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EXTREME STABILITY AND ALMOST PERIODICITY IN A DISCRETE LOGISTIC EQUATION

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Abstract. A sufficient condition is obtained for the existence of a globally asymptotically stable positive almost periodic solution of a discrete logistic equation. The sufficient condition becomes a necessary and sufficient condition for the global asymptotic stability of the positive equilibrium of the corresponding autonomous equation.

1. Introduction. The dynamical characteristics of the elementary autonomous differential equation

$$\frac{dx(t)}{dt} = rx(t)\left[1 - \frac{x(t)}{K}\right], \quad t > 0, \tag{1.1}$$

in which $r, K \in (0, \infty)$ have been utilized in the derivation of a multitude of generalized models to describe the temporal evolution of single and multispecies population systems. One assumes that $x(t)$ denotes the density or biomass of a species, r denotes the intrinsic or Malthusian growth rate and K denotes the carrying capacity of the habitat. In order to incorporate the temporal variations of the environment and resources in the model (1.1), several authors have discussed the dynamics of the nonautonomous counterpart of (1.1) given by

$$\frac{dx(t)}{dt} = r(t)x(t)\left[1 - \frac{x(t)}{K(t)}\right], \quad t > 0, \tag{1.2}$$

where the parameters $r(\cdot)$ and $K(\cdot)$ denote time-dependent coefficients. The dynamics of (1.2) and several of its generalisations have been studied by many authors (Boyce and Daley [1], Coleman [2], Coleman et al. [3], Gopalsamy [10] and many others) under the assumption that $r(\cdot)$ and $K(\cdot)$ periodic and almost periodic.

The purpose of this article is to investigate the dynamics of a discrete analogue of (1.2); discrete analogues of (1.2) correspond to the dynamics of single species populations which have non-overlapping generations. Discrete time models can also provide efficient computational models of continuous time models for numerical simulations.

One of the ways of deriving difference equations modelling the dynamics of populations with non-overlapping generations is based on appropriate modifications of models with overlapping generations. In this approach, differential equations with piecewise constant arguments have been useful. For literature on differential equations with piecewise constant arguments we refer to Wiener [20]. For instance, if we assume that the average growth rate

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in (1.2) changes at regular intervals of time, then we can incorporate this aspect in (1.2) and obtain the modified system

$$\frac{1}{x(t)}\frac{dx(t)}{dt} = r([t])\left\{1 - \frac{x([t])}{K([t])}\right\}, \quad t \neq 0, 1, 2, \ldots, \tag{1.3}$$

where $[t]$ denotes the integer part of t, $t \in (0, \infty)$. Equations of the type (1.3) are known as differential equations with piecewise constant arguments and these equations occupy a position midway between differential equations and difference equations. For applications of differential equations with piecewise constant arguments we refer to Cooke and Huang [4], Cooke and Wiener [5], Gopalsamy [10], [11], [13], Györi and Ladas [14].

By a solution of (1.3) we mean a function x which is defined for $t \in [0, \infty)$ and possesses the following properties:

1. x is continuous on $[0, \infty)$;
2. the derivative $dx(t)/dt$ exists at each point $t \in [0, \infty)$ with the possible exception of the points $t \in \{0, 1, 2, \ldots\}$ where left-sided derivatives exist;
3. the equation (1.3) is satisfied on each interval $[n, n+1)$ with $n = 0, 1, 2, \ldots$.

On any interval of the form $[n, n+1)$, $n = 0, 1, 2, \ldots$, we can integrate (1.3) and obtain for $n \leq t < n+1$, $n = 0, 1, 2, \ldots$

$$x(t) = x(n)\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)(t-n)\right], \quad n \leq t < n+1. \tag{1.4}$$

Letting $t \to n+1$, we obtain from (1.4),

$$x(n+1) = x(n)\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right], \quad n = 0, 1, 2, \ldots. \tag{1.5}$$

If one can obtain the stability characteristics of (1.5), then the stability characteristics of (1.4) can be derived. Thus it is necessary to study the dynamics of (1.5).

The autonomous counterpart of (1.5) in the form

$$x(n+1) = x(n)\exp\left[r\left(1 - \frac{x(n)}{K}\right)\right], \tag{1.6}$$

in which r and K denote positive numbers, has been proposed under certain ad-hoc hypotheses in the literature on single-species population dynamics, and studied by several authors (Fisher and Goh [9], Gopalsamy [10], May [17], [18]). One can simplify (1.6) by letting $y(n) = x(n)/K$ so that (1.6) becomes

$$y(n+1) = y(n)\exp[r(1 - y(n))], \quad n = 0, 1, 2, \ldots. \tag{1.7}$$

It is known (see for instance Fisher and Goh [9], Gopalsamy [10], May [17], [18]) that

$$0 < y(0) \quad \text{and} \quad 0 < r < 2 \Longrightarrow y(n) \to 1 \quad \text{as } n \to \infty. \tag{1.8}$$

Also if $r = 2$, the equilibrium $y(n) = 1$ becomes unstable, and as r increases beyond the value 2, repeated bifurcations occur eventually leading to chaotic dynamics (May and Oster [18]).

We are inspired to study (1.5) from the fact that if $r(t)$ and $K(t)$ are periodic and if the time domain is discretised, the resulting sequences $\{r(n)\}$ and $\{K(n)\}$ are not necessarily periodic sequences, but become almost periodic sequences. Although there exists vast literature on almost periodic differential equations (Corduneanu [7], Fink [8], Yoshizawa [22]), published literature on almost periodic difference equations is comparatively scarce. It is our belief that while differential equations with periodic coefficients have wide ranging applications, difference equations with periodic coefficients are not natural and discretisations of periodic differential equations usually lead to almost periodic difference equations.

In the following, we obtain sufficient conditions on $\{r(n)\}$ and $\{K(n)\}$ for (1.5) to have a globally attracting and locally asymptotically stable positive almost periodic solution assuming that $\{r(n)\}$ and $\{K(n)\}$ are almost periodic sequences. For some literature related to almost periodic sequences and difference equations we refer to Corduneanu [6], [7], Halanay [15], Moadab [16], Samoilenko and Perestyuk [19]. For the convenience of the reader, we will provide certain definitions and properties of almost periodic sequences in Section 4 below.

We remark that the sufficient condition on $\{r(n)\}$ which provides the existence of a globally asymptotically stable almost periodic solution of (1.5) is sharp in the sense that our sufficient condition becomes a necessary and sufficient condition for the global asymptotic stability of the positive equilibrium of the corresponding autonomous equation.

2. Boundedness and persistence.

We consider the dynamics of the logistic equation

$$x(n+1) = x(n)\exp\left[r(n)\left(1-\frac{x(n)}{K(n)}\right)\right], \quad n = 0, 1, 2, \dots \tag{2.1}$$

under the assumptions that $x(0) > 0$, $\{r(n)\}$ and $\{K(n)\}$ are strictly positive sequences of real numbers defined for $n \in \boldsymbol{Z}$, where $\boldsymbol{Z}$ denotes the set of integers. In addition, we assume that

$$0 < r_* \le r(n) \le r^*, \quad 0 < K_* \le K(n) \le K^*, \quad n \in \boldsymbol{Z}, \tag{2.2}$$

where

$$\inf_{n\in\boldsymbol{Z}} r(n) = r_*, \quad \sup_{n\in\boldsymbol{Z}} r(n) = r^*, \quad \inf_{n\in\boldsymbol{Z}} K(n) = K_*, \quad \sup_{n\in\boldsymbol{Z}} K(n) = K^*.$$

If follows from the form of (2.1) that solutions of (2.1) corresponding to positive initial values remain positive for $n \ge 0$. If (2.1) denotes a realistic model for a population system, then solutions of (2.1) have to remain bounded for $n \in \boldsymbol{Z}$. Another important aspect of population models is the persistence of the species; for instance if there exists a positive number $\alpha = \alpha(r_*, r^*, K_*, K^*)$ such that

$$\liminf_{n\to\infty} x(n) \ge \alpha > 0,$$

then the species whose dynamics is modelled by (2.1) is said to be uniformly persistent; note that the constant α is independent of the initial value $x(0)$ of the solution $x(n)$ considered.

LEMMA 2.1. *Assume that $r(n)$ and $K(n)$ satisfy the assumptions in* (2.2). *Then there exist positive constants $x_{\min}$ and $x_{\max}$ such that any positive solution of* (2.1) *will satisfy*

(eventually for all large n)

$$x_{\min} \le x(n) \le x_{\max}\,, \tag{2.3}$$

where

$$x_{\max} = \frac{K^*}{r_*}\exp(r^* - 1)\,, \quad x_{\min} = x_{\max}\exp\left(r_* - \frac{r^*}{K_*}x_{\max}\right). \tag{2.4}$$

PROOF. One can obtain from (2.1) and (2.2) that

$$x(n)\exp\left[r_* - \frac{r^*x(n)}{K_*}\right] \le x(n+1) \le x(n)\exp\left[r^* - \frac{r_*x(n)}{K^*}\right], \quad n \ge 0. \tag{2.5}$$

Define F and G by the following:

$$G(v) = v\exp\left[r_* - \frac{r^*v}{K_*}\right], \quad F(v) = v\exp\left[r^* - \frac{r_*v}{K^*}\right], \quad v \in (0, \infty). \tag{2.6}$$

One can obtain the maximum value of F by elementary methods and obtain that

$$F(v) \le F\left(\frac{K^*}{r_*}\right). \tag{2.7}$$

The following diagram (see Figure 1) illustrates the existence of a trapping region for positive solutions of (2.1) and also demonstrates the existence of the eventual lower and upper bounds of solutions of (2.1).

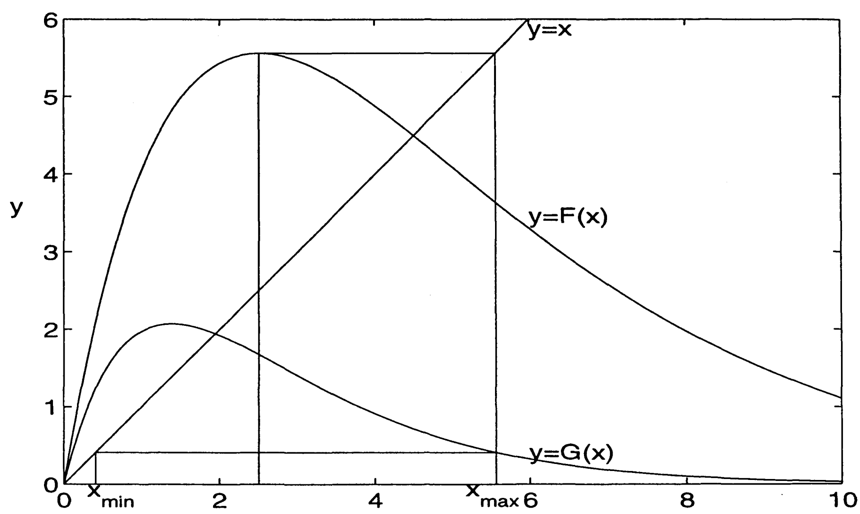

FIGURE 1. The trapping region, $[x_{\min}, x_{\max}]$.

The eventual lower bound $x_{\min}$ is obtained by using the eventual upper bound $x_{\max}$ so that

$$
\begin{aligned}
x_{\min} = G(x_{\max}) &= x_{\max} \exp\left(r_* - \frac{r^*}{K_*} x_{\max}\right) \\
&= \frac{K^*}{r_*} \exp\left[r^* + r_* - 1 - \left(\frac{r^* K^*}{r_* K_*}\right) \exp(r^* - 1)\right].
\end{aligned} \tag{2.8}
$$

The region defined by the interval $[x_{\min}, x_{\max}]$ can be called a trapping region in the sense that this interval is invariant for the dynamical system (2.1) and if $0 < x(0) \notin [x_{\min}, x_{\max}]$, then for some n_0, $x(n_0) \in [x_{\min}, x_{\max}]$ and $x(n) \in [x_{\min}, x_{\max}]$ for $n \geq n_0$. The independence of $x_{\min}$ and $x_{\max}$ on $x(0)$ demonstrates the uniform persistence of the species governed by the dynamics of (2.1). This completes the proof. □

In the next Lemma, we establish certain technical results which will be used in the subsequent investigations of stability and asymptotic behaviour of positive solutions of (2.1).

LEMMA 2.2. *Assume that $r(n)$ and $K(n)$ satisfy the hypotheses in* (2.2). *Let $x(n)$ denote any positive solution of* (2.1). *Then*

$$
\limsup_{n\to\infty} \frac{x(n+1)}{x(n)} \geq 1 \quad \textit{and} \quad 0 < \liminf_{n\to\infty} \frac{x(n+1)}{x(n)} \leq 1. \tag{2.9}
$$

Furthermore

$$
0 < \liminf_{n\to\infty} \frac{x(n)}{K(n)} \leq 1 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{x(n)}{K(n)} \geq 1. \tag{2.10}
$$

PROOF. Since positive solutions $x(n)$ of (2.1) remain bounded and stay bounded away from zero, we have

$$
\frac{x(n+1)}{x(n)} = \exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right], \quad n = 0, 1, 2, \ldots.
$$

By Lemma 2.1, the sequence $x(n+1)/x(n)$ remains bounded for $n \in \mathbf{Z}$. Hence there exists a positive real number ν such that

$$
\liminf_{n\to\infty} \frac{x(n+1)}{x(n)} = \liminf_{n\to\infty} \left\{\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right]\right\} = \nu. \tag{2.11}
$$

We claim that $0 < \nu \leq 1$; if $\nu > 1$, then let $\nu = 1 + \delta$ for some $\delta > 0$. From the properties of limit inferior, it will follow that for a positive number $\varepsilon < \delta$, there will exist a positive integer $N_1(\varepsilon)$ such that

$$
\frac{x(n+1)}{x(n)} > 1 + \delta - \varepsilon \quad \text{for } n \geq N_1(\varepsilon). \tag{2.12}
$$

One can derive from (2.12) that

$$
x(n) > (1 + \delta - \varepsilon)^{n-N_1} x(N_1), \quad n \geq N_1 + 1,
$$

which leads to the conclusion

$$
x(n) \to \infty \quad \text{as } n \to \infty,
$$

and this contradicts the boundedness of positive solutions of (2.1). Hence our claim $0 < \nu < 1$ is valid.

We obtain from (2.11) that

$$\liminf_{n\to\infty} \exp\left[r(n)\left(1-\frac{x(n)}{K(n)}\right)\right] = \nu \leq 1. \tag{2.13}$$

Since $0 < r_* \leq \liminf_{n\to\infty} r(n) \leq r^*$, the validity of (2.13) demands that

$$\liminf_{n\to\infty}\left(1-\frac{x(n)}{K(n)}\right) \leq 0, \tag{2.14}$$

which implies that

$$\limsup_{n\to\infty}\frac{x(n)}{K(n)} \geq 1. \tag{2.15}$$

By considering the $\limsup_{n\to\infty} x(n+1)/x(n)$, one can prove the validity of the first inequality of (2.10). For instance, there exists a positive number, say μ, such that

$$\limsup_{n\to\infty}\frac{x(n+1)}{x(n)} = \limsup_{n\to\infty}\left\{\exp\left[r(n)\left(1-\frac{x(n)}{K(n)}\right)\right]\right\} = \mu. \tag{2.16}$$

We can show that $\mu \geq 1$. If $\mu < 1$, we can let $\mu = 1-\delta$ for some $0 < \delta < 1$; then it will follow from

$$\limsup_{n\to\infty}\frac{x(n+1)}{x(n)} = 1-\delta \tag{2.17}$$

that there exist ε, $0 < \varepsilon < \delta$ and a positive integer $N_2(\varepsilon)$ such that

$$\frac{x(n+1)}{x(n)} < (1-\delta+\varepsilon) \quad \text{for } n \geq N_2(\varepsilon). \tag{2.18}$$

One can show that (2.18) leads to

$$x(n) < (1-\delta+\varepsilon)^{n-N_2}x(N_2), \quad n \geq N_2+1, \tag{2.19}$$

which implies that $x(n) \to 0$ as $n \to \infty$. This contradicts the persistence of the species governed by (2.1). Hence our conclusion $\mu \geq 1$ is valid. As before, we derive from

$$\limsup_{n\to\infty}\left\{\exp\left[r(n)\left(1-\frac{x(n)}{K(n)}\right)\right]\right\} \geq 1,$$

using the nature of $r(n)$, that

$$0 < \liminf_{n\to\infty}\frac{x(n)}{K(n)} \leq 1.$$

The proof of the Lemma 2.2 is complete. □

We have provided several computer simulations of the results corresponding to Lemma 2.2 and graphical results displaying two arbitrary positive sequences $\{x_1(n)/K(n)\}$ and $\{x_2(n)/K(n)\}$; $\{x_1(n)\}$, $\{x_2(n)\}$ are solutions of (2.1), are illustrated below by Figures 2–5.

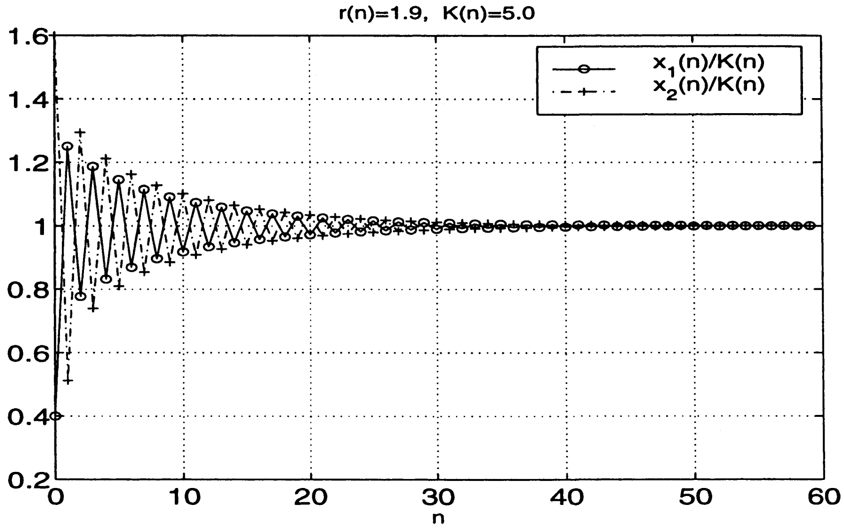

FIGURE 2.

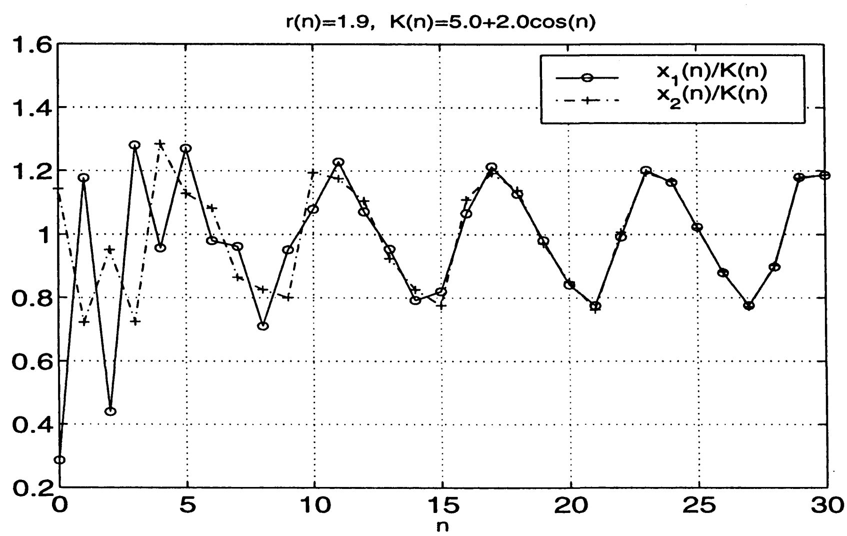

FIGURE 3.

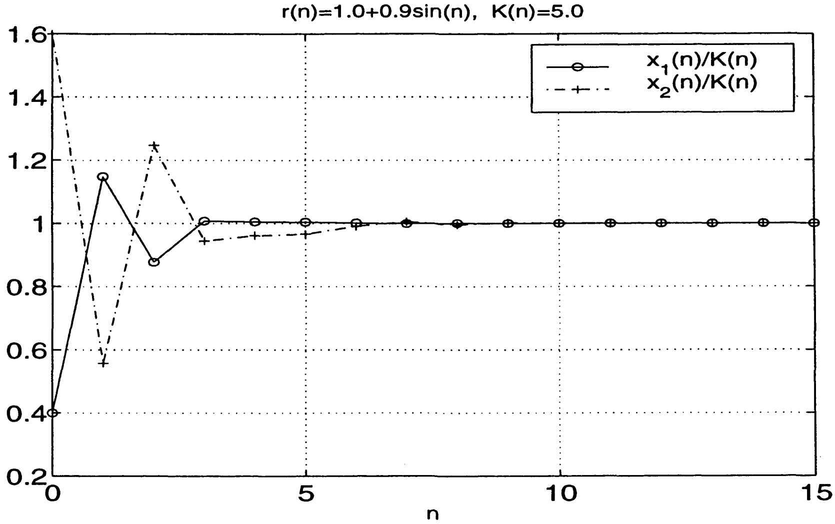

FIGURE 4.

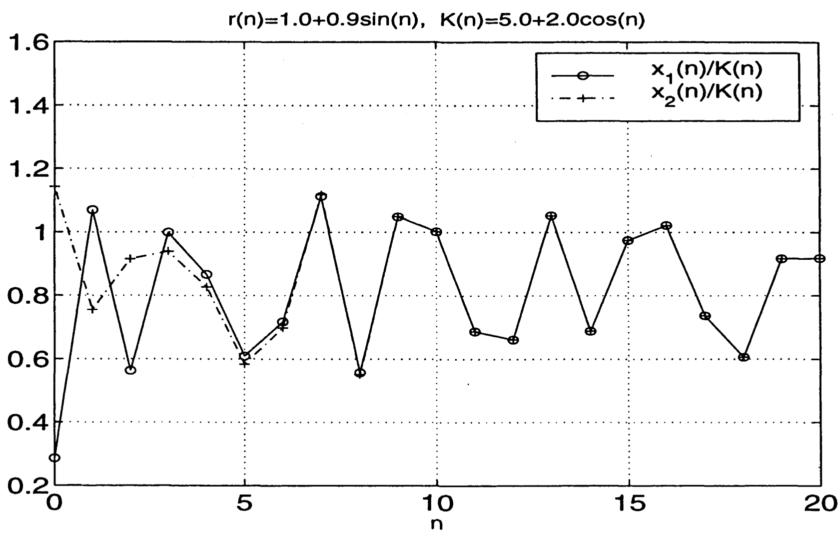

FIGURE 5.

3. Extreme stability of systems. Consider a discrete dynamical system described by a scalar difference equation of the form

$$x(n+1) = F(n, x(n)), \quad n \in \boldsymbol{Z},$$

where $F : \boldsymbol{Z} \times \boldsymbol{R} \to \boldsymbol{R}$ is a suitably defined function. Let $\{x(n)\}$ and $\{y(n)\}$ denote any two arbitrary solutions defined for $n \in \boldsymbol{Z}$. The above dynamical system is said to be "extremely stable" if and only if the following holds:

$$\lim_{n\to\infty} |x(n) - y(n)| = 0.$$

Yoshizawa [21] has utilised the concept of extreme stability of differential and delay-differential equations in the derivation of sufficient conditions for the existence of periodic and almost periodic solutions. Gopalsamy and He [12] have used this type of stability of systems in their discussion of almost periodic solutions of integrodifferential equations. It is the opinion of the authors that "extreme stability" has not been used in the literature on difference equations.

In this section we obtain sufficient conditions for the extreme stability of (2.1). Since only positive solutions of (2.1) are of interest in applications, the system (2.1) is said to be extremely stable if and only if

$$\lim_{n\to\infty} |\tilde{x}(n) - \tilde{y}(n)| = 0,$$

where $\{\tilde{x}(n)\}$ and $\{\tilde{y}(n)\}$ denote any two positive solutions of (2.1) defined for $n \in \{0, 1, 2, \dots\}$.

LEMMA 3.1. *Let $\{x(n)\}$ denote an arbitrary positive solution of* (2.1). *Let $\{r(n)\}$, $\{K(n)\}$ denote strictly positive bounded sequences defined for $n \in \boldsymbol{Z}$. Suppose further that*

$$0 < r_* \le r(n) \le r^* < 2, \quad n \in \boldsymbol{Z}. \tag{3.1}$$

Then there exists a positive integer N such that

$$\left|1 - r(n)\frac{x(n)}{K(n)}\right| < 1, \quad n \ge N, \tag{3.2}$$

$$\left|\left[1 - r(n)\frac{x(n)}{K(n)}\right] \exp\left[r(n)\left\{1 - \frac{x(n)}{K(n)}\right\}\right]\right| < 1, \quad n \ge N. \tag{3.3}$$

PROOF. Since $r(n)$, $K(n)$ and $x(n)$ are strictly positive for all $n \ge 0$,

$$1 - r(n)\frac{x(n)}{K(n)} < 1 \quad \text{for } n \ge 0. \tag{3.4}$$

We know from (2.15) that $\limsup_{n\to\infty} x(n)/K(n) \ge 1$; if we let

$$\limsup_{n\to\infty} \frac{x(n)}{K(n)} = \mu,$$

then there exists a positive integer $N = N(\varepsilon)$ such that for any positive number ε,

$$\frac{x(n)}{K(n)} < \mu + \varepsilon, \quad \mu \ge 1, \quad n \ge N(\varepsilon). \tag{3.5}$$

By hypothesis (3.1) on $\{r(n)\}$, there exists a positive number δ such that

$$r(n) \leq 2(1-\delta), \quad 0 < \delta < 1. \tag{3.6}$$

Corresponding to such a δ, we choose ε, $0 < \varepsilon < 1$, satisfying

$$\delta = 1 - \frac{1}{\mu + \varepsilon} \quad \text{or} \quad \varepsilon = \frac{1}{1-\delta} - \mu. \tag{3.7}$$

It will then follow that

$$1 - r(n)\frac{x(n)}{K(n)} > 1 - 2(1-\delta)(\mu+\varepsilon) > -1 \quad \text{for } n > N. \tag{3.8}$$

From (3.4) and (3.8), one obtains (3.2).

From $\limsup_{n\to\infty} x(n+1)/x(n) \geq 1$, we note that there exists a real number $\nu \geq 1$ such that

$$\limsup_{n\to\infty} \left\{\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right]\right\} = \nu \geq 1. \tag{3.9}$$

It follows that for each positive ε_1, there exists a positive integer $N_1 = N_1(\varepsilon_1)$ such that

$$\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right] < \nu(1+\varepsilon_1) \quad \text{for } n \geq N_1(\varepsilon_1). \tag{3.10}$$

We have from (3.2) that there exists a δ_1, $0 < \delta_1 < 1$, such that

$$1 - r(n)\frac{x(n)}{K(n)} \leq 1 - \delta_1 \quad \text{for } n \geq N_1; \tag{3.11}$$

for such a δ_1, we choose ε_1 satisfying

$$\delta_1 = 1 - \frac{1}{\nu(1+\varepsilon_1)}. \tag{3.12}$$

By using (3.11) and (3.12), we obtain

$$\begin{aligned}\left[1 - r(n)\frac{x(n)}{K(n)}\right]\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right] &< (1-\delta_1)\nu(1+\varepsilon_1) \\ &< 1 \quad \text{for } n \geq N_2,\end{aligned} \tag{3.13}$$

where $N_2 = \max\{N, N_1\}$.

We recall from (2.13) that there exists a positive integer $N_3 = N_3(\varepsilon_2)$ such that for any positive number ε_2,

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(n)} = \liminf_{n\to\infty} \left\{\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right]\right\} = \nu \leq 1, \tag{3.14}$$

and hence

$$\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right] > \nu(1-\varepsilon_2) \quad \text{for } n \geq N_3. \tag{3.15}$$

We also have from (3.2) that there exists a δ_2, $0 < \delta_2 < 1$, such that

$$1 - r(n)\frac{x(n)}{K(n)} > -(1+\delta_2); \tag{3.16}$$

if we choose ε_2 satisfying

$$\delta_2 = \frac{1}{\nu(1-\varepsilon_2)} - 1\,, \tag{3.17}$$

then we have from (3.14)–(3.17) that

$$\begin{aligned}\left[1 - r(n)\frac{x(n)}{K(n)}\right]\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right] &> -(1+\delta_2)\nu(1-\varepsilon_2)\\ &> -1 \quad \text{for } n \geq N_4\,,\end{aligned} \tag{3.18}$$

where $N_4 = \max\{N_3, N\}$. The assertion (3.3) follows from (3.18) and (3.13). The proof is complete. □

THEOREM 3.2. *Let* $\{x(n)\}$ *and* $\{y(n)\}$ *denote arbitrary positive solutions of* (2.1). *Suppose that* $\{r(n)\}$, $\{K(n)\}$ *are strictly positive and suppose* $\{r(n)\}$ *satisfies* (2.2). *Then the system* (2.1) *is extremely stable in the sense that*

$$\lim_{n\to\infty} |x(n) - y(n)| = 0\,. \tag{3.19}$$

PROOF. Due to the positive nature of $\{x(n)\}$ and $\{y(n)\}$, we have from (2.1),

$$\begin{aligned}\ln[x(n+1)] &= \ln[x(n)] + r(n) - \frac{r(n)}{K(n)}x(n)\,,\\ \ln[y(n+1)] &= \ln[y(n)] + r(n) - \frac{r(n)}{K(n)}y(n)\,,\end{aligned} \tag{3.20}$$

for $n = 0, 1, 2, \ldots$. By using the mean value theorem of differential calculus, we derive from (3.20),

$$\begin{aligned}\big|\ln[x(n+1)] - \ln[y(n+1)]\big| &= \left|\ln[x(n)] - \ln[y(n)] - \frac{r(n)}{K(n)}[x(n) - y(n)]\right|\\ &= \left|1 - \frac{r(n)}{K(n)}\theta(n)\right|\big|\ln[x(n)] - \ln[y(n)]\big|\,,\end{aligned} \tag{3.21}$$

where $\theta(n) = \theta(n, x(n), y(n))$ lies between $x(n)$ and $y(n)$ for $n = 0, 1, 2, \ldots$. One can derive from (3.21) that

$$\big|\ln[x(n)] - \ln[y(n)]\big| = \left(\prod_{i=0}^{n-1}|\alpha(i)|\right)\big|\ln[x(0)] - \ln[y(0)]\big|\,, \tag{3.22}$$

where

$$\alpha(i) = 1 - \frac{r(i)}{K(i)}\theta(i)\,, \quad i = 0, 1, 2, \ldots\,. \tag{3.23}$$

Since $\theta(n)$ lies between two solutions of (2.1), by Lemma 3.1 we have $|\alpha(i)| < 1$ for $i \geq N$. It will follow from (3.22) that

(3.24)

$$\big|\ln[x(n)] - \ln[y(n)]\big| = \left(\prod_{i=0}^{N-1}|\alpha(i)|\right)\left(\prod_{i=N}^{n-1}|\alpha(i)|\right)\big|\ln[x(0)] - \ln[y(0)]\big|\,, \quad n \geq N+1\,.$$

By using the fact

$$\prod_{i=N}^{n-1} |\alpha(i)| \to 0 \quad \text{as } n \to \infty$$

in (3.24), we obtain (3.19) and the proof is complete. □

4. Almost periodic solution. In this section we establish the existence and stability of a positive almost periodic solution of (2.1). For convenience of the reader we recall the following definitions.

We first give main properties of almost periodic sequences used below. Let $\{x(n)\}$ be a sequence of real numbers defined for $n \in \mathbf{Z}$. An integer p is called an ε-almost period of the sequence $\{x(n)\}$ if for any $n \in \mathbf{Z}$,

$$|x(n+p) - x(n)| < \varepsilon .$$

If a sequence $\{x(n)\}$ is periodic with period p, i.e., $x(n+p) = x(n)$ for all $n \in \mathbf{Z}$, then for any $\varepsilon > 0$, the numbers jp, $j \in \mathbf{Z}$, are ε-almost periods of this sequence. It is not difficult to see that if p is an ε-almost period of a sequence $\{x(n)\}$, then $-p$ is also an ε-almost period of this sequence.

DEFINITION. A sequence $\{x(n)\}$ is called almost periodic if for any $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods, i.e., there exists a natural number $N(\varepsilon)$ such that for an arbitrary $m \in \mathbf{Z}$, there is at least one integer p in the segment $[m, m+N]$ for which the inequality

$$|x(n+p) - x(n)| < \varepsilon , \quad n \in \mathbf{Z}$$

holds.

We narrate a number of results on almost periodic sequences for the benefit of the reader. The proofs of the following results can be found in Samoilenko and Perestyuk [19].

THEOREM A. *An almost periodic sequence is bounded.*

THEOREM B. *Let for every $m \in \{1, 2, 3, \dots\}$, the sequence $\{x_m(n)\}$, $n \in \mathbf{Z}$, be almost periodic. If the sequence $\{x_m(n)\}$ converges to $\{y(n)\}$ as $m \to \infty$ uniformly for $n \in \mathbf{Z}$, then $\{y(n)\}$ is almost periodic.*

THEOREM C. *A sequence $\{x(n)\}$ is almost periodic if and only if for any sequence of integers $\{m_i\}$ there exists a subsequence $\{m_{k_j}\}$ such that the sequence $\{x(n+m_{k_j})\}$ converges for $j \to \infty$ uniformly with respect to $n \in \mathbf{Z}$.*

THEOREM D. *If $\{x(n)\}$ and $\{y(n)\}$ are almost periodic real sequences, then $\{x(n) + y(n)\}$ and $\{x(n)y(n)\}$ are also almost periodic.*

THEOREM E. *If $\{x(n)\}$ and $\{y(n)\}$ are almost periodic and $\varepsilon > 0$ is an arbitrary real number, then there exists a relatively dense set of their common ε-almost periods.*

DEFINITION. A sequence $\{x(n)\}$ of real numbers is said to be asymptotically almost periodic if and only if there exists two sequences $\{u(n)\}$ and $\{v(n)\}$ such that

$$x(n) = u(n) + v(n), \quad n \in \mathbf{Z},$$

where $\{u(n)\}$ is almost periodic and $v(n) \to 0$ as $n \to \infty$.

We note that the concept of an asymptotically almost periodic sequence has been used by Halanay [15]. Apart from this, asymptotic almost periodic sequences have not been used in the literature on difference equations. However, asymptoticaly almost periodic functions have been used extensively in differential and functional differential equations by many authors.

In this section we assume that the coefficients $\{r(n)\}$ and $\{K(n)\}$ of (2.1) are strictly positive almost periodic sequences. It is of interest to enquire under what conditions, there exists an asymptotically stable almost periodic solution. It is known that if r and K are positive constants, then

$$x(0) > 0 \quad \text{and} \quad 0 < r < 2 \Longrightarrow x(n) \to 1 \quad \text{as } n \to \infty.$$

If $r(\cdot)$ and $K(\cdot)$ are not constants and are almost periodic, it is plausible that (2.1) has an almost periodic solution whose stability characteristics mimic those of the equilibrium solution of the autonomous counterpart. The existence of an almost periodic solution of (2.1) is established in the following:

THEOREM 4.1. *Suppose that* $\{r(n)\}$, $\{K(n)\}$, $n \in \mathbf{Z}$, *are strictly positive almost periodic sequences satisfying* (2.2) *and* (3.1). *Then* (2.1) *has a unique globally asymptotically stable almost periodic solution.*

PROOF. Let $\{x(n)\}$ denote an arbitrary positive solution of (2.1). We remark that since both $\{r(n)\}$ and $\{K(n)\}$ are almost periodic and since $\inf_{n \in Z} K(n) = K_* > 0$, it follows that $\{r(n)/K(n)\}$ is almost periodic (Theorem D above).

Let $\{\tau'_p\}$ and $\{\tau'_q\}$ be sequences of nonnegative integers for all $p, q \geq 0$ such that

$$\tau'_p \to \infty \quad \text{as} \quad p \to \infty, \quad \tau'_q \to \infty \quad \text{as} \quad q \to \infty.$$

Let $\{\tau_p\}$ and $\{\tau_q\}$ denote subsequences of $\{\tau'_p\}$ and $\{\tau'_q\}$, respectively.

The sequences $\{x(n + \tau_p)\}$ and $\{x(n + \tau_q)\}$ are solutions of

$$\begin{aligned} &\ln[x(n + 1 + \tau_p)] = \ln[x(n + \tau_p)] + r(n + \tau_p) - \frac{r(n + \tau_p)}{K(n + \tau_p)} x(n + \tau_p), \\ &\ln[x(n + 1 + \tau_q)] = \ln[x(n + \tau_q)] + r(n + \tau_q) - \frac{r(n + \tau_q)}{K(n + \tau_q)} x(n + \tau_q), \end{aligned} \tag{4.1}$$

where $p, q = 0, 1, 2, \ldots$. We have from (4.1) that

$$\begin{aligned}
&\ln[x(n+1+\tau_p)] - \ln[x(n+1+\tau_q)] \\
&\quad = \ln[x(n+\tau_p)] - \ln[x(n+\tau_q)] + [r(n+\tau_p) - r(n+\tau_q)] \\
&\qquad - \frac{r(n+\tau_p)}{K(n+\tau_p)}[x(n+\tau_p) - x(n+\tau_q)] \\
&\qquad - x(n+\tau_q)\left[\frac{r(n+\tau_p)}{K(n+\tau_p)} - \frac{r(n+\tau_q)}{K(n+\tau_q)}\right]
\end{aligned} \tag{4.2}$$

for all n, τ_p, τ_q, $n+\tau_p > 0$ and $n+\tau_q > 0$. We can rewrite (4.2) by using the mean value theorem of differential calculus as follows:

$$\begin{aligned}
&\ln[x(n+1+\tau_p)] - \ln[x(n+1+\tau_q)] \\
&\quad = \left[1 - \frac{r(n+\tau_p)}{K(n+\tau_q)}\theta(n, \tau_p, \tau_q)\right](\ln[x(n+\tau_p)] - \ln[x(n+\tau_q)]) \\
&\qquad + [r(n+\tau_p) - r(n+\tau_q)] \\
&\qquad - x(n+\tau_q)\left[\frac{r(n+\tau_p)}{K(n+\tau_p)} - \frac{r(n+\tau_q)}{K(n+\tau_q)}\right],
\end{aligned} \tag{4.3}$$

where $\theta(n, \tau_p, \tau_q)$ lies between $x(n+\tau_p)$ and $x(n+\tau_q)$ for all $n+\tau_p \geq 0$, $n+\tau_q \geq 0$. It follows from (4.3) that

$$\begin{aligned}
&\big|\ln[x(n+1+\tau_p)] - \ln[x(n+1+\tau_q)]\big| \\
&\quad \leq \left|1 - \frac{r(n+\tau_p)}{K(n+\tau_q)}\theta(n, \tau_p, \tau_q)\right| \big|\ln[x(n+\tau_p)] - \ln[x(n+\tau_q)]\big| \\
&\qquad + |r(n+\tau_p) - r(n+\tau_q)| \\
&\qquad + x(n+\tau_q)\left|\frac{r(n+\tau_p)}{K(n+\tau_p)} - \frac{r(n+\tau_q)}{K(n+\tau_q)}\right|.
\end{aligned} \tag{4.4}$$

Let ε be an arbitrary positive number. By the almost periodicity of $\{r(n)\}$, $\{K(n)\}$ and the boundedness of $\{x(n)\}$, it will follow (by Theorem E) that there exists a positive integer $N = N(\varepsilon)$ such that for $n+\tau_p \geq N$, $n+\tau_q \geq N$

$$\begin{aligned}
|r(n+\tau_p) - r(n+\tau_q)| &< \frac{\varepsilon}{2}, \\
x(n+\tau_q)\left|\frac{r(n+\tau_p)}{K(n+\tau_p)} - \frac{r(n+\tau_q)}{K(n+\tau_q)}\right| &< \frac{\varepsilon}{2}.
\end{aligned} \tag{4.5}$$

Also, if N is sufficiently large, it follows from Lemma 2.1 that there exists a positive number β so that

$$\left|1 - \frac{r(n+\tau_p)}{K(n+\tau_q)}\theta(n, \tau_p, \tau_q)\right| \leq \beta < 1. \tag{4.6}$$

We have from (4.4)–(4.6) that

$$\big|\ln[x(n+1+\tau_p)] - \ln[x(n+1+\tau_q)]\big| \leq \beta\big|\ln[x(n+\tau_p)] - \ln[x(n+\tau_q)]\big| + \varepsilon,$$

and hence

$$\sup_{\tau_p \geq 0, \tau_q \geq 0} \left| \ln[x(n+1+\tau_p)] - \ln[x(n+1+\tau_q)] \right|$$
$$\leq \beta \sup_{\tau_p \geq 0, \tau_q \geq 0} \left| \ln[x(n+\tau_p)] - \ln[x(n+\tau_q)] \right| + \varepsilon\,,$$

which implies that

$$\sup_{\tau_p \geq 0, \tau_q \geq 0} \left| \ln[x(n+\tau_p)] - \ln[x(n+\tau_q)] \right| < \frac{\varepsilon}{1-\beta} \quad \text{for } n \geq N\,.$$

It follows that the sequence $\{x(n)\}$ is asymptotically almost periodic.

Since $\{x(n)\}$ is asymptotically almost periodic, there exists an almost periodic sequence $\{\tilde{x}(n)\}$ and another sequence $\{X(n)\}$ such that

$$x(n) = \tilde{x}(n) + X(n)\,, \qquad X(n) \to 0 \quad \text{as } n \to \infty\,.$$

The sequence $\{x(n)\}$ is a solution of (2.1) and hence we have

$$\begin{aligned} \tilde{x}(n+1) + X(n+1) &= [\tilde{x}(n) + X(n)] \exp\left\{r(n)\left[1 - \frac{\tilde{x}(n)+X(n)}{K(n)}\right]\right\} \\ &= \tilde{x}(n) \exp\left\{r(n)\left[1 - \frac{\tilde{x}(n)}{K(n)}\right]\right\} \exp\left[-r(n)\frac{X(n)}{K(n)}\right] \\ &\quad + X(n) \exp\left\{r(n)\left[1 - \frac{\tilde{x}(n)+X(n)}{K(n)}\right]\right\}. \end{aligned}$$

By using $X(n) \to 0$ as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \left\{\tilde{x}(n+1) - \tilde{x}(n) \exp\left[r(n)\left(1 - \frac{\tilde{x}(n)}{K(n)}\right)\right]\right\} = 0\,. \tag{4.7}$$

Since $\{\tilde{x}(n)\}$, $\{r(n)\}$, $\{K(n)\}$ are almost periodic, we claim that

$$\tilde{x}(n+1) = \tilde{x}(n) \exp\left[r(n)\left(1 - \frac{\tilde{x}(n)}{K(n)}\right)\right] \quad \text{for all } n \in \mathbf{Z}\,. \tag{4.8}$$

If (4.8) is not valid, we let

$$w(n) = \tilde{x}(n+1) - \tilde{x}(n) \exp\left[r(n)\left(1 - \frac{\tilde{x}(n)}{K(n)}\right)\right], \quad n \in \mathbf{Z}\,,$$

and suppose that $w(n^*) \neq 0$. We let $\tilde{\varepsilon} = |w(n^*)|/2 > 0$ and note that there exists an integer $L = L(\tilde{\varepsilon}) > 0$ such that any set of L consecutive integers contains an integer, say m, for which

$$|w(n^* + m) - w(n^*)| \leq \frac{|w(n^*)|}{2}\,,$$

which then implies that

$$|w(n^* + m)| \geq \frac{|w(n^*)|}{2}\,.$$

Now let $I_j = [jm, (j+1)m]$, where $j = 0, 1, 2, \ldots$, denote intervals of length m. Within each interval I_j there is a corresponding m_j such that

$$|w(n^* + m_j)| \geq \frac{|w(n^*)|}{2}, \quad j = 0, 1, 2, \ldots .$$

We let $m_j \to \infty$ as $j \to \infty$ and note that $\lim_{n\to\infty} w(n) \geq |w(n^*)|/2$, which contradicts the fact that $w(n) \to 0$ as $n \to \infty$. Hence

$$w(n) = \tilde{x}(n+1) - \tilde{x}(n) \exp\left[r(n)\left(1 - \frac{\tilde{x}(n)}{K(n)}\right)\right] \equiv 0 \quad \text{for all } n \in \mathbf{Z},$$

which asserts that $\tilde{x}(n)$ is an almost periodic solution of (2.1).

The global attractivity of the almost periodic solution $\tilde{x}(n)$ is a consequence of the fact established in Theorem 3.2. For instance, if $x(n)$ is an arbitrary positive solution, then by Theorem 3.2, it follows that

$$\lim_{n\to\infty} |\tilde{x}(n) - x(n)| = 0,$$

from which the global attractivity of $\{\tilde{x}(n)\}$ follows. The local asymptotic stability of $\{\tilde{x}(n)\}$ can be established as follows: We let

$$x(n) = \tilde{x}(n) + y(n),$$

where $y(n)$ denotes the deviation of $\tilde{x}(n)$ from an arbitrary solution $\{x(n)\}$ of (2.1). The linear variational system corresponding to $y(n)$ is given by

$$y(n+1) = \beta(n) y(n), \tag{4.9}$$

where

$$\beta(n) = \left[1 - r(n)\frac{\tilde{x}(n)}{K(n)}\right] \exp\left[r(n)\left(1 - \frac{\tilde{x}(n)}{K(n)}\right)\right].$$

From (3.3) of Lemma 3.1, there exists an integer $\tilde{N}$ such that $|\beta(n)| < 1$ for all $n \geq \tilde{N}$. We have from (4.9) that

$$\begin{aligned} |y(n)| &= \left(\prod_{i=0}^{n-1} |\beta(i)|\right) |y(0)| \\ &= \left(\prod_{i=0}^{\tilde{N}-1} |\beta(i)|\right)\left(\prod_{i=\tilde{N}}^{n-1} |\beta(i)|\right) |y(0)|, \quad n \geq 1. \end{aligned} \tag{4.10}$$

Let $\varepsilon > 0$ be given. Choose $\delta = \delta(\varepsilon)$ as follows:

$$\delta = \frac{\varepsilon}{\prod_{i=0}^{\tilde{N}-1} |\beta(i)|}, \quad \text{where} \quad |\beta(i)| > 1 \quad \text{for } i = 0, 1, \ldots, \tilde{N} - 1;$$

if $|y(0)| < \delta$, then it follows from (4.10) that

$$|y(n)| < \left(\prod_{i=0}^{n-1} |\beta(i)|\right) \frac{\varepsilon}{\prod_{i=0}^{\tilde{N}-1} |\beta(i)|} < \varepsilon \quad \text{for } n \geq 1.$$

The local asymptotic stability of $\{\tilde{x}(n)\}$ follows; this together with the global attractivity of $\{\tilde{x}(n)\}$ implies the global asymptotic stability of the almost periodic solution of $\{\tilde{x}(n)\}$, and the proof is complete. □

The results of Theorem 3.2 and Theorem 4.1 are further illustrated by the following figures. Figure 6 displays typical behaviour of positive solutions approaching an almost periodic solution of (2.1). Figure 7 illustrates positive solutions approaching a periodic solution of (2.1) when $\{r(n)\}$, $\{K(n)\}$ are periodic sequences.

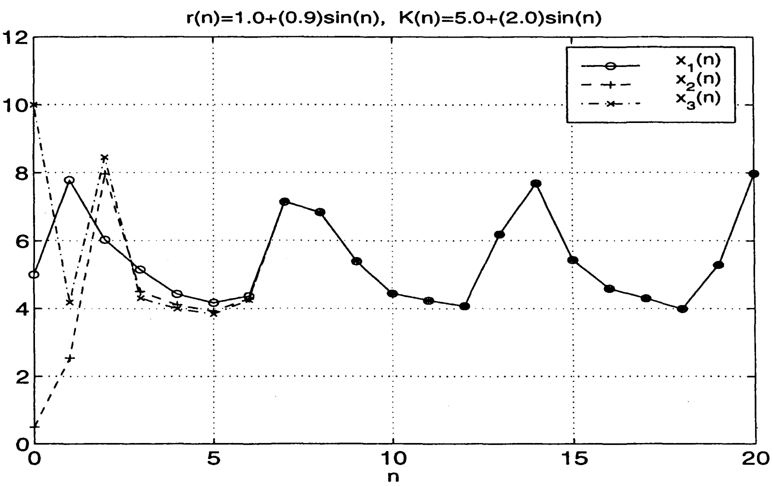

FIGURE 6. Solutions of (2.1) approaching an almost periodic solution.

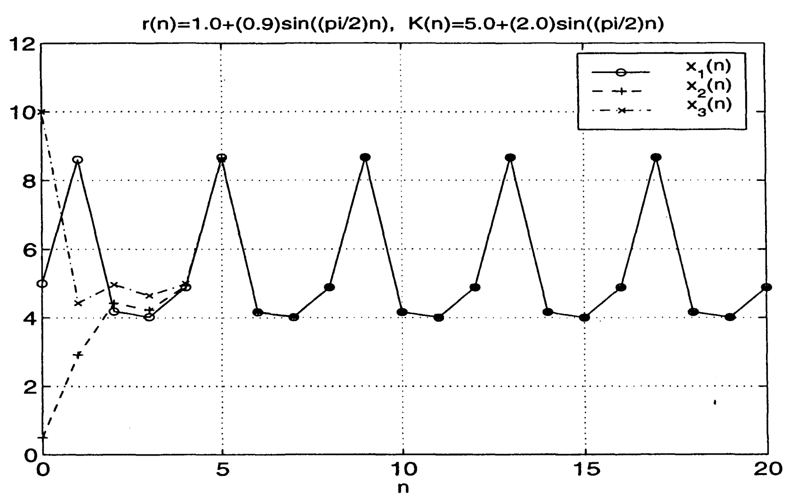

FIGURE 7. Solutions of (2.1) approaching a periodic solution.

5. Conclusion. We have shown that if $\{r(n)\}$ and $\{K(n)\}$ are almost periodic positive sequences such that

$$0 < r_* \le r(n) \le r^* < 2\,, \quad 0 < K_* \le K(n)\,, \quad n \in \mathbf{Z}\,, \tag{5.1}$$

then all positive solutions of the nonautonomous equation

$$x(n+1) = x(n)\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right], \quad n = 0, 1, 2, \ldots \tag{5.2}$$

satisfy

$$\lim_{n\to\infty} |x(n) - \tilde{x}(n)| = 0\,, \tag{5.3}$$

where $\tilde{x}(n)$ is an almost periodic solution of (5.2). As a particular case, if $\{r(n)\}$ and $\{K(n)\}$ are periodic with a common period p so that $r(n+p) = r(n)$ and $K(n+p) = K(n)$, $n \in \mathbf{Z}$, and satisfy (5.1), then the corresponding periodic system (5.2) has a globally asymptotically stable periodic solution of (5.2). This itself is a new result in our opinion. We note that if $f(t)$ is a periodic function of the continuous variable $t \in (-\infty, \infty)$ (or $t \in (0, \infty)$), then a discrete sequence $\{f(n)\}$, $n \in \mathbf{Z}$, is not necessarily periodic, but $\{f(n)\}$ is almost periodic. Thus it is necessary to consider and investigate the dynamics of almost periodic discrete analogues of continuous time systems rather the discrete periodic analogues.

The condition established in (5.1) becomes a necessary and sufficient condition for the global asymptotic stability of the autonomous analogue of (5.2). In this sense our condition (5.1) is sharp; however we conjecture that the condition (5.1) can be replaced by a condition of the form

$$0 < \lim_{n\to\infty}\left[\frac{1}{n}\sum_{j=0}^{n-1} r(j)\right] < 2\,.$$

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