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MAXIMAL ISOTROPY GROUPS OF LIE GROUPS RELATED TO NILRADICALS OF PARABOLIC SUBALGEBRAS

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Abstract. We consider Lie groups whose Lie algebra is the nilradical of a parabolic subalgebra of a complex simple Lie algebra, endowed with left-invariant Hermitian metrics. For such Riemannian Lie groups, we describe the Lie algebras of their maximal isotropy groups.

1. Introduction. Let $\mathfrak{s}$ be a semisimple complex Lie algebra and $\mathfrak{n}$ the nilradical of a parabolic subalgebra of $\mathfrak{s}$. Let N be the unique connected and simply-connected Lie group whose Lie algebra is $\mathfrak{n}$. Let us consider a left-invariant Hermitian metric g on N and denote by K the isotropy group of (N, g) at the identity element of N. Since N is nilpotent, a theorem of Wolf [10] ensures that the isometry group of (N, g) is isomorphic to the semidirect product of N and the compact group K.

In general, the dimension of such isotropy group (and hence of the isometry group itself) varies from one metric to another. The aim of this paper is to determine those isotropy groups of N endowed with a Hermitian metric which have maximal dimension. Since N is connected and simply-connected, we may calculate the identity connected component of such maximal isotropy group via exponentiation of its Lie algebra. Thus, we deal with the problem of determining the compact Lie algebra $\mathfrak{k}$ verifying that $\mathfrak{k}$ is the Lie algebra of a maximal isotropy group K.

In this paper, when $\mathfrak{s}$ is a simple Lie algebra, we study the case in which $\mathfrak{n}$ is defined by a subsystem S_1 of the system of simple roots of $\mathfrak{s}$. We obtain a complete classification, depending of course on $\mathfrak{s}$ and S_1, of those *maximal isotropy algebras* obtained for a Hermitian metric. Our results are listed in Tables III to VII below.

I want to thank Professor J. F. Torres Lopera and Professor Yu. B. Hakimjanov for introducing me to the subject of this paper and for the careful reading of the details. I am also in debt with the referee for drawing [7] to my attention and pointing out Remark 1.

2. Preliminaries. Let $\mathfrak{s}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{s}$ and R, R^+, S, respectively, the systems of roots, positive roots and simple roots of $\mathfrak{s}$ with respect to $\mathfrak{h}$.

It is well-known that $\mathfrak{s} = \mathfrak{h} + \sum_{\alpha \in R} V_\alpha$, where V_α is a complex vector space spanned by a unique element e_α. Moreover, the brackets in $\mathfrak{s}$ are given by $[e_\alpha, H] = \alpha(H)e_\alpha$ for every

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$H \in \mathfrak{h}$ and

$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha,\beta} e_{\alpha+\beta} & \text{if } \alpha + \beta \in R \\ 0 & \text{if } \alpha + \beta \notin R\,, \end{cases}$$

where $N_{\alpha,\beta} \in \boldsymbol{R}$ and $N_{\alpha,\beta} \neq 0$. We will denote by $\delta \in R^+$ the *maximal root*, this is, the only positive root verifying that $\delta + \alpha \notin R$ for every $\alpha \in R^+$. A simple root $\alpha \in S$ is said to be a *singular root* if $\delta - \alpha \in R$ or, equivalently, if α is linked with δ in the extended Dynkin diagram of $\mathfrak{s}$ (see Table II).

Let $S_1 \subset S$ be a non-empty subsystem of S, and let us consider the sets $\Delta_1 = \{\gamma \in R;\, \gamma = \sum_{\alpha \in S \setminus S_1} m_\alpha \alpha\}$ and $\Delta_2^+ = (R \setminus \Delta_1) \cap R^+$. The Lie algebra $\mathfrak{p} = \mathfrak{h} + \sum_{\alpha \in \Delta_1} V_\alpha + \sum_{\alpha \in \Delta_2^+} V_\alpha$ is a parabolic subalgebra of $\mathfrak{s}$. In fact, every parabolic subalgebra of $\mathfrak{s}$ is conjugate to an algebra of this kind. It is well-known that such parabolic subalgebra admits a decomposition $\mathfrak{p} = \mathfrak{r} + \mathfrak{n}$, where $\mathfrak{r} = \mathfrak{h} + \sum_{\alpha \in \Delta_1} V_\alpha$ is its reductive summand and the ideal $\mathfrak{n} = \sum_{\alpha \in \Delta_2^+} V_\alpha$ is the nilpotent radical (which coincides with the nilradical of $\mathfrak{p}$).

DEFINITION 1. We say that $\mathfrak{n}$ is the *standard nil-subalgebra* of $\mathfrak{s}$ with respect to S_1 or, for short, $(\mathfrak{s}, S_1)$*-nilalgebra* if $\mathfrak{n} = \sum_{\alpha \in \Delta_2^+} V_\alpha$, that is, if $\mathfrak{n}$ is the nilradical of the parabolic subalgebra of $\mathfrak{s}$ given by

$$\mathfrak{p} = \mathfrak{h} + \sum_{\alpha \in \Delta_1} V_\alpha + \sum_{\alpha \in \Delta_2^+} V_\alpha\,.$$

Notice that if $S_0 = S \setminus S_1$, then the partition $S = S_0 \cup S_1$ defines a graded Lie algebra $\mathfrak{s} = \sum_{k=-\nu}^{\nu} \mathfrak{s}_k$ (see [7], Theorem 1.7). With this notation, $\mathfrak{p} = \sum_{k \geq 0} \mathfrak{s}_k$, $\mathfrak{r} = \mathfrak{s}_0$ and $\mathfrak{n} = \sum_{k>0} \mathfrak{s}_k$. A Lie group N is called a *standard nil-group* or a $(\mathfrak{s}, S_1)$*-nilgroup* if its Lie algebra is a $(\mathfrak{s}, S_1)$-nilalgebra.

In what follows, two types of $(\mathfrak{s}, S_1)$-nilalgebras will have a different treatment: the abelian ones and those isomorphic to any of the *Heisenberg algebras* $\mathfrak{H}_k$. The Heisenberg algebra $\mathfrak{H}_k$ is the complex Lie algebra spanned by $2k+1$ elements $\{x_1, \cdots, x_k, y_1, \cdots, y_k, z\}$ with non-trivial brackets given by $[x_i, y_i] = z$, $1 \leq i \leq n$. The whole set of $(\mathfrak{s}, S_1)$-nilalgebras which are abelian or Heisenberg have been described by Hakimjanov [4] and Hakimjanov and Onischik [8]. They are listed in Table I, where we have corrected a small erratum in one of the dimensions.

On the other hand, Wolf [10] and later Wilson [9] have proved that if N is a nilpotent real Lie group and g is a left-invariant Riemannian metric on N, then the isometry group $I(N, g)$ of (N, g) is isomorphic to the semidirect product of N (acting on itself by left-translations) and the isotropy group K at the unity of N. Further, if N is connected and simply-connected then K is isomorphic to the group $\mathrm{Aut}(\mathfrak{n}, \langle.,.\rangle)$ of orthogonal automorphisms of $(\mathfrak{n}, \langle.,.\rangle)$, where $\mathfrak{n}$ is the Lie algebra of N and $\langle.,.\rangle$ denotes the euclidean product induced by g on $\mathfrak{n}$. Hence, in this case, the identity connected component of K is completely determined by its Lie algebra, namely the Lie algebra $\mathfrak{aut}(\mathfrak{n}, \langle.,.\rangle)$ of skew-symmetric derivations of $(\mathfrak{n}, \langle.,.\rangle)$. Note that the algebra $\mathfrak{aut}(\mathfrak{n}, \langle.,.\rangle)$ is always a compact Lie algebra.

Since the $(\mathfrak{s}, S_1)$-nilgroups admit an obvious complex structure, all the left-invariant metrics considered in this paper will be Hermitian, that is, the euclidean product $\langle.,.\rangle$ on $\mathfrak{n}$ verifies

$\langle Jx, Jy\rangle = \langle x, y\rangle$ for all $x, y \in \mathfrak{n}$, where J denotes the complex structure $J^2 = -1$. It is important to remark that, in this case, in order to compute the Lie algebra of K it is necessary to calculate not only the skew-symmetric complex derivations of $\mathfrak{n}$ but the skew-symmetric real derivations of the underlying real algebra. In this sense, we will use some of the results given in [1]. To avoid confusions, we will denote by $\mathfrak{Der}_C(\mathfrak{n})$ the set of complex derivations of $\mathfrak{n}$ and by $\mathfrak{Der}(\mathfrak{n})$ the set of real derivations of (the underlying real algebra of) $\mathfrak{n}$. These considerations justify the following:

DEFINITION 2. Let N be a complex Lie group, $\mathfrak{n}$ its Lie algebra and denote by J the complex structure. We say that a compact Lie subalgebra $\mathfrak{k}$ of $\mathfrak{Der}(\mathfrak{n})$ is an *isotropy algebra of* N if there exists a J-invariant euclidean product $\langle .,.\rangle$ on $\mathfrak{n}$ such that $\mathfrak{k} = \mathfrak{aut}(\mathfrak{n}, \langle .,.\rangle)$. An isotropy algebra $\mathfrak{k}$ of N is said to be *maximal* if the dimension of $\mathfrak{k}$ is maximal among the dimensions of all isotropy algebras of N.

In the sequel, if V is a real vector space, we will denote by V^C its complexification. In order to avoid possible confusions, when W is a complex vector space, we will write $\dim_C W$ for its complex dimensions and $\dim W$ for its real dimension (this is, the dimension of the underlying real space).

3. Main results. We begin with the following proposition which, as far as we know, has never been proved.

PROPOSITION 1. *Let* $\mathfrak{s}$ *be a simple complex Lie algebra,* $\mathfrak{p}$ *the parabolic subalgebra defined by the subsystem* S_1 *of simple roots, and let* $\mathfrak{r}, \mathfrak{n}$ *be respectively the reductive summand and the nilradical of* $\mathfrak{p}$. *The representation* $\rho : \mathfrak{r} \to \mathfrak{gl}(\mathfrak{n})$ *obtained as the restriction of the adjoint representation by* $\rho(s)(x) = [s, x]$ *for every* $s \in \mathfrak{r}$, $x \in \mathfrak{n}$ *is faithful.*

PROOF. We first show that if $\beta \in \Delta_1$, then there exists $\alpha \in \Delta_2^+$ such that $\beta + \alpha \in R$.

Suppose that $\beta \in \Delta_1$ is a positive root such that $\beta + \alpha \notin R$ for all $\alpha \in \Delta_2^+$. Clearly, β is not the maximal root, since $\beta \in \Delta_1$. Hence, there exists $\gamma_1 \in R^+$, $\gamma_1 \notin \Delta_2^+$ such that $\beta + \gamma_1 \in R$. Therefore, $\beta + \gamma_1$ is a positive root not belonging to Δ_2^+ and thus $\beta + \gamma_1 \in \Delta_1$. Now, if there exists $\alpha \in \Delta_2^+$ verifying $\beta + \gamma_1 + \alpha \in \Delta_2^+$, then, by the Jacobi identity, we would have $\beta + \alpha \in R$ or $\gamma_1 + \alpha \in \Delta_2^+$, which contradicts the assumption on β. Thus, we have that $\beta_1 = \beta + \gamma_1$ has the same property as β, and we can repeat our argument infinitely. Since the set of roots is finite, we conclude that our assumption on β leads to an absurd.

If $\beta \in \Delta_1$ is a negative root, then $(-\beta) \in \Delta_1 \cap R^+$ and, therefore, there exists $\alpha_1 \in \Delta_2^+$ such that $\alpha = -\beta + \alpha_1 \in \Delta_2^+$ and hence $\beta + \alpha = \alpha_1 \in R$.

Now, let us consider $s = H + \sum_{\beta\in\Delta_1} a_\beta e_\beta \in \mathfrak{r}$ with $H \in \mathfrak{h}$ and $a_\beta \in C$ such that $\rho(s) = 0$. For each $\alpha \in \Delta_2^+$ we have

$$0 = [s, e_\alpha] = \alpha(H)e_\alpha + \sum_{\beta+\alpha\in R} a_\beta N_{\beta,\alpha} e_{\beta+\alpha}\,,$$

where $N_{\beta,\alpha} \neq 0$ and, since $\{e_\alpha\} \cup \{e_{\beta+\alpha} : \beta+\alpha \in R\}$ is a linearly independent set, it follows that $\alpha(H) = a_\beta N_{\beta,\alpha} = 0$ for all $\alpha \in \Delta_2^+$. Hence, $a_\beta = 0$, since otherwise $\beta + \alpha \notin R$ for every $\alpha \in \Delta_2^+$, which may not occur.

These arguments lead us to $s = H \in \mathfrak{h}$ and $\alpha(H) = 0$ for all $\alpha \in \Delta_2^+$. Take an element $\beta \in S$. If $\beta \in S_1$ then $\beta(H) = 0$ since $S_1 \subset \Delta_2^+$. On the other hand, if $\beta \in S \setminus S_1$ then, by the first part of our proof, there exists $\alpha \in \Delta_2^+$ such that $\gamma = \alpha + \beta \in R$. Then $\gamma \in \Delta_2^+$ and thus $\beta(H) = \gamma(H) - \alpha(H) = 0$. Noting that the system S of simple roots of $\mathfrak{s}$ is a basis of the dual space $\mathfrak{h}^*$, we conclude that $s = H = 0$. □

LEMMA 1. *Let $\mathfrak{s}$ be a simple complex Lie algebra, S_1 a subsystem of simple roots of $\mathfrak{s}$ and $\mathfrak{n}$ the corresponding $(\mathfrak{s}, S_1)$-nilalgebra. Let $\langle .,. \rangle$ be an euclidean product on $\mathfrak{n}$ invariant under the complex structure J. If $\mathfrak{n}$ is not abelian then the algebra $\mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)$ of skew-symmetric derivations of $(\mathfrak{n}, \langle .,. \rangle)$ is contained in the algebra $\mathfrak{Der}_{\mathbf{C}}(\mathfrak{n})$ of complex derivations of $\mathfrak{n}$.*

PROOF. For each $\alpha \in \Delta_2^+$, let W_α be the $\mathbf{R}$-vector space spanned by e_α. Then the real algebra $\mathfrak{m} = \sum_{\alpha \in \Delta_2^+} W_\alpha$ is a real form of the nilalgebra $\mathfrak{n}$. Note that, from the definition of a $(\mathfrak{s}, S_1)$-nilalgebra, the center of $\mathfrak{m}$ is always contained in its derived algebra $[\mathfrak{m}, \mathfrak{m}]$ unless $\mathfrak{m}$ (and, hence, $\mathfrak{n}$) is abelian. Thus, $\mathfrak{z}(\mathfrak{m}) \cap [\mathfrak{m}, \mathfrak{m}]^\perp = \{0\}$ and we obtain from the theorem in [1] that every $F \in \mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)$ is skew-hermitian and hence $\mathbf{C}$-linear. □

The following theorem is the main tool for the classification of the maximal isotropy algebras of the $(\mathfrak{s}, S_1)$-nilalgebras given below. In its proof, Hakimjanov's Theorems 2 and 3 in [4] will play an important role.

THEOREM 1. *Let $\mathfrak{s}$ be a simple complex Lie algebra, S_1 a subsystem of simple roots of $\mathfrak{s}$ and N a non-abelian connected and simply-connected $(\mathfrak{s}, S_1)$-nilgroup. Let us denote by $\mathfrak{h}_1$ the complex abelian Lie algebra spanned by elements $\{H_\alpha : \alpha \in S \setminus S_1\}$ and by $\mathfrak{g}_1$ the semisimple algebra $\mathfrak{g}_1 = \mathfrak{h}_1 + \sum_{\alpha \in \Delta_1} V_\alpha$. If $\mathfrak{s}$ and S_1 verify one of the following conditions:*

(i) *S_1 contains at least one non-singular root,*

(ii) *$\mathfrak{s} = C_n$ and $S_1 = \{\alpha_1\}$,*

then the maximal isotropy algebra of N is isomorphic to $\mathfrak{k}_1 \oplus \mathbf{R}^k$, where $\mathfrak{k}_1$ is the compact real form of $\mathfrak{g}_1$ and k is the cardinal number of S_1.

PROOF. Let $\mathfrak{n}$ be the Lie algebra of N, J the complex structure and let $\langle .,. \rangle$ be any J-invariant euclidean product on $\mathfrak{n}$. Let us denote by $\mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)$ the Lie algebra of skew-symmetric derivations of the pair $(\mathfrak{n}, \langle .,. \rangle)$. From the lemma above, $\mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)$ is contained in the complex vector space $\mathfrak{Der}_{\mathbf{C}}(\mathfrak{n})$. Therefore, JF is also a $\mathbf{C}$-linear derivation of $\mathfrak{n}$ for every $F \in \mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)$ and hence $\mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)^{\mathbf{C}} \subset \mathfrak{Der}_{\mathbf{C}}(\mathfrak{n})$.

Clearly, $\mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)^{\mathbf{C}}$ is reductive, since $\mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)$ is compact. It is not difficult to prove that if $\mathfrak{M}$ is the nilpotent radical of $\mathfrak{Der}_{\mathbf{C}}(\mathfrak{n})$, then $\mathfrak{M} \cap \mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)^{\mathbf{C}} = \{0\}$. Thus we have

$$\dim_{\mathbf{C}} \mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)^{\mathbf{C}} + \dim_{\mathbf{C}} \mathfrak{M} \leq \dim_{\mathbf{C}} \mathfrak{Der}_{\mathbf{C}}(\mathfrak{n}),$$

and therefore if $\mathfrak{R}$ is a reductive subalgebra such that $\mathfrak{der}_C(\mathfrak{n}) = \mathfrak{R} + \mathfrak{M}$, then

$$\dim \mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle) = \dim_C \mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)^C \leq \dim_C \mathfrak{R}.$$

Now, since $\mathfrak{s}$ and S_1 verify one of the assumptions (i) or (ii), Theorems 2 and 3 in [4] are applicable and these results together with the proposition above show that $\mathfrak{R}$ is isomorphic to $\mathfrak{r}$. Note that $\mathfrak{r} = \mathfrak{g}_1 \oplus \mathfrak{a}$, where $\mathfrak{a}$ is an abelian direct summand and $\dim_C \mathfrak{a} = \mathrm{card}(S_1) = k$. Hence, its compact real form is isomorphic to $\mathfrak{k}_1 \oplus \boldsymbol{R}^k$ and thus $\dim \mathfrak{aut}(\mathfrak{n}\langle .,. \rangle) \leq \dim \mathfrak{k}_1 \oplus \boldsymbol{R}^k$.

Thus, it suffices to prove that there exists a J-invariant euclidean product $\langle .,. \rangle$ on $\mathfrak{n}$ such that $\mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle) = \mathfrak{k}_1 \oplus \boldsymbol{R}^k$.

As $\mathfrak{k}_1$ is compact and semisimple and the representation ρ in the proposition above is faithful, the algebra $\rho(\mathfrak{k}_1)$ is a compact and semisimple subalgebra of $\mathfrak{gl}(\mathfrak{n})$. Let K_1 be the unique connected subgroup of $GL(\mathfrak{n})$ with Lie algebra $\rho(\mathfrak{k}_1)$. It is clear that K_1 is compact and $K_1 = \{e^{\rho(x)}; x \in \mathfrak{k}_1\}$. Now, let us consider the connected compact Lie group K given by the direct product $K = K_1 A_1 \dots A_k$, where $A_j = \{e^{\rho(J\lambda x_j)}; \lambda \in \boldsymbol{R}\}$ and $\{x_1, x_2, \dots, x_k\}$ is a basis of $\mathfrak{a}$.

Since for every $x \in \mathfrak{r}$ and $y \in \mathfrak{n}$ we have that $\rho(x)(Jy) = J\rho(x)(y)$, it follows that $\rho(x)$ is $\boldsymbol{C}$-linear and hence so is $e^{\rho(x)}$. Clearly, this implies that K is also composed of $\boldsymbol{C}$-linear mappings.

Now, let $(.|.)$ be any J-invariant euclidean product on $\mathfrak{n}$, Ω a bi-invariant volume element on K and define a new euclidean product on $\mathfrak{n}$ by

$$\langle x, y \rangle = \int_K (Fx \mid Fy)\Omega\,.$$

A classical result assures that K is a subgroup of the group $\mathrm{Aut}(\mathfrak{n}, \langle .,. \rangle)$ of orthogonal automorphisms of $(\mathfrak{n}, \langle .,. \rangle)$. Further, since $(.|.)$ is J-invariant and every $F \in K$ is $\boldsymbol{C}$-linear we easily get that $\langle .,. \rangle$ is also J-invariant.

Therefore, the Lie algebra $\mathfrak{k}_1 \oplus \boldsymbol{R}^k$ of K is a subalgebra of $\mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle)$. Recalling that $\dim \mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle) \leq \dim \mathfrak{k}_1 \oplus \boldsymbol{R}^k$, we conclude that both algebras must be equal, which completes the proof. □

REMARK 1. An euclidean product $\langle .,. \rangle$ satisfying $\mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle) = \mathfrak{k}_1 \oplus \boldsymbol{R}^k$ can be obtained explicitly as follows: Let θ be the conjugation of $\mathfrak{s}$ with respect to a compact real form $\mathfrak{u}$, and B the Killing form of the underlying real Lie algebra of $\mathfrak{s}$. Then, $\langle x, y \rangle = -B(x, \theta y)$ defines an euclidean product which satisfies our conditions (see [6], p. 253, Lemma 1.2), since $\mathfrak{k}_1 \oplus \boldsymbol{R}^k = \mathfrak{s}_0 \cap \mathfrak{u}$, where $\mathfrak{s} = \sum_{k=-\nu}^{\nu} \mathfrak{s}_k$ is the attached graded Lie algebra.

THEOREM 2. *Let $\mathfrak{s}$ be a complex simple Lie algebra, S_1 a subsystem of simple roots of $\mathfrak{s}$ and N a connected and simply-connected $(\mathfrak{s}, S_1)$-nilgroup. Then the maximal isotropy algebra of N is one of the algebras listed in Tables* III, IV, V, VI *and* VII.

PROOF. Let $\mathfrak{n}$ be the Lie algebra of N. We first consider the case that $\mathfrak{n}$ is abelian and $\dim_C \mathfrak{n} = n$. It is clear that for any euclidean product $\langle .,. \rangle$ on $\mathfrak{n}$ we have $\mathfrak{aut}(\mathfrak{n}, \langle .,. \rangle) = \mathfrak{so}(2n)$. Thus in this case, the result follows from those obtained by Hakimjanov and Onischik listed in Table I.

On the other hand, if the $(\mathfrak{s}, S_1)$-nilalgebra $\mathfrak{n}$ is a Heisenberg algebra of dimension $2n-1$, then $\mathfrak{n}$ is isomorphic to the standard nil-subalgebra associated to $(C_n, \{\alpha_1\})$ and we may compute its maximal isotropy algebra by means of the theorem above.

We finish pointing out that if $\mathfrak{n}$ is neither abelian nor a Heisenberg algebra, then $(\mathfrak{s}, S_1)$ verifies the Condition (i) of Theorem 1 and $\mathfrak{k}$ is isomorphic to $\mathfrak{k}_1 \oplus \boldsymbol{R}^k$. Recall that k is the cardinal number of S_1 and note that $\mathfrak{k}_1$ may be easily computed, since it is the compact real form of the semisimple Lie algebra whose Dynkin diagram is obtained by elimination of the roots of S_1 in the diagram of $\mathfrak{s}$. □

REMARK 2. In the Tables below we have written $\mathfrak{su}(n+1)$, $\mathfrak{so}(2n+1)$, $\mathfrak{sp}(n)$, $\mathfrak{so}(2n)$, $\mathfrak{e}_i$, $\mathfrak{f}_4$, $\mathfrak{g}_2$ for the compact real forms of the algebras of type A_n, B_n, C_n, D_n, E_i, F_4 and G_2, respectively. The numbering of the roots coincides with that given in Table II and we consider the indices ordered, meaning that if we write $S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}\}$ then $i_1 < i_2 < \cdots < i_k$.

In case that $\mathfrak{s}$ is not a simple Lie algebra, we have a partial result which derives from Theorem 1 and Proposition 2 in [2]. Since its proof is quite standard, we will omit it.

COROLLARY 1. *Let $\mathfrak{s}$ be a complex semisimple Lie algebra, S_1 a subsystem of simple roots of $\mathfrak{s}$ and let N be a connected and simply-connected $(\mathfrak{s}, S_1)$-nilgroup. Suppose that the algebra $\mathfrak{s}$ admits the decomposition into simple ideals $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_q$ and that $S_{1_1}, \dots, S_{1_q}$ are the intersections of S_1 with the respective systems of simple roots of $\mathfrak{s}_1, \dots, \mathfrak{s}_q$. For every $i \le q$, denote by $\mathfrak{k}_i$ the isotropy of a standard nil-group N_i associated with $(\mathfrak{s}_i, S_{1_i})$ with an arbitrary Hermitian metric g_i, and let g be the metric on N obtained as the product of the metrics g_i. Then the following hold*:

(i) *If none of the standard nil-subalgebras associated to $(\mathfrak{s}_i, S_{1_i})$ is abelian, then the isotropy algebra of (N, g) is $\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_q$.*

(ii) *If the standard nil-subalgebras associated to $(\mathfrak{s}_i, S_{1_i})$ are abelian for every $p \le i \le q$ and their respective dimension over $\boldsymbol{C}$ is n_i, then the isotropy algebra of (N, g) is $\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_{p-1} \oplus \mathfrak{so}(2m)$ where $m = n_p + \cdots + n_q$.*

TABLE I. $(\mathfrak{s}, S_1)$-nilalgebras $\mathfrak{n}$ isomorphic to an abelian or Heisenberg algebra.

Abelian			Heisenberg		
$\mathfrak{s}$	S_1	$\mathfrak{n}$	$\mathfrak{s}$	S_1	$\mathfrak{n}$
A_n	$S_1 = \{\alpha_i\}$	$\boldsymbol{C}^{i(n-i+1)}$	A_n	$S_1 = \{\alpha_1, \alpha_n\}$	$\mathfrak{H}_{n-1}$
B_n	$S_1 = \{\alpha_1\}$	$\boldsymbol{C}^{2n-1}$	B_n	$S_1 = \{\alpha_2\}$	$\mathfrak{H}_{2n-3}$
C_n	$S_1 = \{\alpha_n\}$	$\boldsymbol{C}^{(n^2+n)/2}$	C_n	$S_1 = \{\alpha_1\}$	$\mathfrak{H}_{n-1}$
D_n	$S_1 = \{\alpha_1\}$	$\boldsymbol{C}^{2n-2}$	D_n	$S_1 = \{\alpha_2\}$	$\mathfrak{H}_{2n-4}$
	$S_1 = \{\alpha_{n-1}\}$	$\boldsymbol{C}^{(n^2-n)/2}$	E_6	$S_1 = \{\alpha_2\}$	$\mathfrak{H}_{10}$
	$S_1 = \{\alpha_n\}$	$\boldsymbol{C}^{(n^2-n)/2}$	E_7	$S_1 = \{\alpha_1\}$	$\mathfrak{H}_{16}$
E_6	$S_1 = \{\alpha_1\}$	$\boldsymbol{C}^{16}$	E_8	$S_1 = \{\alpha_8\}$	$\mathfrak{H}_{28}$
	$S_1 = \{\alpha_6\}$	$\boldsymbol{C}^{16}$	F_4	$S_1 = \{\alpha_1\}$	$\mathfrak{H}_7$
E_7	$S_1 = \{\alpha_7\}$	$\boldsymbol{C}^{27}$	G_2	$S_1 = \{\alpha_2\}$	$\mathfrak{H}_2$

TABLE II. Dynkin extended Diagrams and maximal roots of simple Lie algebras.

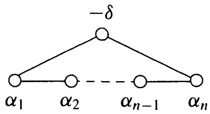
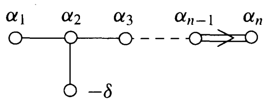
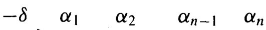
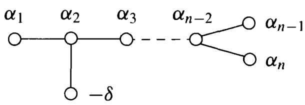
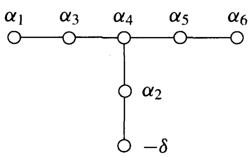
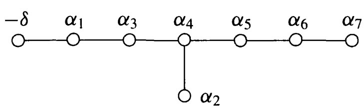
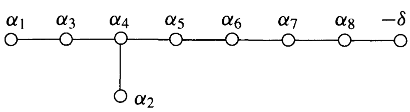
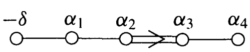
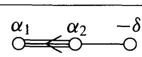
Type of algebra	Dynkin extended diagram	Maximal root
A_n $(n \geq 2)$		$\alpha_1 + \alpha_2 + \cdots + \alpha_n$
B_n $(n \geq 3)$		$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$
C_n $(n \geq 2)$		$2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$
D_n $(n \geq 4)$		$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} +$ $+\alpha_{n-1} + \alpha_n$
E_6		$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 +$ $+2\alpha_5 + \alpha_6$
E_7		$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 +$ $+3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8		$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 +$ $+5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$
F_4		$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2		$3\alpha_1 + 2\alpha_2$

TABLE III. Maximal isotropy algebras $\mathfrak{k}$ of the $(\mathfrak{s}, S_1)$-nilgroups associated to the classical algebras.

$\mathfrak{s}$	S_1	$\mathfrak{k}$
A_n $(n \geq 2)$	$S_1 = \{\alpha_i\}$	$\mathfrak{so}(2ni - 2i^2 + 2i)$
	$S_1 = \{\alpha_1, \alpha_n\}$	$\mathfrak{sp}(n-1) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_i, \alpha_j\}, \{i, j\} \neq \{1, n\}$	$\mathfrak{su}(i) \oplus \mathfrak{su}(j) \oplus \mathfrak{su}(n+1-j) \oplus \boldsymbol{R}^2$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\}, k \geq 3$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{l-1})) \oplus \mathfrak{su}(n+1-i_k) \oplus \boldsymbol{R}^k$
B_n $(n \geq 3)$	$S_1 = \{\alpha_1\}$	$\mathfrak{so}(4n-2)$
	$S_1 = \{\alpha_2\}$	$\mathfrak{sp}(2n-3) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_i\}, i \notin \{1, 2\}$	$\mathfrak{su}(i) \oplus \mathfrak{so}(2n-2i+1) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\}, k \geq 2$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{l-1})) \oplus \mathfrak{so}(2n+1-2i_k) \oplus \boldsymbol{R}^k$
C_n $(n \geq 2)$	$S_1 = \{\alpha_n\}$	$\mathfrak{so}(n^2 + n)$
	$S_1 = \{\alpha_1\}$	$\mathfrak{sp}(n-1) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_i\}, i \notin \{1, n\}$	$\mathfrak{su}(i) \oplus \mathfrak{sp}(n-i) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\}, k \geq 2$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{l-1})) \oplus \mathfrak{sp}(n-i_k) \oplus \boldsymbol{R}^k$
D_n $(n \geq 4)$	$S_1 = \{\alpha_n\}$	$\mathfrak{so}(n^2 - n)$
	$S_1 = \{\alpha_{n-1}\}$	$\mathfrak{so}(n^2 - n)$
	$S_1 = \{\alpha_{n-2}\}$	$\mathfrak{su}(n-2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_1\}$	$\mathfrak{so}(4n-4)$
	$S_1 = \{\alpha_2\}$	$\mathfrak{sp}(2n-4) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_i\}, i \notin \{1, 2, n, n-1, n-2\}$	$\mathfrak{su}(i) \oplus \mathfrak{so}(2n-2i) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\}, k \geq 2, i_k < n-3$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{l-1})) \oplus \mathfrak{so}(2n-2i_k) \oplus \boldsymbol{R}^k$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}, \alpha_{n-3}\}, k \geq 1, i_k < n-3$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{l-1})) \oplus \mathfrak{su}(n-3-i_k) \oplus \mathfrak{su}(4) \oplus \boldsymbol{R}^{k+1}$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}, \alpha_{n-1}\}, k \geq 1, i_k < n-2$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{l-1})) \oplus \mathfrak{su}(n-i_k) \oplus \boldsymbol{R}^{k+1}$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}, \alpha_n\}, k \geq 1, i_k < n-2$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{l-1})) \oplus \mathfrak{su}(n-i_k) \oplus \boldsymbol{R}^{k+1}$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}, \alpha_{n-1}, \alpha_n\}, k \geq 1, i_k < n-2$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{l-1})) \oplus \mathfrak{su}(n-1-i_k) \oplus \boldsymbol{R}^{k+2}$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}, \alpha_{n-2}\}, k \geq 1, i_k < n-2$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{i-1})) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}^{k+1}$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}, \alpha_{n-2}, \alpha_{n-1}\}, k \geq 1, i_k < n-2$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{l-1})) \oplus \mathfrak{su}(n-2-i_k) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}^{k+2}$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}, \alpha_{n-2}, \alpha_n\}, k \geq 1, i_k < n-2$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{l-1})) \oplus \mathfrak{su}(n-2-i_k) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}^{k+2}$
	$S_1 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}, \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}, k \geq 1, i_k < n-2$	$\mathfrak{su}(i_1) \oplus (\bigoplus_{l=2}^k \mathfrak{su}(i_l - i_{l-1})) \oplus \mathfrak{su}(n-2-i_k) \oplus \boldsymbol{R}^{k+3}$

TABLE IV. Maximal isotropy algebras $\mathfrak{k}$ of the (E_6, S_1)-nilgroups.

$\mathfrak{s}$	S_1	$\mathfrak{k}$
E_6	$S_1 = \{\alpha_1\}$	$\mathfrak{so}(32)$
	$S_1 = \{\alpha_2\}$	$\mathfrak{sp}(10) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_3\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_4\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_5\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_6\}$	$\mathfrak{so}(32)$
	$\mathrm{card}(S_1) = 2$	$\begin{cases} \mathfrak{so}(8) \oplus \boldsymbol{R}^2 \\ (\bigoplus_{i=1}^3 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^2,\ p_i \geq 0,\ \sum_{i=1}^3 p_i = 4 \end{cases}$
	$\mathrm{card}(S_1) = 3$	$(\bigoplus_{i=1}^3 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^3,\ p_i \geq 0,\ \sum_{i=1}^3 p_i = 3$
	$\mathrm{card}(S_1) = 4$	$(\bigoplus_{i=1}^2 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^4,\ p_i \geq 0,\ \sum_{i=1}^2 p_i = 2$
	$\mathrm{card}(S_1) = 5$	$\mathfrak{su}(2) \oplus \boldsymbol{R}^5$
	$\mathrm{card}(S_1) = 6$	$\boldsymbol{R}^6$

TABLE V. Maximal isotropy algebras $\mathfrak{k}$ of the (E_7, S_1)-nilgroups.

$\mathfrak{s}$	S_1	$\mathfrak{k}$
E_7	$S_1 = \{\alpha_1\}$	$\mathfrak{sp}(16) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_2\}$	$\mathfrak{su}(7) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_3\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(6) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_4\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(4) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_5\}$	$\mathfrak{su}(3) \oplus \mathfrak{su}(5) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_6\}$	$\mathfrak{so}(10) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_7\}$	$\mathfrak{so}(54)$
	$\mathrm{card}(S_1) = 2$	$\begin{cases} \mathfrak{so}(10) \oplus \boldsymbol{R}^2 \\ \mathfrak{so}(8) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}^2 \\ (\bigoplus_{i=1}^4 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^2,\ p_i \geq 0,\ \sum_{i=1}^4 p_i = 5 \end{cases}$
	$\mathrm{card}(S_1) = 3$	$\begin{cases} \mathfrak{so}(8) \oplus \boldsymbol{R}^2 \\ (\bigoplus_{i=1}^4 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^3,\ p_i \geq 0,\ \sum_{i=1}^4 p_i = 4 \end{cases}$
	$\mathrm{card}(S_1) = 4$	$(\bigoplus_{i=1}^3 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^4,\ p_i \geq 0,\ \sum_{i=1}^3 p_i = 3$
	$\mathrm{card}(S_1) = 5$	$(\bigoplus_{i=1}^2 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^4,\ p_i \geq 0,\ \sum_{i=1}^2 p_i = 2$
	$\mathrm{card}(S_1) = 6$	$\mathfrak{su}(2) \oplus \boldsymbol{R}^6$
	$\mathrm{card}(S_1) = 7$	$\boldsymbol{R}^7$

TABLE VI. Maximal isotropy algebras $\mathfrak{k}$ of the (E_8, S_1)-nilgroups.

$\mathfrak{s}$	S_1	$\mathfrak{k}$
E_8	$S_1 = \{\alpha_1\}$	$\mathfrak{so}(14) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_2\}$	$\mathfrak{su}(8) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_3\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(7) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_4\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(5) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_5\}$	$\mathfrak{su}(4) \oplus \mathfrak{su}(5) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_6\}$	$\mathfrak{so}(10) \oplus \mathfrak{su}(3) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_7\}$	$\mathfrak{e}_6 \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_8\}$	$\mathfrak{sp}(28) \oplus \boldsymbol{R}$
	$\operatorname{card}(S_1) = 2$	$\begin{cases} \mathfrak{e}_6 \oplus \boldsymbol{R}^2 \\ \mathfrak{so}(12) \oplus \boldsymbol{R}^2 \\ \mathfrak{so}(10) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}^2 \\ \mathfrak{so}(8) \oplus \mathfrak{su}(3) \oplus \boldsymbol{R}^2 \\ (\bigoplus_{i=1}^4 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^2,\ p_i \geq 0,\ \sum_{i=1}^4 p_i = 6 \end{cases}$
	$\operatorname{card}(S_1) = 3$	$\begin{cases} \mathfrak{so}(10) \oplus \boldsymbol{R}^3 \\ \mathfrak{so}(8) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}^3 \\ (\bigoplus_{i=1}^4 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^3,\ p_i \geq 0,\ \sum_{i=1}^4 p_i = 5 \end{cases}$
	$\operatorname{card}(S_1) = 4$	$\begin{cases} \mathfrak{so}(8) \oplus \boldsymbol{R}^4 \\ (\bigoplus_{i=1}^4 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^4,\ p_i \geq 0,\ \sum_{i=1}^4 p_i = 4 \end{cases}$
	$\operatorname{card}(S_1) = 5$	$(\bigoplus_{i=1}^3 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^5,\ p_i \geq 0,\ \sum_{i=1}^3 p_i = 3$
	$\operatorname{card}(S_1) = 6$	$(\bigoplus_{i=1}^2 \mathfrak{su}(p_i+1)) \oplus \boldsymbol{R}^6,\ p_i \geq 0,\ \sum_{i=1}^2 p_i = 2$
	$\operatorname{card}(S_1) = 7$	$\mathfrak{su}(2) \oplus \boldsymbol{R}^7$
	$\operatorname{card}(S_1) = 8$	$\boldsymbol{R}^8$

TABLE VII. Maximal isotropy algebras $\mathfrak{k}$ of the $(\mathfrak{s}, S_1)$-nilgroups associated to F_4 and G_2.

$\mathfrak{s}$	S_1	$\mathfrak{k}$
F_4	$S_1 = \{\alpha_1\}$	$\mathfrak{sp}(7) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_4\}$	$\mathfrak{so}(7) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_2\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_3\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_1, \alpha_2\}$	$\mathfrak{su}(3) \oplus \boldsymbol{R}^2$
	$S_1 = \{\alpha_3, \alpha_4\}$	$\mathfrak{su}(3) \oplus \boldsymbol{R}^2$
	$S_1 = \{\alpha_1, \alpha_4\}$	$\mathfrak{sp}(2) \oplus \boldsymbol{R}^2$
	$S_1 = \{\alpha_1, \alpha_3\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}^2$
	$S_1 = \{\alpha_2, \alpha_3\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}^2$
	$S_1 = \{\alpha_2, \alpha_4\}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \boldsymbol{R}^2$
	$S_1 = \{\alpha_i, \alpha_j, \alpha_k\}$	$\mathfrak{su}(2) \oplus \boldsymbol{R}^3$
	$S_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$	$\boldsymbol{R}^4$
G_2	$S_1 = \{\alpha_1\}$	$\mathfrak{su}(2) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_2\}$	$\mathfrak{sp}(2) \oplus \boldsymbol{R}$
	$S_1 = \{\alpha_1, \alpha_2\}$	$\boldsymbol{R}^2$

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