. Write $k := -\alpha(h_{\boldsymbol{R}}) \geq 0$ and define $u_{-k}, u_{-k+2}, \dots, u_{k-2}, u_k \in \mathfrak{u}(\alpha)$ by $u_{-k} = X_\alpha$ and $x_{\boldsymbol{R}} u_j = (k - j)u_{j+2}/2$. Then we have the following:*

(i) $h_{\boldsymbol{R}} u_j = ju_j$, $\sigma u_j = i^k u_{-j}$, $u_k \in \mathfrak{g}_{\alpha^h}$.

(ii) *There exists $C_\alpha = C_\alpha(x_{\boldsymbol{R}}) = \pm 1$, which depends on the choice of $x_{\boldsymbol{R}} \in C$ (cf. Lemma 2.12), such that $\theta(u_{-k}) = C_\alpha i^k u_k$.*

(iii) $\sigma^{-1} \circ \theta(X_\alpha) = C_\alpha X_\alpha$.

(iv) $\theta(u_{-k+2j}) = (-1)^j C_\alpha i^k u_{k-2j}$.

PROOF. (i) Write $S := \boldsymbol{C}h_{\boldsymbol{R}} + \boldsymbol{C}x_{\boldsymbol{R}} + \boldsymbol{C}y_{\boldsymbol{R}} \simeq \mathfrak{sl}(2, \boldsymbol{C})$. It is easy to see that $-k$ is the $h_{\boldsymbol{R}}$-lowest weight in $\mathfrak{u}(\alpha)$, and that u_{-k} is a non-zero $h_{\boldsymbol{R}}$-weight vector of weight $-k$, which is unique up to a constant. By the representation theory of $\mathfrak{sl}(2, \boldsymbol{C})$, $\boldsymbol{C}u_{-k} + \boldsymbol{C}u_{-k+2} +$

$\cdots + \boldsymbol{C}u_{k-2} + \boldsymbol{C}u_k$ is the unique irreducible S-submodule of $\mathfrak{u}(\alpha)$ of dimension $k+1$ and $u_{-k}, u_{-k+2}, \ldots, u_{k-2}, u_k$ satisfy the same relation as in (2.5).

Since $\alpha^h(h_{\boldsymbol{R}})$ is the $h_{\boldsymbol{R}}$-highest weight of multiplicity one in $\mathfrak{u}(\alpha)$, $\alpha^h(h_{\boldsymbol{R}}) = k$ and $u_k \in \mathfrak{g}_{\alpha^h}$. $\sigma u_j = i^k u_{-j}$ follows from Lemma 2.10.

(ii) Since $h_{\boldsymbol{R}}\theta(u_{-k}) = \theta(-h_{\boldsymbol{R}}u_{-k}) = k\theta(u_{-k})$, there exists $c \in \boldsymbol{C}^\times$ such that $\theta(u_{-k}) = cu_k$. By the relation like (2.5), we have

$$u_{-k} = \frac{1}{k!}(y_{\boldsymbol{R}})^k u_k \quad \text{and} \quad u_k = \frac{1}{k!}(x_{\boldsymbol{R}})^k u_{-k}\,.$$

Then

$$cu_k = \theta(u_{-k}) = \frac{1}{k!}\theta(y_{\boldsymbol{R}})^k\theta(u_k) = \frac{1}{k!}(-x_{\boldsymbol{R}})^k\left(\frac{1}{c}u_{-k}\right) = \frac{(-1)^k}{c}\left(\frac{1}{k!}(x_{\boldsymbol{R}})^k u_{-k}\right) = \frac{(-1)^k}{c}u_k$$

and hence $c = \pm i^k$.

(iii) follows from (i) and (ii).

(iv) We notice that

$$u_{-k+2j} = \frac{1}{k(k-1)\cdots(k-j+1)}(x_{\boldsymbol{R}})^j u_{-k} \text{ and } u_{k-2j} = \frac{1}{k(k-1)\cdots(k-j+1)}(y_{\boldsymbol{R}})^j u_k\,.$$

Then

$$\begin{aligned}\theta(u_{-k+2j}) &= \frac{1}{k(k-1)\cdots(k-j+1)}(-y_{\boldsymbol{R}})^j\theta(u_{-k}) \\ &= \frac{C_\alpha(-1)^j i^k}{k(k-1)\cdots(k-j+1)}(y_{\boldsymbol{R}})^j u_k = (-1)^j C_\alpha i^k u_{k-2j}\,.\end{aligned}$$

q.e.d.

LEMMA 2.12. *Suppose that* $\mathfrak{u}(\alpha)(\alpha \in LW(\mathfrak{u}, \mathfrak{h}))$ *is θ-stable.*

(i) $C_\alpha(ax_{\boldsymbol{R}}) = \alpha(a^2)C_\alpha(x_{\boldsymbol{R}})$ $(a \in F_L)$.

(ii) *If* $R(\mathfrak{u}(\alpha), \mathfrak{h})_{i\boldsymbol{R}} \neq \emptyset$, $\alpha(a^2) = 1$ *for any* $a \in F_L$. *In particular,* $C_\alpha(x_{\boldsymbol{R}})$ *is independent of the choice of* $x_{\boldsymbol{R}} \in \mathcal{C}$.

PROOF. (i) $C_\alpha(ax_{\boldsymbol{R}})X_\alpha = \sigma_{ax_{\boldsymbol{R}}}^{-1}\theta(X_\alpha) = (a\sigma_{x_{\boldsymbol{R}}}a^{-1})^{-1}\theta(X_\alpha) = a\sigma_{x_{\boldsymbol{R}}}^{-1}a^{-1}\theta(X_\alpha) = a\sigma_{x_{\boldsymbol{R}}}^{-1}\theta(aX_\alpha) = \alpha(a)a\sigma_{x_{\boldsymbol{R}}}^{-1}\theta(X_\alpha) = \alpha(a)a\{C_\alpha(x_{\boldsymbol{R}})X_\alpha\} = \alpha(a^2)C_\alpha(x_{\boldsymbol{R}})X_\alpha$.

(ii) Take $\gamma \in R(\mathfrak{u}(\alpha), \mathfrak{h})_{i\boldsymbol{R}}$. Then γ can be written as $\gamma = \alpha + \sum_{\beta\in\Delta(\mathfrak{p}_L,\mathfrak{h})} n_\beta\beta$ for some $n_\beta \in \boldsymbol{Z}_{\geq 0}$. Since $\gamma(a) = 1$ and $\beta(a) = \pm 1$ $(\beta \in \Delta(\mathfrak{p}_L, \mathfrak{h}))$, we have $\alpha(a) = \gamma(a)\prod_{\beta\in\Delta(\mathfrak{p}_L,\mathfrak{h})}\beta(a)^{-n_\beta} = \pm 1$. q.e.d.

To prove Theorem 2.7, we have to show that $\mathfrak{b}$ is of large type with respect to $\theta' = \mathrm{Ad}(\sigma_{x_{\boldsymbol{R}}}^{-1}) \circ \theta$ for some $x_{\boldsymbol{R}} \in \mathcal{C}$. The set $\Delta(\mathfrak{b}, \mathfrak{h})$ of simple roots corresponding to $\mathfrak{b}$ is decomposed as $\Delta(\mathfrak{b}, \mathfrak{h}) = \Delta(\mathfrak{p}_L, \mathfrak{h}) \cup (\Delta(\mathfrak{b}, \mathfrak{h}) \cap R(\mathfrak{u}, \mathfrak{h}))$ and clearly $\Delta(\mathfrak{b}, \mathfrak{h}) \cap R(\mathfrak{u}, \mathfrak{h}) \subset LW(\mathfrak{u}, \mathfrak{h})$. Since $x_{\boldsymbol{R}} = \sum_{\alpha\in\Delta(\mathfrak{p}_L,\mathfrak{h})} X_\alpha$ for some $X_\alpha \in \mathfrak{g}_\alpha$ and $\sigma_{x_{\boldsymbol{R}}}^{-1}\theta(x_{\boldsymbol{R}}) = \sigma_{x_{\boldsymbol{R}}}^{-1}(-y_{\boldsymbol{R}}) = -x_{\boldsymbol{R}}$, we have $\sigma_{x_{\boldsymbol{R}}}^{-1}\theta(X_\alpha) = -X_{\sigma_{x_{\boldsymbol{R}}}^{-1}\theta(\alpha)}$ for $\alpha \in \Delta(\mathfrak{p}_L, \mathfrak{h})$. Hence the roots in $\Delta(\mathfrak{p}_L, \mathfrak{h})$ are complex or non-compact imaginary with respect to $\mathrm{Ad}(\sigma_{x_{\boldsymbol{R}}}^{-1}) \circ \theta$.

Take $\alpha \in \Delta(\mathfrak{b}, \mathfrak{h}) \cap R(\mathfrak{u}, \mathfrak{h})$. If $\mathfrak{u}(\alpha)$ is not θ-stable, α is complex with respect to $\mathrm{Ad}(\sigma_{x_R}^{-1}) \circ \theta$. Suppose that $\mathfrak{u}(\alpha)$ is θ-stable. Then $\sigma_{x_R}^{-1}\theta(\alpha) = \alpha$, that is, α is imaginary with respect to $\mathrm{Ad}(\sigma_{x_R}^{-1}) \circ \theta$. By Lemma 2.11, we have $\sigma_{x_R}^{-1}\theta(X_\alpha) = C_\alpha(x_R)X_\alpha$. Therefore, to prove Theorem 2.7, it is sufficient to prove the following proposition. In particular, for $x_R \in \mathcal{C} = F_L x_R^0$, Theorem 2.7 holds for x_R if and only if x_R satisfies the condition of Proposition 2.13(ii).

PROPOSITION 2.13. *Suppose that $\Sigma = R(\mathfrak{u}, \mathfrak{h})_{iR}$ is of large type.*

(i) *For a root $\alpha \in \Delta(\mathfrak{b}, \mathfrak{h}) \cap R(\mathfrak{u}, \mathfrak{h})$, assume that $\mathfrak{u}(\alpha)$ is θ-stable, and that $R(\mathfrak{u}(\alpha), \mathfrak{h})_{iR} \neq \emptyset$. Then $C_\alpha(x_R) = -1$ for any $x_R \in \mathcal{C} = F_L x_R^0$.*

(ii) *We can choose $x_R \in \mathcal{C}$ such that $C_\alpha(x_R) = -1$ for any root $\alpha \in \Delta(\mathfrak{b}, \mathfrak{h}) \cap R(\mathfrak{u}, \mathfrak{h})$ with the properties that $\mathfrak{u}(\alpha)$ is θ-stable and $R(\mathfrak{u}(\alpha), \mathfrak{h})_{iR} = \emptyset$.*

The proof of this proposition will be given in Subsection 2.3.

Now suppose that $\mathcal{B}_\mathfrak{q}^L / K \neq \emptyset$. We would like to construct a subgroup $F_L(\mathfrak{q})$ of F_L which acts on $\mathcal{B}_\mathfrak{q}^L / K$ transitively.

Let $F_L(\mathfrak{q})$ be the subgroup of F_L consisting of elements $a \in F_L$ satisfying the following condition:

$$\alpha(a) = \pm 1 \text{ for } \alpha \in \Delta(\mathfrak{b}, \mathfrak{h}) \cap R(\mathfrak{u}, \mathfrak{h}) \text{ such that } \theta(\mathfrak{u}(\alpha)) = \mathfrak{u}(\alpha)\,. \tag{2.6}$$

By Lemma 2.5, we can choose $x_R^1 \in \mathcal{C} = F_L x_R^0$ such that $s_{x_R^1}\mathfrak{b} \in \mathcal{B}_\mathfrak{g}^L$. Then for $a \in F_L$, $a(s_{x_R^1}\mathfrak{b}) = s_{ax_R^1}\mathfrak{b} \in \mathcal{B}_\mathfrak{g}^L$ if and only if the condition (2.6) is satisfied (i.e., $a \in F_L(\mathfrak{q})$). Hence $F_L(\mathfrak{q})$ acts on $\{s_{x_R}\mathfrak{b};\, x_R \in \mathcal{C}, s_{x_R}\mathfrak{b} \in \mathcal{B}_\mathfrak{g}^L\}$ transitively: $\{s_{x_R}\mathfrak{b};\, x_R \in \mathcal{C}, s_{x_R}\mathfrak{b} \in \mathcal{B}_\mathfrak{g}^L\} = F_L(\mathfrak{q})\{s_{x_R^1}\mathfrak{b}\}$. Again by Lemma 2.5, $F_L(\mathfrak{q})\{s_{x_R^1}\mathfrak{b}\}$ contains representatives of $\mathcal{B}_\mathfrak{q}^L/K$. Therefore we have a bijection $F_L(\mathfrak{q})\{s_{x_R^1}\mathfrak{b}\}/\overset{K}{\sim} \xrightarrow{\sim} \mathcal{B}_\mathfrak{q}^L/K$. Here the equivalence relation $\overset{K}{\sim}$ in $F_L(\mathfrak{q})\{s_{x_R^1}\mathfrak{b}\}$ is defined as follows: for $\mathfrak{b}_1^c, \mathfrak{b}_2^c \in F_L(\mathfrak{q})\{s_{x_R^1}\mathfrak{b}\}$, we write $\mathfrak{b}_1^c \overset{K}{\sim} \mathfrak{b}_2^c$ if there exists $k \in K$ such that $\mathfrak{b}_2^c = k\mathfrak{b}_1^c$.

PROPOSITION 2.14. *Suppose that $\mathcal{B}_\mathfrak{q}^L / K \neq \emptyset$. The quotient set $F_L(\mathfrak{q})\{s_{x_R^1}\mathfrak{b}\}/\overset{K}{\sim}$ has an action of $F_L(\mathfrak{q})$ which is induced from that of $F_L(\mathfrak{q})$ on $F_L(\mathfrak{q})\{s_{x_R^1}\mathfrak{b}\}$. Therefore, via the bijection $F_L(\mathfrak{q})\{s_{x_R^1}\mathfrak{b}\}/\overset{K}{\sim} \xrightarrow{\sim} \mathcal{B}_\mathfrak{q}^L/K$, $F_L(\mathfrak{q})$ acts on $\mathcal{B}_\mathfrak{q}^L/K$ transitively.*

PROOF. Suppose that $\mathfrak{b}_1^c \overset{K}{\sim} \mathfrak{b}_2^c$ ($\mathfrak{b}_1^c, \mathfrak{b}_2^c \in F_L(\mathfrak{q})\{s_{x_R^1}\mathfrak{b}\}$). We first show that $\mathfrak{b}_1^c$ and $\mathfrak{b}_2^c$ are $L \cap K$-conjugate. Let $\mathfrak{t}_j$ ($j = 1, 2$) be a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing a Cartan subalgebra of $\mathfrak{k}$ such that $\mathfrak{t}_j \subset \mathfrak{b}_j^c \cap \mathfrak{l}$ (cf. Lemma 2.4). Since $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are conjugate by an element of $L \cap K$, we may assume that $\mathfrak{t}_1 = \mathfrak{t}_2 =: \mathfrak{t}$. Since $\mathfrak{b}_1^c \cap \mathfrak{l}$ and $\mathfrak{b}_2^c \cap \mathfrak{l}$ are Borel subalgebra of $\mathfrak{l}$ containing $\mathfrak{t}$, there exists $g \in N_L(\mathfrak{t})$ such that $\mathfrak{b}_2^c \cap \mathfrak{l} = g(\mathfrak{b}_1^c \cap \mathfrak{l})$. Then $g\mathfrak{b}_1^c = g(\mathfrak{b}_1^c \cap \mathfrak{l} + \mathfrak{u}) = \mathfrak{b}_2^c \cap \mathfrak{l} + g\mathfrak{u} = \mathfrak{b}_2^c \cap \mathfrak{l} + \mathfrak{u} = \mathfrak{b}_2^c$.

On the other hand, since $\mathfrak{b}_1^c \overset{K}{\sim} \mathfrak{b}_2^c$, there exists $k \in N_K(\mathfrak{t})$ such that $\mathfrak{b}_2^c = k\mathfrak{b}_1^c$. Thus $g^{-1}k\mathfrak{b}_1^c = \mathfrak{b}_1^c$ and hence the element of $W_G(\mathfrak{t}) = N_G(\mathfrak{t})/T$ defined by $g^{-1}k$ is 1: $g^{-1}k \in T$. Therefore $k \in gT \subset L$ and $k \in K \cap L$.

Then for $a \in F_L(\mathfrak{q})$, we have $a\mathfrak{b}_2^c = (aka^{-1})(a\mathfrak{b}_1^c)$. Since $F_L \subset N_L(L \cap K)$, $a\mathfrak{b}_1^c$ and $a\mathfrak{b}_2^c$ are also K-conjugate. q.e.d.

Suppose that θ is of inner type. Then the action of $\sigma^{-1} \circ \theta$ on $R(\mathfrak{b}, \mathfrak{h})$, which is the graph automorphism defined by θ, is trivial. It follows that $\theta(\mathfrak{u}(\alpha)) = \mathfrak{u}(\alpha)$ for any $\alpha \in \Delta(\mathfrak{b}, \mathfrak{h}) \cap R(\mathfrak{u}, \mathfrak{h})$. Therefore, if $a \in F_L(\mathfrak{q})$, $\alpha(a) = \pm 1$ for any $\alpha \in \Delta(\mathfrak{p}_L, \mathfrak{h}) \cup (\Delta(\mathfrak{b}, \mathfrak{h}) \cap R(\mathfrak{u}, \mathfrak{h}))$ and hence $\mathrm{Ad}(a^2) = \mathrm{id}_{\mathfrak{g}}$. This implies $F_L(\mathfrak{q}) = F_L^G$. Since F_L^G naturally acts on $\mathcal{B}_{\mathfrak{q}}^L/K$, we have the following

COROLLARY TO PROPOSITION 2.14. *Suppose that $\mathcal{B}_{\mathfrak{q}}^L/K \neq \emptyset$ and θ is of inner type. Then the action of F_L^G on $\mathcal{B}_{\mathfrak{q}}^L/K$ is transitive.*

REMARK 2.15. Let $(\mathfrak{q}, H(\boldsymbol{R}), \delta, \nu)$ be a quadruplet of θ-stable data and $X = X_{G(\boldsymbol{R})}(\mathfrak{q}, H(\boldsymbol{R}), \delta, \nu)$ the corresponding standard $(\mathfrak{g}, K)$-module. Suppose that Σ is of large type. Then, via the identification $\mathfrak{g} \simeq \mathfrak{g}^*$ by a G-invariant bilinear form on $\mathfrak{g}$, the set $\mathrm{Ass}(X)^{\mathfrak{g}-\mathrm{pr}}/K$ of $\mathfrak{g}$-principal K-orbits contained in the associated variety $\mathrm{Ass}(X)$ of X is described in [AV] and coincides with $[\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K]_{\mathfrak{q}^-}$:

$$\mathrm{Ass}(X)^{\mathfrak{g}-\mathrm{pr}}/K = [\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-\mathrm{pr}}/K]_{\mathfrak{q}^-} \simeq \mathcal{B}_{\mathfrak{q}^-}^L/K\,,$$

where we write $\mathfrak{q}^-$ the opposite paraboric subalgebra of $\mathfrak{q}$([O2]). Hence, via the above identification, $F_L(\mathfrak{q}) = F_L(\mathfrak{q}^-)$ acts on $\mathrm{Ass}(X)^{\mathfrak{g}-\mathrm{pr}}/K$ transitively. In particular, if θ is of inner type, $F_L^G = F_L(\mathfrak{q}) = F_L(\mathfrak{q}^-)$ naturally acts on $\mathrm{Ass}(X)^{\mathfrak{g}-\mathrm{pr}}/K$ transitively.

In order to reduce the proof of Proposition 2.13, we need the following lemma.

LEMMA 2.16. (i) *There exists a θ-stable parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}'$ of $\mathfrak{g}$ which belongs to $(\Sigma, \mathfrak{h})$ such that $\mathfrak{q}' \subset \mathfrak{q}$ and $R(\mathfrak{l}', \mathfrak{h}) = R(\mathfrak{l}, \mathfrak{h})_{\boldsymbol{R}} = R(\mathfrak{g}, \mathfrak{h})_{\boldsymbol{R}}$.*

(ii) *If $\mathcal{B}_{\mathfrak{q}'}^L/K \neq \emptyset$ for $\mathfrak{q}'$ of* (i), Proposition 2.13 *holds for* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$.

PROOF. (i) Define the Levi subalgebra $\mathfrak{l}'$ by $R(\mathfrak{l}', \mathfrak{h}) = R(\mathfrak{l}, \mathfrak{h})_{\boldsymbol{R}}$ and take $A \in i(\mathfrak{h}(\boldsymbol{R}) \cap \mathfrak{k})$ such that $\alpha(A) \neq 0$ for any $\alpha \in R(\mathfrak{l}, \mathfrak{h}) \setminus R(\mathfrak{l}, \mathfrak{h})_{\boldsymbol{R}}$. Let $\mathfrak{q}_1 = \mathfrak{l}' + \mathfrak{u}_{\mathfrak{l}}$ the parabolic subalgebra of $\mathfrak{l}$ defined by $R(\mathfrak{q}_1, \mathfrak{h}) = \{\alpha \in R(\mathfrak{l}, \mathfrak{h}); \alpha(A) \geq 0\}$. Then $\mathfrak{q}' = \mathfrak{q}_1 + \mathfrak{u}$ satisfies the condition of (i).

(ii) Suppose that $\mathcal{B}_{\mathfrak{q}'}^L/K \neq \emptyset$. It follows from $\mathfrak{q}' \subset \mathfrak{q}$ that $\mathcal{B}_{\mathfrak{q}}^L/K \neq \emptyset$. Since $\{s_{x_{\boldsymbol{R}}}\mathfrak{b}; x_{\boldsymbol{R}} \in \mathcal{C}\}$ contains a representative of $\mathcal{B}_{\mathfrak{q}}^L/K$, there exists $x_{\boldsymbol{R}} \in \mathcal{C}$ such that $s_{x_{\boldsymbol{R}}}\mathfrak{b}$ is of large type with respect to θ. By the remark after Proposition 2.8, $\mathfrak{b}$ is of large type with respect to $\mathrm{Ad}(\sigma_{x_{\boldsymbol{R}}}^{-1}) \circ \theta$. Therefore Proposition 2.13 holds for $\mathfrak{q}$. q.e.d.

By Lemma 2.16, to prove Proposition 2.13, it is sufficient to prove Proposition 2.13 for $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ such that $R(\mathfrak{l}, \mathfrak{h}) = R(\mathfrak{g}, \mathfrak{h})_{\boldsymbol{R}}$.

2.3. The proof of Proposition 2.13. Throughout this subsection, we assume that $\mathfrak{h}$, Σ, $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, $\mathfrak{p}_L$, $\mathfrak{b} \ldots$ are as in Subsection 2.1.

We first give the proof of Proposition 2.13(i), which is based on the following lemma.

LEMMA 2.17. *Suppose that $\mathfrak{u}(\alpha)$ $(\alpha \in LW(\mathfrak{u}, \mathfrak{h}))$ is θ-stable, and that $R(\mathfrak{u}(\alpha), \mathfrak{h})$ contains an imaginary root γ. Let us write the highest weight α^h of $\mathfrak{u}(\alpha)$ as $\alpha^h = \sum_{\beta \in \Delta(\mathfrak{b}, \mathfrak{h})} n_\beta \beta$*

$(n_\beta \in \mathbf{Z}_{\geq 0})$, *where* $\Delta(\mathfrak{b}, \mathfrak{h})$ *is the set of simple roots in* $R(\mathfrak{b}, \mathfrak{h})$. *Put*

$$D_\alpha := \{\beta \in \Delta(\mathfrak{b}, \mathfrak{h}); n_\beta \neq 0\}, \qquad D_\alpha \cap \Delta(\mathfrak{p}_L, \mathfrak{h}) = \{\beta_1, \beta_2, \dots, \beta_r\},$$

$$\gamma = \alpha + \sum_{j=1}^{r} c_j \beta_j \quad (c_j \in \mathbf{Z}_{\geq 0})$$

and $c = \sum_{j=1}^{r} c_j$. *Let us write* $x_{\boldsymbol{R}}^c X_\alpha = (\operatorname{ad} x_{\boldsymbol{R}})^c X_\alpha$ *as a sum of root vectors*:

$$x_{\boldsymbol{R}}^c X_\alpha = \sum_{\delta \in R(\mathfrak{u}(\alpha), \mathfrak{h})} Y_\delta \quad (Y_\delta \in \mathfrak{g}_\delta).$$

If $Y_\gamma \neq 0$, *and* γ *is non-compact, then* $C_\alpha(x_{\boldsymbol{R}}) = -1$ (i.e., $\sigma_{x_{\boldsymbol{R}}}^{-1} \circ \theta(X_\alpha) = -X_\alpha$).

PROOF. Define $u_{-k}, u_{-k+2}, \dots, u_{k-2}, u_k \in \mathfrak{u}(\alpha)$ $(k := -\alpha(h_{\boldsymbol{R}}) \geq 0)$ as in Lemma 2.11. Since $\beta_j(h_{\boldsymbol{R}}) = 2$ and $\gamma(h_{\boldsymbol{R}}) = 0$, we have $k = 2c$. We can write $x_{\boldsymbol{R}}^c X_\alpha = Au_0$ for some constant $A \neq 0$: $Au_0 = \sum_{\delta \in R(\mathfrak{u}(\alpha), \mathfrak{h})} Y_\delta$. By Lemma 2.11(iv), we have

$$\theta(u_0) = \theta(u_{-k+2(k/2)}) = (-1)^{k/2} C_\alpha(x_{\boldsymbol{R}}) i^k u_{k-2(k/2)} = C_\alpha(x_{\boldsymbol{R}}) u_0,$$

and hence

$$\sum_{\delta \in R(\mathfrak{u}(\alpha), \mathfrak{h})} \theta(Y_\delta) = C_\alpha(x_{\boldsymbol{R}}) \left(\sum_{\delta \in R(\mathfrak{u}(\alpha), \mathfrak{h})} Y_\delta \right).$$

Since γ is non-compact imaginary, we have $-Y_\gamma = \theta(Y_\gamma) = C_\alpha(x_{\boldsymbol{R}}) Y_\gamma$. Hence $C_\alpha(x_{\boldsymbol{R}}) = -1$. q.e.d.

LEMMA 2.18. *In the setting of Lemma* 2.17, *suppose that the Dynkin diagram of* D_α *is of type* A, B, D, E *or* G. *Then* $Y_\gamma \neq 0$.

To prove Lemma 2.18, We need the following two lemmas.

LEMMA 2.19. *Let* $\mathfrak{g}$ *be a simple Lie algebra of type* A, B, D, E *or* G *and* $\mathfrak{h}$ *a Cartan subalgebra of* $\mathfrak{g}$. *Let* R^+ *be a positive system of* $R := R(\mathfrak{g}, \mathfrak{h})$ *and* Δ *the base of* R^+. *Suppose that a root* $\beta \in R^+$ *and simple roots* $\beta_1, \beta_2 \in \Delta$ $(\beta_1 \neq \beta_2)$ *satisfy* $\operatorname{ht}\beta \geq 3$ *and that* $\beta - \beta_1, \beta - \beta_2 \in R^+$, *where we write* $\operatorname{ht}\beta$ *the height of* β. *Then* $\beta_1 + \beta_2 \notin R^+$ *and* $\beta - \beta_1 - \beta_2 \in R^+$.

PROOF. First suppose that R has only one root length. We may assume that $\langle \delta, \delta \rangle = 2$ for any $\delta \in R$, where $\langle , \rangle$ denotes the inner product on R. It follows that $\beta + \beta_j, \beta - 2\beta_j \notin R$ and hence the β_j-string roots through β are $\beta - \beta_j$ and β. By considering the action of the simple reflection s_{β_j} defined by β_j on $\beta - \beta_j$ and β, we have $s_{\beta_j}(\beta) = \beta - 2\langle \beta_j, \beta \rangle / \langle \beta_j, \beta_j \rangle \cdot \beta_j = \beta - \beta_j$ and hence $\langle \beta_j, \beta \rangle = 1$.

If $\beta_1 + \beta_2 \in R$, it follows that $\langle \beta, \beta_1 + \beta_2 \rangle = 2$. Hence we have $\beta = \beta_1 + \beta_2$ which contradicts the assumption $\operatorname{ht}\beta \geq 3$. Therefore $\beta_1 + \beta_2 \notin R$. Since $\beta_1 \pm \beta_2 \notin R$, we have $\langle \beta_1, \beta_2 \rangle \geq 0$ and $\langle \beta_1, \beta_2 \rangle \leq 0$. Hence $\langle \beta_1, \beta_2 \rangle = 0$. This implies $s_{\beta_2}(\beta - \beta_1) = \beta - \beta_1 - \beta_2 \in R$.

For the cases of type B and G_2, we can verify Lemma 2.19 directly. q.e.d.

LEMMA 2.20. *In the setting of Lemma 2.19, suppose that $\alpha \in R^+$, and $\beta_1, \dots, \beta_p \in \Delta$ which are not necessarily distinct. Let $X_\delta \in \mathfrak{g}_\delta \setminus \{0\}$ ($\delta \in R$) and write $X_i = X_{\beta_i}$. Then for two permutations $(i_1, i_2, \dots, i_p)$, $(j_1, j_2, \dots, j_p)$ of $1, 2, \dots, p$ such that*

$$\alpha + \beta_{i_1} + \dots + \beta_{i_k}, \quad \alpha + \beta_{j_1} + \dots + \beta_{j_k} \in R^+$$

for $1 \le k \le p$, it holds that

$$\mathrm{ad}(X_{i_p}) \circ \mathrm{ad}(X_{i_{p-1}}) \circ \dots \circ \mathrm{ad}(X_{i_1})(X_\alpha) = \mathrm{ad}(X_{j_p}) \circ \mathrm{ad}(X_{j_{p-1}}) \circ \dots \circ \mathrm{ad}(X_{j_1})(X_\alpha). \tag{2.7}$$

PROOF. We prove (2.7) by induction on p.

If $\beta_{i_p} = \beta_{j_p}$, (2.7) holds by induction. Suppose that $\beta_{i_p} \neq \beta_{j_p}$ and write

$$\beta := \alpha + \beta_{i_1} + \beta_{i_2} + \dots + \beta_{i_p} = \alpha + \beta_{j_1} + \beta_{j_2} + \dots + \beta_{j_p} \in R^+.$$

Since $\beta - \beta_{i_p}, \beta - \beta_{j_p} \in R^+$, we have $(\beta - \beta_{i_p}) - \beta_{j_p} \in R^+$ and $\beta_{i_p} + \beta_{j_p} \notin R$ by Lemma 2.19. Hence there exists a permutation $(i'_1, i'_2, \dots, i'_{p-1})$ of $i_1, i_2, \dots, i_{p-1}$ such that $i'_{p-1} = j_p$ and $\alpha + \beta_{i_1'} + \beta_{i_2'} + \dots + \beta_{i_k'} \in R^+$ $(1 \le k \le p-1)$. By induction, we have

$$\mathrm{ad}(X_{i_{p-1}}) \circ \mathrm{ad}(X_{i_{p-2}}) \circ \dots \circ \mathrm{ad}(X_{i_1})(X_\alpha) = \mathrm{ad}(X_{i'_{p-1}}) \circ \mathrm{ad}(X_{i'_{p-2}}) \circ \dots \circ \mathrm{ad}(X_{i'_1})(X_\alpha).$$

Similarly, there exists a permutation $(j'_1, j'_2, \dots, j'_{p-1})$ of $j_1, j_2, \dots, j_{p-1}$ such that $j'_{p-1} = i_p$, $\alpha + \beta_{j_1'} + \beta_{j_2'} + \dots + \beta_{j_k'} \in R^+$ $(1 \le k \le p-1)$ and

$$\mathrm{ad}(X_{j_{p-1}}) \circ \mathrm{ad}(X_{j_{p-2}}) \circ \dots \circ \mathrm{ad}(X_{j_1})(X_\alpha) = \mathrm{ad}(X_{j'_{p-1}}) \circ \mathrm{ad}(X_{j'_{p-2}}) \circ \dots \circ \mathrm{ad}(X_{j'_1})(X_\alpha).$$

Notice that $\{i'_1, i'_2, \dots, i'_{p-2}\} = \{j'_1, j'_2, \dots, j'_{p-2}\} = \{1, 2, \dots, p\} \setminus \{i_p, j_p\}$. Then by induction, we have

$$Y := \mathrm{ad}(X_{i'_{p-2}}) \circ \dots \circ \mathrm{ad}(X_{i'_1})(X_\alpha) = \mathrm{ad}(X_{j'_{p-2}}) \circ \dots \circ \mathrm{ad}(X_{j'_1})(X_\alpha).$$

Consequently, $\mathrm{ad}(X_{i_p}) \circ \mathrm{ad}(X_{i_{p-1}}) \circ \dots \circ \mathrm{ad}(X_{i_1})(X_\alpha) - \mathrm{ad}(X_{j_p}) \circ \mathrm{ad}(X_{j_{p-1}}) \circ \dots \circ \mathrm{ad}(X_{j_1})(X_\alpha)$ $= \mathrm{ad}(X_{i_p}) \circ \mathrm{ad}(X_{i'_{p-1}}) \circ \mathrm{ad}(X_{i'_{p-2}}) \circ \dots \circ \mathrm{ad}(X_{i'_1})(X_\alpha) - \mathrm{ad}(X_{j_p}) \circ \mathrm{ad}(X_{j'_{p-1}}) \circ \mathrm{ad}(X_{j'_{p-2}}) \circ \dots \circ \mathrm{ad}(X_{j'_1})(X_\alpha) = \mathrm{ad}([X_{i_p}, X_{j_p}])(Y) = 0$. q.e.d.

Then Lemma 2.18 is an immediate consequence of Lemma 2.19 and Lemma 2.20.

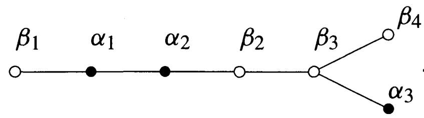
DEFINITION 2.21. (i) A Dynkin diagram Δ, which is attached (white node) $\circ$ or (black node) $\bullet$ to each node, is called a WB-Dynkin diagram. We write Δ_W (resp. Δ_B) the set of roots in Δ to which white (resp. black) nodes are attached: $\Delta = \Delta_W \cup \Delta_B$.

(ii) WB-Dynkin diagram Δ is called connected if Δ is connected as an ordinary Dynkin diagram, and any two black nodes in Δ are not connected with an edge.

(iii) For a WB-Dynkin diagram Δ, the connected WB-Dynkin diagrams, which are obtained from Δ by erasing the edges connecting two black nodes, are called connected components of Δ. For $\alpha \in \Delta$, we write $\Delta(\alpha)$ the connected component of Δ containing α.

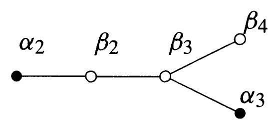
(iv) For a WB-Dynkin diagram Δ, Δ_W is considered as a sum of connected Dynkin diagrams. For $\beta \in \Delta_W$, we write $\Delta^\circ(\beta)$ the connected Dynkin diagram in Δ_W containing β.

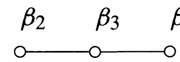
EXAMPLE. Suppose that Δ is the following WB-Dynkin diagram:

$\Delta =$ 

.

Then

$\Delta(\alpha_3) = \Delta(\beta_4) =$ 

, $\Delta^\circ(\beta_4) =$ 

.

We write $\Delta_{\mathfrak{b}}$ the WB-Dynkin diagram which we obtain from the Dynkin diagram of $\Delta(\mathfrak{b}, \mathfrak{h})$ by attaching white nodes to the roots in $\Delta(\mathfrak{p}_L, \mathfrak{h})$ and black nodes to those in $\Delta(\mathfrak{b}, \mathfrak{h}) \cap R(\mathfrak{u}, \mathfrak{h})$.

Now let us give a proof of Proposition 2.13(i) in the case that $\Delta_{\mathfrak{b}}(\alpha)$ ($\alpha \in (\Delta_{\mathfrak{b}})_B = \Delta(\mathfrak{b}, \mathfrak{h}) \cap R(\mathfrak{u}, \mathfrak{h})$) is of type A, B, D, E or G.

In the setting of Proposition 2.13(i), suppose that $\Delta_{\mathfrak{b}}(\alpha)$ is of type A, B, D, E or G_2. Take a θ-imaginary root $\gamma \in R(\mathfrak{u}(\alpha), \mathfrak{h})_{i\boldsymbol{R}}$. Suppose that $\gamma \in \Sigma$ can be written as $\gamma = \gamma_1 + \gamma_2$ for some $\gamma_1, \gamma_2 \in R(\mathfrak{b}, \mathfrak{h})$. Since γ is of the form $\gamma = \alpha + \sum_{\beta \in \Delta(\mathfrak{p}_L, \mathfrak{h})} n_\beta \beta$ ($n_\beta \in \boldsymbol{Z}_{\geq 0}$), we may assume that γ_2 is written as $\gamma_2 = \sum_{\beta \in \Delta(\mathfrak{p}_L, \mathfrak{h})} n'_\beta \beta$ for some $n'_\beta \in \boldsymbol{Z}_{\geq 0}$. Then $\theta(\gamma_2) \in -R(\mathfrak{p}_L, \mathfrak{h})$ and hence γ_2 cannot be imaginary. This means that γ is simple in Σ. Since Σ is of large type, γ is non-compact. It is easily verified that $D_\alpha \subset \Delta_{\mathfrak{b}}(\alpha)$ (Lemma 2.17). Hence D_α is also of type A, B, D, E or G. Then $C_\alpha(x_{\boldsymbol{R}}) = -1$ follows from Lemma 2.17 and Lemma 2.18.

According to Lemma 2.16, to prove Proposition 2.13(ii) and the remaining cases of Proposition 2.13(i), we assume the following.

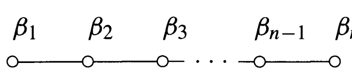
ASSUMPTION 2.22. Every root of $R(\mathfrak{l}, \mathfrak{h})$ is θ-real.

It is easy to see that $\Delta_{\mathfrak{b}}$, $(\Delta_{\mathfrak{b}})_W$ and $(\Delta_{\mathfrak{b}})_B$ are stable under the action of $\sigma^{-1} \circ \theta$. Suppose that $\Delta^\circ(\beta)$ ($\beta \in (\Delta_{\mathfrak{b}})_W = \Delta(\mathfrak{p}_L, \mathfrak{h})$) is not $\sigma^{-1} \circ \theta$-stable. Then $\sigma^{-1} \circ \theta(\Delta^\circ(\beta)) \cap \Delta^\circ(\beta) = \emptyset$ and this implies $\theta(\Delta^\circ(\beta)) \cap (-\Delta^\circ(\beta)) = \emptyset$ (cf. Lemma 2.3(i)). Hence $\Delta^\circ(\beta)$ has no real roots. This contradicts Assumption 2.22.

REMARK 2.23. Under Assumption 2.22, $\Delta^\circ(\beta)$ ($\beta \in (\Delta_{\mathfrak{b}})_W$) is $\sigma^{-1} \circ \theta$-stable.

LEMMA 2.24. *Under Assumption* 2.22, $\Delta^\circ(\beta)$ ($\beta \in (\Delta_{\mathfrak{b}})_W$) *and the action of* $\sigma^{-1} \circ \theta$ *on* $\Delta^\circ(\beta)$ *are given in the following list.*

(1)

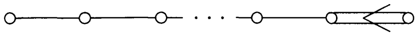


, $\sigma^{-1} \circ \theta(\beta_i) = \beta_{n-i+1}$

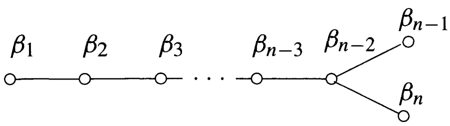
(2)


, $\sigma^{-1} \circ \theta = \mathrm{id}$

(3)

 , $\sigma^{-1} \circ \theta = \mathrm{id}$

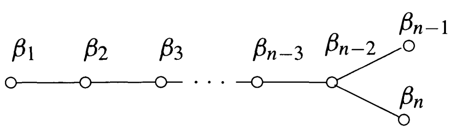
(4)



(n : even) ,

$$\sigma^{-1} \circ \theta = \mathrm{id}$$

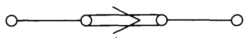
(5)



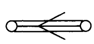
(n : odd) ,

$$\sigma^{-1} \circ \theta(\beta_i) = \beta_i \ (1 \leq i \leq n-2) , \quad \sigma^{-1} \circ \theta(\beta_{n-1}) = \beta_n$$

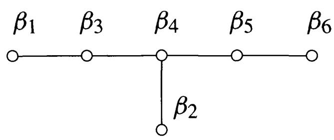
(6)

 , $\sigma^{-1} \circ \theta = \mathrm{id}$

(7)

 , $\sigma^{-1} \circ \theta = \mathrm{id}$

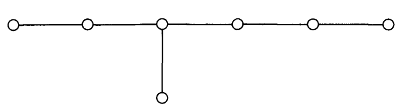
(8)



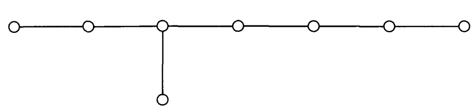
,

$$\sigma^{-1} \circ \theta(\beta_1) = \beta_6 , \quad \sigma^{-1} \circ \theta(\beta_3) = \beta_5 , \quad \sigma^{-1} \circ \theta(\beta_4) = \beta_4 , \quad \sigma^{-1} \circ \theta(\beta_2) = \beta_2$$

(9)

 , $\sigma^{-1} \circ \theta = \mathrm{id}$

(10)

 , $\sigma^{-1} \circ \theta = \mathrm{id}$

PROOF. Since all roots in $\Delta^\circ(\beta)$ are real and σ defines the longest element of the Weyl group of $\Delta^\circ(\beta)$ (cf. Lemma 2.3(i)), Lemma 2.24 follows. q.e.d.

For a root $\alpha \in (\Delta_{\mathfrak{b}})_B = \Delta(\mathfrak{b}, \mathfrak{h}) \cap R(\mathfrak{u}, \mathfrak{h})$ which is considered in Proposition 2.13, (i) or (ii), it holds that $\sigma^{-1} \circ \theta(\alpha) = \alpha$. Hence the connected component $\Delta_{\mathfrak{b}}(\alpha)$ of $\Delta_{\mathfrak{b}}$ containing α is $\sigma^{-1} \circ \theta$-stable. In the following lemma, we list up the $\sigma^{-1} \circ \theta$-stable connected components of $\Delta_{\mathfrak{b}}$.

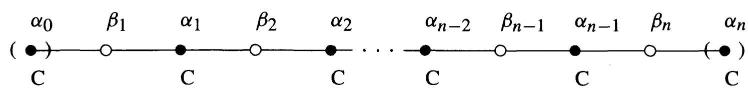
LEMMA 2.25. *Under Assumption* 2.22, *a $\sigma^{-1} \circ \theta$-stable connected component Δ of $\Delta_{\mathfrak{b}}$ is contained in the following Table* I, *where we have to consider the cases*; *the roots in Δ_B, which are put in ( ), are omitted. We attach* C (*resp.* I) *to a root $\alpha \in \Delta_B$ such that $\sigma^{-1} \circ \theta(\alpha) = \alpha$ and that $R(\mathfrak{u}(\alpha), \mathfrak{h})_{i\boldsymbol{R}} = \emptyset$ (resp. $R(\mathfrak{u}(\alpha), \mathfrak{h})_{i\boldsymbol{R}} \neq \emptyset$). If $\sigma^{-1} \circ \theta$ is not the identity, we show the action of $\sigma^{-1} \circ \theta$ on Δ. We also show the action of θ on Δ_B. If there is a root $\alpha \in \Delta_B$ such that $\sigma^{-1} \circ \theta(\alpha) = \alpha$ and $R(\mathfrak{u}(\alpha), \mathfrak{h})_{i\boldsymbol{R}} \neq \emptyset$, we show the set $R(\mathfrak{u}(\alpha), \mathfrak{h})_{i\boldsymbol{R}}$.*

TABLE I.

(A.i)

, $\theta(\alpha_0) = \alpha_0$, $R(\mathfrak{u}(\alpha), \mathfrak{h})_{i\boldsymbol{R}} = \{\alpha_0\}$.

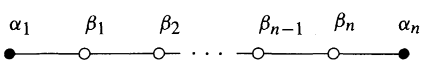
(A.ii)



$(n \geq 1)$,

$\theta(\alpha_0) = \alpha_0 + \beta_1, \theta(\alpha_i) = \beta_i + \alpha_i + \beta_{i+1}$ $(1 \leq i \leq n-1), \theta(\alpha_n) = \beta_n + \alpha_n$.

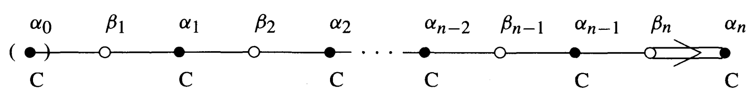
(A.iii)



,

$\sigma^{-1} \circ \theta(\alpha_1) = \alpha_2, \sigma^{-1} \circ \theta(\beta_i) = \beta_{n-i+1}$ $(1 \leq i \leq n), \theta(\alpha_1) = \alpha_2 + \beta_1 + \cdots + \beta_n, \theta(\alpha_2) = \alpha_1 + \beta_1 + \cdots + \beta_n$.

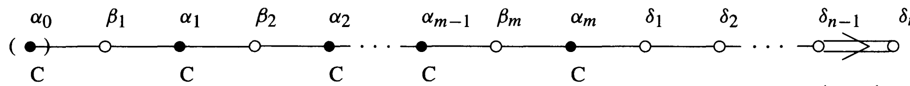
(B.i)



$(n \geq 1)$,

$\theta(\alpha_0) = \alpha_0 + \beta_1, \theta(\alpha_i) = \beta_i + \alpha_i + \beta_{i+1}$ $(1 \leq i \leq n-1), \theta(\alpha_n) = \alpha_n + \beta_n$.

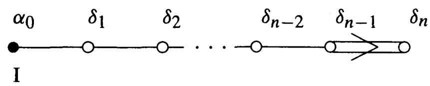
$(B.\mathrm{ii})$



$(m \geq 1, n \geq 2)$,

$\theta(\alpha_0) = \alpha_0 + \beta_1, \theta(\alpha_i) = \beta_i + \alpha_i + \beta_{i+1}$ $(1 \leq i \leq m-1), \theta(\alpha_m) = \beta_m + \alpha_m + 2(\delta_1 + \delta_2 + \cdots + \delta_n)$.

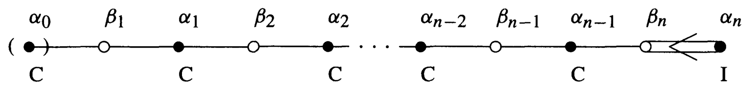
$(B.\mathrm{iii})$



$(n \geq 2)$,

$\theta(\alpha_0) = \alpha_0 + 2(\delta_1 + \delta_2 + \cdots + \delta_n)$, $R(\mathfrak{u}(\alpha_0), \mathfrak{h})_{iR} = \{\alpha_0 + \delta_1 + \delta_2 + \cdots + \delta_n\}$.

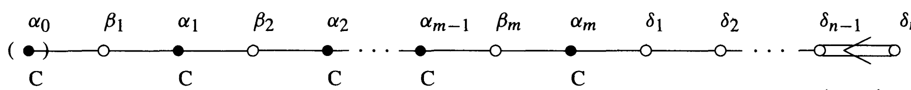
$(C.\mathrm{i})$



$(n \geq 1)$,

$\theta(\alpha_0) = \alpha_0 + \beta_1, \theta(\alpha_i) = \beta_i + \alpha_i + \beta_{i+1}$ $(1 \leq i \leq n-1), \theta(\alpha_n) = \alpha_n + 2\beta_n$, $R(\mathfrak{u}(\alpha_n), \mathfrak{h})_{iR} = \{\alpha_n + \beta_n\}$.

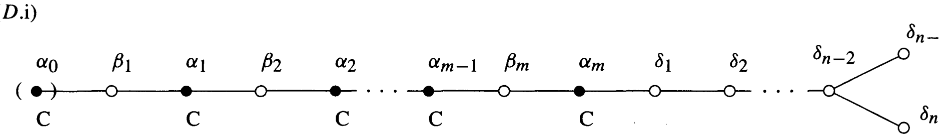
$(C.\mathrm{ii})$



$(m \geq 1, n \geq 2)$,

$\theta(\alpha_0) = \alpha_0 + \beta_1, \theta(\alpha_i) = \beta_i + \alpha_i + \beta_{i+1}$ $(1 \leq i \leq m-1), \theta(\alpha_m) = \beta_m + \alpha_m + 2(\delta_1 + \delta_2 + \cdots + \delta_{n-1}) + \delta_n$.

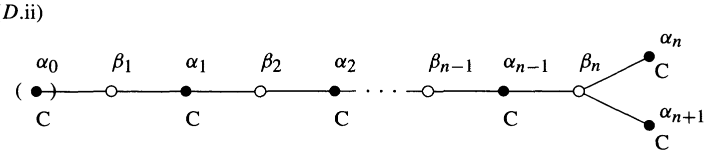
$(D.\mathrm{i})$



$(m \geq 1, n \geq 2, n$: even) ,

$\theta(\alpha_0) = \alpha_0 + \beta_1, \theta(\alpha_i) = \beta_i + \alpha_i + \beta_{i+1}$ $(1 \leq i \leq m-1), \theta(\alpha_m) = \beta_m + \alpha_m + 2(\delta_1 + \cdots + \delta_{n-2}) + \delta_{n-1} + \delta_n$.

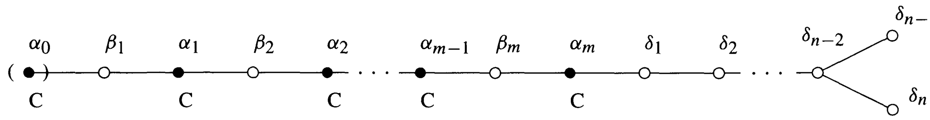
$(D.\mathrm{ii})$



$(n \geq 1)$,

$\theta(\alpha_0) = \alpha_0 + \beta_1, \theta(\alpha_i) = \beta_i + \alpha_i + \beta_{i+1}$ $(1 \leq i \leq n-1), \theta(\alpha_n) = \alpha_n + \beta_n, \theta(\alpha_{n+1}) = \alpha_{n+1} + \beta_n$.

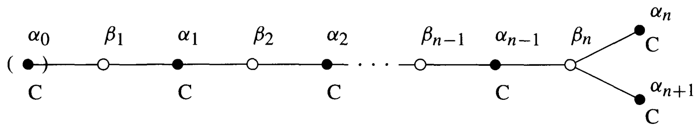
$(D.\mathrm{iii})$



$(m \geq 1, n \geq 3, n : \text{odd})$,

$\sigma^{-1} \circ \theta(\delta_{n-1}) = \delta_n$, $\sigma^{-1} \circ \theta(\delta_n) = \delta_{n-1}$, ($\sigma^{-1} \circ \theta$ acts trivially on the other simple roots.) $\theta(\alpha_0) = \alpha_0 + \beta_1$, $\theta(\alpha_i) = \beta_i + \alpha_i + \beta_{i+1}$ $(1 \leq i \leq m-1)$, $\theta(\alpha_m) = \beta_m + \alpha_m + 2(\delta_1 + \cdots + \delta_{n-2}) + \delta_{n-1} + \delta_n$.

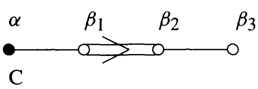
$(D.\mathrm{iv})$



$(n \geq 1)$,

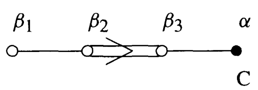
$\sigma^{-1} \circ \theta(\alpha_n) = \alpha_{n+1}$, $\sigma^{-1} \circ \theta(\alpha_{n+1}) = \alpha_n$ ($\sigma^{-1} \circ \theta$ acts trivially on the other simple roots.) $\theta(\alpha_0) = \alpha_0 + \beta_1$, $\theta(\alpha_i) = \beta_i + \alpha_i + \beta_{i+1}$ $(1 \leq i \leq n-1)$, $\theta(\alpha_n) = \alpha_{n+1} + \beta_n$, $\theta(\alpha_{n+1}) = \alpha_n + \beta_n$.

$(F_4.\mathrm{i})$



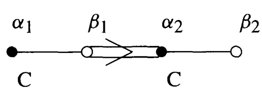
, $\theta(\alpha) = \alpha + 3\beta_1 + 4\beta_2 + 2\beta_3$.

$(F_4.\mathrm{ii})$



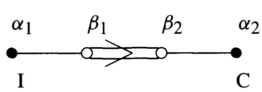
, $\theta(\alpha) = \beta_1 + 2\beta_2 + 3\beta_3 + \alpha$.

$(F_4.\mathrm{iii})$



, $\theta(\alpha_1) = \alpha_1 + \beta_1$, $\theta(\alpha_2) = \beta_1 + \alpha_2 + \beta_2$.

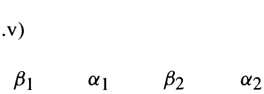
$(F_4.\mathrm{iv})$



,

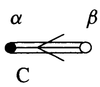
$\theta(\alpha_1) = \alpha_1 + 2\beta_1 + 2\beta_2$, $\theta(\alpha_2) = \alpha_2 + 2\beta_2 + \beta_1$, $R(\mathfrak{u}(\alpha_1), \mathfrak{h})_{i\boldsymbol{R}} = \{\alpha_1 + \beta_1 + \beta_2\}$.

$(F_4.\mathrm{v})$



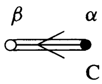
, $\theta(\alpha_1) = \beta_1 + \alpha_1 + \beta_2$, $\theta(\alpha_2) = \alpha_2 + \beta_2$.

$(G_2.\text{i})$



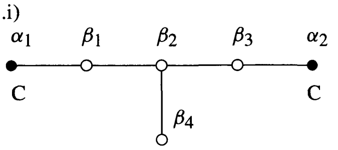
, $\theta(\alpha) = \alpha + \beta$.

$(G_2.\text{ii})$



, $\theta(\alpha) = \alpha + 3\beta$.

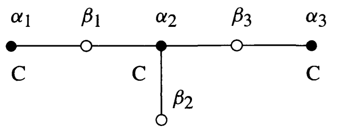
$(E_6.\text{i})$



,

$\theta(\alpha_1) = \alpha_1 + 2\beta_1 + 2\beta_2 + \beta_3 + \beta_4, \theta(\alpha_2) = \alpha_2 + 2\beta_3 + 2\beta_2 + \beta_1 + \beta_4.$

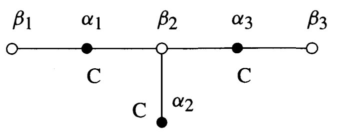
$(E_6.\text{ii})$



,

$\theta(\alpha_1) = \alpha_1 + \beta_1, \theta(\alpha_2) = \alpha_2 + \beta_1 + \beta_2 + \beta_3, \theta(\alpha_3) = \alpha_3 + \beta_3.$

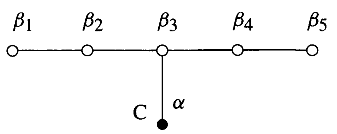
$(E_6.\text{iii})$



,

$\theta(\alpha_1) = \alpha_1 + \beta_1 + \beta_2, \theta(\alpha_2) = \alpha_2 + \beta_2, \theta(\alpha_3) = \alpha_3 + \beta_2 + \beta_3.$

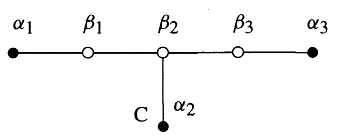
$(E_6.\text{iv})$



,

$\sigma^{-1} \circ \theta(\beta_i) = \beta_{6-i}, \sigma^{-1} \circ \theta(\alpha) = \alpha, \theta(\alpha) = \alpha + \beta_1 + 2\beta_2 + 3\beta_3 + 2\beta_4 + \beta_5.$

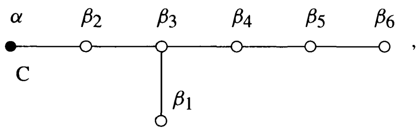
$(E_6.\text{v})$



,

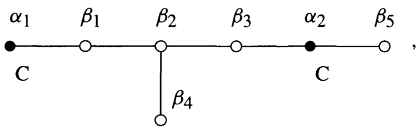
$\sigma^{-1} \circ \theta(\alpha_1) = \alpha_3, \sigma^{-1} \circ \theta(\alpha_2) = \alpha_2, \sigma^{-1} \circ \theta(\beta_i) = \beta_{4-i}\ (1 \leq i \leq 3), \theta(\alpha_2) = \alpha_2 + 2\beta_2 + \beta_1 + \beta_3.$

(E_7.i)



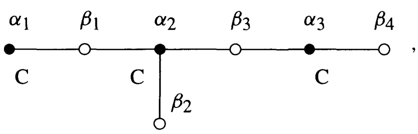
$\theta(\alpha) = \alpha + 2\beta_1 + 3\beta_2 + 4\beta_3 + 3\beta_4 + 2\beta_5 + \beta_6.$

(E_7.ii)



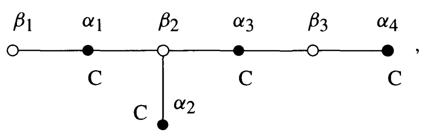
$\theta(\alpha_1) = \alpha_1 + 2\beta_1 + 2\beta_2 + \beta_3 + \beta_4, \theta(\alpha_2) = \alpha_2 + 2\beta_3 + 2\beta_2 + \beta_1 + \beta_4 + \beta_5.$

(E_7.iii)



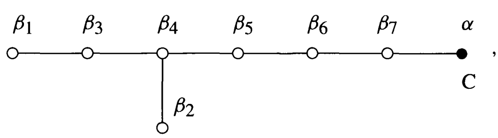
$\theta(\alpha_1) = \alpha_1 + \beta_1, \theta(\alpha_2) = \alpha_2 + \beta_1 + \beta_2 + \beta_3, \theta(\alpha_3) = \alpha_3 + \beta_3 + \beta_4.$

(E_7.iv)



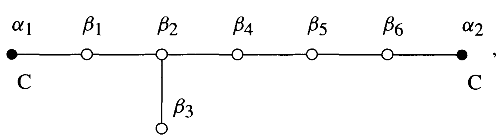
$\theta(\alpha_1) = \alpha_1 + \beta_1 + \beta_2, \theta(\alpha_2) = \alpha_2 + \beta_2, \theta(\alpha_3) = \alpha_3 + \beta_2 + \beta_3, \theta(\alpha_4) = \alpha_4 + \beta_3.$

(E_8.i)



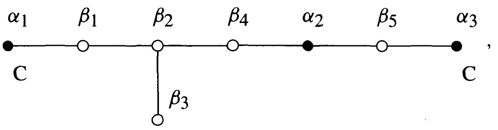
$\theta(\alpha) = \alpha + 2\beta_1 + 3\beta_2 + 4\beta_3 + 6\beta_4 + 5\beta_5 + 4\beta_6 + 3\beta_7.$

(E_8.ii)



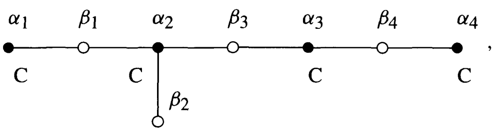
$\theta(\alpha_1) = \alpha_1 + 3\beta_1 + 4\beta_2 + 2\beta_3 + 3\beta_4 + 2\beta_5 + \beta_6, \theta(\alpha_2) = \alpha_2 + \beta_1 + 2\beta_2 + \beta_3 + 2\beta_4 + 2\beta_5 + 2\beta_6.$

(E_8.iii)



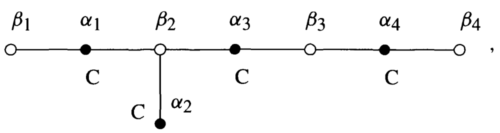
$\theta(\alpha_1) = \alpha_1 + 2\beta_1 + 2\beta_2 + \beta_3 + \beta_4, \theta(\alpha_2) = \alpha_2 + \beta_1 + 2\beta_2 + \beta_3 + 2\beta_4 + \beta_5, \theta(\alpha_3) = \alpha_3 + \beta_5.$

(E_8.iv)



$\theta(\alpha_1) = \alpha_1 + \beta_1, \theta(\alpha_2) = \alpha_2 + \beta_1 + \beta_2 + \beta_3, \theta(\alpha_3) = \alpha_3 + \beta_3 + \beta_4, \theta(\alpha_4) = \alpha_4 + \beta_3.$

(E_8.v)



$\theta(\alpha_1) = \alpha_1 + \beta_1 + \beta_2, \theta(\alpha_2) = \alpha_2 + \beta_2, \theta(\alpha_3) = \alpha_3 + \beta_2 + \beta_3, \theta(\alpha_4) = \alpha_4 + \beta_3 + \beta_4.$

PROOF. Lemma 2.25 can be checked, case by case, by noticing the following facts:

(a) Δ is a connected Dynkin diagram.

(b) $\sigma^{-1} \circ \theta$ is an involution of Δ which stabilizes Δ_B, Δ_W and $\Delta^\circ(\beta)$ for any $\beta \in \Delta_W$.

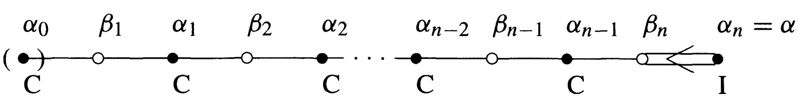
(c) $\Delta^\circ(\beta)$ ($\beta \in \Delta_W$) is a Dynkin diagram (with an action of $\sigma^{-1} \circ \theta$) in Lemma 2.24.

(d) For $\alpha \in \Delta_B$, $\sigma(\alpha)$ is highest in the roots of the form $\alpha + \sum_{\beta \in \Delta_W} c_\beta \beta$ ($c_\beta \in \mathbf{Z}_{\geq 0}$).

(e) If $\sigma^{-1} \circ \theta(\alpha) = \alpha$ for $\alpha \in \Delta_B$, $\theta(\alpha)$ is highest in the roots of the form $\alpha + \sum_{\beta \in \Delta_W} c_\beta \beta$ ($c_\beta \in \mathbf{Z}_{\geq 0}$). q.e.d.

Now let us give the proof of Proposition 2.13(i) in the remaining cases. We have to show Proposition 2.13(i) in the cases that $\Delta_{\mathfrak{b}}(\alpha)$ is of type C or F_4. By Lemma 2.25, we can assume that $\Delta_{\mathfrak{b}}(\alpha)$ is of type (C.i) with $\alpha = \alpha_n$, or that $\Delta_{\mathfrak{b}}(\alpha)$ is of type (F_4.iv) with $\alpha = \alpha_1$ in Table I. We use the notation in Lemma 2.17.

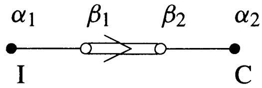
First suppose that $\Delta_{\mathfrak{b}}(\alpha)$ is of type (C.i):



($n \geq 1$).

Then $\gamma := \alpha + \beta_n$ is an imaginary root in $R(\mathfrak{u}(\alpha), \mathfrak{h})$. Since $x_{\mathbf{R}}$ is a sum of root vectors $X_\beta \in \mathfrak{g}_\beta \setminus \{0\}$ ($\beta \in (\Delta_{\mathfrak{b}})_W$) and $[X_\beta, X_\alpha] = 0$ for $\beta \in (\Delta_{\mathfrak{b}})_W \setminus \{\beta_n\}$, we have $x_{\mathbf{R}} X_\alpha = Y_\gamma \in \mathfrak{g}_\gamma \setminus \{0\}$. Clearly γ is a simple root in $\Sigma = R(\mathfrak{u}, \mathfrak{h})_{i\mathbf{R}}$. Since Σ is of large type, γ is non-compact. Therefore we have $C_\alpha(x_{\mathbf{R}}) = -1$ by Lemma 2.17.

Second suppose that $\Delta_{\mathfrak{b}}(\alpha)$ is of type (F_4.iv):



$(\alpha = \alpha_1)$.

Then $\gamma := \alpha + \beta_1 + \beta_2$ is an imaginary root in $R(\mathfrak{u}(\alpha), \mathfrak{h})$. Since $\alpha + \beta_2$ and $\alpha + 2\beta_1$ are not roots, we have $x_{\boldsymbol{R}}^2 X_\alpha = [X_{\beta_2}, [X_{\beta_1}, X_\alpha]] = Y_\gamma \in \mathfrak{g}_\gamma \setminus \{0\}$. It is easy to see that γ is a simple root in $\Sigma = R(\mathfrak{u}, \mathfrak{h})_{i\boldsymbol{R}}$. As before, we have $C_\alpha(x_{\boldsymbol{R}}) = -1$. Therefore the proof of Proposition 2.13(i) is completed.

As a preparation for the proof of Proposition 2.13(ii), we define an element $a_\beta \in F_L$ ($\beta \in \Delta(\mathfrak{p}_L, \mathfrak{h})$) as follows. For $\beta \in \Delta(\mathfrak{p}_L, \mathfrak{h})$, which is real by Assumption 2.22, we write A_β the element of $\mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}]$ such that $\gamma(A_\beta) = \delta_{\beta,\gamma}$ ($\gamma \in \Delta(\mathfrak{p}_L, \mathfrak{h})$), where $\delta_{\beta,\gamma}$ is Kronecker's symbol. We write $a_\beta := \exp(\pi i A_\beta)$.

LEMMA 2.26. (i) *$a_\beta \in F_L$ for any $\beta \in \Delta(\mathfrak{p}_L, \mathfrak{h})$.*

(ii) *Suppose that* $\mathfrak{u}(\alpha)$ *($\alpha \in LW(\mathfrak{u}, \mathfrak{h})$) is θ-stable. Define $n_\gamma \in \boldsymbol{Z}_{\geq 0}$ ($\gamma \in \Delta(\mathfrak{p}_L, \mathfrak{h})$) by $\theta(\alpha) - \alpha = \sum_{\gamma \in \Delta(\mathfrak{p}_L, \mathfrak{h})} n_\gamma \gamma$ (note that $\theta(\alpha)$ is the highest weight of the* $\mathfrak{l}$*-module* $\mathfrak{u}(\alpha)$*). Then we have*

$$\alpha(a_\beta^2) = \begin{cases} 1 & (n_\beta \text{ is even}), \\ -1 & (n_\beta \text{ is odd}). \end{cases}$$

In particular, we have

$$C_\alpha(a_\beta x_{\boldsymbol{R}}) = \begin{cases} C_\alpha(x_{\boldsymbol{R}}) & (n_\beta \text{ is even}), \\ -C_\alpha(x_{\boldsymbol{R}}) & (n_\beta \text{ is odd}). \end{cases}$$

PROOF. Since $\Delta(\mathfrak{p}_L, \mathfrak{h})$ consists of real roots, we have $A_\beta \in \mathfrak{h} \cap \mathfrak{s}$. For $\gamma \in \Delta(\mathfrak{p}_L, \mathfrak{h})$,

$$a_\beta X_\gamma = e^{\pi i \gamma(A_\beta)} X_\gamma = e^{\pi i \delta_{\beta,\gamma}} X_\gamma = \begin{cases} X_\gamma & (\gamma \neq \beta), \\ -X_\gamma & (\gamma = \beta). \end{cases}$$

Hence $\mathrm{Ad}(a_\beta^2)\,|_{\mathfrak{l}} = \mathrm{id}_{\mathfrak{l}}$.

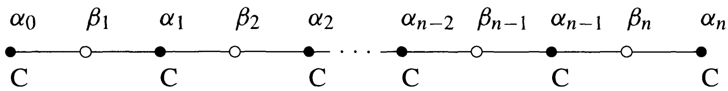
(ii) $\{\theta(\alpha) - \alpha\}(a_\beta) = \prod_{\gamma \in \Delta(\mathfrak{p}_L, \mathfrak{h})} \gamma(a_\beta)^{n_\gamma} = \beta(a_\beta)^{n_\beta} = (-1)^{n_\beta}$.

On the other hand, we have $\{\theta(\alpha) - \alpha\}(a_\beta) = \alpha(\theta(a_\beta))\alpha(a_\beta^{-1}) = \alpha(a_\beta^{-2}) = \{\alpha(a_\beta^2)\}^{-1}$. Hence $\alpha(a_\beta^2) = (-1)^{n_\beta}$. q.e.d.

REMARK 2.27. Suppose that Δ and Δ' ($\Delta \neq \Delta'$) are $\sigma^{-1} \circ \theta$-stable connected components of $\Delta_{\mathfrak{b}}$, and α is a root as in Lemma 2.26(ii). Suppose that α is contained in the root system R_Δ generated by Δ, and $\beta \in \Delta'_W$. By the definition of the connected components of $\Delta_{\mathfrak{b}}$, β is not connected by edges to any roots in Δ. Since R_Δ is θ-stable, $\theta(\alpha) \in R_\Delta$. Hence β does not appear in $\theta(\alpha) - \alpha = \sum_{\gamma \in \Delta(\mathfrak{p}_L, \mathfrak{h})} n_\gamma \gamma$: $n_\beta = 0$. Therefore we have $C_\alpha(a_\beta x_{\boldsymbol{R}}) = C_\alpha(x_{\boldsymbol{R}})$.

PROOF OF PROPOSITION 2.13(ii). It is sufficient to prove Proposition 2.13(ii) under Assumption 2.22. Hence connected components of $\Delta_{\mathfrak{b}}$ are WB-Dynkin diagrams in Table I. For any $x_{\boldsymbol{R}} \in C$ and any connected component Δ of $\Delta_{\mathfrak{b}}$, we will construct an element $a \in F_L$, which is a product of a_β's ($\beta \in \Delta_W$), such that $C_\alpha(ax_{\boldsymbol{R}}) = -1$ for any $\alpha \in \Delta_B$ with the properties that $\sigma^{-1} \circ \theta(\alpha) = \alpha$ and $R(\mathfrak{u}(\alpha), \mathfrak{h})_{i\boldsymbol{R}} = \emptyset$ (i.e., roots in Table I to which C's are attached). Then Proposition 2.13(ii) follows, since $C_\alpha(a_\gamma x_{\boldsymbol{R}}) = C_\alpha(x_{\boldsymbol{R}})$ for $\gamma \in (\Delta_{\mathfrak{b}})_W$, which is contained in another connected component of $\Delta_{\mathfrak{b}}$ (cf. Remark 2.27). Since there exists no root to which C is attached in the cases $(A.\mathrm{i})$, $(A.\mathrm{iii})$, $(B.\mathrm{iii})$, we do not need to consider these cases.

Let us consider the case when Δ is of type $(A.\mathrm{ii})$:



$(n \geq 1)$.

We first show that $\prod_{i=0}^n C_{\alpha_i}(x_{\boldsymbol{R}}) = (-1)^{n+1}$ for any $x_{\boldsymbol{R}} \in C$. Write the root $\gamma := \alpha_0 + \beta_1 + \alpha_1 + \beta_2 + \alpha_2 + \cdots + \beta_n + \alpha_n$. Since $\gamma - \beta_j$ $(1 \leq j \leq n)$ are not roots, $\gamma \in LW(\mathfrak{u}, \mathfrak{h})$. By the action of θ on α_i and β_j, we have $\theta(\gamma) = \gamma$ and hence $\gamma \in \Sigma$. Notice that $R(\mathfrak{u}(\gamma), \mathfrak{h}) = R(\mathfrak{u}(\gamma), \mathfrak{h})_{i\boldsymbol{R}} = \{\gamma\}$. It is easily verified that γ can not be written as a sum $\gamma = \gamma_1 + \gamma_2$ for $\gamma_1, \gamma_2 \in \Sigma$. Hence γ is simple in Σ. Since Σ is of large type, γ is non-compact. Then we have $C_\gamma(x_{\boldsymbol{R}}) = -1$ by Lemma 2.17.

Write $x_{\boldsymbol{R}} = \sum_{j=1}^n X_{\beta_j}$ and $X_\gamma = [X_{\alpha_0}, [X_{\beta_1}, [X_{\alpha_1}, [\ldots [X_{\beta_n}, X_{\alpha_n}] \ldots] \in \mathfrak{g}_\gamma \setminus \{0\}$, where X_δ $(\delta \in \Delta)$ is a non-zero root vector in $\mathfrak{g}_\delta$. Since the action of $\sigma^{-1} \circ \theta$ on Δ is trivial and $\sigma^{-1} \circ \theta(x_{\boldsymbol{R}}) = -x_{\boldsymbol{R}}$, we have $\sigma^{-1} \circ \theta(X_{\beta_j}) = -X_{\beta_j}$. By $\sigma^{-1} \circ \theta(X_{\alpha_i}) = C_{\alpha_i}(x_{\boldsymbol{R}})X_{\alpha_i}$ and $\sigma^{-1} \circ \theta(X_\gamma) = C_\gamma(x_{\boldsymbol{R}})X_\gamma = -X_\gamma$, we have

$$-X_\gamma = \sigma^{-1} \circ \theta(X_\gamma) = (-1)^n \prod_{i=0}^n C_{\alpha_i}(x_{\boldsymbol{R}})X_\gamma\,.$$

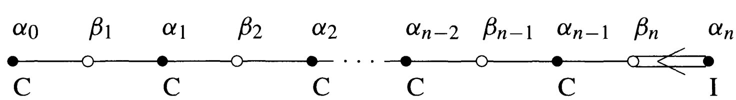
Hence $\prod_{i=0}^n C_{\alpha_i}(x_{\boldsymbol{R}}) = (-1)^{n+1}$. We notice that

$$\begin{aligned}(C_{\alpha_0}(a_{\beta_j}x_{\boldsymbol{R}}), C_{\alpha_1}(a_{\beta_j}x_{\boldsymbol{R}}), \ldots, C_{\alpha_{j-1}}(a_{\beta_j}x_{\boldsymbol{R}}), C_{\alpha_j}(a_{\beta_j}x_{\boldsymbol{R}}), \ldots, C_{\alpha_n}(a_{\beta_j}x_{\boldsymbol{R}}))\\ = (C_{\alpha_0}(x_{\boldsymbol{R}}), C_{\alpha_1}(x_{\boldsymbol{R}}), \ldots, -C_{\alpha_{j-1}}(x_{\boldsymbol{R}}), -C_{\alpha_j}(x_{\boldsymbol{R}}), \ldots, C_{\alpha_n}(x_{\boldsymbol{R}}))\end{aligned}$$

by Lemma 2.26. Therefore we can take an element a of the subgroup $\langle a_{\beta_j}; 1 \leq j \leq n\rangle$ of F_L generated by $\{a_{\beta_j}; 1 \leq j \leq n\}$ such that $C_{\alpha_i}(ax_{\boldsymbol{R}}) = -1$ for $0 \leq i \leq n$. In the case when α_0 or α_n is omitted, the proof is easier.

Similar proofs can be done in the cases $(B.\mathrm{i})$ and $(B.\mathrm{ii})$.

Consider the case when Δ is of type $(C.\mathrm{i})$:



$(n \geq 1)$.

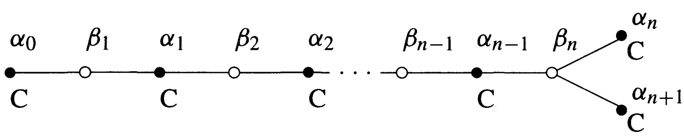
By Lemma 2.26,

$$(C_{\alpha_0}(a_{\beta_j}x_{\boldsymbol{R}}), C_{\alpha_1}(a_{\beta_j}x_{\boldsymbol{R}}), \dots , C_{\alpha_{j-1}}(a_{\beta_j}x_{\boldsymbol{R}}), C_{\alpha_j}(a_{\beta_j}x_{\boldsymbol{R}}), \dots , C_{\alpha_{n-1}}(a_{\beta_j}x_{\boldsymbol{R}}))$$
$$= \begin{cases} (C_{\alpha_0}(x_{\boldsymbol{R}}), C_{\alpha_1}(x_{\boldsymbol{R}}), \dots , -C_{\alpha_{j-1}}(x_{\boldsymbol{R}}), -C_{\alpha_j}(x_{\boldsymbol{R}}), \dots , C_{\alpha_{n-1}}(x_{\boldsymbol{R}})) & (1 \le j \le n-1), \\ (C_{\alpha_0}(x_{\boldsymbol{R}}), C_{\alpha_1}(x_{\boldsymbol{R}}), \dots , C_{\alpha_{n-2}}(x_{\boldsymbol{R}}), -C_{\alpha_{n-1}}(x_{\boldsymbol{R}})) & (j = n). \end{cases}$$

Therefore we can take $a \in \langle a_{\beta_j}; 1 \le j \le n \rangle$ such that $C_{\alpha_i}(ax_{\boldsymbol{R}}) = -1$ for $0 \le i \le n-1$. The proof of the case in $(C.\mathrm{i})$, when α_0 is omitted, is similar.

The proofs of the remaining cases except $(D.\mathrm{ii})$, $(E_7.\mathrm{iv})$ and $(E_8.\mathrm{v})$ are similar to that of the case $(C.\mathrm{i})$.

Consider the case when Δ is of type $(D.\mathrm{ii})$:



$(n \ge 1)$,

Notice the WB-Dynkin diagram obtained by omitting α_{n+1}. Then we can take $a \in \langle a_{\beta_j}; 1 \le j \le n \rangle$ such that $C_{\alpha_i}(ax_{\boldsymbol{R}}) = -1$ $(0 \le i \le n)$ by the case $(A.\mathrm{ii})$. Noticing the WB-Dynkin diagram consisting of α_n, β_n, α_{n+1}, we have $C_{\alpha_n}(x_{\boldsymbol{R}})C_{\alpha_{n+1}}(x_{\boldsymbol{R}}) = (-1)^2$ by the case $(A.\mathrm{ii})$. Hence we have $C_{\alpha_{n+1}}(x_{\boldsymbol{R}}) = -1$.

The proofs of the cases $(D.\mathrm{ii})$ when α_0 is omitted, $(E_7.\mathrm{iv})$ and $(E_8.\mathrm{v})$ are similar to the above one.

Therefore the proof of Proposition 2.13(ii) is completed.

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