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KENMOTSU TYPE REPRESENTATION FORMULA FOR SURFACES WITH PRESCRIBED MEAN CURVATURE IN THE 3-SPHERE

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Abstract. Our primary object of this paper is to give a representation formula for a surface with prescribed mean curvature in the (metric) 3-sphere by means of a single component of the generalized Gauss map. For a CMC (constant mean curvature) surface, we derive another representation formula by means of the adjusted Gauss map. These formulas are spherical versions of the Kenmotsu representation formula for surfaces in the Euclidean 3-space. Spin versions of them are obtained as well.

Introduction. Let $\mathbb{S}^3(c^2)$ be the 3-sphere of constant curvature c^2 $(c > 0)$. In this paper, we give a representation formula for surfaces with prescribed (not necessarily constant) mean curvature in $\mathbb{S}^3(c^2)$, as a spherical version of the Kenmotsu representation formula [8] for surfaces in the Euclidean 3-space $\mathbb{E}^3$. Hence we call it the *Kenmotsu type representation formula*. This is given via an integrable differential equation of first order in terms of the mean curvature and a single component of the generalized Gauss map.

In Section 1, we review the generalized Gauss map of a surface in $\mathbb{S}^3(c^2)$ (cf. [7]). This is decomposed into two maps from M to the unit 2-sphere $\mathbb{S}^2$. Using a single component, we describe explicitly the induced metric and the Hopf differential. In Section 2, we show that each component of the generalized Gauss map satisfies a nonlinear partial differential equation of second order. We call each of them the spherical GH equation, because if we put $c = 0$, it reduces to the generalized harmonic map equation (abbreviated to GH equation) for the Gauss map of a surface in $\mathbb{E}^3$. When H is constant, these turn out to be the harmonic map equation. Each spherical GH equation is the integrability condition for a surface with mean curvature H in $\mathbb{S}^3(c^2)$, from which we obtain the Kenmotsu type representation formula in $\mathbb{S}^3(c^2)$. In [2], we clarified the mechanism of obtaining the Kenmotsu representation formula in $\mathbb{E}^3$. This mechanism is now valid for the case of surfaces in non-flat 3-space forms. In Section 3, we concentrate on constant mean curvature (abbreviated to CMC) surfaces in $\mathbb{S}^3(c^2)$. By the Lawson correspondence [11], any CMC surface M in $\mathbb{S}^3(c^2)$ locally corresponds to an isometric non-minimal CMC surface M_0 in $\mathbb{E}^3$, together with its associated S^1-family $\{M_\theta\}_{\theta\in[-\pi,\pi)}$. Bobenko [4] also gave the correspondence at 'adapted frame level'. In terms of the Gauss map of M_θ, we shall derive another representation formula for M from these

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results. When $\theta = 0$ particularly, we call it the *Kenmotsu-Bryant type representation formula* (see Remark 3.2).

The Kenmotsu type and Kenmotsu-Bryant type representation formulas (locally) describe a CMC surface in $\mathbb{S}^3(c^2)$ by a single non-holomorphic harmonic map to $\mathbb{S}^2$, in contrast to the study in [7], [14] and [12], where a specific pair of non-holomorphic harmonic maps to $\mathbb{S}^2$ is used to describe these surfaces. In [3], we study some global properties of this correspondence.

Recently, many mathematicians (cf. [9], [10]) have applied spinor representations to the study of surfaces in $\mathbb{E}^3$. The spinor representation adopted in [10] is a spin version of the Kenmotsu representation formula for surfaces in $\mathbb{E}^3$ (including the Weierstrass representation formula for minimal surfaces in $\mathbb{E}^3$). In Section 4, we give spin versions of the Kenmotsu type and the Kenmotsu-Bryant type representation formulas in $\mathbb{S}^3(c^2)$.

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1. Generalized Gauss maps of surfaces in $\mathbb{S}^3(c^2)$. Let M be a Riemann surface with an isothermal coordinate $z = x + \sqrt{-1}y$, and f a conformal immersion from M into the Euclidean 4-space $\mathbb{E}^4 = (\mathbb{R}^4, \langle\ ,\ \rangle)$. The *generalized Gauss map* of f is defined by

$$\mathcal{G} = [f_z] : M \to G_{2,2}\,,$$

where $G_{2,2}$ stands for the Grassmann manifold of oriented 2-planes in $\mathbb{E}^4$, and at each point, the oriented complex null line $[f_z]$ in $\mathbb{C}^4$ is identified with the oriented 2-plane spanned by f_x and f_y. Identify $\mathbb{E}^4$ with the linear hull $\mathbb{R}\cdot SU(2)$ of the special unitary group $SU(2)$ by the map

$$\mathbf{x} = (x_1, x_2, x_3, x_4) \mapsto \underline{\mathbf{x}} = x_1\underline{\mathbf{e}_1} + x_2\underline{\mathbf{e}_2} + x_3\underline{\mathbf{e}_3} + x_4\underline{\mathbf{e}_4}\,,$$

where

$$\underline{\mathbf{e}_1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \underline{\mathbf{e}_2} = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \underline{\mathbf{e}_3} = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \underline{\mathbf{e}_4} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $\langle \mathbf{x}, \mathbf{x}\rangle = \det \underline{\mathbf{x}}$. Note that the unit 2-sphere $\mathbb{S}^2$ is realized in $\mathfrak{su}(2) = \operatorname{span}\{\underline{\mathbf{e}_1}, \underline{\mathbf{e}_2}, \underline{\mathbf{e}_3}\} \cong \mathbb{E}^3$ by

$$\mathbb{S}^2 = \{[\mathrm{h}] := \mathrm{h}\underline{\mathbf{e}_3}\mathrm{h}^* \mid \mathrm{h} \in SU(2)\} \cong SU(2)/U(1)\,,$$

where $U(1) = \{\mathrm{h}_\theta = (\sin\theta)\underline{\mathbf{e}_3} + (\cos\theta)\underline{\mathbf{e}_4} \mid \theta \in [-\pi, \pi)\}$. The linear Lie group $SU(2) \times SU(2)$ acts isometrically on $\mathbb{E}^4$ by

$$\mathrm{g}\cdot\mathbf{x} = \mathrm{g}_1\underline{\mathbf{x}}\mathrm{g}_2^* \quad (\mathrm{g} = (\mathrm{g}_1, \mathrm{g}_2) \in SU(2) \times SU(2), \mathbf{x} \in \mathbb{E}^4)\,.$$

Since it acts also transitively on $G_{2,2}$, $\mathbb{S}^2 \times \mathbb{S}^2$ is identified with $G_{2,2}$ through

$$\mathbb{S}^2 \times \mathbb{S}^2 \ni ([\mathrm{g}_1], [\mathrm{g}_2]) \mapsto [\mathrm{g}_1 E_{12}\mathrm{g}_2^*] \in G_{2,2}\,,$$

where $E_{12} = (1/2)(\underline{\mathbf{e}_1} - \sqrt{-1}\underline{\mathbf{e}_2})$. Therefore, we can decompose the generalized Gauss map $\mathcal{G}$ of f into $(\mathcal{G}_1, \mathcal{G}_2) : M \to \mathbb{S}^2 \times \mathbb{S}^2$.

From now on, let f be a conformal immersion from M into $\mathbb{S}^3(c^2) = (1/c)SU(2)$. Regard $\mathbb{S}^3(c^2)$ as the symmetric space

$$\begin{aligned}\mathbb{S}^3(c^2) &= (SU(2) \times SU(2))/\Delta \\ &= \left\{ \frac{1}{c}\mathrm{g} \cdot \underline{\mathbf{e}_4} = \frac{1}{c}\mathrm{g}_1\mathrm{g}_2^* \,\middle|\, \mathrm{g} = (\mathrm{g}_1, \mathrm{g}_2) \in SU(2) \times SU(2) \right\},\end{aligned}$$

where $\Delta = \{(\mathrm{h}, \mathrm{h}) \,|\, \mathrm{h} \in SU(2)\}$. A map $\mathcal{F} : M \to SU(2) \times SU(2)$ is called a *framing* of f if $f = (1/c)\mathcal{F} \cdot \underline{\mathbf{e}_4}$. On every contractible open set U of M, we can uniquely choose a framing map $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2)$ of f such that $N := \mathcal{E} \cdot \underline{\mathbf{e}_3}$ is a unit normal vector field and $\mathcal{E} \cdot E_{12}$ is a vector field of type $(1, 0)$, up to the right action of a $U(1)$-valued function. We call it the *adapted framing* of f on U. Since $[\mathcal{E}_1 E_{12} \mathcal{E}_2^*] = [f_z]$, we conclude that

$$\mathcal{G}_1 = [\mathcal{E}_1] = cNf^* = -cfN^*, \quad \text{and} \quad \mathcal{G}_2 = [\mathcal{E}_2] = cf^*N = -cN^*f.$$

Regarding $\mathbb{S}^2$ as the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by the stereographic projection

$$\Psi : \mathbb{S}^2 \setminus \{\underline{\mathbf{e}_3}\} \ni \left[\begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix}\right] \mapsto \frac{q}{p} \in \mathbb{C},$$

we have

$$\mathcal{G}_i = \frac{Q_i}{P_i}, \quad \text{where} \quad \mathcal{E}_i = \begin{pmatrix} Q_i & -\overline{P_i} \\ P_i & \overline{Q_i} \end{pmatrix}.$$

Let $\phi = e^{\lambda}dz$ be the dual $(1, 0)$-form to $\mathcal{E} \cdot E_{12}$ on U. Then the induced metric is given by $f^*ds^2 = \phi \cdot \bar{\phi} = e^{2\lambda}|dz|^2$ and $d\phi = -\sqrt{-1}\rho \wedge \phi$, where ρ stands for the connection form of f^*ds^2. We denote by H the mean curvature of f and by $\Phi = Q\phi \cdot \phi$ its Hopf differential, where $\cdot$ stands for the complex bilinear inner product on $\mathbb{C}^2$. The pullback $\mathcal{E}^{-1}d\mathcal{E} = \mathcal{E}_1^{-1}d\mathcal{E}_1 \oplus \mathcal{E}_2^{-1}d\mathcal{E}_2$ of the Maurer-Cartan form on $SU(2) \times SU(2)$ by the adapted framing $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2)$ is given by

$$\begin{aligned}\mathcal{E}_1^{-1}d\mathcal{E}_1 &= -\frac{\sqrt{-1}}{2}\begin{pmatrix} -\rho & (H + \sqrt{-1}c)\phi + \overline{Q\phi} \\ (H - \sqrt{-1}c)\bar{\phi} + Q\phi & \rho \end{pmatrix}, \\ \mathcal{E}_2^{-1}d\mathcal{E}_2 &= -\frac{\sqrt{-1}}{2}\begin{pmatrix} -\rho & (H - \sqrt{-1}c)\phi + \overline{Q\phi} \\ (H + \sqrt{-1}c)\bar{\phi} + Q\phi & \rho \end{pmatrix}.\end{aligned} \tag{1.1}$$

For a map $g : M \to \mathbb{S}^2 \cong \hat{\mathbb{C}}$, put $\varphi(g) = [4g_z(\bar{g})_z/(1 + |g|^2)^2]dz \cdot dz$, which is called the *Hopf differential* of g.

PROPOSITION 1.1. *The induced metric f^*ds^2 and the Hopf differential Φ of f are given by*

$$f^*ds^2 = \frac{4|(\mathcal{G}_1)_{\bar{z}}|^2}{(c^2 + H^2)(1 + |\mathcal{G}_1|^2)^2}dz \cdot d\bar{z} = \frac{4|(\mathcal{G}_2)_{\bar{z}}|^2}{(c^2 + H^2)(1 + |\mathcal{G}_2|^2)^2}dz \cdot d\bar{z}, \tag{1.2}$$

$$\Phi = (H + \sqrt{-1}c)^{-1}\varphi(\mathcal{G}_1) = (H - \sqrt{-1}c)^{-1}\varphi(\mathcal{G}_2), \tag{1.3}$$

and the Gauss curvature K of f is given by

$$K = (H^2 + c^2)\left(1 - \left|\frac{(\mathcal{G}_1)_z}{(\mathcal{G}_1)_{\bar z}}\right|^2\right) = (H^2 + c^2)\left(1 - \left|\frac{(\mathcal{G}_2)_z}{(\mathcal{G}_2)_{\bar z}}\right|^2\right).$$

PROOF. Because of

$$d\mathcal{G}_i = \frac{P_i dQ_i - Q_i dP_i}{P_i^2} \quad \text{and} \quad \mathcal{E}_i^{-1} d\mathcal{E}_i = \begin{pmatrix} * & * \\ Q_i dP_i - P_i dQ_i & * \end{pmatrix},$$

it follows from (1.1) that

$$(1.4) \qquad \begin{aligned} (\mathcal{G}_1)_{\bar z} &= \frac{1}{2P_1^2} e^{\lambda}(c + \sqrt{-1}H), \quad (\mathcal{G}_1)_z = \frac{\sqrt{-1}}{2P_1^2} e^{\lambda} Q, \\ (\mathcal{G}_2)_{\bar z} &= \frac{1}{2P_2^2} e^{\lambda}(-c + \sqrt{-1}H), \quad (\mathcal{G}_2)_z = \frac{\sqrt{-1}}{2P_2^2} e^{\lambda} Q. \end{aligned}$$

Then the equation (1.4) with $\mathcal{G}_i = Q_i/P_i$ implies the expressions (1.2) and (1.3). □

REMARK 1.2. Since f^*ds^2 is positive definite on M, $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$ is nowhere-holomorphic. From (1.3), anti-holomorphic points of $\mathcal{G}$ correspond to umbilic points of f.

COROLLARY 1.3. *Each component $\mathcal{G}_i$ $(i = 1, 2)$ of the generalized Gauss map $\mathcal{G}$ satisfies the following Beltrami equations*:

$$\Phi(\mathcal{G}_1)_{\bar z} = (H - \sqrt{-1}c)(\mathcal{G}_1)_z, \quad \Phi(\mathcal{G}_2)_{\bar z} = (H + \sqrt{-1}c)(\mathcal{G}_2)_z.$$

2. Kenmotsu type representation formula in $\mathbb{S}^3(c^2)$. In this section, we give an integrability condition for a conformal immersion $f : M \to \mathbb{S}^3(c^2)$ in terms of the mean curvature H and a single component of the generalized Gauss map $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$.

For every contractible open set U of M and a map $\mathcal{H} = (h, h) : U \to \Delta$, $\mathcal{E}\mathcal{H}^{-1} = (\mathcal{E}_1 h^{-1}, \mathcal{E}_2 h^{-1}) : U \to SU(2) \times SU(2)$ is a framing of f. When we choose $h = \mathcal{E}_2$ (resp. $h = \mathcal{E}_1$), the new framing is given by $(cf, \underline{\mathbf{e}_4})$ (resp. $(\underline{\mathbf{e}_4}, cf^*)$). Therefore, the map $\mathcal{S} = cf : M \to SU(2)$ satisfies

$$(2.1) \qquad \mathcal{S}^{-1}d\mathcal{S} = \mathcal{E}_2(\mathcal{E}_1^{-1}d\mathcal{E}_1 - \mathcal{E}_2^{-1}d\mathcal{E}_2)\mathcal{E}_2^{-1} = c(\alpha_2 - \alpha_2^*),$$

$$(2.2) \qquad (\mathcal{S}^*)^{-1}d\mathcal{S}^* = \mathcal{E}_1(\mathcal{E}_2^{-1}d\mathcal{E}_2 - \mathcal{E}_1^{-1}d\mathcal{E}_1)\mathcal{E}_1^{-1} = -c(\alpha_1 - \alpha_1^*),$$

where

$$\alpha_i = \mathcal{E}_i E_{12} \mathcal{E}_i^* \phi = -\begin{pmatrix} \mathcal{G}_i & -\mathcal{G}_i^2 \\ 1 & -\mathcal{G}_i \end{pmatrix} P_i^2 \phi \quad (i = 1, 2).$$

It follows from (1.4) combined with (1.2) that

$$(2.3) \qquad P_1^2\phi = \frac{2\sqrt{-1}(\overline{\mathcal{G}_1})_z}{(H + \sqrt{-1}c)(1 + |\mathcal{G}_1|^2)^2} dz, \quad P_2^2\phi = \frac{2\sqrt{-1}(\overline{\mathcal{G}_2})_z}{(H - \sqrt{-1}c)(1 + |\mathcal{G}_2|^2)^2} dz.$$

We note that each α_i is a global section of $T^{*(1,0)}M \otimes \mathcal{G}_i^{-1}T^{(1,0)}\mathbb{S}^2$. Put $\mu_1 = -c(\alpha_1 - \alpha_1^*)$ and $\mu_2 = c(\alpha_2 - \alpha_2^*)$. Then these are $\mathfrak{su}(2)$-valued 1-forms on M. The equations $dd\mathcal{S} = dd\mathcal{S}^* = 0$, equivalently, $d\mu_i + \mu_i \wedge \mu_i = 0$ $(i = 1, 2)$ imply the following

THEOREM 2.1. *The generalized Gauss map* $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$ *of a conformal immersion* $f : M \to \mathbb{S}^3(c^2)$ *with mean curvature* H *satisfies the following equations*:

$$(\mathcal{G}_1)_{z\bar{z}} - \frac{2\overline{\mathcal{G}_1}}{1+|\mathcal{G}_1|^2}(\mathcal{G}_1)_z(\mathcal{G}_1)_{\bar{z}} = \frac{1}{H-\sqrt{-1}c}H_z(\mathcal{G}_1)_{\bar{z}}\,, \tag{2.4}$$

$$(\mathcal{G}_2)_{z\bar{z}} - \frac{2\overline{\mathcal{G}_2}}{1+|\mathcal{G}_2|^2}(\mathcal{G}_2)_z(\mathcal{G}_2)_{\bar{z}} = \frac{1}{H+\sqrt{-1}c}H_z(\mathcal{G}_2)_{\bar{z}}\,. \tag{2.5}$$

Conversely, the equation (2.5) (resp. (2.4)) is the integrability condition for the equation (2.1) (resp. (2.2)).

THEOREM 2.2 (Kenmotsu type representation formula in $\mathbb{S}^3(c^2)$). *Let* M *be a simply connected Riemann surface with a reference point* z_0 *and* H *a real-valued smooth function on* M. *For a non-holomorphic smooth map* $\nu : M \to \hat{\mathbb{C}}$ *satisfying*

$$\nu_{z\bar{z}} - \frac{2\bar{\nu}}{1+|\nu|^2}\nu_z\nu_{\bar{z}} = \frac{1}{H+\sqrt{-1}c}H_z\nu_{\bar{z}} \quad \left(\text{resp. } \nu_{z\bar{z}} - \frac{2\bar{\nu}}{1+|\nu|^2}\nu_z\nu_{\bar{z}} = \frac{1}{H-\sqrt{-1}c}H_z\nu_{\bar{z}}\right),$$

define a smooth 1*-form* ω *on* M *by*

$$\omega = \frac{-2\sqrt{-1}(\bar{\nu})_z}{(H-\sqrt{-1}c)(1+|\nu|^2)^2}dz \quad \left(\text{resp. } \omega = \frac{-2\sqrt{-1}(\bar{\nu})_z}{(H+\sqrt{-1}c)(1+|\nu|^2)^2}dz\right).$$

Also define an $\mathfrak{sl}(2;\mathbb{C})$*-valued* 1*-form* α *and an* $\mathfrak{su}(2)$*-valued* 1*-form* μ *on* M *by*

$$\alpha = \begin{pmatrix} \nu & -\nu^2 \\ 1 & -\nu \end{pmatrix}\omega\,, \quad \mu = c(\alpha - \alpha^*)\,.$$

Then there exists uniquely a smooth map $\mathcal{S} : M \to SU(2)$ *such that* $\mathcal{S}(z_0) = \underline{\mathbf{e}_4}$ *and* $\mathcal{S}^{-1}d\mathcal{S} = \mu$ (*resp.* $\mathcal{S}^{-1}d\mathcal{S} = -\mu$). *Put* $f = (1/c)\mathcal{S}$ (*resp.* $f = (1/c)\mathcal{S}^*$). *Then* $f : M \to \mathbb{S}^3(c^2)$ *is a conformal immersion outside* $\{w \in M \mid \nu_{\bar{z}}(w) = 0\}$ *with prescribed mean curvature* H *and the generalized Gauss map* $\mathcal{G} = (\mathcal{S}[\nu], \nu)$ (*resp.* $\mathcal{G} = (\nu, \mathcal{S}^*[\nu])$). *Moreover, the induced metric is given by* $f^*ds^2 = (1+|\nu|^2)^2\omega\cdot\bar{\omega}$ *and the Hopf differential by* $\Phi = 2\sqrt{-1}\nu_z\omega\cdot dz = (H-\sqrt{-1}c)^{-1}\varphi(\nu)$ (*resp.* $= (H+\sqrt{-1}c)^{-1}\varphi(\nu)$).

In the above theorem, $\mathcal{S}[\nu]$ (at $z \in M$) stands for the linear fractional transformation of $\nu(z) \in \hat{\mathbb{C}}$ by $\mathcal{S}(z) \in SU(2)$.

REMARK 2.3. Putting $c = 0$ in (2.4) and (2.5), we obtain the generalized harmonic map (GH) equation for Gauss maps of surfaces in $\mathbb{E}^3$. In the Kenmotsu representation formula [8] in $\mathbb{E}^3$, the GH equation gives the compatibility condition for existence of surfaces. When H is constant, the equations (2.4), (2.5) and the GH equation are the equation for harmonic maps into the standard metric 2-sphere $\mathbb{S}^2$ (cf. [7]).

REMARK 2.4. In [7], Theorem 4.10 combined with Proposition 4.11 asserts the existence of a surface in $\mathbb{S}^3(c^2)$ with prescribed generalized Gauss map, which is a pair of complex functions satisfying some relations. On the other hand, the Kenmotsu type representation formula is given via the integrable differential equation of first order by means of a single complex function satisfying the spherical GH equation (2.4) (or (2.5)).

3. Kenmotsu-Bryant type representation formula for CMC surfaces in $\mathbb{S}^3(c^2)$. In this section, we confine our argument to CMC surfaces, and prove that the Kenmotsu type representation formula can be adjusted to the Kenmotsu-Bryant type representation formula through the Lawson correspondence at adapted frame level. For a harmonic map $\nu : M \to \mathbb{S}^2$, the Kenmotsu type formula represents a pair of CMC H surfaces in $\mathbb{S}^3(c^2)$ with Hopf differentials $(H \pm \sqrt{-1}c)^{-1}\varphi(\nu)$, but, in contrast, the Kenmotsu-Bryant type formula below represents another CMC H surface with the Hopf differential $(H^2 + c^2)^{-1/2}\varphi(\nu)$.

Let H_0 $(> c)$ be a positive constant, and put $H_c = \sqrt{H_0^2 - c^2}$. Let M be a contractible Riemann surface with an isothermal coordinate z and a metric $ds^2 = e^{2\lambda}|dz|^2$. We denote its connection form by ρ and put $\phi = e^\lambda dz$. Let $\Phi = Q\phi \cdot \phi$ be a holomorphic quadratic differential form on M. Lawson [11] and Bobenko [5, Theorem 14.1] proved that the integrability conditions for the following differential equations are identical (cf. [6]):

$$(3.1)\quad \mathcal{E}_{[\theta]}^{-1} d\mathcal{E}_{[\theta]} = -\frac{\sqrt{-1}}{2}\begin{pmatrix} -\rho & H_0 e^{\sqrt{-1}\theta}\phi + \overline{Q\phi} \\ H_0 e^{-\sqrt{-1}\theta}\bar{\phi} + Q\phi & \rho \end{pmatrix} \quad (\theta \in [-\pi, \pi)),$$

and the pair of solutions $\mathcal{E}_1 = \mathcal{E}_{[\theta_c]}$, $\mathcal{E}_2 = \mathcal{E}_{[-\theta_c]} : M \to SU(2)$ for $\theta = \theta_c := \arg(H_c + \sqrt{-1}c)$ is the adapted framing of an isometric immersion $f : M \to \mathbb{S}^3(c^2)$ with CMC $H = H_c$ and the Hopf differential Φ. Moreover, a solution $\mathcal{E}_{[0]}$ for $\theta = 0$ is the adapted framing of an isometric immersion $f_0 : M \to \mathbb{E}^3$ with CMC H_0 and the Hopf differential Φ, that is, $[\mathcal{E}_{[0]}]$ is the normal vector field of f_0 and $\mathcal{E}_{[0]} E_{12} \mathcal{E}_{[0]}^*$ is a vector field of type $(1, 0)$. We note that for any θ, a solution $\mathcal{E}_{[\theta]}$ of (3.1) gives the adapted framing $\mathcal{E}_{[\theta]}\mathrm{h}_{\theta/2}$ of an isometric immersion $f_0^\theta : M \to \mathbb{E}^3$ with CMC H_0 and the Hopf differential $\Phi_{[\theta]} = e^{\sqrt{-1}\theta}\Phi$. The Gauss map $\mathcal{G}_{[\theta]} = [\mathcal{E}_{[\theta]}\mathrm{h}_{\theta/2}] = [\mathcal{E}_{[\theta]}] : M \to \mathbb{S}^2$ of f_0^θ is a harmonic map satisfying $\varphi(\mathcal{G}_{[\theta]}) = H_0 e^{\sqrt{-1}\theta}\Phi$ (cf. [8], [12], [14]).

By a similar argument to that in Section 2, if we choose $h = \mathcal{E}_{[\theta]}$ for any θ, we can obtain a representation formula for f in terms of $\mathcal{G}_{[\theta]}$.

THEOREM 3.1. *Let M be a simply connected Riemann surface with a reference point z_0, and let H be a non-negative constant. Put $H_0 = \sqrt{H^2 + c^2}$ and $\theta_c = \arg(H + \sqrt{-1}c)$. For a non-holomorphic harmonic map $\nu : M \to \mathbb{S}^2 \cong \hat{\mathbb{C}}$ and $\theta \in [-\pi, \pi)$, define an $\mathfrak{sl}(2; \mathbb{C})$-valued 1-form $\alpha_{[\theta]}$ by*

$$\alpha_{[\theta]} = \begin{pmatrix} \nu & -\nu^2 \\ 1 & -\nu \end{pmatrix}\omega_{[\theta]}, \quad \omega_{[\theta]} = \frac{-2\sqrt{-1}e^{-\sqrt{-1}\theta}(\bar{\nu})_z}{H_0(1 + |\nu|^2)^2}dz.$$

Also define $\mathfrak{su}(2)$-valued 1-forms $\mu_{[\theta],1}$, $\mu_{[\theta],2}$ on M by

$$\mu_{[\theta],1} = \sqrt{-1}(H_0/2)\{(e^{\sqrt{-1}\theta} - e^{\sqrt{-1}\theta_c})\alpha_{[\theta]} + (e^{-\sqrt{-1}\theta} - e^{-\sqrt{-1}\theta_c})\alpha_{[\theta]}^*\},$$
$$\mu_{[\theta],2} = \sqrt{-1}(H_0/2)\{(e^{\sqrt{-1}\theta} - e^{-\sqrt{-1}\theta_c})\alpha_{[\theta]} + (e^{-\sqrt{-1}\theta} - e^{\sqrt{-1}\theta_c})\alpha_{[\theta]}^*\}.$$

Then there exists uniquely a smooth map $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : M \to SU(2) \times SU(2)$ such that $\mathcal{F}_1^{-1}d\mathcal{F}_1 = \mu_{[\theta],1}$, $\mathcal{F}_2^{-1}d\mathcal{F}_2 = \mu_{[\theta],2}$ and $\mathcal{F}(z_0) = (\underline{\mathbf{e}}_4, \underline{\mathbf{e}}_4)$. Put $f = (1/c)\mathcal{F}_1\mathcal{F}_2^*$. Then $f : M \to \mathbb{S}^3(c^2)$ is a conformal CMC H immersion outside isolated degenerate points

$\{z \in M \mid \omega(z) = 0\}$ *with the Hopf differential* $\Phi = (1/H_0)e^{-\sqrt{-1}\theta}\varphi(\nu)$, *where* $f^*ds^2 = [2|\nu_{\bar{z}}|/(H_0(1+|\nu|^2))]^2 dz\cdot d\bar{z}$. *The generalized Gauss map of f is given by* $\mathcal{G} = (\mathcal{F}_1[\nu], \mathcal{F}_2[\nu])$.

REMARK 3.2. The Bryant formula ([5], [15]) represents a CMC c surface M in the hyperbolic 3-space of constant curvature $-c^2$ by means of the pair of a holomorphic 1-form and a holomorphic map G to $\hat{\mathbb{C}}$, where G is the Gauss map of a minimal surface M_0 in $\mathbb{E}^3$, and M_0 corresponds to M through the canonical Lawson correspondence (see [15, Theorem 3.1]). Hence, in the case of $\theta = 0$ in Theorem 3.1, we call it the *Kenmotsu-Bryant representation formula* and $\mathcal{G}_{[0]}$ the *adjusted Gauss map* of f (cf. [1]).

4. Spin versions of representation formulas. In this section, we give spin versions of the Kenmotsu type and the Kenmotsu-Bryant type representation formulas for surfaces in $\mathbb{S}^3(c^2)$. Using the framing method, we will modify the approach by Kusner and Schmitt [10] in spin calculus. We then treat a spin structure on a Riemann surface M as a complex line bundle whose square is the holomorphic tangent bundle $T^{(1,0)}M$ of M. Given a conformal immersion f from M into $\mathbb{S}^3(c^2)$, a spin structure $\mathrm{Spin}(M)$ on M is induced canonically from the pullback of the unique spin structure $\mathrm{Spin}(\mathbb{S}^2)$ on $\mathbb{S}^2$ via a single component of the generalized Gauss map or the adjusted Gauss map. We give the condition that the lift $\psi : \mathrm{Spin}(M) \to \mathrm{Spin}(\mathbb{S}^2)$ induces the integrable differential equation for f.

4.1. The spin structure on the Riemann 2-sphere $\mathbb{S}^2$. First, we review the spin structure on the Riemann 2-sphere $\mathbb{S}^2$ (see [13] for the general theory of spin bundles over Riemann surfaces).

Let $P \to \mathbb{S}^2$ be the unitary frame bundle. The fiber $P_{\mathbf{x}}$ on $\underline{\mathbf{x}} = [\mathrm{g}] \in \mathbb{S}^2$ ($\mathrm{g} \in SU(2)$) is given by

$$P_{\mathbf{x}} = \left\{e^{\sqrt{-1}\theta}\mathrm{g}E_{12}\mathrm{g}^* \,\middle|\, e^{\sqrt{-1}\theta} \in \boldsymbol{S}^1\right\}.$$

Then the $U(1)$-bundle P can be regarded as $SU(2)/\pm\underline{\mathbf{e}_4}$. Recall that the group Spin(2) is considered as the double cover of $U(1)$, and $SU(2)$ is the (unique) principal Spin(2)-bundle $\tilde{P}$ on $\mathbb{S}^2$.

For the representation $\rho_- : \mathrm{Spin}(2) \to \mathrm{Aut}(\mathbb{C})$; $\rho_-(\mathrm{h}_\theta) = e^{\sqrt{-1}\theta}$, there is an associated complex line bundle

$$\mathrm{Spin}(\mathbb{S}^2) = S^-_{\mathbb{C}}(\tilde{P}) := \tilde{P} \times_{\rho_-} \mathbb{C},$$

which is called the minus spin bundle associated to $\tilde{P}$ and we will regard it as the *spin structure* on $\mathbb{S}^2$. Excepting the image $\mathbf{0}(\mathbb{S}^2)$ of the zero-section, we can identify $\mathrm{Spin}(\mathbb{S}^2)$ with $\mathbb{R}^*\cdot SU(2)$ ($\mathbb{R}^* = \mathbb{R} \setminus \{0\}$) by

$$\begin{aligned}
&\mathrm{Spin}(\mathbb{S}^2) \setminus \mathbf{0}(\mathbb{S}^2) \to \mathbb{R}^*\cdot SU(2),\\
&\quad [(\mathrm{g}, w)] = [(\mathrm{g}\mathrm{h}_\theta, r)] \mapsto r\mathrm{g}\mathrm{h}_\theta = \mathrm{g}\begin{pmatrix} w & 0 \\ 0 & \bar{w} \end{pmatrix}\\
&\qquad (\mathrm{g} \in SU(2),\, w = re^{\sqrt{-1}\theta} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}).
\end{aligned}$$

We then obtain the following projections

$$\begin{array}{llll}
\mathrm{Spin}(\mathbb{S}^2) & \supset \mathbb{R}^*\cdot SU(2) & \ni & \mathrm{s} = \begin{pmatrix} s_1 & -\overline{s_2} \\ s_2 & \overline{s_1} \end{pmatrix} \\
\downarrow \sigma & & & \\
T^{(1,0)}\mathbb{S}^2 & & \ni & \mathrm{s}E_{12}\mathrm{s}^* = \begin{pmatrix} -s_1 s_2 & {s_1}^2 \\ -{s_2}^2 & s_1 s_2 \end{pmatrix} \\
\downarrow \pi & & & \\
\mathbb{S}^2\ (\cong \hat{\mathbb{C}}) & \ni & & [(\det(\mathrm{s})^{-1}\mathrm{s}] = (\det \mathrm{s})^{-2}\mathrm{s}\underline{\mathrm{e}}_3\mathrm{s}^*(= s_1/s_2)\,.
\end{array}$$

Moreover, $\mathrm{Spin}(\mathbb{S}^2) \setminus \mathbf{0}(\mathbb{S}^2)$ is considered as a $\mathbb{Z}_2$-bundle on $T^{(1,0)}\mathbb{S}^2 \setminus \mathbf{0}(\mathbb{S}^2)$.

4.2. A spinor representation of a bundle map of $T^{(1,0)}M$ into $T^{(1,0)}\mathbb{S}^2$. Let (M, ds^2) be an oriented connected Riemannian 2-manifold and ν a smooth map from M to $\mathbb{S}^2$. Let h be a local lift of ν, that is, h is a smooth map on every contractible open set U of M into $SU(2)$ satisfying $\nu = [h]$. Take an isothermal coordinate z on U with $ds^2 = e^{2\lambda}|dz|^2$ and put $\phi = e^{\lambda}dz$. Put

$$\alpha = hE_{12}h^*\phi = \begin{pmatrix} \nu & -\nu^2 \\ 1 & -\nu \end{pmatrix}\omega\,,$$

where

$$h = \begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix}, \quad \nu(= \Psi \circ \nu) = \frac{q}{p}, \quad \omega = -p^2\phi\,.$$

If (α, ν) is a (fiber metric preserving) bundle map of the holomorphic tangent bundle $T^{(1,0)}M$ into $T^{(1,0)}\mathbb{S}^2$ (that is, $\alpha \in \Gamma(T^{*(1,0)}M \otimes \nu^{-1}T^{(1,0)}\mathbb{S}^2)$), α is locally described as above.

For a bundle map (α, ν) of $T^{(1,0)}M$ into $T^{(1,0)}\mathbb{S}^2$, let $S = S^-$ be the (unique) pullback bundle of $\mathrm{Spin}(\mathbb{S}^2)$ under α. Then S defines a spin structure on M, that is, S is the minus spin bundle associated to the spin bundle $\tilde{P}_M$ on M defined uniquely from S. Moreover, the lift $\psi : S \to \mathrm{Spin}(\mathbb{S}^2)$ of α is described by a pair $(\psi_1(z,\bar{z})\sqrt{dz}, \psi_2(z,\bar{z})\sqrt{dz})$ of smooth sections of the plus spin bundle S^+ associated to $\tilde{P}_M$, where we consider ψ merely as the map from $S \setminus \mathbf{0}(M)$ to $\mathbb{C}^2 \setminus \{\mathbf{0}\}$ identified with $\mathbb{R}^*\cdot SU(2)$ by

$$\mathbb{C}^2 \ni (s_1, s_2) \mapsto \begin{pmatrix} s_1 & -\overline{s_2} \\ s_2 & \overline{s_1} \end{pmatrix} \in \mathbb{R}\cdot SU(2)\,.$$

(We remark that ψ maps the zero spinor on M to the zero spinor on $\mathbb{S}^2$.) We call $\psi = (\psi_1\sqrt{dz}, \psi_2\sqrt{dz})$ the *spinor representation* of the bundle map $(\alpha, \nu) : T^{(1,0)}M \to T^{(1,0)}\mathbb{S}^2$.

For the dual ϕ^* to ϕ, $\sqrt{\phi^*}$ is considered as a basic local section of the minus spin bundle S. Since ψ is the lift of $\alpha = hE_{12}h\phi$, we obtain $\psi(\sqrt{\phi^*}) = h$. Then

$$\psi_1 = e^{\lambda/2}q\,, \quad \psi_2 = e^{\lambda/2}p\,, \tag{4.1}$$

and hence

$$\nu = \psi_1/\psi_2\,, \quad \omega = -{\psi_2}^2dz\,. \tag{4.2}$$

Now we define the *Dirac operator* $\not{D}$ for the spinor representation $\psi = (\psi_1(z,\bar{z})\sqrt{dz}, \psi_2(z,\bar{z})\sqrt{dz})$ which is a smooth section of $S^+ \oplus S^+$, as the original Dirac operator $\not{\partial}$ for the section $\iota \circ \psi = (\psi_1(z, \bar{z})\sqrt{dz}, \overline{\psi_2(z, \bar{z})}\sqrt{d\bar{z}})$ of $S^+ \oplus S^-$:

$$\partial\!\!\!/ = 2\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix}, \quad \partial\!\!\!/ \begin{pmatrix} \psi_1\sqrt{dz} \\ \overline{\psi_2}\sqrt{d\bar{z}} \end{pmatrix} = 2\begin{pmatrix} \partial\overline{\psi_2}\sqrt{dz} \\ -\bar{\partial}\psi_1\sqrt{d\bar{z}} \end{pmatrix},$$

$$D\!\!\!/\psi = \iota^{-1}(\partial\!\!\!/(\iota \circ \psi)) = 2\begin{pmatrix} \partial\overline{\psi_2}\sqrt{dz} \\ -\partial\overline{\psi_1}\sqrt{dz} \end{pmatrix}.$$

4.3. Spinor representation for surfaces in $\mathbb{S}^3(c^2)$. Let $f : M \to \mathbb{S}^3(c^2)$ be an isometric immersion, and ν the second (resp. first) component $\mathcal{G}_2$ (resp. $\mathcal{G}_1$) of the generalized Gauss map of f. Recall that the bundle map (α, ν) is given by

$$\alpha = \alpha_2 = (1/c)f^{-1}\partial f \quad (\text{resp. } \alpha = \alpha_1 = -(1/c)(f^*)^{-1}\partial f^*)\,. \tag{4.3}$$

By using the spinor representation $\psi = (\psi_1\sqrt{dz}, \psi_2\sqrt{dz})$ of (α, ν) combined with (4.1), we obtain

$$f^*ds^2 = (|\psi_1|^2 + |\psi_2|^2)^2|dz|^2 = |\psi|^4|dz|^2\,.$$

It follows from (2.3) combined with (4.2) that

$$\bar{\psi} \cdot D\!\!\!/\psi = (c + \sqrt{-1}H)|\psi|^4 \quad (\text{resp. } \bar{\psi} \cdot D\!\!\!/\psi = (-c + \sqrt{-1}H)|\psi|^4)\,, \tag{4.4}$$

where H is the mean curvature of f. Moreover, from (1.3) combined with (4.2), the Hopf differential Φ is given by

$$\Phi = \sqrt{-1}(\psi \cdot D\!\!\!/\bar{\psi})dz \cdot dz\,.$$

The integrability condition for (4.3) is the following

$$\bar{\partial}\alpha - \partial\alpha^* = c[\alpha \wedge \alpha^*] \quad (\text{resp. } \bar{\partial}\alpha - \partial\alpha^* = -c[\alpha \wedge \alpha^*])\,. \tag{4.5}$$

We remark that α can be described by

$$\alpha = \begin{pmatrix} -\psi_1\psi_2 & \psi_1{}^2 \\ -\psi_2{}^2 & \psi_1\psi_2 \end{pmatrix} dz = \underline{\psi} E_{12}\underline{\psi}^* dz\,, \quad \underline{\psi} := \begin{pmatrix} \psi_1 & -\overline{\psi_2} \\ \psi_2 & \overline{\psi_1} \end{pmatrix}.$$

Then the above equation (4.5) combined with (4.4) implies the following results.

PROPOSITION 4.1. *The spinor representation ψ of $\alpha_i \in \Gamma(T^{*(1,0)}M \otimes \mathcal{G}_i^{-1}T^{(1,0)}\mathbb{S}^2)$ satisfies the following non-linear Dirac equation*:

$$D\!\!\!/\psi = (c + \sqrt{-1}H)|\psi|^2\psi \quad (\textit{resp. } D\!\!\!/\psi = (-c + \sqrt{-1}H)|\psi|^2\psi)\,.$$

THEOREM 4.2 (Spin version of Kenmotsu type representation formula). *Let M be a simply connected Riemann surface with a reference point z_0 and H a real-valued smooth function on M. For a nowhere-vanishing $\mathbb{C}^2$-valued smooth function $\psi = (\psi_1, \psi_2) : M \to \mathbb{C}^2$ satisfying*

$$D\!\!\!/\psi = (c + \sqrt{-1}H)|\psi|^2\psi \quad (\textit{resp. } D\!\!\!/\psi = (-c + \sqrt{-1}H)|\psi|^2\psi)\,,$$

define an $\mathfrak{sl}(2;\mathbb{C})$-valued 1-form α and an $\mathfrak{su}(2)$-valued 1-form μ by

$$\alpha = \begin{pmatrix} -\psi_1\psi_2 & \psi_1{}^2 \\ -\psi_2{}^2 & \psi_1\psi_2 \end{pmatrix} dz\,, \quad \mu = c(\alpha - \alpha^*)\,.$$

Then there exists uniquely a smooth map $\mathcal{S} : M \to SU(2)$ *such that* $\mathcal{S}(z_0) = \underline{\mathbf{e}_4}$ *and* $\mathcal{S}^{-1}d\mathcal{S} = \mu$ *(resp.* $\mathcal{S}^{-1}d\mathcal{S} = -\mu$*). Put* $f = (1/c)\mathcal{S}$ *(resp.* $f = (1/c)\mathcal{S}^*$*). Then* $f : M \to \mathbb{S}^3(c^2)$ *is a conformal immersion with prescribed mean curvature H, and* $\nu = \psi_1/\psi_2$ *is the second (resp. first) component of the generalized Gauss map* $\mathcal{G}$ *of f.*

Similarly, we can represent a CMC surface in $\mathbb{S}^3(c^2)$ with the adjusted Gauss map $\nu = \mathcal{G}_{[0]}$ is terms of the spinor representation ψ of $\alpha \in \Gamma(T^{*(1,0)}M \otimes \mathcal{G}_{[0]}^{-1}T^{(1,0)}\mathbb{S}^2)$.

THEOREM 4.3 (Spin version of Kenmotsu-Bryant type representation formula). *Let M be a simply connected Riemann surface with a reference point* z_0*, and let H be a non-negative constant. Put* $H_0 = \sqrt{H^2 + c^2}$*. For a nowhere-vanishing* $\mathbb{C}^2$*-valued smooth function* $\psi = (\psi_1, \psi_2) : M \to \mathbb{C}^2$ *satisfying*

$$\not{D}\psi = \sqrt{-1}H_0|\psi|^2\psi\,,$$

define an $\mathfrak{sl}(2;\mathbb{C})$*-valued* 1*-form* α *and two* $\mathfrak{su}(2)$*-valued* 1*-forms* μ_1, μ_2 *by*

$$\alpha = \begin{pmatrix} -\psi_1\psi_2 & {\psi_1}^2 \\ -{\psi_2}^2 & \psi_1\psi_2 \end{pmatrix} dz\,,$$

$$\mu_1 = \frac{1}{2}[\{c - \sqrt{-1}(H - H_0)\}\alpha - \{c + \sqrt{-1}(H - H_0)\}\alpha^*]\,,$$
$$\mu_2 = \frac{1}{2}[-\{c + \sqrt{-1}(H - H_0)\}\alpha + \{c - \sqrt{-1}(H - H_0)\}\alpha^*]\,.$$

Then there exists uniquely a smooth map $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : M \to SU(2) \times SU(2)$ *such that* $\mathcal{F}_1^{-1}d\mathcal{F}_1 = \mu_1$, $\mathcal{F}_2^{-1}d\mathcal{F}_2 = \mu_2$ *and* $\mathcal{F}(z_0) = (\underline{\mathbf{e}_4}, \underline{\mathbf{e}_4})$*. Put* $f = (1/c)\mathcal{F}_1\mathcal{F}_2^*$*. Then* $f : M \to \mathbb{S}^3(c^2)$ *is a conformal CMC H immersion. The induced metric* f^*ds^2 *on M and the Hopf differential* Φ *are given by*

$$f^*ds^2 = |\psi|^4|dz|^2\,, \quad \Phi = \sqrt{-1}(\psi \cdot \not{D}\bar{\psi})dz \cdot dz\,.$$

The generalized Gauss map of f is given by $\mathcal{G} = (\mathcal{F}_1[\psi_1/\psi_2], \mathcal{F}_2[\psi_1/\psi_2])$*. (*$\psi_1/\psi_2$ *is the adjusted Gauss map* $\mathcal{G}_{[0]}$ *of* f*.)*

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