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ON THE SECOND VARIATION OF THE IDENTITY MAP OF A PRODUCT MANIFOLD

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Abstract. The main aim of this paper is to compute the index and the nullity of the identity map of $S^n \times S^m$ and $S^n \times T^m$. In order to obtain this we establish a rather general result on the spectrum of the Hodge-Laplacian on k-forms on a product manifold, which could prove useful in other contexts.

1. Introduction. If $f : (M, g) \to (N, h)$ is a smooth map between compact oriented Riemannian manifolds, its energy is defined by

$$E(f) = \frac{1}{2} \int_M |df|^2 dv_M \,. \tag{1.1}$$

Then a harmonic map is defined to be a smooth critical point of the functional (1.1). We refer to the surveys [3], [4] for motivations and background on harmonic maps.

Let $f^{-1}TN$ be the induced vector bundle by f over M and $C(f^{-1}TN)$ the space of all sections of $f^{-1}TN$. The second variation formula for a harmonic map f was first obtained in [8] and [9]. It is given by

$$\begin{aligned} H_f(v, w) &= \int_M \langle \nabla^{f^{-1}TN} v, \nabla^{f^{-1}TN} w \rangle - \mathrm{Trace}\, \langle R^N(df, v)df, w \rangle dv_M \\ &= \int_M \langle J_f v, w \rangle dv_M \,, \end{aligned} \tag{1.2}$$

for all $v, w \in C(f^{-1}TN)$, where $\nabla^{f^{-1}TN} = \nabla$ is the connection on $f^{-1}TN$ induced by the Riemannian connections of M and N, and R^N denotes the curvature tensor of N. Setting $\Delta v = -\mathrm{Trace}\,(\nabla^2 v)$, we have $J_f = \Delta - \mathrm{Trace}\, R^N(df, \,)df$. The operator J_f is called the *Jacobi operator*, which is linear, elliptic and self-adjoint.

The variational significance of the formula (1.2) can be expressed as follows: Let $f_{s,t}$ be a smooth, 2-parameter variation of f with $\partial f_{s,t}/\partial s|_{s,t=0} = v$ and $\partial f_{s,t}/\partial t|_{s,t=0} = w$. Then we have

$$\left. \frac{\partial^2 E(f_{s,t})}{\partial s \partial t} \right|_{s,t=0} = H_f(v, w) \,. \tag{1.3}$$

This motivates the following definitions. The *index* of f, denoted by $\mathrm{Ind}(f)$, is the dimension of the largest subspace of $C(f^{-1}TN)$ on which H_f is negative definite. The *nullity*, denoted

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by $\mathrm{Null}(f)$, is the dimension of the kernel of J_f (Note that the spectral properties of elliptic operators on compact manifolds imply that both $\mathrm{Ind}(f)$ and $\mathrm{Null}(f)$ are finite). Vector fields v along f which satisfy $J_f v = 0$ are called *Jacobi fields.*

A harmonic map f is said to be *stable* if $\mathrm{Ind}(f) = 0$ (resp. *unstable* if $\mathrm{Ind}(f) > 0$). For instance, harmonic maps into nonpositively curved manifolds are absolute minima of the energy in their homotopy class and thus are stable. Similarly, holomorphic mappings between Kähler manifolds are stable. On the other hand, any harmonic map from or to S^n, $n \geq 3$, is unstable (see [3], [5], [11]).

As for the nullity of a harmonic map, it is convenient to introduce the notion of *Killing nullity* [9] defined by

$$\mathrm{Null}_k(f) = \dim(\mathrm{span}(i(N), df(i(M))) , \tag{1.4}$$

where the elements of $i(N)$ (infinitesimal isometries) and $df(i(M))$ are considered as variation fields along f. Clearly, they are Jacobi fields and give rise to variations of f through harmonic maps. This leads us to define the *reduced nullity* by

$$\mathrm{Null}_r(f) = \mathrm{Null}(f) - \mathrm{Null}_k(f) . \tag{1.5}$$

The following qualitative problem can now be stated:

QUESTION (see [9], [12]). Do all Jacobi fields along a harmonic map f arise from a variation of f through harmonic maps? We say that f is a *generating* harmonic map if this is the case.

In general, the computation of the index and the nullity of a harmonic map is a difficult task; in particular, to compute them for the (apparently innocuous) identity map of a compact manifold M is geometrically significant. Indeed, in this case the Jacobi operator becomes

$$J_{Id} v = Jv = \Delta v - \mathrm{Ricci}^M v , \tag{1.6}$$

where the Ricci tensor Ricci^M of M is regarded as a linear map on TM. Recall that the Hodge-Laplacian Δ_H (acting on vector fields via the musical isomorphisms) satisfies

$$\Delta_H v = \Delta v + \mathrm{Ricci}^M v . \tag{1.7}$$

Suppose now that (M, g) is an Einstein manifold, say $\mathrm{Ricci}^M = cg$ for some constant $c > 0$. Then (1.6) and (1.7) yield

$$J = \Delta_H - 2cI , \tag{1.8}$$

where I denotes the identity operator. This observation led Smith to relate the index and the nullity of the identity map to the spectrum of M. In particular, he obtained

$$(1.9)\qquad \begin{aligned} &\text{(i)}\quad \mathrm{Ind}(Id_{S^n}) = \begin{cases} 0 & \text{if } n = 1, 2, \\ n+1 & \text{if } n \geq 3, \end{cases} \\ &\text{(ii)}\quad \mathrm{Null}_r(Id_{S^n}) = \begin{cases} 0 & \text{if } n \neq 2, \\ n+1 & \text{if } n = 2, \end{cases} \\ &\text{(iii)}\quad \mathrm{Null}_k(Id_{S^n}) = \frac{n(n+1)}{2} \quad (n \geq 1). \end{aligned}$$

As for the flat torus T^n, we also have

$$(1.10)\qquad \begin{aligned} &\text{(i)}\quad \mathrm{Ind}(Id_{T^n}) = 0 && \text{for all } n \geq 2, \\ &\text{(ii)}\quad \mathrm{Null}_r(Id_{T^n}) = 0 && \text{for all } n \geq 2, \\ &\text{(iii)}\quad \mathrm{Null}_k(Id_{T^n}) = \dim H^1_{\mathbf{R}} = n && \text{for all } n \geq 2. \end{aligned}$$

Certainly, the method of relating the spectra of J and Δ_H relies heavily on the assumption that (M, g) is Einstein. In this paper we show that some of these ideas can be adapted to the case of a product of Einstein manifolds. In particular, we shall compute explicitly the index and the nullity of the identity map of $S^n \times S^m$ and $S^n \times T^m$.

Our paper is organised as follows:

2. Statement of the results
3. Preliminaries
4. Proofs of the results
5. Conclusions

All manifolds are supposed to be smooth and without boundary.

2. Statement of the results. Let $f_i : (M_i, g_i) \to (N_i, h_i)$, $i = 1, 2$, be two harmonic maps. We shall consider the product map

$$(2.1)\qquad f : M = (M_1 \times M_2, g_1 \times g_2) \to N = (N_1 \times N_2, h_1 \times h_2),$$

which sends (x, y) to $(f_1(x), f_2(y))$. The map f is harmonic and one may hope to be able to estimate (or compute) $\mathrm{Ind}(f)$ and $\mathrm{Null}(f)$ in terms of the $\mathrm{Ind}(f_i)$'s and $\mathrm{Null}(f_i)$'s. For instance, a first easy result in this direction is:

PROPOSITION 2.1. (i) *If f_1 and f_2 are stable, then f is stable.*

(ii) $\mathrm{Ind}(f) \geq \mathrm{Ind}(f_1) + \mathrm{Ind}(f_2)$.

(iii) $\mathrm{Null}(f) \geq \mathrm{Null}(f_1) + \mathrm{Null}(f_2)$.

REMARK 2.2. In several cases, equality holds in (ii) and (iii) above. For instance, we have (compare with (1.9))

$$(2.2)\qquad \mathrm{Ind}(Id_{S^n \times S^n}) = \begin{cases} 0 & \text{if } n = 1, 2, \\ 2(n+1) & \text{if } n \geq 3. \end{cases}$$

$$(2.3)\qquad \mathrm{Null}_k(Id_{S^n \times S^n}) = n(n+1) \quad (n \geq 1).$$

$$\text{Null}_r(Id_{S^n\times S^n}) = \begin{cases} 0 & \text{if } n\neq 2, n\geq 1\,, \\ 6 & \text{if } n=2\,. \end{cases} \tag{2.4}$$

(Note that (2.2)–(2,4) are an easy consequence of the fact that $S^n \times S^n$ is Einstein.)

However, the following result shows that in general equality does not hold, even if $M = N$ is a product of two Einstein manifolds and f is the identity map. More precisely, we obtain:

THEOREM 2.3. *Suppose, for convenience, that $m > n \geq 1, m, n \in N$.*

Part (i) [*Index*] *Let $A_{m,n} = \{k \in N : k(k+n-1) < m-2\}$, $N_m = \text{Card}A_{m,1}$. Then*

$$\text{Ind}(Id_{S^1\times S^m}) = \begin{cases} 0 & \text{if } m=2\,, \\ 2(m+1)N_m & \text{if } m\geq 3\,. \end{cases} \tag{2.5}$$

$$\text{Ind}(Id_{S^2\times S^m}) = (m+1)\sum_{k=0}^{\text{Max}A_{m,2}}(2k+1)\,, \quad m\geq 3\,. \tag{2.6}$$

$$\text{Ind}(Id_{S^n\times S^m}) = (n+1)+(m+1)\sum_{k=0}^{\text{Max}A_{m,n}}\frac{(2k+n-1)(k+n-2)!}{k!(n-1)!}\,, m>n>2\,. \tag{2.7}$$

Part (ii) [*Killing Nullity*]

$$\text{Null}_k(Id_{S^n\times S^m}) = \frac{m(m+1)}{2}+\frac{n(n+1)}{2}\,, \quad m>n\geq 1\,. \tag{2.8}$$

Part (iii) [*Reduced Nullity*] *As for the reduced nullity, we need to separate two cases*:

Case (a) *There exists $k \in N-\{0\}$ such that $k(k+n-1) = m-2$. In this case we have*

$$\text{Null}_r(Id_{S^n\times S^m}) = 2(m+1)\,, \quad m>n=1\,. \tag{2.9}$$

$$\text{Null}_r(Id_{S^n\times S^m}) = 3+(2k+1)(m+1)\,, \quad m>n+2\,. \tag{2.10}$$

$$\text{Null}_r(Id_{S^n\times S^m}) = (m+1)\frac{(2k+n-1)(k+n-2)!}{k!(n-1)!}\,, \quad m>n>2\,. \tag{2.11}$$

Case (b) *There exists no $k \in N-\{0\}$ such that $k(k+n-1) = m-2$. In this case we have*

$$\text{Null}_r(Id_{S^n\times S^m}) = \begin{cases} 3 & \text{if } m=2,\ n=1\,, \\ 0 & \text{if } m>2,\ n=1\,. \end{cases} \tag{2.12}$$

$$\text{Null}_r(Id_{S^n\times S^m}) = 3 \quad m>n=2\,. \tag{2.13}$$

$$\text{Null}_r(Id_{S^n\times S^m}) = 0 \quad m>n>2\,. \tag{2.14}$$

REMARK 2.4. As we shall see in Section 5 below, in (2.5)–(2.7) and (2.9)–(2.11) we find Jacobi fields depending on *both* variables.

Based on similar principles, we also obtain the next result.

THEOREM 2.5. *Part* (i) [*Index*]

$$\text{Ind}(Id_{S^2\times T^m}) = 0\,, \quad m\geq 2\,. \tag{2.15}$$

$$\text{Ind}(Id_{S^n\times T^m}) = (n+1)\sum_{k=1}^{N}\text{Card}\{z\in \mathbf{Z}^m : |z|^2 = k^2\}\,, \quad m\geq 2\ n>2\,, \tag{2.16}$$

where $N = \text{Max}\{k \in N - \{0\} : 4\pi^2 k^2 < n-2\}$.

Part (ii) [*Killing Nullity*]

$$\text{Null}_k(Id_{S^n\times T^m}) = m + \frac{n(n+1)}{2}\,, \quad m,n\geq 2\,. \tag{2.17}$$

Part (iii) [*Reduced Nullity*]

$$\text{Null}_r(Id_{S^n\times T^m}) = \begin{cases} 3 & n=2\,,\ m\geq 2\,,\\ 0 & n>2\,,\ m\geq 2\,. \end{cases} \tag{2.18}$$

In the proof of Theorem 2.3 and Theorem 2.5, we shall need to know the spectrum of the Hodge-Laplacian on 1-forms. Here we elaborate ideas of [1] and establish a rather general result on k-forms. Let us begin with several notations. Let $A^k(M)$ be the space of smooth k-forms on M. We denote by $d : A^k(M) \to A^{k+1}(M)$ the exterior differential, while $d^* : A^{k+1}(M) \to A^k(M)$ is its formal adjoint with respect to the L^2-inner product induced on differential forms by the Riemannian structure of M. Thus the Hodge-Laplacian on k-forms is defined by

$$\Delta_H = -(dd^* + d^*d)\,. \tag{2.19}$$

We shall be concerned with the case that M is a product manifold with the product metric

$$M = (M_1 \times M_2, g_1\times g_2)\,. \tag{2.20}$$

In this context, a superscript M_i indicates that the operator under consideration is to be taken with respect to variables of M_i only; d^{M_i}, d^{*M_i}, $\Delta_H^{M_i}$, for instance.

Let $\mathcal{P}^k_\lambda(M_i, g_i)$ be the eigenspace consisting of k-eigenforms associated to the eigenvalue λ of $\Delta_H^{M_i}$. We set

$$\mathcal{P}^k(M_i, g_i) = \sum_{\lambda\in \text{Spec}^k(M_i,g_i)} \mathcal{P}^k_\lambda(M_i, g_i)\,, \tag{2.21}$$

where $\text{Spec}^k(M_i, g_i)$ is the spectrum of $\Delta_H^{M_i}$ acting on $A^k(M_i)$. The projection maps

$$p : (M_1\times M_2, g_1\times g_2) \to (M_1, g_1)\,, \quad q : (M_1\times M_2, g_1\times g_2)\to (M_2, g_2) \tag{2.22}$$

induce, for $0\leq i\leq m_1 = \dim M_1$ and $0\leq j\leq m_2 = \dim M_2$, the following maps

$$p_i^* : A^i(M_1)\to A^i(M_1\times M_2)\,, \quad q_j^* : A^j(M_2)\to A^j(M_1\times M_2)\,. \tag{2.23}$$

We denote by $p_i^*(A^i(M_1))\wedge q_j^*(A^j(M_2))$ the linear subspace generated by the exterior product forms $p_i^*(\alpha_i)\wedge q_j^*(\beta_j)$, where $\alpha_i \in A^i(M_1)$, $\beta_j \in A^j(M_2)$. (If $i=0$ or $j=0$, we get exterior product forms of the type $p_0^*(\alpha_0)q_j^*(\beta_j)$ or $p_i^*(\alpha_i)q_0^*(\beta_0)$ respectively.)

Finally, we find it convenient to adopt the following convention. The greek letters λ, α will be used to represent eigenvalues and differential forms respectively on the factor M_1. Moreover, λ^i, α_i denote the fact that $\alpha_i \in A^i(M_1)$ and $\lambda^i \in \text{Spec}^i(M_1, g_1)$. As for the factor M_2, we use in a similar way the greek letters μ, β.

We are now in a right position to state our result

THEOREM 2.6. *Consider a compact, oriented product manifold* $(M, g) = (M_1 \times M_2, g_1 \times g_2)$. *Then the following hold.*

(i) *The spectrum of the Hodge-Laplacian* Δ_H *acting on k-forms is given by*

$$\mathrm{Spec}^k(M, g) = \{\lambda^i + \mu^j : \lambda^i \in \mathrm{Spec}^i(M_1, g_1), \mu^j \in \mathrm{Spec}^j(M_2, g_2), i + j = k\}. \tag{2.24}$$

(ii) *Let* $\gamma \in \mathrm{Spec}^k(M, g)$. *Then the eigenspace associated to* γ *is*

$$\mathcal{P}^k_\gamma(M, g) = \bigoplus_{i+j=k} \bigoplus_{\substack{\lambda^i+\mu^j=\gamma \\ \lambda^i \in \mathrm{Spec}^i(M_1,g_1) \\ \mu^j \in \mathrm{Spec}^j(M_2,g_2)}} p_i^*(\mathcal{P}^i_{\lambda^i}(M_1, g_1)) \wedge q_j^*(\mathcal{P}^j_{\mu^j}(M_2, g_2)). \tag{2.25}$$

(iii) *Moreover,*

$$\mathcal{P}^k(M, g) = \bigoplus_{i+j=k} p_i^*(\mathcal{P}^i(M_1, g_1)) \wedge q_j^*(\mathcal{P}^j(M_2, g_2)). \tag{2.26}$$

3. Preliminaries. In this section we collect some known facts which will be used in the proof of our results. We begin with some properties of infinitesimal conformal fields and infinitesimal isometries. Let (M, g) be a Riemannian manifold with metric g. We denote by δv the divergence of the vector field v on M. Note that the relation between δ and d^* is simply given by

$$\delta v = d^*(v^\flat), \tag{3.1}$$

where the superscript $\flat$ denotes the operation of lowering indices using the metric g (classically, the inverse of this operator is indicated by $\sharp$). Infinitesimal conformal fields are characterized by the following formula

$$L_v g = \frac{2}{\dim M}(\delta v) g, \tag{3.2}$$

where $L_v g$ is the Lie derivative of the metric g in the direction v. In the special case that $L_v g = 0$, v is called a *Killing field* (or, equivalently, v is an infinitesimal isometry). Next, the following useful integral formula holds on a compact oriented manifold M (see [2])

$$\int_M \langle Jv, v\rangle dv_M = \frac{1}{2}\int_M \{|L_v g|^2 - 2(\delta v)^2\} dv_M. \tag{3.3}$$

Putting together (3.2) and (3.3) we obtain:

(3.4) *Let* (M, g) *be a compact, oriented Riemannian manifold. Then the Killing fields are precisely the divergence-free Jacobi fields.*

Each eigenspace of Δ_H on 1-forms decomposes into an orthogonal sum as follows:

$$\mathcal{P}^1_\lambda(M, g) = (\mathcal{P}^1_\lambda(M, g) \cap \mathrm{Ker} d) \oplus (\mathcal{P}^1_\lambda(M, g) \cap \mathrm{Ker} d^*). \tag{3.5}$$

The previous discussion suggests that determining Null_k and Null_r may require a complete knowledge of the spectrum of Δ_H *on each* of the subspaces on the right-hand side of (3.5).

More precisely, we shall need the following facts (see [1], [6], [7]) (S^n, T^m are given the standard metric):

$$(3.6)\quad \begin{aligned} &\text{(i)} \quad \mathrm{Spec}^0(S^n) = \{\lambda_k^0 = k(k+n-1) : k \in N\}\,.\\ &\text{(ii)} \quad \dim \mathcal{P}^0_{\lambda_k^0}(S^n) = (2k+n-1)(k+n-2)!/k!(n-1)!\,.\\ &\text{(iii)} \quad \mathrm{Spec}^1(S^n)|_{\mathrm{Ker}d} = \{\lambda_k^1 = k(k+n-1) : k \in N - \{0\}\}\,.\\ &\text{(iv)} \quad \dim(\mathcal{P}^1_{\lambda_k^1}(S^n) \cap \mathrm{Ker}d) = \dim \mathcal{P}^0_{\lambda_k^0}(S^n)\,.\\ &\text{(v)} \quad \mathrm{Inf}\{\mathrm{Spec}^1(S^n)|_{\mathrm{Ker}d^*}\} = 2(n-1)\,.\\ &\text{(vi)} \quad \dim(\mathcal{P}^1_{2(n-1)}(S^n) \cap \mathrm{Ker}d^*) = n(n+1)/2\,. \end{aligned}$$

(The symbol | denotes "restriction". Note also that elements in $\mathcal{P}^1_{\lambda_k^1}(S^n) \cap \mathrm{Ker}d$ correspond precisely to the differentials of elements in $\mathcal{P}^0_{\lambda_k^0}(S^n)$.) As for the torus, it will suffice to know that

$$(3.7)\quad \begin{aligned} &\text{(i)} \quad \mathrm{Spec}^0(T^m) = \{\mu_k^0 = 4\pi^2k^2 : k \in N\}\,.\\ &\text{(ii)} \quad \dim(\mathcal{P}^0_{\mu_k^0}(T^m)) = \mathrm{Card}\{z \in \mathbf{Z}^m : |z|^2 = k^2\}\,.\\ &\text{(iii)} \quad \dim(\mathcal{P}^1_0(T^m)) = \dim(\mathcal{P}^1_0(T^m) \cap \mathrm{Ker}d^*) = m\,. \end{aligned}$$

Let us end this section by fixing some convenient notations concerning a product map f as in (2.1). With reference to the canonical orthogonal sum decomposition

$$(3.8)\quad T_{f(x,y)}N = T_{f_1(x)}N_1 \oplus T_{f_2(y)}N_2\,,$$

we decompose the vector field v along f as follows:

$$(3.9)\quad v = {}^1v + {}^2v\,.$$

In the special case that ${}^2v \equiv 0$ and ${}^1v$ depends only on the x-variables, we say that $v \in C(f_1^{-1}TN_1)$. Similarly, we define the subspace $C(f_2^{-1}TN_2)$. With this preparation, we can now proceed to the proofs.

4. Proof of the results.

Proof of Proposition 2.1: A computation shows that the second variation formula (1.2) for a product map f as in (2.1) becomes:

$$(4.1)\quad \begin{aligned} H_f(v,w) = &\int_{M_1} H_{f_2}({}^2v, {}^2w)dv_{M_1} + \int_{M_2} H_{f_1}({}^1v, {}^1w)dv_{M_2}\\ &+ \int_M \left(\sum_{i=1}^{m_1} \left\langle \sum_{k=1}^{n_2} e_i({}^2v^k)\frac{\partial}{\partial \bar{u}^k}, \sum_{k=1}^{n_2} e_i({}^2w^k)\frac{\partial}{\partial \bar{u}^k} \right\rangle \right.\\ &\left. + \sum_{j=1}^{m_2} \left\langle \sum_{l=1}^{n_1} \bar{e}_j({}^1v^l)\frac{\partial}{\partial u^l}, \sum_{l=1}^{n_1} \bar{e}_j({}^1w^l)\frac{\partial}{\partial u^l} \right\rangle \right) dv_M\,, \end{aligned}$$

where $m_i = \dim M_i$, $n_i = \dim N_i$, $i = 1, 2$. The e_i's (resp. $\bar{e}_j$'s) are a local orthonormal frame on TM_1 (resp. TM_2); and the $\partial/\partial u^l$'s (resp. $\partial/\partial \bar{u}^k$'s) are a local basis for TN_1 (resp. TN_2). Now the Proposition follows. Indeed, (i) follows from the facts that the H_{f_i}'s are positive definite, $i = 1, 2$, and the terms in the last integral of (4.1) are nonnegative too. As for (ii) and (iii), it suffices to consider energy decreasing or Jacobi fields in $C(f_1^{-1}TN_1)$ or $C(f_2^{-1}TN_2)$ (for these vector fields the last integral in (4.1) vanishes).

Proof of Theorem 2.6: It is convenient to divide the proof into 5 steps.

Step 1. *The space $\bigoplus_{i+j=k} p_i^*(A^i(M_1, g_1)) \wedge q_j^*(A^j(M_2, g_2))$ is dense in $A^k(M)$ for the L^2-norm.*

PROOF. We fix $k_1, k_2 \in N$ such that $k_1 + k_2 = k$. Let (U, x) (resp. (U', y)) be a local chart on M_1 (resp. M_2). First, we consider a k-form w on M which is compactly supported on the product chart $(U \times U', (x, y))$ and has the form

$$w = a dx^{i_1} \wedge \cdots \wedge dx^{i_{k_1}} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_{k_2}}, \tag{4.2}$$

where a is a smooth function whose support lies on $(U \times U')$. (Note that the indices $i_1, \ldots, i_{k_1}$, $j_1, \ldots, j_{k_2}$ are fixed.) We know from [1] that $p_0^*(A^0(M_1)) \otimes q_0^*(A^0(M_2))$ is dense for the uniform convergence norm in $A^0(M)$. Thus we can assume the existence of $\sum_{r=0}^{R}(h^r \circ p)(g^r \circ q) \in p_0^*(A^0(M_1)) \otimes q_0^*(A^0(M_2))$ such that

$$\left\| a - \sum_{r=0}^{R}(h^r \circ p)(g^r \circ q) \right\|_\infty^2 \le \varepsilon, \quad \varepsilon > 0, \tag{4.3}$$

and $\operatorname{supp}((h^r \circ p)(g^r \circ q)) \subset \operatorname{supp}(a) = \operatorname{supp}(w)$ (Here $\| \ \|_\infty$ denotes the uniform convergence norm).

Next, we define

$$\bar{w} = \sum_{r=0}^{R}((h^r \circ p)dx^{i_1} \wedge \cdots \wedge dx^{i_{k_1}}) \wedge ((g^r \circ q)dy^{j_1} \wedge \cdots \wedge dy^{j_{k_2}}), \tag{4.4}$$

where $\bar{w} \in p_{k_1}^*(A^{k_1}(M_1)) \wedge q_{k_2}^*(A^{k_2}(M_2))$ and $\operatorname{supp}(\bar{w}) \subset \operatorname{supp}(w)$. We get

$$\|w - \bar{w}\|_{L^2}^2 \le K\varepsilon, \tag{4.5}$$

where K is a constant which depends only on the maximum of the coefficients of the Riemannian metric on $\operatorname{supp}(w)$. Now the conclusion of Step 1 follows from the facts that ε in (4.3) can be chosen arbitrarily, and each element of $A^k(M)$ can be written (by means of a convenient partition of unity) as a finite sum of k-forms of the type (4.2).

Step 2. *The space $\bigoplus_{i+j=k} p_i^*(\mathcal{P}^i(M_1, g_1)) \wedge q_j^*(\mathcal{P}^j(M_2, g_2))$ is dense in $A^k(M)$ for the L^2-norm.*

PROOF. Owing to Step 1, it is enough to check that $\bigoplus_{i+j=k} p_i^*(\mathcal{P}^i(M_1, g_1)) \wedge q_j^*(\mathcal{P}^j(M_2, g_2))$ is dense in $\bigoplus_{i+j=k} p_i^*(A^i(M_1, g_1)) \wedge q_j^*(A^j(M_2, g_2))$ in the L^2-norm. Now this follows from the fact that $\mathcal{P}^i(M_1, g_1)$ (resp. $\mathcal{P}^j(M_2, g_2)$) is dense in $A^i(M_1)$ (resp. $A^j(M_2)$).

Step 3. *Let* $\lambda \in \mathrm{Spec}^i(M_1, g_1)$, $\mu \in \mathrm{Spec}^j(M_2, g_2)$ *and* $\alpha \in \mathcal{P}^i_\lambda(M_1, g_1)$, $\beta \in \mathcal{P}^j_\mu(M_2, g_2)$. *Then*

$$\Delta_H(p_i^*(\alpha) \wedge q_j^*(\beta)) = (\lambda + \mu) p_i^*(\alpha) \wedge q_j^*(\beta)\,. \tag{4.6}$$

PROOF. The key point is to show that

$$d^*(p_i^*(\alpha) \wedge q_j^*(\beta)) = p_{i-1}^*(d^{*M_1}\alpha) \wedge q_j^*(\beta) + (-1)^i p_i^*(\alpha) \wedge q_{j-1}^*(d^{*M_2}\beta)\,. \tag{4.7}$$

First we observe that (4.7) is equivalent to

$$\begin{aligned} &\int_M \langle p_{i-1}^*(d^{*M_1}\alpha) \wedge q_j^*(\beta) + (-1)^i p_i^*(\alpha) \wedge q_{j-1}^*(d^{*M_2}\beta), w\rangle dv_M \\ &\qquad = \int_M \langle p_i^*(\alpha) \wedge q_j^*(\beta), dw\rangle dv_M \end{aligned} \tag{4.8}$$

for all $w \in A^{i+j-1}(M)$. Using a partition of unity and the linearity of d, it is enough to prove the equality in (4.8) when w is a differential form as in (4.2), with $k_1 + k_2 = i + j - 1$.

Then we have

$$\begin{aligned} &\int_M \langle p_{i-1}^*(d^{*M_1}\alpha) \wedge q_i^*(\beta), a dx^{i_1} \wedge \cdots \wedge dx^{i_{k_1}} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_{k_2}}\rangle dv_M \\ &= \int_{M_2} \left(\langle \beta, dy^{j_1} \wedge \cdots \wedge dy^{j_{k_2}}\rangle \int_{M_1} \langle d^{*M_1}\alpha, a dx^{i_1} \wedge \cdots \wedge dx^{i_{k_1}}\rangle dv_{M_1} \right) dv_{M_2} \\ &= \int_{M_2} \left(\langle \beta, dy^{j_1} \wedge \cdots \wedge dy^{j_{k_2}}\rangle \int_{M_1} \langle \alpha, d^{M_1}(a dx^{i_1} \wedge \cdots \wedge dx^{i_{k_1}})\rangle dv_{M_1} \right) dv_{M_2} \\ &= \int_M \langle p_i^*(\alpha) \wedge q_j^*(\beta), d^{M_1}(a dx^{i_1} \wedge \cdots \wedge dx^{i_{k_1}} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_{k_2}})\rangle dv_M\,, \end{aligned} \tag{4.9}$$

where the first and the third equalities are due to Fubini's Theorem, while the second is obtained using the fact that d^{*M_1} is the formal adjoint of d^{M_1} provided that the y-variables are considered as fixed. It should also be noticed that if $k_1 \neq i - 1$ or $k_2 \neq j$, then the first and the last term of (4.9) are both equal to 0. In a similar way, we also obtain

$$\begin{aligned} &\int_M (-1)^i \langle p_i^*(\alpha) \wedge q_{j-1}^*(d^{*M_2}\beta), a dx^{i_1} \wedge \cdots \wedge dx^{i_{k_1}} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_{k_2}}\rangle dv_M \\ &\quad = \int_M \langle p_i^*(\alpha) \wedge q_j^*(\beta), d^{M_2}(a dx^{i_1} \wedge \cdots \wedge dx^{i_{k_1}} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_{k_2}})\rangle dv_M\,. \end{aligned} \tag{4.10}$$

Now we add (4.9) and (4.10) and obtain (4.8). Finally, a short calculation using (4.7) gives us

$$\begin{aligned} \Delta_H(p_i^*(\alpha) \wedge q_j^*(\beta)) &= p_i^*(\Delta_H^{M_1}\alpha) \wedge q_j^*(\beta) + p_i^*(\alpha) \wedge q_j^*(\Delta_H^{M_2}\beta) \\ &= (\lambda + \mu) p_i^*(\alpha) \wedge q_j^*(\beta)\,, \end{aligned} \tag{4.11}$$

completing the proof of Step 3.

REMARK 4.1. It is easy to see that, in general, the following formula

$$d^*(\omega_1 \wedge \omega_2) = d^*(\omega_1) \wedge \omega_2 + (-1)^i \omega_1 \wedge d^*(\omega_2)\,, \tag{4.12}$$

where $\omega_1 \in A^i(M)$, $\omega_2 \in A^j(M)$, does not hold (even if M is compact). So the validity of (4.7) is due to the very special type of forms under consideration.

Step 4. *Suppose that for any $r \in N$, there exists a linear subspace W_r of $A^k(M)$ such that the following two conditions are satisfied*:

(A) *There exists $\gamma_r \in \mathbf{R}$ such that $\Delta_H w = \gamma_r w$ for all $w \in W_r$*;

(B) *$\sum_{r \in N} W_r$ is dense in $A^k(M)$ for the L^2-norm.*

Then $\mathrm{Spec}^k(M, g) = \{\gamma_r\}_{r \in N}$ *and* $\mathcal{P}^k_{\gamma_r}(M, g) = W_r$.

PROOF. The proof of this Step is done by an adaptation of Lemma A II 1 of [1], so we omit it.

Step 5. *End of the proof of Theorem* 2.6: Let

$$S_k = \{\lambda^i + \mu^j : \lambda^i \in \mathrm{Spec}^i(M_1, g_1), \mu^j \in \mathrm{Spec}^j(M_2, g_2), i + j = k\}. \tag{4.13}$$

We order the elements of S_k in such a way that we obtain an increasing sequence $(\gamma_r)_{r \in N}$ so that

$$S_k = \{\gamma_r\}_{r \in N}. \tag{4.14}$$

Now, let $r \in N$. We set

$$W_r = \bigoplus_{i+j=k} \bigoplus_{\substack{\lambda^i + \mu^j = \gamma_r \\ \lambda^i \in \mathrm{Spec}^i(M_1, g_1) \\ \mu^j \in \mathrm{Spec}^j(M_2, g_2)}} p_i^*(\mathcal{P}^i_{\lambda^i}(M_1, g_1)) \wedge q_j^*(\mathcal{P}^j_{\mu^j}(M_2, g_2)). \tag{4.15}$$

Because of Step 4, it is sufficient to show that conditions (A) and (B) above are satisfied. Indeed, Step 3 implies that (A) is fulfilled. Moreover, from the definition of W_r and from Step 3, we obtain

$$\sum_{r \in N} W_r = \bigoplus_{i+j=k} p_i^*(\mathcal{P}^i(M_1, g_1)) \wedge q_j^*(\mathcal{P}^j(M_2, g_2)). \tag{4.16}$$

On the other hand, it follows from (4.16) and Step 2 that (B) holds too. This completes the proof.

Proof of Theorem 2.3: Let $v \in C(T(S^n \times S^m))$. Refering to the decomposition (3.9), we have

$$\mathrm{Ricci}^{S^n \times S^m}(v) = \mathrm{Ricci}^{S^n}({}^1v) + \mathrm{Ricci}^{S^m}({}^2v) = (n-1)^1v + (m-1)^2v. \tag{4.17}$$

Using this, together with (1.6) and (1.7), we find that the second variation for $Id_{S^n \times S^m}$ takes the following form:

$$H_{Id_{S^n \times S^m}}(v, w) = H_1(v, w) + 2(m-n) \int_{S^n \times S^m} \langle {}^1v, {}^1w \rangle dv_{S^n \times S^m} \tag{4.18}$$

for all $v, w \in C(T(S^n \times S^m))$, where

$$H_1(v, w) = \int_{S^n \times S^m} \langle \Delta_H v - 2(m-1)v, w \rangle dv_{S^n \times S^m}. \tag{4.19}$$

(Note that the use of musical isomorphisms to pass from vector fields to 1-forms and its converse is always tacitly assumed.)

Now the idea of the proof is to control the term H_1 in (4.18) due to the spectral result in Theorem 2.6. The appearance of an additional term in (4.18) makes it necessary to develop some further technicalities. More precisely, *keeping notation as in Step* 5 *above*, let $\gamma_r \in \mathrm{Spec}^1(S^n \times S^m)$. Its associated eigenspace is

$$(4.20)\quad W_r = \bigoplus_{\substack{\lambda^0+\mu^1=\gamma_r \\ \lambda^1+\mu^0=\gamma_r}} [p_0^*(\mathcal{P}^0_{\lambda^0}(S^n)) \otimes q_1^*(\mathcal{P}^1_{\mu^1}(S^m))] \oplus [p_1^*(\mathcal{P}^1_{\lambda^1}(S^n)) \otimes q_0^*(\mathcal{P}^0_{\mu^0}(S^m))]\,.$$

(Here and below, we omit to write $\lambda^i \in \mathrm{Spec}^i(S^n)$, $\mu^j \in \mathrm{Spec}^j(S^m)$, $i = 0, 1$.)

If we prove that the eigenspace W_r is a sum of subspaces orthogonal to $H_{Id_{S^n \times S^m}}$ (denoted by H from now on), then we can apply standard properties of bilinear forms to deduce that

$$(4.21)\quad \begin{aligned} \mathrm{Ind}(H|_{W_r}) = &\sum_{\lambda^0+\mu^1=\gamma_r} \mathrm{Ind}(H|_{p_0^*(\mathcal{P}^0_{\lambda^0}(S^n))\otimes q_1^*(\mathcal{P}^1_{\mu^1}(S^m))}) \\ &+ \sum_{\lambda^1+\mu^0=\gamma_r} \mathrm{Ind}(H|_{p_1^*(\mathcal{P}^1_{\lambda^1}(S^n))\otimes q_0^*(\mathcal{P}^0_{\mu^0}(S^m))})\,, \end{aligned}$$

and, similarly,

$$(4.22)\quad \begin{aligned} \mathrm{Null}(H|_{W_r}) = &\sum_{\lambda^0+\mu^1=\gamma_r} \mathrm{Null}(H|_{p_0^*(\mathcal{P}^0_{\lambda^0}(S^n))\otimes q_1^*(\mathcal{P}^1_{\mu^1}(S^m))}) \\ &+ \sum_{\lambda^1+\mu^0=\gamma_r} \mathrm{Null}(H|_{p_1^*(\mathcal{P}^1_{\lambda^1}(S^n))\otimes q_0^*(\mathcal{P}^0_{\mu^0}(S^m))})\,, \end{aligned}$$

where $\mathrm{Ind}(H|_V)$ (resp. $\mathrm{Null}(H|_V)$) denotes the index (resp. the nullity) of the restriction of H to the subspace V.

To sum up, (4.21) and (4.22) hold provided that we show that the various subspaces which appear on the right-hand side of (4.20) are orthogonal to H. For this purpose, we consider the following two cases:

Case 1. Let $v = (f_{\lambda^0} \circ p)q_1^*(\beta_{\mu^1}) \in p_0^*(\mathcal{P}^0_{\lambda^0}(S^n)) \otimes q_1^*(\mathcal{P}^1_{\mu^1}(S^m))$ and $w = (g_{\mu^0} \circ q)p_1^*(\alpha_{\lambda^1}) \in q_0^*(\mathcal{P}^0_{\mu^0}(S^m)) \otimes p_1^*(\mathcal{P}^1_{\lambda^1}(S^n))$, where $\lambda^0 + \mu^1 = \lambda^1 + \mu^0 = \gamma_r$. In this case we find:

$$(4.23)\quad \begin{aligned} &\text{(i)}\quad \int_{S^n \times S^m} \langle v, w\rangle dv_{S^n\times S^m} = 0, \\ &\text{(ii)}\quad H_1(v, w) = (\gamma_r - 2(m-1)) \int_{S^n \times S^m} \langle v, w\rangle dv_{S^n\times S^m}\,, \end{aligned}$$

from which it follows immediately that $H(v, w) = 0$.

Case 2. Let v be as in Case 1. Also, let $w = (f_{\bar{\lambda}^0} \circ p) q_1^*(\beta_{\bar{\mu}^1}) \in p_0^*(\mathcal{P}^0_{\bar{\lambda}^0}(S^n)) \otimes q_1^*(\mathcal{P}^1_{\bar{\mu}^1}(S^m))$, where $\bar{\lambda}^0 + \bar{\mu}^1 = \lambda^0 + \mu^1 = \gamma_r$, $\bar{\lambda}^0 \neq \lambda^0$. In this case we find:

$$\text{(4.24)}\quad \begin{aligned} \text{(i)}\quad \int_{S^n \times S^m} \langle v, w \rangle dv_{S^n \times S^m} &= \int_{S^n \times S^m} \langle {}^2v, {}^2w \rangle dv_{S^n \times S^m} \\ &= \int_{S^n} \langle f_{\bar{\lambda}^0}, f_{\lambda^0} \rangle dv_{S^n} \int_{S^m} \langle \beta_{\bar{\mu}^1}, \beta_{\mu^1} \rangle dv_{S^m} = 0\,, \\ \text{(ii)}\quad H_1(v, w) = (\gamma_r - 2(m-1)) &\int_{S^n \times S^m} \langle v, w \rangle dv_{S^n \times S^m}\,, \end{aligned}$$

from which again it follows that $H(v, w) = 0$. The remaining cases are similar, so we conclude that (4.21) and (4.22) hold. Moreover, in a similar spirit, it is easy to show that if W_r and $W_{r'}$ are two distinct eigenspaces of Δ_H, then they are orthogonal to H.

Next, let $\gamma_r \in \mathrm{Spec}^1(S^n \times S^m)$. We observe that, if $\gamma_r \geq 2(m-1)$ (resp. $\gamma_r > 2(m-1)$), then

$$H(v, v) \geq 0 \quad (\text{resp. } H(v, v) > 0)\,, \tag{4.25}$$

for all $v \in W_r$. Putting these facts together, we conclude that

$$\text{(4.26)}\quad \begin{aligned} \mathrm{Ind}(H) = \sum_{\gamma_r < 2(m-1)} \mathrm{Ind}(H|_{W_r}) &= \sum_{\lambda^0 + \mu^1 < 2(m-1)} \mathrm{Ind}(H|_{p_0^*(\mathcal{P}^0_{\lambda^0}(S^n)) \otimes q_1^*(\mathcal{P}^1_{\mu^1}(S^m))}) \\ &+ \sum_{\lambda^1 + \mu^0 < 2(m-1)} \mathrm{Ind}(H|_{p_1^*(\mathcal{P}^1_{\lambda^1}(S^n)) \otimes q_0^*(\mathcal{P}^0_{\mu^0}(S^m))})\,, \end{aligned}$$

and similarly,

$$\text{(4.27)}\quad \begin{aligned} \mathrm{Null}(H) = \sum_{\gamma_r \leq 2(m-1)} \mathrm{Null}(H|_{W_r}) &= \sum_{\lambda^0 + \mu^1 = \leq 2(m-1)} \mathrm{Null}(H|_{p_0^*(\mathcal{P}^0_{\lambda^0}(S^n)) \otimes q_1^*(\mathcal{P}^1_{\mu^1}(S^m))}) \\ &+ \sum_{\lambda^1 + \mu^0 \leq 2(m-1)} \mathrm{Null}(H|_{p_1^*(\mathcal{P}^1_{\lambda^1}(S^n)) \otimes q_0^*(\mathcal{P}^0_{\mu^0}(S^m))})\,. \end{aligned}$$

Each eigenspace decomposes into the sum of two subspaces which are orthogonal to H, that is,

$$\mathcal{P}^1_{\lambda^1}(S^n) = (\mathcal{P}^1_{\lambda^1}(S^n) \cap \mathrm{Ker} d^{S^n}) \oplus (\mathcal{P}^1_{\lambda^1}(S^n) \cap \mathrm{Ker} d^{*S^n}) \tag{4.28}$$

(similarly for $\mathcal{P}^1_{\mu^1}(S^m)$, of course). On the other hand, a simple computation yields

$$H(v, v) = (\lambda^0 + \mu^1 - 2(m-1)) \int_{S^n \times S^m} \langle v, v \rangle dv_{S^n \times S^m} \tag{4.29}$$

for all $v \in p_0^*(\mathcal{P}^0_{\lambda^0}(S^n)) \otimes q_1^*(\mathcal{P}^1_{\mu^1}(S^m))$. Similarly,

$$H(v, v) = (\lambda^1 + \mu^0 - 2(n-1)) \int_{S^n \times S^m} \langle v, v \rangle dv_{S^n \times S^m} \tag{4.30}$$

for all $v \in p_1^*(\mathcal{P}^1_{\lambda^1}(S^n)) \otimes q_0^*(\mathcal{P}^0_{\mu^0}(S^m))$.

Now we use (3.6) (i), (iii), (v), together with $m > n$, to deduce from (4.29) and (4.30) that

$$\operatorname{Ind}(H) = \sum_{\lambda^0+\mu^1<2(m-1)} \dim(\mathcal{P}^0_{\lambda^0}(S^n)) \dim(\mathcal{P}^1_{\mu^1}(S^m) \cap \operatorname{Ker} d^{S^m}) + \sum_{\lambda^1+\mu^0<2(n-1)} \dim(\mathcal{P}^1_{\lambda^1}(S^n) \cap \operatorname{Ker} d^{S^n}) \dim(\mathcal{P}^0_{\mu^0}(S^m)), \tag{4.31}$$

and

$$\begin{aligned} \operatorname{Null}(H) = & \sum_{\lambda^0+\mu^1=2(m-1)} \dim(\mathcal{P}^0_{\lambda^0}(S^n)) \dim(\mathcal{P}^1_{\mu^1}(S^m) \cap \operatorname{Ker} d^{S^m}) \\ & + \dim(\mathcal{P}^1_{2(m-1)}(S^m) \cap \operatorname{Ker} d^{*S^m}) \\ & + \sum_{\lambda^1+\mu^0=2(n-1)} \dim(\mathcal{P}^1_{\lambda^1}(S^n) \cap \operatorname{Ker} d^{S^n}) \dim(\mathcal{P}^0_{\mu^0}(S^m)) \\ & + \dim(\mathcal{P}^1_{2(n-1)}(S^n) \cap \operatorname{Ker} d^{*S^n}) . \end{aligned} \tag{4.32}$$

Finally, a long but straightforward computation of (4.31) and (4.32) using (3.6) enables us to complete the theorem. (Note that, in order to separate the Killing nullity from the reduced nullity, we use (3.4).)

Proof of Theorem 2.5: The arguments follow closely those of the previous theorem, so we confine ourselves to indicate how the main steps have to be modified. Since T^m is Ricci flat, the analogue of (4.18) reads here

$$H_{S^n\times T^m}(v, w) = H_2(v, w) + 2(n-1) \int_{S^n\times T^m} \langle {}^2v, {}^2w \rangle dv_{S^n\times T^m} , \tag{4.18$'$}$$

where

$$H_2(v, w) = \int_{S^n\times T^m} \langle \Delta_H v - 2(n-1)v, w \rangle dv_{S^n\times T^m} .$$

Also, instead of (4.29) and (4.30), we find

$$H(v, v) = (\lambda^0 + \mu^1) \int_{S^n\times T^m} \langle v, v \rangle dv_{S^n\times T^m} \tag{4.29$'$}$$

for all $v \in p_0^*(\mathcal{P}^0_{\lambda^0}(S^n)) \otimes q_1^*(\mathcal{P}^1_{\mu^1}(T^m))$, and

$$H(v, v) = (\lambda^1 + \mu^0 - 2(n-1)) \int_{S^n\times T^m} \langle v, v \rangle dv_{S^n\times T^m} \tag{4.30$'$}$$

for all $v \in p_1^*(\mathcal{P}^1_{\lambda^1}(S^n)) \otimes q_0^*(\mathcal{P}^0_{\mu^0}(T^m))$.

The rest is the same. Indeed, we just use (3.7) to have the required information on the spectrum of $\Delta_H^{T^m}$.

5. Conclusions. (I) The proofs of Theorem 2.3 and Theorem 2.5 tell us explicitly how to construct energy decreasing or Jacobi fields in terms of eigenfunctions and 1-eigenforms. We can also point out some cases where $Id_{S^n\times S^m}$ is certainly a *generating* harmonic map: more precisely, that happens when the conditions in (2.12), (2.13) or (2.14) apply. Similarly, we find that $Id_{T^n\times S^m}$ is a generating harmonic map for all $n, m (n \geq 2)$.

(II) Here we compute explicitly the flow of a certain Jacobi field v along $Id_{S^1\times S^6}$. This flow does not consist of harmonic maps (i.e., $Id_{S^1\times S^6}$ is not a generating harmonic map. Moreover, the same occurs in all cases included in (2.9), and the proof is similar). Let us first introduce a convenient set of coordinates on $S^1 \times S^6$:

$$S^1 \times S^6 \ni (\theta, \sin r\omega, \cos r) \overset{Id}{\longmapsto} (\theta, \sin r\omega, \cos r) \in S^1 \times S^6\,, \tag{5.1}$$

where $0 \leq \theta \leq 2\pi$, $0 \leq r \leq \pi$, $\omega \in S^5$ (the obvious immersion into $\boldsymbol{R}^9$ is tacitly assumed). Because of (2.9), we know that $\mathrm{Null}_r(Id_{S^1\times S^6}) = 14$. Moreover, the associated Jacobi fields are of the form

$$v = f(\theta)\tilde{v}(r, \omega)\,, \tag{5.2}$$

where f is an eigenfunction of $\Delta_H^{S^1}$ associated to the eigenvalue $k^2 = m - 2 = 4$, while $\tilde{v}$ is a conformal field on S^6, i.e., $\tilde{v}$ is the projection on TS^6 of the gradient of a linear function on $\boldsymbol{R}^7$. A simple choice for such a v is

$$v = -\left(\frac{1}{2}\sin(2\theta)\right)\sin r\frac{\partial}{\partial r}\,. \tag{5.3}$$

Because of the symmetries in the construction, we expect the flow of v to have the following form

$$S^1 \times S^6 \ni (\theta, \sin r\omega, \cos r) \overset{F_t}{\longmapsto} (\theta, \sin h(t, r, \theta)\omega, \cos h(t, r, \theta)) \in S^1 \times S^6\,, \tag{5.4}$$

where $|t| < \varepsilon$ and the function $h(t, r, \theta)$ must be such that

$$(i)\ F_0 = Id \qquad (ii)\ \left.\frac{\partial F_t}{\partial t}\right|_{t=0} = v\,. \tag{5.5}$$

In terms of h, conditions (5.5) become

$$(i)\ h(0, r, \theta) = r \qquad (ii)\ \left.\frac{\partial h(t, r, \theta)}{\partial t}\right|_{t=0} = -\left(\frac{1}{2}\sin(2\theta)\right)\sin r\,. \tag{5.6}$$

Then it is straightforward to check that

$$h(t, r, \theta) = \operatorname{Arccos}\left(\cos t\cos r + \frac{t}{2}\sin(2\theta)\sin^2 r\right)\,, \quad |t| < \varepsilon \tag{5.7}$$

fulfills both conditions in (5.6). Moreover, it is easy to see that (5.4), with h as in (5.7), is smooth across $r = 0$, $r = \pi$ for all $|t| < \varepsilon$ (ε small), just by showing that these F_t's are restrictions to $S^1 \times S^6$ of smooth maps of $\boldsymbol{R}^9$ into itself. Finally, a routine verification shows that a diffeomorphism as in (5.4) can be harmonic only if h does not depend on θ, a fact which is false for h as in (5.7). This shows that the flow of such a v does not consist of harmonic maps.

(III) The method of proof of Theorem 2.3 and Theorem 2.5 works, more generally, when one wants to compute $\mathrm{Ind}(Id_{M_1\times M_2})$ or $\mathrm{Null}(Id_{M_1\times M_2})$, provided that one knows $\mathrm{Spec}^k(\Delta_H^{M_i}|_{\mathrm{Ker}d^{M_i}})$ and $\mathrm{Spec}^k(\Delta_H^{M_i}|_{\mathrm{Ker}d^{*M_i}})$, $i = 1, 2$, and (M_i, g_i) are compact Einstein manifolds. For instance, spheres of radius $r \neq 1$ or other tori $\boldsymbol{R}^m/\Gamma$ could be easily treated in this way. We leave the details to the interested reader.

REFERENCES

[1] M. BERGER, P. GAUDUCHON AND E. MAZET, Le spectre d'une varieté Riemannienne, Lecture Notes in Math., Vol. 194, Springer-Verlag, Berlin-New York, 1971.

[2] S. BOCHNER AND K. YANO, Curvature and Betti numbers, Annals of Math. Studies 32, Princeton Univ. Press, Princeton, 1953.

[3] J. EELLS AND L. LEMAIRE, Selected topics in harmonic maps, C.B.M.S. Regional Conf. Series 50, Amer. Math. Soc., Providence, 1983.

[4] J. EELLS AND L. LEMAIRE, Another report on harmonic maps, Bull. London Math. Soc. 20 (1988), 385–524.

[5] A. EL-SOUFI, Indice de Morse des applications harmoniques de la sphère, Compositio Math. 95 (1995), 343–362.

[6] S. GALLOT AND D. MEYER, Opérateur de courbure et laplacien des formes différentielles d'une variété Riemannienne, J. Math. Pures Appl. 54 (1975), 259–284.

[7] I. IWASAKI AND K. KATASE, On the spectra of Laplace operator on $\Lambda^*(S^n)$, Proc. Japan Acad. Ser. A Math. Sci. 55 (1979), 141–145.

[8] E. MAZET, La formule de la variation seconde de l'énergie au voisinage d'une application harmonique, J. Differential Geom. 8 (1973), 279–296.

[9] R. T. SMITH, The second variation for harmonic mappings, Proc. Amer. Math. Soc. 47 (1975), 229–236.

[10] R. T. SMITH, Harmonic maps between spheres, Ph. D. Thesis, Univ. of Warwick, 1972.

[11] Y. L. XIN, Some results on stable harmonic maps, Duke Math. J. 47 (1980), 609–613.

[12] K. YANO AND T. NAGANO, On geodesic vector fields in a compact orientable Riemannian space, Comment. Math. Helv. 35 (1965), 55–64.

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