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INVARIANT SUBVARIETIES OF LOW CODIMENSION IN THE AFFINE SPACES

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Abstract. Let W be an irreducible subvariety of codimension r in a smooth affine variety X of dimension n defined over the complex field $\boldsymbol{C}$. Suppose that W is left pointwise fixed by an automorphism of X of infinite order or by a one-dimensional algebraic torus action on X. In the present article, we consider whether or not X is then an affine space bundle over W of fiber dimension $n-r$. Our results concern the case $r=1$ or the case $r=2$ and $n \leq 3$. As by-products, we obtain algebro-topological characterizations of the affine 3-space.

0. Introduction. Let k be an algebraically closed field of characteristic zero, which we fix as the ground field throughout the present article and assume to be the complex field $\boldsymbol{C}$ whenever we have to depend on the topological arguments. Let β be an algebraic automorphism of the affine space $\boldsymbol{A}^n$ of dimension n and W an irreducible hypersurface of $\boldsymbol{A}^n$. We call W a *coordinate hyperplane* if there exists a system of coordinates $\{x_1, \dots, x_n\}$ of $\boldsymbol{A}^n$ such that W is defined by $x_1 = 0$. We first pose the following question:

QUESTION. *If β is of infinite order and leaves W pointwise fixed, is W a coordinate hyperplane after a suitable change of coordinates on $\boldsymbol{A}^n$?*

Indeed, the answer is affirmative if $n=2$ (see Corollary 1.10).

We consider the question in the case $n=3$ with an additional hypothesis. Namely, we prove the following (see Corollary 2.9):

THEOREM. *Suppose $n=3$. If β is diagonalizable* (*see Section* 2 *below for the definition*), *then W is a coordinate hyperplane after a suitable change of coordinates on $\boldsymbol{A}^3$.*

As a by-product, we obtain the following algebraic characterization of the affine space of dimension 3 (see Theorem 2.10).

THEOREM. *Let $X = \operatorname{Spec} A$ be a nonsingular affine threefold. Then X is isomorphic to the affine space of dimension* 3 *if and only if the following conditions are satisfied*:

(1) $\operatorname{Pic} X = (0)$ *and* $A^* = k^*$, *where A^* is the set of invertible elements of A.*

(2) *There exist an irreducible hypersurface W of X and a diagonalizable automorphism β of infinite order such that β leaves W pointwise fixed and that W has Kodaira dimension* $-\infty$.

We next consider the case of codimension two. Let W be an irreducible subvariety of codimension 2 in a nonsingular affine variety X of dimension n defined over the complex field

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C. Suppose that a one-dimensional algebraic torus G_m acts on X in such a way that W is the fixed-point locus X^{G_m}. Our main result in the codimension two case is Theorem 4.2, which characterizes the affine 3-space among the acyclic affine threefolds. In this article, we say that a nonsingular algebraic variety X is *acyclic* if all the reduced integral homology groups of X vanish. An acyclic surface is called a *homology plane*.

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1. The case $n = 2$. Let C be an irreducible curve on the affine plane $\boldsymbol{A}^2 = \operatorname{Spec} k[x, y]$ and $f \in k[x, y]$ an element which generates the defining ideal of C. Let X be the complement of C in $\boldsymbol{A}^2$. So, $X = \operatorname{Spec} k[x, y, f^{-1}]$. Let β be an algebraic automorphism of $\boldsymbol{A}^2$ of infinite order which stabilizes the curve C, i.e., $\beta(C) = C$. Then β induces an automorphism on X and on the coordinate ring $k[x, y, f^{-1}]$ of X. We denote the induced k-algebra automorphism of $k[x, y, f^{-1}]$ by the same symbol β. We denote by $\bar{\kappa}(X)$ the Kodaira dimension of X. First of all, we note the following result (cf. Iitaka [6, Theorem 11.12]).

LEMMA 1.1. *If $\bar{\kappa}(X) = 2$, then* Aut (X) *is a finite group.*

Since X has an automorphism β of infinite order, it follows that $\bar{\kappa}(X) \leq 1$.

LEMMA 1.2. *If $\bar{\kappa}(X) = -\infty$, then $f = x$ after a suitable change of coordinates. The automorphism β is written as*

$$\beta(x) = ax\,, \quad \beta(y) = by + g(x)$$

with $a, b \in k^*$ and $g(x) \in k[x]$.

PROOF. Since $\bar{\kappa}(X) = -\infty$, there exists an $\boldsymbol{A}^1$-fibration $\varphi' : X \to B'$, which extends naturally to an $\boldsymbol{A}^1$-fibration $\varphi : \boldsymbol{A}^2 \to B$, where B' is an open set of a smooth curve B. Then the curve C is contained in a fiber of φ. Hence C is isomorphic to $\boldsymbol{A}^1$, since every fiber of φ is a disjoint union of finitely many smooth components which are isomorphic to $\boldsymbol{A}^1$ (cf. [12, Lemma 4.4]). By a theorem of Abhyankar-Moh-Suzuki (cf. [11]), we may and shall put $f = x$ after a change of coordinates. Since $\beta(C) = C$, it follows that $\beta(x) = ax$ with $a \in k[x, y]$. Since $\beta^{-1}(C) = C$, we have $\beta^{-1}(x) = bx$ with $b \in k[x, y]$. Then a is an invertible element of $k[x, y]$, i.e., $a \in k^*$. Write

$$\beta(y) = g_0(x)y^n + g_1(x)y^{n-1} + \cdots + g_n(x)$$

with $g_i(x) \in k[x]$. Considering the Jacobian determinant J of $\beta(x), \beta(y)$ with respect to x, y, we have

$$J = a(ng_0(x)y^{n-1} + \cdots + g_{n-1}(x)) \in k^*\,.$$

This implies that $n = 1$ and $g_0(x) = b \in k^*$. So we are done. Q.E.D.

LEMMA 1.3. *Suppose $\bar{\kappa}(X) = 0$ and X is NC-minimal (see* [4] *for the definition). Then $f = xy + 1$ after a suitable change of coordinates. The automorphism β is written as*

$$\beta(x) = ax, \beta(y) = a^{-1}y \quad or \quad \beta(x) = ay, \beta(y) = a^{-1}x$$

with $a \in k^*$.

PROOF. By Fujita [4, (8.13), (8.64)], X is isomorphic to either $\boldsymbol{P}^2-(\ell_1+\ell_2+\ell_3)$ with non-confluent lines ℓ_1, ℓ_2, ℓ_3 or $\boldsymbol{P}^2-(C+\ell)$ with a smooth conic C and a line ℓ meeting each other in two distinct points. In the former case, X is isomorphic to $\boldsymbol{A}^1_* \times \boldsymbol{A}^1_*$, where $\boldsymbol{A}^1_*$ denotes the affine line $\boldsymbol{A}^1$ with one point deleted off and the reduced multiplicative group $\Gamma(X)^*/k^*$ is a free abelian group of rank two, where $\Gamma(X)$ is the coordinate ring of X. Meanwhile, since $\Gamma(X)=k[x,y,f^{-1}]$ with an irreducible element f, $\Gamma(X)^*/k^*$ has rank one. So, the latter case takes place. Then $f=xy+1$ after a suitable change of coordinates. We shall determine the automorphism β. Since $\beta(f)=cf$ with $c\in k^*$, we have

$$\beta(x)\beta(y)+1=c(xy+1)$$

or

$$\beta(x)\beta(y)=cxy+(c-1),$$

where the right side is irreducible unless $c=1$. So, $c=1$ and $\beta(x)\beta(y)=xy$. The result follows readily from the unique irreducible decomposition of $\beta(x)\beta(y)$. Q.E.D.

If X is not NC-minimal and $\bar{\kappa}(X)=0$, then X is obtained from an NC-minimal one by applying the sub-divisional blowing-ups or half-point attachments (cf. [4]). Then it is easy to see that X has an $\boldsymbol{A}^1_*$-fibration. In the case of $\bar{\kappa}(X)=1$, by Kawamata's theorem [7, 12], X has an $\boldsymbol{A}^1_*$-fibration. So we consider the case where X has an $\boldsymbol{A}^1_*$-fibration $\rho: X\to B$. Considering the possible extensions of ρ on $\boldsymbol{A}^2$ and also making use of the classification of the standard forms of generically rational polynomials with two places at infinity (cf. [20,16]), we have the following result (see [1] for the detail).

LEMMA 1.4. *Let X be the complement in $\boldsymbol{A}^2$ of an irreducible curve C defined by $f=0$. Suppose that $\bar{\kappa}(X)\geq 0$ and X has an $\boldsymbol{A}^1_*$-fibration $\rho: X\to B$. Then, after a suitable change of coordinates, the polynomial f is written in one of the following forms*:

(I) *Case where the given $\boldsymbol{A}^1_*$-fibration $\rho: X\to B$ extends to an $\boldsymbol{A}^1_*$-fibration $\tilde{\rho}: \boldsymbol{A}^2\to\tilde{B}$*:

(1) $f=x^my^n+1$, *where $m,n>0$ and* $\gcd(m,n)=1$. *In this case, $B\cong\boldsymbol{A}^1_*$ and $\tilde{B}\cong\boldsymbol{A}^1$.*

(2) $f=x^m(x^ly+p(x))^n+1$, *where $l,m,n>0$,* $\gcd(m,n)=1$ *and $p(x)\in k[x]$ with* $\deg p(x)<l$ *and $p(0)\neq 0$. In this case, $B\cong\boldsymbol{A}^1_*$ and $\tilde{B}\cong\boldsymbol{A}^1$.*

(II) *Case where the given $\boldsymbol{A}^1_*$-fibration $\rho: X\to B$ is not extended to an $\boldsymbol{A}^1_*$-fibration on $\boldsymbol{A}^2$*:

(3) $f=a_0(x)y+a_1(x)$, *where $a_0(x), a_1(x)\in k[x]$,* $\gcd(a_0(x),a_1(x))=1$, $\deg a_1(x)<\deg a_0(x)$ *and $a_0(x)$ has two or more distinct linear factors. In this case, the $\boldsymbol{A}^1_*$-fibration $\rho: X\to B$ extends to an $\boldsymbol{A}^1$-fibration $\tilde{\rho}: \boldsymbol{A}^2\to\tilde{B}$, where $B=\tilde{B}\cong\boldsymbol{A}^1$.*

(4) $f=x^m-y^n$ *with $m,n>0$ and* $\gcd(m,n)=1$. *In this case, the closures of the fibers of the $\boldsymbol{A}^1_*$-fibration $\rho: X\to B$ form a linear pencil $\{x^m-\lambda y^n\}$ parametrized by $\lambda\in\boldsymbol{P}^1=k\cup\{\infty\}$, which has the point of origin as a base point. Furthermore, $B\cong\boldsymbol{A}^1$.*

Note that the case (4) above is obtained by Lin-Zaidenberg's theorem [5] which asserts that an irreducible curve C on $\boldsymbol{A}^2$, defined over the complex field $\boldsymbol{C}$, which is topologically contractible is defined by $x^m = y^n$ in terms of a suitable system of coordinates $\{x, y\}$ on $\boldsymbol{A}^2$. We shall look into the automorphism β in each of the above four cases.

LEMMA 1.5. *In the case* (1) *in Lemma* 1.4, *an automorphism β stabilizing the curve C is written as*

$$\beta(x) = ax\,, \quad \beta(y) = by$$

with $a, b \in k^*$ and $a^m b^n = 1$. We can write $a = u^n, b = \zeta^m u^{-m}$ with $u \in k^*$ and an mn-th root of unity ζ. So, β is of finite order if and only if u is a root of unity.

PROOF. As in the proof of Lemma 1.3, we have

$$\beta(x)^m \beta(y)^n + 1 = c(x^m y^n + 1)$$

with $c \in k^*$. So,

$$\beta(x)^m \beta(y)^n = cx^m y^n + (c-1)\,,$$

where the right side is irreducible unless $c = 1$. Hence $c = 1$ and $\beta(x)^m \beta(y)^n = x^m y^n$. Since $\gcd(m, n) = 1$, we have

$$\beta(x) = ax, \beta(y) = by \quad \text{with} \quad a, b \in k^*\,,$$

where $a^m b^n = 1$. The rest of the assertion is readily verified. Q.E.D.

LEMMA 1.6. *In the case* (2) *of Lemma* 1.4, *an automorphism β stabilizing the curve C is written as*

$$\beta(x) = ax\,, \quad \beta(y) = a^{-l} y$$

with $a^m = 1$. So, β is of finite order.

PROOF. Note that $x^l y + p(x)$ is an irreducible polynomial. Write

$$p(x) = c_0 x^{l-1} + c_1 x^{l-2} + \cdots + c_{l-1}$$

with $c_{l-1} \neq 0$. As in the proof of Lemmas 1.3 and 1.5, we have

$$\beta(x)^m (\beta(x)^l \beta(y) + p(\beta(x)))^n = x^m (x^l y + p(x))^n\,.$$

Since $\gcd(m, n) = 1$, we have $\beta(x) = ax$ with $a \in k^*$, and

$$a^{m/n}(a^l x^l \beta(y) + p(ax)) = \zeta(x^l y + p(x))\,,$$

where $\zeta^n = 1$. Hence it follows that

$$a^{l+m/n}\beta(y) = \zeta y\,, \quad \text{i.e.,} \ \beta(y) = a^{-(l+m/n)}\zeta y\,.$$

Furthermore, by comparing constant terms, we have

$$a^{m/n} c_{l-1} = \zeta c_{l-1}\,, \quad \text{i.e.,} \ a^{m/n} = \zeta\,,$$

whence $a^m = 1$, and $\beta(x) = ax$, $\beta(y) = a^{-l} y$. Then $\beta^m = 1$, and β is of finite order. Q.E.D.

LEMMA 1.7. *In the case* (3) *in Lemma* 1.4, *an automorphism* β *stabilizing the curve* C *is of finite order.*

PROOF. Note that $\bar{\kappa}(X) = 1$ (cf. [1, Lemma 3.11]) and that the $\boldsymbol{A}^1_*$-fibration $\rho : X \to B$ is canonical for the surface X in the sense that it is determined by a log pluri-canonical system $|n(D + K_V)|$ for $n \gg 0$, if (V, D) is a smooth compactification of X with boundary divisor D of simple normal crossings. Hence the automorphism β preserves the $\boldsymbol{A}^1_*$-fibration ρ (cf. [1, Lemma 3.3] for the detail). This implies that a fiber $x = \lambda$ of ρ is transformed to a fiber $x = \mu$. Namely,

$$\beta(x - \lambda) = c(x - \mu) \quad \text{and} \quad c \in k^* .$$

Hence we have

$$\beta(x) = cx + d \quad \text{with} \quad c, d \in k \quad \text{and} \quad c \neq 0 .$$

The fibration ρ has singular fibers, which are by definition not isomorphic to $\boldsymbol{A}^1_*$, over the points α with $a_0(\alpha) = 0$. If β is of infinite order and if $a_0(x) \notin k$, then there would be infinitely many singular fibers. Hence $a_0(x) = a_0 \in k$ or β is of finite order. In the former case, the curve C is isomorphic to $\boldsymbol{A}^1$, and $\bar{\kappa}(X) = -\infty$ by a theorem of Abhyankar-Moh-Suzuki. So, β is of finite order. Q.E.D.

LEMMA 1.8. *In the case* (4) *of Lemma* 1.4, *an automorphism* β *stabilizing the curve* C *is written as*

$$\beta(x) = ax , \quad \beta(y) = by ,$$

where $a, b \in k$, $ab \neq 0$ *and* $a^m = b^n$.

PROOF. Note that β preserves the pencil $\{x^m - \lambda y^n\}$ with $\lambda \in \boldsymbol{P}^1$ by the same reason as in the proof of Lemma 1.7. The pencil has two multiple fibers mA and nB, where A and B are defined by $x = 0$ and $y = 0$, respectively. Since $\gcd(m, n) = 1$, it follows that $\beta(x) = ax$ and $\beta(y) = by$ with $a, b \in k$ and $ab \neq 0$. Since $\beta(f) = cf$ with $c \neq 0$, we have $a^m = b^n$. Q.E.D.

Summarizing the above results, we obtain the following result:

THEOREM 1.9. *Let* β *be an automorphism of* $\boldsymbol{A}^2$ *of infinite order such that* β *stabilizes an irreducible curve* C *defined by* $f = 0$. *Then, after a suitable change of coordinates,* β *and* f *are written in one of the following forms*:

(1) $f = x$; $\beta(x) = ax$, $\beta(y) = by + g(x)$ *with* $a, b \in k^*$ *and* $g(x) \in k[x]$.

(2) $f = xy + 1$; $\beta(x) = ax$, $\beta(y) = a^{-1}y$ *or* $\beta(x) = ay$, $\beta(y) = a^{-1}x$, *where* $a \in k^*$.

(3) $f = x^m y^n + 1$; $\beta(x) = ax$, $\beta(y) = by$, *where* $mn > 1$, $\gcd(m, n) = 1$, $a, b \in k^*$ *and* $a^m b^n = 1$.

(4) $f = x^m - y^n$, $\gcd(m, n) = 1$; $\beta(x) = ax$, $\beta(y) = by$ *with* $a, b \in k^*$ *and* $a^m = b^n$.

COROLLARY 1.10. *Let* β *be as in Theorem* 1.9. *Suppose, furthermore, that* β *leaves* C *pointwise fixed. Then* β *and* f *are written as*

$$f = x ; \quad \beta(x) = ax , \quad \beta(y) = y + xh(x) ,$$

where $h(x) \in k[x]$. In particular, the curve C is a coordinate line after a change of coordinates on $\mathbf{A}^2$.

2. Higher-dimensional case. Let $X = \operatorname{Spec} A$ be a nonsingular affine variety of dimension n such that $\operatorname{Pic} X = (0)$ and $A^* = k^*$. We shall begin with the following result:

LEMMA 2.1. *Let W be an irreducible hypersurface of X, and let β be a nontrivial automorphism of X such that*

(1) *β leaves W pointwise fixed, and*

(2) *β induces a nontrivial action on I/I^2, where I is the defining ideal of W.*

Then W is nonsingular.

PROOF. (I) Since A is factorial, the ideal I is principal. Let $u \in A$ be an element such that $I = (u)$. Since $\beta(W) = W$, one may write $\beta(u) = au$ with $a \in A$. Since β^{-1} also leaves W pointwise fixed, one may write $\beta^{-1}(u) = bu$. Then we have

$$u = \beta^{-1}(\beta(u)) = \beta^{-1}(au) = \beta^{-1}(a)\beta^{-1}(u) = \beta^{-1}(a)bu\,,$$

whence $\beta^{-1}(a) \in A^* = k^*$. So, $a \in k^*$. Since β induces a nontrivial action on I/I^2, it follows that $a \neq 1$.

(II) Let $Q \in W$ be a closed point and $\{x_1, \dots, x_n\}$ a system of local coordinates of X at Q. In the completion $\hat{\mathcal{O}}_{X,Q} = k[[x_1, \dots, x_n]]$, write

$$u = \sum_{i \geq m} u_i(x_1, \dots, x_n)\,,$$

where u_i is the i-th homogeneous part and $m \geq 1$. Since $\beta(Q) = Q$, one can write

$$\beta(x_i) = \sum_{j=1}^{n} b_{ij}x_j + (\text{terms of degree} \geq 2)\,.$$

Then we have

$$\begin{aligned}\beta(u) &= u_m\left(\sum_{j=1}^{n} b_{1j}x_j, \dots, \sum_{j=1}^{n} b_{nj}x_j\right) + (\text{terms of degree} \geq m+1)\\ &= a\sum_{i \geq m} u_i(x_1, \dots, x_n)\,.\end{aligned}$$

Hence

$$u_m\left(\sum_{j=1}^{n} b_{1j}x_j, \dots, \sum_{j=1}^{n} b_{nj}x_j\right) = au_m(x_1, \dots, x_n)\,.$$

This implies that the matrix $B = (b_{ij})$ is not the identity matrix.

(III) Suppose that Q is a singular point of W. Then we have

$$\underline{m}_{W,Q}/\underline{m}^2_{W,Q} = \underline{m}_{X,Q}/\underline{m}^2_{X,Q}\,,$$

where $\underline{m}_{W,Q}, \underline{m}_{X,Q}$ are the maximal ideals of the local rings $\mathcal{O}_{W,Q}, \mathcal{O}_{X,Q}$, respectively, and the automorphism β induces the identity automorphism on $\underline{m}_{W,Q}/\underline{m}^2_{W,Q}$, while β acts on

$\underline{m}_{X,Q}/\underline{m}_{X,Q}^2$ via the matrix B. This is a contradiction to a conclusion in the step (II). Hence W is nonsingular. Q.E.D.

We denote by G_m a one-dimensional algebraic torus.

PROPOSITION 2.2. *Let G_m act nontrivially on an n-dimensional nonsingular affine variety X = Spec A defined over the complex field $\boldsymbol{C}$ with* Pic $X = (0)$ *and let W be an irreducible hypersurface such that the G_m-action leaves W pointwise fixed. Then W is nonsingular. Suppose, furthermore, that X is a contractible threefold with $A^* = \boldsymbol{C}^*$. Then $X \cong W \times \boldsymbol{A}^1$. If $\bar{\kappa}(W) = -\infty$ or $X = \boldsymbol{A}^3$ in particular, we have $W \cong \boldsymbol{A}^2$, and X is isomorphic to the affine space of dimension* 3 *with W as a coordinate hyperplane.*

PROOF. Let u be a generator of the defining ideal I of W. Then we have $t \cdot u = \chi(t)u$ for $t \in G_m$ with $\chi(t) \in A^* = \boldsymbol{C}^*$. Then χ is a multiplicative character of G_m. Write $\chi(t) = t^m$, where $m \neq 0$. In fact, if $m = 0$, then the G_m-action is trivial near the points of W. But this is not the case. Hence W is nonsingular by Lemma 2.1 (see also Fogarty [3]).

For any point $P \in X$, we have

$$\lim_{t\to 0} t \cdot P \in W \quad \text{if } m > 0$$

and

$$\lim_{t\to\infty} t \cdot P \in W \quad \text{if } m < 0.$$

Hence W is the fixedpoint locus X^{G_m} and, by Bialynicki-Birula [2], X is an $\boldsymbol{A}^1$-bundle over W. Meanwhile, W is also the algebraic quotient $X//G_m$, since G_m acts on X along the fibers of the $\boldsymbol{A}^1$-bundle. So, W is a contractible surface by Kraft-Petrie-Randall [9], because so is X by the hypothesis. Then Pic$(W) = (0)$ by [4, 1.20]. This implies that the $\boldsymbol{A}^1$-bundle over W is trivial. Namely, we have $X \cong W \times \boldsymbol{A}^1$. Write W = Spec B, where B is identified with the G_m-invariant subalgebra of A. Note then that B is a factorial domain with $B^* = \boldsymbol{C}^*$. If $\bar{\kappa}(W) = -\infty$ in particular, W is isomorphic to $\boldsymbol{A}^2$ by the characterization of the affine plane (cf. [12]). If $X = \boldsymbol{A}^3$, then $W \cong \boldsymbol{A}^2$ by the cancellation theorem [12]. Q.E.D.

We extend Proposition 2.2 to a case where G_m is replaced by a single automorphism of infinite order. Let A be an affine domain over k, i.e., a k-algebra domain which is finitely generated over k. A k-automorphism β of A is called *rational* if, for every $w \in A$, the k-vector space $\sum_{i\geq 0} k\beta^i(w)$ is finite-dimensional. A k-automorphism β of A is called *diagonalizable* if β is rational and if the action of β on $\sum_{i\geq 0} k\beta^i(w)$ is diagonalizable, i.e., there exists a certain k-basis $\{v_1, \dots, v_r\}$ of $\sum_{i\geq 0} k\beta^i(w)$ such that $\beta(v_i) = a_i v_i$ with $a_i \in k^*$ for $1 \leq i \leq r$. Note that given a G_m action on X = Spec A the automorphism $x \mapsto t \cdot x$ of X, with t a general point of G_m, induces a diagonalizable k-automorphism of A. We shall begin with the following simple but useful result.

LEMMA 2.3. *Let A be an affine domain and β a diagonalizable automorphism of A. Let I be an ideal of A such that $\beta(I) \subseteq I$. Then, for any element $v \in A$ such that $\beta(v) \equiv v$* (mod I), *there exists an element $v' \in A$ such that $\beta(v') = v'$ and $v' \equiv v$* (mod I).

PROOF. Let $V = \sum_{i\geq 0} k\beta^i(v)$. Then V is finite-dimensional. Since β is diagonalizable, we may choose a k-basis $\{v_1, \dots, v_r\}$ of V such that $\beta(v_j) = a_j v_j$ $(1 \leq j \leq r)$ for $a_j \in k^*$. Note that $\beta^i(v) \equiv v \pmod{I}$ for every $i \geq 0$. Since v_j is a k-linear combination of $\{\beta^i(v)\}_{i\geq 0}$, it follows that $\beta(v_j) \equiv v_j \pmod{I}$ for every $1 \leq j \leq r$. Let $\overline{v_j}$ be the residue class of v_j modulo I. Since $\beta(v_j) = a_j v_j$, we have $a_j = 1$ provided $\overline{v_j} \neq 0$. After a change of indices, suppose that $\overline{v_j} \neq 0$ for $1 \leq j \leq s$ and $\overline{v_j} = 0$ for $s+1 \leq j \leq r$. Write

$$v = c_1 v_1 + \cdots + c_s v_s + c_{s+1} v_{s+1} + \cdots + c_s v_r ,$$

and let

$$v' = c_1 v_1 + \cdots + c_s v_s .$$

Then $\beta(v') = v'$ and $v' \equiv v \pmod{I}$. Q.E.D.

We need the following lemma in the subsequent argument.

LEMMA 2.4. *Let C be an irreducible nonsingular affine curve with an automorphism β of infinite order. If β has a fixed point, then C is isomorphic to $\boldsymbol{A}^1$. Furthermore, if we write $\boldsymbol{A}^1 = \operatorname{Spec} k[t]$, then β is given as $\beta(t) = ct$ with $c \in k^*$.*

PROOF. If $\bar{\kappa}(C) = 1$, then $\operatorname{Aut}(C)$ is a finite group. Hence $\bar{\kappa}(C) \leq 0$. If $\bar{\kappa}(C) = 0$, then C is either a complete elliptic curve or is isomorphic to G_m. The first case is obviously not the case. In the second case, every automorphism β of G_m of infinite order is a translation. Hence it has no fixed points. So, the second case is not the case either, and we have $\bar{\kappa}(C) = -\infty$. Then $C \cong \boldsymbol{A}^1$. The last assertion is clear. Q.E.D.

In what follows in this section, we shall work in the following set-up:

Let $X = \operatorname{Spec} A$ be a nonsingular affine variety of dimension n with $\operatorname{Pic}(X) = (0)$ and $A^* = k^*$. Let W be an irreducible hypersurface of X and β a nontrivial automorphism of X of infinite order. Assume that

(i) *β leaves W pointwise fixed, and*

(ii) *the induced k-automorphism β on A is diagonalizable.*

Let $L = Q(A)$ be the function field of X. Then the automorphism β extends to L in a natural fashion. We define a subalgebra B of A and a subfield K of L by

$$B = \{a \in A;\ \beta^m(a) = a \ \text{ for some } m > 0\}$$

and

$$K = \{\xi \in Q(A);\ \beta^m(\xi) = \xi \ \text{ for some } m > 0\} .$$

It is clear that $B = A \cap K$. Since $\operatorname{Pic}(X) = (0)$, the defining ideal I of W is principal. Let u be a generator of the ideal I. Then $\beta(u) = au$ with $a \in k^*$.

LEMMA 2.5. *The following assertions hold:*

(1) *The element a is not a root of unity, and β acts nontrivially on I/I^2. Hence W is nonsingular.*

(2) *K is the quotient field $Q(B)$ of B, and u is transcendental over K. Furthermore, K is algebraically closed in L.*

(3) *B is k-isomorphic to A/I. In particular, B is finitely generated over k.*

(4) *B is a normal subalgebra of A of dimension* $n-1$.

PROOF. (1) Let P be a smooth point of W and let $v_1, \dots, v_{n-1} \in A$ be the elements such that the residue classes $\bar{v}_1, \dots, \bar{v}_{n-1}$ form a local system of parameters of W at P. Then $\beta(v_i) \equiv v_i \pmod{I}$ for $1 \le i \le n-1$. By virtue of Lemma 2.3, we may assume that $\beta(v_i) = v_i$ after a suitable change of the elements v_i. Then $\{v_1, \dots, v_{n-1}, u\}$ is a local system of parameters of X at P such that $\beta(v_i) = v_i$ for $1 \le i \le n-1$ and $\beta(u) = au$ with $a \in k^*$. We shall show that a is not a root of unity. Indeed, the function field L of X is a finite algebraic extension of the field $k(v_1, \dots, v_{n-1}, u)$. If a is a root of unity, we may replace β by some power β^m and assume that β acts on L as an $k(v_1, \dots, v_{n-1}, u)$-automorphism. This is impossible because β is of infinite order. Hence a is not a root of unity. Then β acts nontrivially on I/I^2. By Lemma 2.1, W is nonsingular.

(2) We shall first show that u is transcendental over the field K. Indeed, if u were algebraic over K, u satisfies a nontrivial algebraic equation

$$(\dagger) \qquad u^N + \xi_1 u^{N-1} + \cdots + \xi_N = 0 \quad \text{with} \quad \xi_i \in K\,.$$

By replacing β by β^m with some $m > 0$, we may assume that $\beta(\xi_i) = \xi_i$ for $1 \le i \le N$. Then β permutes the roots of the above equation (†). But this is impossible because $\beta(u) = au$, where a is not a root of unity. Hence u is transcendental over K. On the other hand, we may choose a system of elements $\{v_1, \dots, v_{n-1}\}$ of B such that $\{\bar{v}_1, \dots, \bar{v}_{n-1}\}$ is a local system of parameters of W at a point Q. This implies that $k(v_1, \dots, v_{n-1}) \subseteq K$ and $\operatorname{tr.deg}_k K = n-1$. Hence K is algebraic over $Q(B)$. Let η be an element of L such that η is algebraic over $Q(B)$. Then η satisfies a relation

$$(\dagger\dagger) \qquad a_0\eta^N + a_1\eta^{N-1} + \cdots + a_N = 0 \quad \text{with} \quad a_i \in B\,.$$

Replacing β by β^m for some $m > 0$, we may assume that $\beta(a_j) = a_j$ for every j. Then $\beta(\eta)$ is also a solution of (††). Since there are finitely many solutions of (††), we have $\beta^m(\eta) = \eta$ for some $m > 0$. Namely $\eta \in K$. Hence K is algebraically closed in L. If $\eta \in L$ is, in particular, integral over B, then we have $\eta \in A \cap K = B$ because A is normal. The relation (††) implies that $a_0\eta$ is integral over B and hence $a_0\eta \in B$. Therefore $\eta \in Q(B)$. This implies that $K = Q(B)$.

(3) Restricting the residue homomorphism $A \to A/I$ onto B, we have a k-algebra homomorphism $\rho : B \to A/I$. Since β induces a trivial automorphism on A/I, it follows from Lemma 2.3 that ρ is surjective. We shall show that ρ is injective. Namely, we show that $I \cap B = (0)$. Let $w \in I \cap B$, and write $w = uw_1$ with $w_1 \in A$. Then $\beta^m(w) = w$ for some $m > 0$. This implies that $\beta^m(w_1) = a^{-m}w_1$. Meanwhile, since $\beta(w_1) \equiv w_1 \pmod{I}$, we may express $\beta^m(w_1) = w_1 + uz$ with $z \in A$. Hence we obtain $(a^m - 1)w_1 = -a^m uz$. Since a is not a root of unity, $a^m - 1 \ne 0$. So, we have $w_1 = uw_2$ with $w_2 \in A$ and $w = u^2 w_2$. Applying the same argument as above to the expression $w = u^2 w_2$, we can show that $w = u^3 w_3$ with $w_3 \in A$. Thus $w \in \bigcap_{i \ge 0} I^i$. Now, applying the intersection theorem of Krull [18, Theorem 3.11], we know that $\bigcap_{i \ge 0} I^i = (0)$. Hence $w = 0$. Alternatively, we could argue that since A is a factorial domain, w cannot be divided infinitely many times by

an irreducible element u unless $w = 0$. We have thus shown that B is isomorphic to A/I. In particular, B is finitely generated over k. If $n = 3$, Zariski's lemma [17] also implies that B is finitely generated over k because $B = A \cap K$.

(4) Since we know that B is an affine domain and $B = A \cap Q(B)$, it is clear that B is a normal k-subalgebra of dimension $n - 1$. Q.E.D.

Since B is finitely generated over k, there exists an integer $m > 0$ such that $\beta^m(b) = b$ for every $b \in B$. By replacing β by β^m, we may and shall assume without loss of generality that $\beta(b) = b$ for every $b \in B$. Let $Y = \mathrm{Spec}(B)$ and $\pi : X \to Y$ a morphism induced by the inclusion $B \hookrightarrow A$. Then the general fibers of π are nonsingular irreducible curves. The automorphism β acts on X along the fibers of π.

LEMMA 2.6. *The morphism $\pi : X \to Y$ is an $\boldsymbol{A}^1$-fibration, and the generic fiber of π is given as* Spec $K[u]$.

PROOF. It follows from the assertion (3) of Lemma 2.5 that W is a cross-section of the morphism π. Let C be a general fiber of π. Then C meets W in one point transversally, and the automorphism β induces an automorphism of C of infinite order. The intersection point of C with W is a fixed point under this automorphism. By Lemma 2.4, C is then isomorphic to $\boldsymbol{A}^1$. Hence π is an $\boldsymbol{A}^1$-fibration.

Write the generic fiber $X_K := \mathrm{Spec}\, A \otimes_B K$ as Spec $K[t]$ with some parameter t. Then β acts on X_K by $\beta(t) = \xi t$ with $\xi \in K^*$. We shall show that $t = \eta u$ with $\eta \in K^*$. Write u as

$$u = \eta_0 t^m + \eta_1 t^{m-1} + \cdots + \eta_m \quad \text{with} \quad \eta_i \in K ,$$

where $\eta_0 \neq 0$. Since $\beta(u) = au$ and $\beta(\eta_i) = \eta_i$, we can readily show that $u = \eta_0 t^m$. Choose a general fiber C of π so that the function η_0 is regular and nonzero at the intersection point $P = C \cap W$. The argument in the proof of Lemma 2.5, about lifting a local system of parameters $\{\bar{v}_1, \ldots, \bar{v}_{n-1}\}$ of W at the point P to a system of elements $\{v_1, \ldots, v_{n-1}\}$ of B, shows that

$$\underline{m}_{X,P} = (u, v_1, \ldots, v_{n-1}) \quad \text{and} \quad \underline{m}_{W,P} = (v_1, \ldots, v_{n-1}) ,$$

where $\underline{m}_{X,P}$ and $\underline{m}_{W,P}$ are the maximal ideals of the local rings $\mathcal{O}_{X,P}$ and $\mathcal{O}_{W,P}$, respectively. Since $u \notin \underline{m}^2_{X,P}$, it follows that $m = 1$. Hence we conclude that $X_K = \mathrm{Spec}\, K[u]$. Q.E.D.

Note that $\beta(b) = b$ for every element $b \in B$. For $c \in k^*$, set

$$M_c = \{w \in A \mid \beta(w) = cw\} ,$$

and let

$$\Phi = \{c \in k^* \mid M_c \neq (0)\} .$$

LEMMA 2.7. *The following assertions hold*:

(1) $\Phi = \{a^l \mid l \geq 0\}$.

(2) $M_{a^l} = Bu^l$ *for every* $l \geq 0$.

(3) $A = \bigoplus_{l \geq 0} M_{a^l} \cong B[u]$.

PROOF. By Lemma 2.6, $A \otimes_B K = K[u]$. Suppose $w \in M_c$. Then $w = \xi u^l$ for some $\xi \in K$ and $l \geq 0$. Hence $c = a^l$ for some $l \geq 0$. This implies that

$$\Phi = \{a^l \mid l \geq 0\}.$$

Write $\xi = z_2/z_1$ with $z_1, z_2 \in B$. Then we have

$$(*) \qquad z_1 w = z_2 u^l .$$

Note that u is an irreducible element of A. Suppose u is a factor of z_1 and write $z_1 = uz_1'$. Then $\beta(z_1') = a^{-1}z_1'$. So, $a^{-1} \in \Phi$, i.e., $a^{-1} = a^m$ with $m \geq 0$. Hence $a^{m+1} = 1$, a contradiction. So, u^l divides w in the equality $(*)$. Hence $\xi \in A \cap K = B$. Namely, $w \in Bu^l$. It then follows that $M_c = Bu^l$, where $c = a^l$.

Now we shall show that $A = \bigoplus_{l \geq 0} M_{a^l}$. Let w be anew any nonzero element of A. Since β is diagonalizable, we have

$$w = c_1 w_1 + \cdots + c_r w_r$$

with $\beta(w_i) = a_i w_i$ and $a_i \in \Phi$. So, $w \in \bigoplus_{l \geq 0} M_{a^l}$. Hence $A \subseteq \bigoplus_{l \geq 0} M_{a^l}$. The converse inclusion $\bigoplus_{l \geq 0} M_{a^l} \subseteq A$ is clear. Q.E.D.

Summarizing the above lemmas, we have shown the following result:

THEOREM 2.8. *Let $X =$ Spec A be a nonsingular affine variety of dimension n with* Pic $X = (0)$ *and $A^* = k^*$. Let W be an irreducible hypersurface of X and β a nontrivial automorphism of X of infinite order. Assume that*

(i) *β leaves W pointwise fixed, and*

(ii) *β is diagonalizable.*

Then $X \cong W \times \mathbf{A}^1$. Hence W is a coordinate hyperplane after a suitable change of coordinates of X if W is isomorphic to $\mathbf{A}^{n-1}$, and X is accordingly isomorphic to $\mathbf{A}^n$.

Hence Theorem 2.8 implies the next result:

COROLLARY 2.9. *Let $X = \mathbf{A}^3$ be the affine space of dimension* 3. *Let W be an irreducible hypersurface of X and β a nontrivial automorphism of X of infinite order. Assume that*

(i) *β leaves W pointwise fixed, and*

(ii) *β is diagonalizable.*

Then $X \cong W \times \mathbf{A}^1$ and W is a coordinate hyperplane after a suitable change of coordinates.

PROOF. If X is the affine space of dimension 3, the cancellation theorem (cf. [12]) implies that W is isomorphic to the affine plane $\mathbf{A}^2$. Hence W becomes a coordinate plane after a suitable choice of the coordinates. Q.E.D.

REMARK. Theorem 2.8 shows that an automorphism β on X extends to a G_m-action on X which has W as the fixed-point locus. In fact, the property of β being diagonalizable is immediate if β extends to a G_m-action. We do not know, in general, under which conditions β extends to a G_m-action.

As stated in the introduction, we obtain an algebraic characterization of the affine space of dimension 3.

THEOREM 2.10. *Let $X = \operatorname{Spec} A$ be a nonsingular affine threefold. Then X is the affine space of dimension* 3 *if and only if the following conditions are satisfied*:

(1) $\operatorname{Pic}(X) = (0)$ *and* $A^* = k^*$.

(2) *There exist an irreducible hyperplane W and a nontrivial automorphism β of X of infinite order such that*

 (a) *β leaves W pointwise fixed,*

 (b) *β is diagonalizable,*

 (c) *W has Kodaira dimension $-\infty$.*

PROOF. Suppose X is the affine space of dimension 3 with the coordinates x, y, z. Then we can take a linear hyperplane $x = 0$ as W and an automorphism β defined by $\beta(x) = ax$, $\beta(y) = y$ and $\beta(z) = z$ with some $a \in k^*$ which is not a root of unity. We shall show the converse. By Theorem 2.8, $X \cong W \times \boldsymbol{A}^1$. Write $W = \operatorname{Spec} B$. Then $\operatorname{Pic}(W) = (0)$ and $B^* = k^*$. If W has Kodaira dimension $-\infty$, then $W \cong \boldsymbol{A}^2$ (cf. [12]). Hence $X \cong \boldsymbol{A}^3$.

Q.E.D.

We note that there is an algebraic characterization of the affine space of dimension 3 obtained by the second author [13]. The hypersurface W has Kodaira dimension $-\infty$, for example, provided there is a G_a-action commuting with the given automorphism β.

3. An algebro-topological characterization of the affine plane. In the present and next sections, W is an irreducible subvariety in a non-singular affine variety X of codimension two such that W is the fixed-point locus under a given effective G_m-action on X. A closed orbit O is called a *multiple orbit* if the isotropy group is a nontrivial finite group. We consider first the case where X is a surface and W is a point P. Considering the tangential representation of G_m at the point P, let a and b be the weights. Then $ab \neq 0$ because the fixed-point locus consists only of P. We have the *unmixed* case $ab > 0$ and the *mixed* case $ab < 0$. We obtain the following algebro-topological characterization of the affine plane.

THEOREM 3.1. *Let X be a nonsingular affine surface with an effective G_m-action. Assume that the fixed-point locus consists of a single point P. If one of the following conditions is satisfied, X is then isomorphic to the affine plane.*

(1) *The G_m-action is unmixed.*

(2) *The G_m-action is mixed and X is a homology plane.*

(3) *The G_m-action is mixed, the algebraic quotient $T := X//G_m$ is a curve isomorphic to the affine line and any closed orbit is not a multiple orbit.*

PROOF. (1) If the G_m-action is unmixed, the result is immediate by [2]. An elementary proof is given as follows. We may assume that $a > 0$ and $b > 0$. Let A be the coordinate ring of X. Then A is a graded k-algebra

$$A = \bigoplus_{i \geq 0} A_i \,.$$

Let $A^+ = \bigoplus_{i>0} A_i$. The fixed-point locus is defined by the ideal A^+. Hence $A_0 = A/A^+ = k$, where k is the ground field. By the hypothesis, $A^+/(A^+)^2 = k\bar{x} + k\bar{y}$ with G_m-action given by

$$t \cdot \bar{x} = t^a \bar{x}\,, \quad t \cdot \bar{y} = t^b \bar{y}\,.$$

By the complete reducibility of the G_m-action, we find elements $x \in A_a$ and $y \in A_b$ such that $t \cdot x = t^a x$ and $t \cdot y = t^b y$.

We shall show that A is generated over k by these elements x and y. The proof proceeds by induction on the weight of each element of A. Let z be an element of A. We may assume that z is homogeneous because z is a sum of homogeneous elements. Then the residue class $\bar{z}$ of z by $(A^+)^2$ is a linear combination

$$\bar{z} = c\bar{x} + d\bar{y} \quad \text{with} \quad c, d \in k\,.$$

Hence $z - (cx + dy) \in (A^+)^2$. So, we may write

$$z - (cx + dy) = \sum_i z_i z'_i\,,$$

where $z_i, z'_i \in A^+$ are homogeneous elements with $\deg(z_i) < \deg(z)$ and $\deg(z'_i) < \deg(z)$. Here the degree of each element is the one in the graded ring A. Hence it is the weight of a semi-invariant element. By the induction hypothesis, we may assume that $z_i, z'_i \in k[x, y]$. Then $z \in k[x, y]$. Thus $A = k[x, y]$ and X is isomorphic to the affine plane $\boldsymbol{A}^2$.

(2) Note that X is then a homology plane with $\boldsymbol{A}^1_*$-fibration. Since the G_m-action is mixed, by [2], there exist two curves C_1 and C_2 isomorphic to $\boldsymbol{A}^1$ and meeting each other transversally in the point P. By a general result on the number of the lines contained in a homology plane [15, Theorem 13], we conclude that X is isomorphic to the affine plane $\boldsymbol{A}^2$.

(3) Since the G_m-action is mixed, as in the case (2) above, there are two affine lines C_1, C_2 meeting transversally in P. Let $\pi : X \to T$ be the quotient morphism, and let a_1, a_2 be the multiplicities of C_1, C_2 in the fiber $\pi^{-1}(Q)$, where $Q = \pi(P)$. We *claim* that $d := \gcd(a_1, a_2) = 1$. Suppose otherwise that $d > 1$. Choose a parameter t of T so that Q is defined by $t = 0$. Let $T' \to T$ be the branched covering of degree d which totally ramifies at the point Q and the point at infinity. Then T' is the affine line. Let X' be the normalization of the fiber product $T' \times_T X$. Then $X' \to X$ is étale, the projection $\pi_{T'} : X' \to T'$ is an $\boldsymbol{A}^1_*$-fibration, and the fiber $\pi_{T'}^{-1}(Q')$ is a disjoint sum of d copies of $a'_1 C'_1 + a'_2 C'_2$, where C'_1 and C'_2 are affine lines meeting transversally in one point and $a'_i = a_i/d$ for $i = 1, 2$. This is, however, impossible by [14, Lemma 4]. So, $d = 1$.

We shall next *show* that $\pi_1(X) = (1)$. For this purpose, set $X_1 = X - C_2$ and $X_2 = X - C_1$. Let $p_i := \pi\,|_{X_i} : X_i \to T$ and $C_i^* = C_i - \{P\}$ for $i = 1, 2$. By the hypothesis that any closed orbit is not a multiple orbit, $p_i : X_i \to T$ is then an $\boldsymbol{A}^1_*$-fibration with only one singular fiber which is $a_i C_i^*$. Consider $p_1 : X_1 \to T$, and let $T_1 \to T$ be the branched covering of degree a_1 which totally ramifies at the point Q and the point at infinity. Let X'_1 be the normalization of $T_1 \times_T X_1$. Then $(p_1)_{T_1} : X'_1 \to T_1$ is an $\boldsymbol{A}^1_*$-bundle over T_1. Indeed, the natural morphism $X'_1 \to X_1$ is a finite étale covering and the inverse image of the multiple fiber $a_1 C_1^*$ is a reduced fiber of the $\boldsymbol{A}_*^1$-fibration $X_1' \to T_1$, which consists of several connected components isomorphic to $\boldsymbol{A}_*^1$. By [14, Lemma 4], it consists of only one connected reduced fiber isomorphic to $\boldsymbol{A}_*^1$. So, $X_1' \to T_1$ is an $\boldsymbol{A}_*^1$-bundle over T_1. Since any $\boldsymbol{A}_*^1$-bundle over the affine line T_1 is trivial, we have $\pi_1(X_1') \cong \pi_1(\boldsymbol{A}_*^1) \cong \boldsymbol{Z}$. Since $X_1' \to X_1$ is a cyclic étale covering of degree a_1, we obtain an exact sequence:

$$\pi_1(C_1^*) \to \pi_1(X_1) \to \boldsymbol{Z}/m_1\boldsymbol{Z} \to 0\,,$$

where $m_1 \mid a_1$. (We may apply also a result of Nori [19] to $(p_1)_{T_1} : X_1' \to T_1$ to obtain the above exact sequence.) This yields an exact sequence

$$\pi_1(C_1) \to \pi_1(X) \to \boldsymbol{Z}/m\boldsymbol{Z} \to 0$$

with $m \mid a_1$ because the natural homomorphism $\pi_1(X_1) \to \pi_1(X)$ is a surjection. Similarly, we have an exact sequence

$$\pi_1(C_2) \to \pi_1(X) \to \boldsymbol{Z}/n\boldsymbol{Z} \to 0\,,$$

where $n \mid a_2$. Since $\gcd(a_1, a_2) = 1$, we end up with a surjection

$$\pi_1(C_1 \cup C_2) \to \pi_1(X) \to 0\,.$$

Since $C_1 \cup C_2$ is simply connected, we have $\pi_1(X) = (1)$. On the other hand, it is easy to see that the Euler number $e(X) = 1$. Hence X is a contractible surface. Since X contains two affine lines, X is isomorphic to the affine plane (cf. [15, Theorem 13]). Q.E.D.

REMARK. In the unmixed case, we have only to assume that X is a reduced algebraic k-scheme with a G_m-action and that P is the unique fixed point at which X is nonsingular. In fact, let Q be an arbitrary point of X. Then the closure of the G_m-orbit $\overline{G_m \cdot Q}$ passes through the point P. Hence the orbit $G_m \cdot Q$ contains a nonsingular point, whence Q is nonsingular on X.

4. The affine 3-space as an acyclic threefold. We extend the unmixed case of Theorem 3.1 to the higher-dimensional case.

LEMMA 4.1. *Let X be a reduced affine algebraic k-scheme and let W be an irreducible closed subscheme of X of codimension two. Suppose the algebraic torus G_m acts on X in such a way that W is the fixed-point locus. Furthermore, assume that W is nonsingular and X is nonsingular near W. We assume that every orbit of a point not in W is non-closed. Then X is an $\boldsymbol{A}^2$-bundle over W.*

PROOF. Note that the hypothesis implies the smoothness of X. Let A be the coordinate ring of X. Then we may assume that A is a graded ring

$$A = \bigoplus_{i \geq 0} A_i\,.$$

Then A_0 is the coordinate ring of W. Let P be a point of W and $\underline{p}$ the prime ideal of A_0 corresponding to P. Then $A_P := A \otimes_{A_0} A_0/\underline{p}$ is the coordinate ring of the fiber $\pi^{-1}(P)$, where $\pi : X \to W$ is the morphism associated with the inclusion $A_0 \hookrightarrow A$. By Theorem

3.1 and the subsequent remark, $\pi^{-1}(P)$ is nonsingular and is isomorphic to $\boldsymbol{A}^2$. Now X is nonsingular and is an $\boldsymbol{A}^2$-bundle over W by [2]. Q.E.D.

The arguments using the acyclicity and a G_m-action lead us to an algebro-topological characterization of the affine 3-space among the acyclic threefolds.

THEOREM 4.2. *Let X be a nonsingular affine threefold defined over the complex field* $\boldsymbol{C}$. *Then X is isomorphic to the affine* 3-*space* $\boldsymbol{A}^3$ *if and only if the following conditions are satisfied*:

(1) *X is acyclic and endowed with an effective* G_m*-action.*

(2) *There exists a nonsingular irreducible subvariety W of codimension two which is the fixed-point locus under the given* G_m*-action.*

(3) *X has the logarithmic Kodaira dimension* $\bar{\kappa}(X) = -\infty$.

The subvariety W then becomes a coordinate line.

PROOF. The "only if" part is clear. We have only to consider a G_m-action on $\boldsymbol{A}^3 = \operatorname{Spec} k[x, y, z]$ given by

$$t \cdot (x, y, z) = (tx, ty, z) \quad \text{or} \quad t \cdot (x, y, z) = (t^{-1}x, ty, z),$$

where $t \in G_m$. So, we prove the "if" part. Our proof consists of several steps.

STEP (I). *W is an affine line and any closed orbit has the trivial isotropy group unless it is a fixed point.*

Indeed, let p be a prime number and H_n the subgroup of G_m consisting of p^n-th roots of the unity. Let W_n be the fixed-point locus of X under the induced H_n-action. Then W_n is a closed subset and $W = \bigcap_{n \geq 1} W_n$. Hence $W = W_n$ for some $n > 0$. By the Smith theory applied to the H_n-action on X with p varying, it follows from the acyclicity of X that W_n is connected and acyclic. Since W is a curve, W is then an affine line. Suppose that there exists a closed orbit $O = G_m \cdot P$ with a nontrivial finite isotropy group G. Let p be a prime number dividing the order of G. Again, by the Smith theory, the acyclicity of X implies that the fixed-point locus under the H_1-action on X is connected. Hence we may assume that there exists an irreducible subvariety, say V, of codimension one such that V contains W and the orbit O and that V is left pointwise fixed by H_1. Let P be a point of W and let $t \cdot (u, v, w) = (t^a u, t^b v, w)$ be the induced G_m-action on the tangent space $T_{X,P}$ (cf. the step (II) below). Since W is contained in V, it follows that p divides both a and b. Then G_m acts non-effectively on an open neighborhood of P, hence everywhere on X. This is a contradiction on the effectiveness of the G_m-action. So, we conclude that there are no multiple orbits.

STEP (II). *Let P be a point of W and let a, b be the weights of the induced representation of* G_m *on the tangent space* $T_{X,P}$. *Namely, after diagonalizing the representation, it is given as*

$$t \cdot (u, v, w) = (t^a u, t^b v, w).$$

Then the weights a, b are independent of the choice of P, and $\gcd(a, b) = 1$. *Furthermore, if* $ab > 0$, *then X is isomorphic to the affine 3-space $\boldsymbol{A}^3$.*

Indeed, by Luna [10, Lemme, p. 96], there exists a G_m-equivariant morphism $\varphi : X \to T_{X,P}$ such that φ is étale in P and $\varphi(P) = 0$. Then we may assume that the affine line W is mapped isomorphically to the w-axis according to the above notation. Then the tangential actions of G_m at the points on W near P are the same as the one at the point P. So, the weights a, b are constant in a neighborhood of the point P on W. Since W is connected, they are constant on W. Suppose $\gcd(a, b) = d > 1$. This implies that there exists an orbit whose isotropy group is a finite nontrivial group. But this is not the case by Step (I). If $ab > 0$, then X is an $\boldsymbol{A}^2$-bundle by [2]. Since W is isomorphic to $\boldsymbol{A}^1$, the $\boldsymbol{A}^2$-bundle is trivial, and X is isomorphic to $\boldsymbol{A}^1 \times \boldsymbol{A}^2 \cong \boldsymbol{A}^3$.

Hereafter we assume that $ab < 0$ and call the G_m-action mixed.

STEP (III). *Let Y be the quotient variety $X//G_m$ and $\pi : X \to Y$ the quotient morphism. Then we have*:

(1) *Y is a nonsingular, acyclic surface,*

(2) *$\pi|_W : W \to \pi(W)$ is an isomorphism and $\pi(W)$ is a closed subvariety of Y,*

(3) *Y is an affine plane and $\pi(W)$ is a coordinate line.*

With the notations in Step (II), we may and shall assume that $a > 0$ and $b < 0$. Then the completion of the local ring $\mathcal{O}_{Y,\pi(P)}$ is isomorphic to $\boldsymbol{C}[[u^{-b}v^a, w]]$. It then follows that π embeds W in Y as a nonsingular closed subvariety and that Y is nonsingular near $\pi(W)$. This proves the assertion (2). The smoothness of Y follows from Luna's étale slice theorem [10] if one notes that every closed orbit has a trivial isotropy group unless it is a fixed point. The acyclicity of Y follows from [9]. This proves the assertion (1). For the proof of the assertion (3), we apply Kawamata's addition theorem [8] for $\pi : X \to Y$

$$\bar{\kappa}(X) \geq \bar{\kappa}(F) + \bar{\kappa}(Y),$$

where F is a general closed orbit. Since F is isomorphic to G_m, we have $\bar{\kappa}(F) = 0$. Since $\bar{\kappa}(X) = -\infty$ by the hypothesis, it follows that $\bar{\kappa}(Y) = -\infty$. Then Y is an affine plane [15]. By a theorem of Abhyankar-Moh-Suzuki (cf. [11]), $\pi(W)$ is a coordinate line in Y. We write $Y = \operatorname{Spec} \boldsymbol{C}[\xi, \eta]$ with $\pi(W)$ defined by $\eta = 0$.

STEP (IV). *Let $\rho : X \to \pi(W)$ be the composite of π and the projection $(\xi, \eta) \mapsto \xi$ from Y to $\pi(W)$. Let Q be a point of $\pi(W)$, $Z = \rho^{-1}(Q)$ the fiber over Q, and let P be the intersection point of Z and W. Then the following assertions hold*:

(1) *Z is a nonsingular affine surface with a G_m-action.*

(2) *The point P is the unique fixed point on Z and the induced G_m-action on the tangent space $T_{Z,P}$ has weights a, b.*

(3) *Z has no multiple orbits.*

Let L be the line $\xi = \xi(Q)$ on Y. Consider $\pi|_Z : Z \to L$. Since π is a smooth morphism outside $\pi(W) \subset Y$, Z is nonsingular outside $\pi^{-1}(Q)$. With the notations in (II) and (III), we may assume that $\xi = w$ near the point Q and that Z is a hypersurface $u^{-b}v^a = \eta$

in the (u, v, η)-space near the point P. Then it follows that P is a nonsingular point of Z. Note that $\pi^{-1}(Q)$ is a union of two affine lines meeting at the point P and that two affine lines with the point P removed off are the G_m-orbits. Hence it follows that Z is nonsingular along $\pi^{-1}(Q)$. This proves the assertion (1). The assertion (2) is now clear. The assertion (3) follows from the corresponding property of X.

STEP (V). By the step (IV) and Theorem 3.1, we know that each fiber of $\rho : X \to \pi(W)$ is the affine plane. By Sathaye [21], it is then an A^2-bundle. Since any A^2-bundle over the affine line is trivial, we conclude that X is isomorphic to the affine 3-space.

This completes the proof of Theorem 4.2.

REFERENCES

[1] H. AOKI, Étale endomorphisms of smooth affine surfaces, J. Algebra, to appear.
[2] A. BIALYNICKI-BIRULA, Some theorems on the actions of algebraic groups, Ann. of Math. 98 (1973), 480–497.
[3] J. FOGARTY, Fixed point schemes, Amer. J. Math. 95 (1973), 35–51.
[4] T. FUJITA, On the topology of non-complete algebraic surfaces, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 29 (1982), 503–566.
[5] R. V. GURJAR AND M. MIYANISHI, On contractible curves in the complex affine plane, Tôhoku Math. J. 48 (1996), 459–469.
[6] S. IITAKA, Algebraic Geometry, Grad. Texts in Math. 76, Springer, New York-Berlin, 1982.
[7] Y. KAWAMATA, On the classification of non-complete algebraic surfaces, Proc. Copenhagen summer meeting in algebraic geometry, 215–232, Lecture Notes in Math. 732, Springer, Berlin, 1979.
[8] Y. KAWAMATA, Addition formula of logarithmic Kodaira dimensions for morphisms of relative dimension one, Int. Symp. on Algebraic Geometry, Kyoto, 207–217, Kinokuniya, Tokyo, 1977.
[9] H. KRAFT, T. PETRIE AND R. RANDALL, Quotient varieties, Adv. in Math. 74 (1989), 145–162.
[10] D. LUNA, Slices étales, Bull. Soc. Math. France Mémoire 33 (1973), 81–105.
[11] M. MIYANISHI, Curves on rational and unirational surfaces, Tata Inst. Fund. Res. Lectures on Math. and Phys. 60, the Narosa Publishing House, New Delhi, 1978.
[12] M. MIYANISHI, Non-complete algebraic surfaces, Lecture Notes in Math. 857, Springer, New York-Berlin, 1981.
[13] M. MIYANISHI, An algebro-topological characterization of the affine space of dimension three, Amer. J. Math. 106 (1984), 1469–1486.
[14] M. MIYANISHI, Etale endomorphisms of algebraic varieties, Osaka J. Math. 22 (1985), 345–364.
[15] M. MIYANISHI, Recent topics on open algebraic surfaces, Amer. Math. Soc. Transl. Ser. 2, 172 (1996), 61–76.
[16] M. MIYANISHI AND T. SUGIE, Generically rational polynomials, Osaka J. Math. 17 (1980), 339–362.
[17] M. NAGATA, Lectures on the fourteenth problem of Hilbert, Tata Institute of Fundamental Research, Bombay, 1965.
[18] M. NAGATA, Local rings, John Wiley, New York-London, 1962.
[19] M. NORI, Zariski's conjecture and related problems, Ann. Sci. École Norm. Sup. (4) 16 (1983), 305–344.
[20] H. SAITO, Fonctions entières qui se reduisent à certains polynômes, I, Osaka J. Math. 9 (1972), 293–332.
[21] A. SATHAYE, Polynomial ring in two variables over a D.V.R.: A criterion, Invent. Math. 74 (1983), 159–168.

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