Tohoku Math. J.
52 (2000), 199–233

LIE SPHERE GEOMETRY AND INTEGRABLE SYSTEMS

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(Received September 10, 1998, revised October 1, 1999)

Abstract. Two basic Lie-invariant forms uniquely defining a generic (hyper)surface in Lie sphere geometry are introduced. Particularly interesting classes of surfaces associated with these invariants are considered. These are the diagonally cyclidic surfaces and the Lie-minimal surfaces, the latter being the extremals of the simplest Lie-invariant functional generalizing the Willmore functional in conformal geometry.

Equations of motion of a special Lie sphere frame are derived, providing a convenient unified treatment of surfaces in Lie sphere geometry. In particular, for diagonally cyclidic surfaces this approach immediately implies the stationary modified Veselov-Novikov equation, while the case of Lie-minimal surfaces reduces in a certain limit to the integrable coupled Tzitzeica system.

In the framework of the canonical correspondence between Hamiltonian systms of hydrodynamic type and hypersurfaces in Lie sphere geometry, it is pointed out that invariants of Lie-geometric hypersurfaces coincide with the reciprocal invariants of hydrodynamic type systems.

Integrable evolutions of surfaces in Lie sphere geometry are introduced. This provides an interpretation of the simplest Lie-invariant functional as the first local conservation law of the $(2+1)$-dimensional modified Veselov-Novikov hierarchy.

Parallels between Lie sphere geometry and projective differential geometry of surfaces are drawn in the conclusion.

1. Introduction. Lie sphere geometry dates back to the dissertation of Lie in 1872 [27]. After that the subject was extensively developed by Blaschke and his coworkers and resulted in publication in 1929 of Blaschke's "Vorlesungen über Differentialgeometrie" [2], entriely devoted to the Lie sphere geometry of curves and surfaces. The modern multidimensional period of the theory was initiated by Pinkall's classification of Dupin hypersurfaces in E^4 [30], [31]. We refer also to Cecil's book [7] with the review of the last results in this direction. Since most of the recent research in Lie sphere geometry is concentrated around Dupin hypersurfaces and Dupin submanifolds, the general theory of Lie-geometric hypersurfaces seems not to be constructed so far. The aim of this paper is to shed some new light on Lie sphere geometry of (hyper)surfaces and to reveal its remarkable interrelations with the modern theory of integrable systems.

Lie $M^2 \subset E^3$ be a surface in the 3-dimensional Euclidean space E^3 parametrized by the coordinates R^1, R^2 of the lines of curvature. Let k^1, k^2 and $g_{11}(dR^1)^2 + g_{22}(dR^2)^2$ be the principal curvatures and the induced metric of M^2, respectively. In Section 2 we introduce

1991 *Mathematics Subject Classification.* Primary 53A40; Secondary 53A20, 35Q58, 35L60.

two basic Lie sphere invariants of the surface M^2, namely, the symmetric 2-form

$$\frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2} dR^1 dR^2 \tag{1}$$

(which can be viewed as the Lie-invariant metric of the Lorentzian signature) and the conformal class of the cubic form

$$\partial_1 k^1 g_{11} (dR^1)^3 + \partial_2 k^2 g_{22} (dR^2)^3 , \tag{2}$$

$\partial_i = \partial/\partial R^i$, which define generic surface M^2 uniquely up to Lie sphere equivalence. We recall that the group of Lie sphere transformations in E^{n+1} is a contact group, generated by conformal transformations and normal shifts, translating each point of the surface to a fixed distance $a =$ const along the normal direction. Conformal transformations and normal shifts generate in E^{n+1} a finite-dimensional Lie group isomorphic to $SO(n+2, 2)$. Lie sphere transformations can be equivalently characterized as those contact transformations, which map spheres into spheres and preserve their oriented contact. In an implicit form the objects (1) and (2) have been introduced already in [2]. Particular classes of surfaces in Lie sphere geometry can be specified by certain restrictions on (1), (2).

In Section 3 we discuss diagonally cyclidic surfaces (diagonalzyklidische flächen in the terminology of [2, p. 406]), which can be characterized as the surfaces M^2 possessing parametrization R^1, R^2 by the coordinates of lines of curvature such that

$$\partial_1 k^1 g_{11} = \partial_2 k^2 g_{22} ,$$

so that the cubic form (2) becomes proportional to $(dR^1)^3 + (dR^2)^3$. This class of surfaces is a straightforward generalization of isothermic surfaces in conformal differential geometry (characterized by the condition $g_{11} = g_{22}$).

Quadratic form (1) gives rise to the Lie-invariant functional

$$\iint \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2} dR^1 dR^2 , \tag{3}$$

whose extremals are known as minimal surfaces in Lie sphere geometry (K-minimalflächen in the terminology of [2, §94]). Some of the most important geometric properties of Lie-minimal surfaces are reviewed in Section 4.

In Section 5 a Lie sphere frame associated with a surface M^2 is constructed. Up to certain normalizations this construction follows that proposed by Blaschke in [2]. However, our final formulae prove to be more appropriate for the purposes of the theory of integrable systems. In particular, they immediately imply that in the case of diagonally cyclidic surfaces the Lie sphere density p defined by

$$p^2 = \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2}$$

satisfies the stationary modified Veselov-Novikov (mVN) equation

$$\partial_1^3 p - 2V\partial_1 p - p\partial_1 V = \partial_2^3 p - 2W\partial_2 p - p\partial_2 W ,$$
$$\partial_1 W = -(3/2)\partial_2(p^2) , \quad \partial_2 V = -(3/2)\partial_1(p^2) .$$

The Euler-Lagrange equations of the functional (3) governing Lie-minimal surfaces are written down explicitly. It is demonstrated that in a certain limit they reduce to the integrable coupled Tzitzeica system

$$\partial_1 \partial_2 \ln p = pq + 1/p\,, \quad \partial_1 \partial_2 \ln q = pq + 1/q\,,$$

which is a special reduction of periodic Toda lattice of period 6 (geometric meaning of V, W, p, q is clarified in Section 5).

In Section 6 we introduce Lie sphere invariants of multidimensional hypersurfaces $M^n \subset E^{n+1}$, namely, the symmetric 2-form

$$\sum_{i \neq j} \frac{k^i_i k^j_j}{(k^i - k^j)^2} \omega^i \omega^j \tag{4}$$

and the conformal class of the cubic form

$$\sum_i k^i_i g_{ii} (\omega^i)^3\,, \tag{5}$$

which define "generic" hypersurface uniquely up to Lie sphere equivalence. Here k^i are principal curvatures, ω^i are principal covectors, $\sum g_{ii}(\omega^i)^2$ is the first fundamental form and the coefficients k^i_i are defined by the expansions $dk^i = k^i_j \omega^j$ (we emphasize that hypersurface M^n of dimension $n \geq 3$ does not necessarily possess parametrization by the coordinates of curvature lines). Objects (4) and (5) are the Lie-geometric analogs of the second fundamental form and the Darboux cubic form in projective differential geometry of hypersurfaces.

In Sections 7–10 the interrelations between Lie sphere invariants and reciprocal invariants of hydrodynamic type systems

$$u^i_t = v^i_j(u) u^j_x\,, \quad i, j = 1, \dots, n \tag{6}$$

are discussed. We recall that reciprocal transformations are transformations from x, t to the new independent variables X, T defined by the formulae

$$dX = B(u)dx + A(u)dt\,, \quad dT = N(u)dx + M(u)dt\,,$$

where $Bdx + Adt$ and $Ndx + Mdt$ are two integrals of the system (6). Reciprocal transformations originated from gas dynamics and have been extensively investigated in [32], [33]. In [10], [11] we introduced reciprocal invariants, defining a hydrodynamic type system uniquely up to reciprocal equivalence. The summary of these results in the 2-component case is given in Section 7. In Sections 8–9 we recall the necessary information about Hamiltonian systems of hydrodynamic type and describe the general construction of [14], [12], relating Hamiltonian systems (6) and hypersurfaces in E^{n+1}. The main property of this correspondence is the "equivariance" in the sense that Lie sphere transformations of hypersurfaces correspond to "canonical" reciprocal transformations, that is, reciprocal transformations preserving the Hamiltonian structure. In this approach Lie sphere invariants of hypersurfaces correspond to reciprocal invariants of hydrodynamic type systems, providing thus their differential-geometric interpretation.

Integrable evolutions of surfaces in Lie sphere geometry are introduced in Section 10. This construction provides a simple interpretation of the Lie-invariant functional (3) as the first conservation law of the integrable $(2+1)$-dimensional mVN hierarchy.

In Section 11 we write down reciprocal invariants of n-component systems for arbitrary $n \geq 3$, since they differ from those in the case $n = 2$.

Parallels between Lie sphere geometry and projective differential geometry are drawn in the Appendix. Projective duals of diagonally cyclidic and Lie-minimal surfaces are discussed: these are the so-called isothermally asymptotic and projectively minimal surfaces, respectively. It is demonstrated that isothermally asymptotic surfaces are also described by the stationary mVN equation (however, with a different real reduction). The case of projectively minimal surfaces is even more surprising: this class is described by exactly the same integrable system as the Lie-minimal surfaces. These analogies can be viewed as an analytic manifestation of Lie's famous line-sphere correspondence.

2. Invariants of surfaces in Lie sphere geometry. In [2, p. 392] Blaschke introduced the Lie-invariant differentials ω^1, ω^2 (in Blaschke's notation, $d\psi, d\bar{\psi}$) which assume the following form in the coordinates R^1, R^2 of the lines of curvature:

$$\omega^1 = \frac{\partial_1 k^1}{k^1 - k^2}\left(\frac{(\partial_2 k^2)^2 g_{11}}{(\partial_1 k^1)^2 g_{22}}\right)^{1/6} dR^1,$$

(7)

$$\omega^2 = \frac{\partial_2 k^2}{k^2 - k^1}\left(\frac{(\partial_1 k^1)^2 g_{22}}{(\partial_2 k^2)^2 g_{11}}\right)^{1/6} dR^2.$$

REMARK 1. In order to check the Lie sphere invariance of the differentials ω^1, ω^2 it is sufficient to check their invariance under the inversions and normal shifts, which can be done by direct calculation. Moreover, forms (7) do not change if the principal curvatures k^i and the metric coefficients g_{ii} are replaced by the radii of principal curvature $w^i = 1/k^i$ and the coefficients of the third fundamental form $G_{ii} = (k^i)^2 g_{ii}$, respectively.

REMARK 2. Similar invariant differentials arise in Möbius (conformal) geometry:

$$\frac{\partial_1 k^1}{k^1 - k^2} dR^1, \quad \frac{\partial_2 k^2}{k^2 - k^1} dR^2.$$

The conformally invariant metric

$$(k^1 - k^2)^2 (g_{11}(dR^1)^2 + g_{22}(dR^2)^2)$$

is an analog of the Lie invariant form (1).

As long as ω^1 and ω^2 are invariant under Lie sphere transformations, so are the quadratic form

$$-\omega^1 \omega^2 = \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2} dR^1 dR^2$$

and the cubic form

$$(\omega^1)^3 - (\omega^2)^3 = \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^3 \sqrt{g_{11} g_{22}}} (\partial_1 k^1 g_{11} (dR^1)^3 + \partial_2 k^2 g_{22} (dR^2)^3)\,, \tag{8}$$

which give rise to (1) and (2), respectively. The reason for introducing these objects is their simple representation in terms of the familiar Euclidean invariants as well as their additional symmetry under the interchange of indices 1 and 2.

It was proved in [2, §85] that up to certain exeptional cases a generic surface in 3-space is determined by the invariant differentials ω^1, ω^2 uniquely up to Lie sphere transformations. Since ω^1, ω^2 can be reconstructed from the quadratic form (1) and the conformal class of the cubic form (2) (the multiple in (8) is not essential), we can formulate the following:

THEOREM 1. *A generic surface $M^2 \subset E^3$ is defined by the quadratic form*

$$\frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2} dR^1 dR^2$$

and the conformal class of the cubic form

$$\partial_1 k^1 g_{11} (dR^1)^3 + \partial_2 k^2 g_{22} (dR^2)^3$$

uniquely up to Lie sphere transformations.

The vanishing of the cubic form is equivalent to the conditions $\partial_1 k^1 = \partial_2 k^2 = 0$ which specify the so-called cyclids of Dupin. We recall that the vanishing of the Darboux cubic form in projective differential geometry specifies quadrics, which are thus projective duals of cyclids of Dupin.

Principal directions of the surface M^2 can be characterized as the zero directions of quadratic form (1). On the other hand, they are exactly those directions, where cubic form (2) reduces to the sum of pure cubes (without mixed terms). It should be pointed out that any cubic form on the plane can be reduced to the sum of cubes, and the directions where it assumes the desired form are defined uniquely (in the nondegenerate case).

In the next sections we discuss two particularly interesting Lie-invariant classes of surfaces which are naturally defined in terms of the invariants (1) and (2).

3. Diagonally cyclidic surfaces. With any surface M^2 we can associate a 3-web (that is, three one-parameter families of curves) formed by the lines of curvature and cyclidic curves (zyklidische kurven in the terminology of Blaschke [2, §86]) which are the zero directions of cubic form (2) and hence are obviously Lie-invariant. In view of (8) the curves of this 3-web can be defined as follows:

$$\omega^1 = 0\,, \quad \omega^2 = 0\,, \quad \omega^1 - \omega^2 = 0\,. \tag{9}$$

Geometric meaning of cyclidic curves has been clarified in [2, §86]. Cyclidic curves are the natural Lie sphere analogs of the Darboux curves in projective differential geometry. Let us compute the connection form of the 3-web (9), that is, the 1-form ω which is uniquely determined by the equations

$$d\omega^1 = \omega \wedge \omega^1 \,, \quad d\omega^2 = \omega \wedge \omega^2 \,,$$

(see [3], [4] for the introduction in web geometry). A direct computation results in

$$\omega = \frac{1}{3}\left(\frac{\partial_1\partial_2 k^2}{\partial_2 k^2} + \frac{\partial_1 k^1}{k^1 - k^2}\right) dR^1 + \frac{1}{3}\left(\frac{\partial_1\partial_2 k^1}{\partial_1 k^1} + \frac{\partial_2 k^2}{k^2 - k^1}\right) dR^2 \tag{10}$$
$$+ \frac{1}{3} d \ln \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^5 \sqrt{g_{11} g_{22}}} \,.$$

Since both ω^1, ω^2 are invariant under Lie sphere transformations, so is the connection 1-form ω. From (10) it immediately follows that the curvature form $d\omega$ of the 3-web (9) is given by

$$d\omega = \frac{1}{3} d\Omega \,,$$

where

$$\Omega = \left(\frac{\partial_1\partial_2 k^2}{\partial_2 k^2} + \frac{\partial_1 k^1}{k^1 - k^2}\right) dR^1 + \left(\frac{\partial_1\partial_2 k^1}{\partial_1 k^1} + \frac{\partial_2 k^2}{k^2 - k^1}\right) dR^2 \,. \tag{11}$$

As we will see in Section 6, an object analogous to (11) arises in the theory of reciprocal invariants of hydrodynamic type systems.

The class of diagonally cyclidic surfaces [2, p. 406] is specified by the requirement, that the 3-web (9) is hexagonal or, equivalently, has zero curvature:

$$d\omega = 0 \,.$$

In this case there exist coordinates R^1, R^2 along the lines of curvature (note that we have a reparametrization freedom $R^i \to \varphi^i(R^i)$), where ω^1, ω^2 assume the form

$$\omega^1 = p dR^1 \,, \quad \omega^2 = -p dR^2$$

with nonzero common multiple p. In these coordinates the cubic form $(\omega^1)^3 - (\omega^2)^3$ becomes proportional to $(dR^1)^3 + (dR^2)^3$.

Another important geometric property of diagonally cyclidic surfaces is the existence of Ribaucour transformations, which preserve the cyclidic curves. In fact, this provides an explicit Bäcklund transformation for diagonally cyclidic surfaces with all familiar features of Bäcklund transformations like the existence of spectral parameters, permutability theorem, etc.

From the point of view of Euclidean differential geometry the class of diagonally cyclidic surfaces is specified by the requirement

$$\partial_1 k^1 g_{11} = \partial_2 k^2 g_{22} \,,$$

which, upon substitution in the Gauss-Codazzi equations, results in a nonlinear system for the Euclidean invariants k^1, k^2, g_{11}, g_{22}, which we do not write down here because of its complexity (see [17] where this system was investigated directly). Instead of this, in Section 5 we derive equations of motion of a special Lie sphere frame, which provides a convenient unified treatment of Lie-geometric surfaces. In particular, for diagonally cyclidic surfaces this approach immediately implies the stationary mVN equation.

4. Minimal surfaces in Lie sphere geometry. Lie-minimal surfaces are defined as the extremals of the functional (3)

$$\iint \frac{\partial_1 k^1 \partial_2 k^2}{(k^1 - k^2)^2} dR^1 dR^2 ,$$

which is a natural Lie sphere analog of the conformally invariant Willmore functional

$$\iint (k^1 - k^2)^2 \sqrt{g_{11} g_{22}}\, dR^1 dR^2 .$$

Lie-minimal surfaces arise also in the theory of Lie cyclids of the surface M^2. For brevity, we only recall the necessary definitions. The details can be found in [2]. Thus, let us consider a point p^0 on the surface M^2 and the R^1-curvature line passing through p^0. Let us take three additional points $p^i, i = 1, 2, 3$ on this curvature line close to p^0 and consider three R^2-curvature lines γ^i passing through p^i. The three curvature spheres of γ^i through the points p^i uniquely define a cyclid of Dupin Q containing them as the generators. As p^i tend to p^0, the cyclid Q tends to a limiting cyclid, the so-called Lie cyclid of the surface M^2 at the point p^0. Even though this construction depends on the initial choice of either the R^1- or the R^2-curvature line through p^0, the resulting cyclid Q is independent of that choice. Thus, we arrive at a two-parameter family of cyclids of Dupin associated with the surface M^2. Now, in a neighbourhood of a generic point p^0 on M^2, the envelopes of the family of Lie cyclids consist of the surface M^2 itself and four, in general, distinct sheets. The case of Lie-minimal surfaces is characterized by the additional property that the curvature lines on all these sheets correspond to the curvature lines of the surface M^2 itself. Moreovere, for Lie-minimal surface all four sheets of the envelope will be Lie-minimal as well. In a sense, it is natural to call the family of Lie cyclids with this property a Ribaucour congruence of cyclids of Dupin.

Particular Lie-minimal surfaces are characterized by the degenerate case of two distinct sheets (the sheets coincide pairwise; this class of surfaces is a straightforward analog of the Godeaux-Rozet surfaces in projective differential geometry—see the Appendix) or even one sheet (all four sheets coincide; this class is an analog of the surfaces of Demoulin).

The analytic treatment of all these cases is given in Section 5.

5. Lie sphere frame. Let M^2 be a surface in E^3 parametrized by the coordinates R^1, R^2 of the curvature lines. The radius-vector $\mathbf{r}$ and the unit normal $\mathbf{n}$ of the surface M^2 satisfy the Weingarten equations

$$\partial_1 \mathbf{r} = w^1 \partial_1 \mathbf{n} , \quad \partial_2 \mathbf{r} = w^2 \partial_2 \mathbf{n} , \tag{12}$$

where w^1, w^2 are the radii of principal curvature.

Let us recall the definition of the Lie sphere map. With any sphere $S(R, \mathbf{r})$ of radius R and center $\mathbf{r} = (r^1, r^2, r^3)$ this map associates a 6-vector $\{y_0, y_1, y_2, y_3, y_4, y_5\}$ with the coordinates y_i defined by

$$\left\{\frac{1+\mathbf{r}^2-R^2}{2},\ \frac{1-\mathbf{r}^2+R^2}{2},\ \mathbf{r},\ R\right\}.$$

These coordinates satisfy the equation

$$-y_0^2+y_1^2+y_2^2+y_3^2+y_4^2-y_5^2=0 \tag{13}$$

defining the so-called Lie quadric. Thus with any sphere $S(R,\mathbf{r})$ in E^3 we associate a point lying on the Lie quadric (13). The reader may consult [2], [7] for the properties of this construction.

Applyhing Lie sphere map to the curvature sphres $S(w^1,\mathbf{r}-w^1\mathbf{n})$, $S(w^2,\mathbf{r}-w^2\mathbf{n})$ of the surface M^2, we obtrain a pair of two-dimensional submanifolds of the Lie quadric with the radius-vectors

$$U=\left\{\frac{1+\mathbf{r}^2-2w^1(\mathbf{r},\mathbf{n})}{2},\ \frac{1-\mathbf{r}^2+2w^1(\mathbf{r},\mathbf{n})}{2},\ \mathbf{r}-w^1\mathbf{n},\ w^1\right\},$$

$$V=\left\{\frac{1+\mathbf{r}^2-2w^2(\mathbf{r},\mathbf{n})}{2},\ \frac{1-\mathbf{r}^2+2w^2(\mathbf{r},\mathbf{n})}{2},\ \mathbf{r}-w^2\mathbf{n},\ w^2\right\},$$

respectively. It can be verified that

$$(U,U)=(U,V)=(V,V)=0, \tag{14}$$

where the scalar product of 6-vectors is defined by the indefinite quadratic form (13). In what follows we use the same notation $(\,,)$ for both the scalar product defined by (13) and the standard Euclidean scalar product in E^3; however, the dimension of vectors will clearly indicate which one has to be chosen.

A direct computation gives

$$\begin{aligned}
\partial_1 U &= \partial_1 w^1\{-(\mathbf{r},\mathbf{n}),(\mathbf{r},\mathbf{n}),-\mathbf{n},1\}\\
\partial_2 U &= \partial_2 w^1\{-(\mathbf{r},\mathbf{n}),(\mathbf{r},\mathbf{n}),-\mathbf{n},1\}+\frac{w^2-w^1}{w^2}\{(\partial_2\mathbf{r},\mathbf{r}),-(\partial_2\mathbf{r},\mathbf{r}),\partial_2\mathbf{r},0\},\\
\partial_1 V &= \partial_1 w^2\{-(\mathbf{r},\mathbf{n}),(\mathbf{r},\mathbf{n}),-\mathbf{n},1\}+\frac{w^1-w^2}{w^1}\{(\partial_1\mathbf{r},\mathbf{r}),-(\partial_1\mathbf{r},\mathbf{r}),\partial_1\mathbf{r},0\},\\
\partial_2 V &= \partial_2 w^2\{-(\mathbf{r},\mathbf{n}),(\mathbf{r},\mathbf{n}),-\mathbf{n},1\},
\end{aligned} \tag{15}$$

implyhing

$$\partial_1 U=\frac{\partial_1 w^1}{w^1-w^2}(U-V),\quad \partial_2 V=\frac{\partial_2 w^2}{w^2-w^1}(V-U). \tag{16}$$

Differentiating (14) and taking into account (15), (16), we conclude that the only nonzero scalar products among the vectors $U, V, \partial_1 U, \partial_2 U, \partial_1 V, \partial_2 V$ are the following:

$$(\partial_2 U,\partial_2 U)=(w^1-w^2)^2G_{22},\quad (\partial_1 V,\partial_1 V)=(w^1-w^2)^2G_{11}.$$

Here $G_{11} = (\partial_1 \mathbf{n}, \partial_1 \mathbf{n})$, $G_{22} = (\partial_2 \mathbf{n}, \partial_2 \mathbf{n})$ are the components of the third fundamental form of the surface M^2. Differentiating the zero scalar products among U, V, $\partial_1 U$, $\partial_2 U$, $\partial_1 V$, $\partial_2 V$ and keeping in mind (16), one can show that the triple U, $\partial_2 U$, $\partial_2^2 U$ is orthogonal to the triple V, $\partial_1 V$, $\partial_1^2 V$. In order to complete the vectors U, V to a frame with the simplest possible table of scalar products we will choose appropriate combinations among the triples U, $\partial_2 U$, $\partial_2^2 U$ and V, $\partial_1 V$, $\partial_1^2 V$, separately. Up to certain normalization the choice described below coincides with that from [2]. However, our final formulae prove to be more appropriate for the purposes of the theory of integrable systems.

Let us introduce the normalized vectors

$$\mathcal{U} = \frac{U}{\sqrt{G_{22}}(w^2 - w^1)}, \quad \mathcal{V} = \frac{V}{\sqrt{G_{11}}(w^1 - w^2)}. \tag{17}$$

This normalization is convenient for several reasons: first of all, the equations (16) reduce to the Dirac equation

$$\partial_1 \mathcal{U} = p\mathcal{V}, \quad \partial_2 \mathcal{V} = q\mathcal{U}, \tag{18}$$

with the coefficients p and q given by

$$p = \frac{\partial_1 w^1}{w^1 - w^2} \frac{\sqrt{G_{11}}}{\sqrt{G_{22}}}, \quad q = \frac{\partial_2 w^2}{w^2 - w^1} \frac{\sqrt{G_{22}}}{\sqrt{G_{11}}}.$$

It is important that both p and q are Lie-invariant (we emphasize that coefficients in (16) are not Lie-invariant). The reparametrization of coordinates

$$(R^1)^* = f(R^1), \quad (R^2)^* = g(R^2)$$

induces the transformation of p and q as follows:

$$p^* = pg'/(f')^2, \quad q^* = qf'/(g')^2. \tag{19}$$

In terms of p, q the invariant differentials (7) assume the form

$$\omega^1 = (qp^2)^{1/3} dR^1, \quad \omega^2 = (pq^2)^{1/3} dR^2,$$

so that the invariant forms (1) and (2) can be rewritten as

$$-pq\, dR^1 dR^2$$

and

$$p(dR^1)^3 - q(dR^2)^3, \tag{20}$$

respectively (note that only the conformal class of the cubic form does make an invariant sense).

There exists one more important property of the normalized vector $\mathcal{U}$ (resp. $\mathcal{V}$). It turns out that the action of the Lie sphere group in E^3 induces linear transformations of the coordinates of $\mathcal{U}$ (resp. $\mathcal{V}$). Since this linear action should necessarily preserve the Lie quadric (13), we arrive at the well-known isomorphism of the Lie sphere group and $SO(4, 2)$. To prove the above statement it is sufficient to consider separately the two building blocks of the Lie sphere group, namely

1. Normal shifts

$$\mathbf{r} \to \mathbf{r} + c\mathbf{n}, \quad \mathbf{n} \to \mathbf{n}, \quad w^i \to w^i + c, \quad G_{ii} \to G_{ii}, \quad c = \text{const};$$

2. Conformal transformations: in fact, it suffices to consider the standard inversion

$$\mathbf{r} \to 2\frac{\mathbf{r}}{(\mathbf{r}, \mathbf{r})}, \quad \mathbf{n} \to \mathbf{n} - 2\frac{(\mathbf{r}, \mathbf{n})}{(\mathbf{r}, \mathbf{r})}\mathbf{r},$$

$$w^i \to \frac{2w^i}{(\mathbf{r}, \mathbf{r}) - 2w^i(\mathbf{r}, \mathbf{n})}, \quad G_{ii} \to G_{ii}\frac{((\mathbf{r}, \mathbf{r}) - 2w^i(\mathbf{r}, \mathbf{n}))^2}{(\mathbf{r}, \mathbf{r})^2}.$$

It can be checked directly that both these transformations induce linear transformations of $\mathcal{U}$ (resp. $\mathcal{V}$). Since conformal transformations and normal shifts span the full Lie sphere group, the statement follows.

REMARK. For the unnormalized vector U (resp. V) the linearization result is no longer valid: in the transformation formulae for U (resp. V) there always arises a nonconstant factor, which breaks the linearity. Thus the normalization (17) linearizes the action of the Lie sphere group.

Using the known scalar products among the vectors $U, V, \partial_1 U, \partial_2 U, \partial_1 V, \partial_2 V$, we immediately see that the only nonzero scalar products among the normalized vectors $\mathcal{U}, \mathcal{V}$, $\partial_1\mathcal{U}, \partial_2\mathcal{U}, \partial_1\mathcal{V}, \partial_2\mathcal{V}$ are the following:

$$(\partial_2\mathcal{U}, \partial_2\mathcal{U}) = (\partial_1\mathcal{V}, \partial_1\mathcal{V}) = 1.$$

Obviously, the normalized triples $\mathcal{U}, \partial_2\mathcal{U}, \partial_2^2\mathcal{U}$ and $\mathcal{V}, \partial_1\mathcal{V}, \partial_1^2\mathcal{V}$ remain mutually orthogonal. Let us introduce the following vectors $\mathcal{A}, \mathcal{P}$ from the first triple:

$$\mathcal{A} = \partial_2\mathcal{U} - \frac{\partial_2 p}{p}\mathcal{U}, \quad \mathcal{P} = \partial_2\mathcal{A} - a\mathcal{U},$$

which we require to have the following nonzero scalar products:

$$(\mathcal{A}, \mathcal{A}) = 1, \quad (\mathcal{U}, \mathcal{P}) = -1.$$

This uniquely specifies

$$a = -\frac{1}{2}(\partial_2\mathcal{A}, \partial_2\mathcal{A})$$

(in principle it is possible to derive an explicit formula for a in terms of the Euclidean invariant; however this is not necessary for what follows). Similarly, we can choose

$$\mathcal{B} = \partial_1\mathcal{V} - \frac{\partial_1 q}{q}\mathcal{V}, \quad \mathcal{Q} = \partial_1\mathcal{B} - b\mathcal{V}$$

with the nonzero scalar products

$$(\mathcal{B}, \mathcal{B}) = 1, \quad (\mathcal{V}, \mathcal{Q}) = -1,$$

which fixes

$$b = -\frac{1}{2}(\partial_1\mathcal{B}, \partial_1\mathcal{B}).$$

Vectors $\mathcal{U}$, $\mathcal{A}$, $\mathcal{P}$ and $\mathcal{V}$, $\mathcal{B}$, $\mathcal{Q}$ constitute the Lie sphere frame with the following simple table of scalar products

$$(\mathcal{A},\mathcal{A})=1\,,\quad (\mathcal{U},\mathcal{P})=-1\,,\quad (\mathcal{B},\mathcal{B})=1\,,\quad (\mathcal{V},\mathcal{Q})=-1\,, \tag{21}$$

(all other scalar products are zero) which is of the desired signature (4, 2).

Our next aim is to derive equations of motion of the Lie sphere frame. What we have at the moment are the following equations:

equations for $\mathcal{U}$:

$$\partial_1\mathcal{U}=p\mathcal{V}\,,\quad \partial_2\mathcal{U}=\frac{\partial_2 p}{p}\mathcal{U}+\mathcal{A}\,; \tag{22}$$

equations for $\mathcal{V}$:

$$\partial_1\mathcal{V}=\frac{\partial_1 q}{q}\mathcal{V}+\mathcal{B}\,,\quad \partial_2\mathcal{V}=q\mathcal{U}\,; \tag{23}$$

equations for $\mathcal{A}$, $\mathcal{B}$:

$$\partial_2\mathcal{A}=a\mathcal{U}+\mathcal{P}\,,\quad \partial_1\mathcal{B}=b\mathcal{V}+\mathcal{Q}\,. \tag{24}$$

It turns out that (21) and (22), (23), (24) completely determine the missing equations of motion. Cross-differentiating, for instance, the equations (22), we obtain

$$\partial_1\mathcal{A}=(pq-\partial_1\partial_2\ln p)\mathcal{U}\,.$$

Introducing k by the formula $\partial_1\partial_2\ln p=pq-k$, we obtain $\partial_1\mathcal{A}=k\mathcal{U}$. Similarly, $\partial_2\mathcal{B}=l\mathcal{V}$, where l is introduced by the formula $\partial_2\partial_2\ln q=pq-l$. Thus, equations of motion of $\mathcal{A}$ and $\mathcal{B}$ assume the form

equations for $\mathcal{A}$:

$$\partial_1\mathcal{A}=k\mathcal{U}\,,\quad \partial_2\mathcal{A}=a\mathcal{U}+\mathcal{P}\,; \tag{25}$$

equations for $\mathcal{B}$:

$$\partial_1\mathcal{B}=b\mathcal{V}+\mathcal{Q}\,,\quad \partial_2\mathcal{B}=l\mathcal{V}\,. \tag{26}$$

Let us demonstrate, for instance, how to find $\partial_1\mathcal{P}$. Representing $\partial_1\mathcal{P}$ in the form

$$\partial_1\mathcal{P}=\alpha\mathcal{U}+\beta\mathcal{A}+\gamma\mathcal{P}+\delta\mathcal{V}+\mu\mathcal{B}+\nu\mathcal{Q}$$

and consequently differentiating the relations

$$(\mathcal{U},\mathcal{P})=-1\,,\quad (\mathcal{A},\mathcal{P})=0\,,\quad (\mathcal{P},\mathcal{P})=0\,,\quad (\mathcal{V},\mathcal{P})=0\,,\quad (\mathcal{B},\mathcal{P})=0\,,$$

we arrive at

$$\gamma=0\,,\quad \beta=k\,,\quad \alpha=0\,,\quad \nu=0\,,\quad \mu=0\,,$$

respectively (in these calculations we always keep in mind (21), (22), (23), (25), (26)). Thus, $\partial_1\mathcal{P}=k\mathcal{A}+\delta\mathcal{V}$, where the coefficient δ is yet undetermined. It can be fixed by cross-differentiating (25): $\delta=-pa$. Moreover, cross-differentiation of (25) produces the compatibility condition

$$\partial_1 a=\partial_2 k+\frac{\partial_2 p}{p}k\,.$$

Proceeding in this way we find the missing equations of motion of $\mathcal{P}$, $\mathcal{Q}$:

$$\partial_1\mathcal{P} = k\mathcal{A} - pa\mathcal{V}, \quad \partial_2\mathcal{P} = a\mathcal{A} - \frac{\partial_2 p}{p}\mathcal{P} + qb\mathcal{V} - q\mathcal{Q},$$

$$\partial_1\mathcal{Q} = pa\mathcal{U} - p\mathcal{P} + b\mathcal{B} - \frac{\partial_1 q}{q}\mathcal{Q}, \quad \partial_2\mathcal{Q} = -qb\mathcal{U} + l\mathcal{B}.$$

Equations of motion of the Lie sphere frame can be conveniently represented in the matrix form

$$\partial_1\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & p & 0 & 0\\ k & 0 & 0 & 0 & 0 & 0\\ 0 & k & 0 & -pa & 0 & 0\\ 0 & 0 & 0 & \partial_1 q/q & 1 & 0\\ 0 & 0 & 0 & b & 0 & 1\\ pa & 0 & -p & 0 & b & -\partial_1 q/q\end{pmatrix}\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix},$$

(27)

$$\partial_2\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix} = \begin{pmatrix} \partial_2 p/p & 1 & 0 & 0 & 0 & 0\\ a & 0 & 1 & 0 & 0 & 0\\ 0 & a & -\partial_2 p/p & qb & 0 & -q\\ q & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & l & 0 & 0\\ -qb & 0 & 0 & 0 & l & 0\end{pmatrix}\begin{pmatrix}\mathcal{U}\\ \mathcal{A}\\ \mathcal{P}\\ \mathcal{V}\\ \mathcal{B}\\ \mathcal{Q}\end{pmatrix}.$$

The compatibility conditions of (27) produce the equations

$$\begin{gathered}\partial_1\partial_2 \ln p = pq - k, \quad \partial_1\partial_2 \ln q = pq - l,\\ \partial_1 a = \partial_2 k + \frac{\partial_2 p}{p}k, \quad \partial_2 b = \partial_1 l + \frac{\partial_1 q}{q}l,\\ p\partial_2 a + 2a\partial_2 p + q\partial_1 b + 2b\partial_1 q = 0,\end{gathered} \tag{28}$$

which can be viewed as the Gauss-Codazzi equations in Lie sphere geometry. Another (equivalent) form of the equations (28) can be obtained by introducing V and W by the formulae

$$V = b + \partial_1^2 \ln q + \frac{1}{2}(\partial_1 \ln q)^2, \quad W = a + \partial_2^2 \ln p + \frac{1}{2}(\partial_2 \ln p)^2,$$

which, upon the substitution in (28), imply

$$\begin{gathered}\partial_2^3 p - 2W\partial_2 p - p\partial_2 W + \partial_1^3 q - 2V\partial_1 q - q\partial_1 V = 0,\\ \partial_1 W = 2q\partial_2 p + p\partial_2 q, \quad \partial_2 V = 2p\partial_1 q + q\partial_1 p.\end{gathered} \tag{29}$$

As follows from (20),

diagonally cyclidic surfaces correspond to the choice $p = -q$, which, upon the substitution in (29), results in the stationary mVN equation

$$\begin{gathered}\partial_1^3 p - 2V\partial_1 p - p\partial_1 V = \partial_2^3 p - 2W\partial_2 p - p\partial_2 W,\\ \partial_1 W = -(3/2)\partial_2(p^2), \quad \partial_2 V = -(3/2)\partial_1(p^2)\end{gathered}$$

(we recall that the modified Veselov-Novikov equation has been introduced in [5]). It can be derived from [2, §94], that

Lie minimal surfaces are characterized by the equations

$$p\partial_2 a + 2a\partial_2 p = 0\,, \quad q\partial_1 b + 2b\partial_1 q = 0\,,$$

(in view of $(28)_5$ any one of these equations implies another). In this case one can introduce parameters λ, μ in the frame equations (27) without violating their consistency:

$$\partial_1 \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \mu p & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -\mu pa & 0 & 0 \\ 0 & 0 & 0 & \partial_1 q/q & 1 & 0 \\ 0 & 0 & 0 & b & 0 & 1 \\ \lambda pa & 0 & -\lambda p & 0 & b & -\partial_1 q/q \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}$$

(30)

$$\partial_2 \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix} = \begin{pmatrix} \partial_2 p/p & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 \\ 0 & a & -\partial_2 p/p & qb/\lambda & 0 & -q/\lambda \\ q/\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ -qb/\mu & 0 & 0 & 0 & l & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}$$

This implies, that the Gauss-Codazzi equations of Lie-minimal surfaces constitute an integrable system (in view of the obvious symmetry $p \to cp, q \to (1/c)q$ of the equations governing Lie-minimal surfaces, only one spectral parameter is really essential).

Setting

$$a = \frac{\varphi(R^1)}{p^2}\,, \quad b = \frac{\psi(R^2)}{q^2}\,,$$

we have three cases to distinguish:

Case I (General case). Both $\varphi(R^1)$ and $\psi(R^2)$ are nonzero. It corresponds to the case when all four sheets of the envelope of the family of Lie cyclids are distinct (see Section 5). Then, we can always normalize $\varphi(R^1)$, $\psi(R^2)$ to ± 1 by means of the transformations (19). Let us assume, for instance, that $\varphi(R^1) = \psi(R^2) = 1$. With this normalization the equations (28) assume the form

(31)
$$\partial_1\partial_2 \ln p = pq - k\,, \quad \partial_1\partial_2 \ln q = pq - l\,,$$
$$\partial_2(pk) + 2\frac{\partial_1 p}{p^2} = 0\,, \quad \partial_1(ql) + 2\frac{\partial_2 q}{q^2} = 0\,.$$

Case II (Degenerate case). Here $\varphi = 0$, and hence $a = 0$, while ψ is nonzero and may be normalized to ± 1. Assuming $\psi = 1$ (this corresponds to the case where four sheets of the envelope coincide in pairs) and inserting this ansatz in (28) we obtain

$$k = \frac{s(R^1)}{p}\,.$$

Hence, if $s(R^1)$ is nonzero, it may be reduced to -1 by means of (19) so that the resulting equations take the form

$$\partial_1\partial_2 \ln p = pq + \frac{1}{p}, \quad \partial_1\partial_2 \ln q = pq - l, \quad \partial_1(ql) + 2\frac{\partial_2 q}{q^2} = 0. \tag{32}$$

Case III. In this case, both φ and ψ are zero and hence $a = b = 0$, so that

$$k = \frac{s(R^1)}{p}, \quad l = \frac{t(R^2)}{q}$$

(geometrically, all four sheets of the envelope coincide). Once again, the analysis falls into three subcases depending on whether s, t are zero or not. In the generic situation $s \neq 0$, $t \neq 0$, both s and t may be normalized to -1 and the resulting equations assume the form

$$\partial_1\partial_2 \ln p = pq + \frac{1}{p}, \quad \partial_1\partial_2 \ln q = pq + \frac{1}{q}. \tag{33}$$

The same system has been presented in [28] as a reduction of the two-dimensional Toda lattice.

REMARK. The specialization $p = q$ reduces (33) to the Tzitzeica equation

$$\partial_1\partial_2 \ln p = p^2 + \frac{1}{p},$$

which governs affine spheres in affine differential geometry [40]. It is quite surprising that the same equation has a precise geometric meaning in Lie sphere geometry as well: according to our discussion, it describes surfaces which are simultaneously diagonally cyclidic and Lie-minimal.

6. Invariants of multidimensional hypersurfaces. In this section we announce several results on Lie sphere geometry on multidimensional hypersurfaces, postponing the detailed proofs to a separate publication.

Let M^n be a hypersurface with principal curvatures k^i and principal covectors ω^i, so that the i-th principal direction of M^n is defined by the equations $\omega^j = 0$, $j \neq i$. It should be pointed out that a generic hypersurface of dimension ≥ 3 does not possess parametrization R^i by the lines of curvature. This is in contrast with the 2-dimensional case where such parametrization is always possible. Differentiating covectors ω^i and the principal curvatures k^i we arrive at the structure equations

$$d\omega^i = c^i_{jk}\omega^j \wedge \omega^k, \quad dk^i = k^i_j\omega^j. \tag{34}$$

Let also

$$ds^2 = \sum_i g_{ii}(\omega^i)^2$$

be the first fundamental form of a hypersurface M^n.

THEOREM 2. *A generic hypersurface M^n is defined by the quadratic form*

$$\sum_{i\neq j} \frac{k_i^i k_j^j}{(k^i - k^j)^2} \omega^i \omega^j$$

and the conformal class of the cubic form

$$\sum_i k_i^i g_{ii} (\omega^i)^3$$

uniquely up to Lie sphere equivalence.

By being generic it is sufficient to understand a hypersurface with $k_i^i \neq 0$. This genericity assumption is essential since, for instance, there do exist examples of Dupin hypersurfaces (that is, hypersurfaces with $k_i^i = 0$) which are not Lie-equivalent. Theorem 2 is an analog of the corresponding theorem in projective differential geometry stating that a hypersurface of the projective space P^n of dimension $n \geq 4$ is uniquely determined by the conformal classes of its second fundamental form and the Darboux cubic form (see [1] for the exact statements and further references).

REMARK 1. In the case $k_i^i \neq 0$, the cubic form (5) encodes all the information about the lines of curvature of a hypersurface M^n. Indeed, the principal directions are uniquely defined as those directions where the cubic form (5) reduces to the sum of pure cubes (without mixed terms). Moreover, principal directions are the zero directions of quadratic form (4). However, this last condition does not fix them uniquely as in the 2-dimensional situation.

REMARK 2. For $n = 3$ the invariant quadratic form (4) gives rise to the Lie-invariant functional

$$\iiint \frac{k_1^1 k_2^2 k_3^3}{(k^1 - k^2)(k^1 - k^3)(k^2 - k^3)} \omega^1 \omega^2 \omega^3 , \tag{35}$$

the extremals of which should be called minimal hypersurfaces in Lie sphere geometry. It does not look like that this functional has been investigated so far.

REMARK 3. In principle, for $n \geq 3$ there exist additional Lie sphere invariants besides those mentioned in Theorem 2, namely:

1. The cross-ratios

$$\frac{(k^i - k^j)(k^n - k^l)}{(k^n - k^j)(k^i - k^l)} \tag{36}$$

of any four principal curvatures.

2. The differentials

$$\frac{k_i^i (k^j - k^l)}{(k^i - k^j)(k^i - k^l)} \omega^i , \quad i \neq j \neq l . \tag{37}$$

For instance, in the case $n = 3$ we have three invariant differentials

$$\Omega^1 = \frac{k_1^1(k^2-k^3)}{(k^1-k^2)(k^1-k^3)}\omega^1, \quad \Omega^2 = \frac{k_2^2(k^3-k^1)}{(k^2-k^1)(k^2-k^3)}\omega^2,$$

$$\Omega^3 = \frac{k_3^3(k^1-k^2)}{(k^3-k^1)(k^3-k^2)}\omega^3,$$

giving rise to the invariant quadratic form

$$(\Omega^1)^2 + (\Omega^2)^2 + (\Omega^3)^2$$

whose volume functional

$$\iiint \Omega^1\Omega^2\Omega^3$$

coincides with (35).

3. Conformal class of the quadratic form

$$g_{11}\left(\prod_{l\neq 1}(k^1-k^l)\right)^{2/(n-2)}(\omega^1)^2 + \cdots + g_{nn}\left(\prod_{l\neq n}(k^n-k^l)\right)^{2/(n-2)}(\omega^n)^2. \tag{38}$$

Lie-invariant class of hypersurfaces with conformally flat quadratic form (38) deserves a special investigation. Note that (38) is an object from the second differential neighbourhood of the surface M^n.

4. Differential $d\Omega$ of the 1-form

$$\Omega = \left(\sum_{l\neq 1}\frac{k_1^l - k_1^1/(n-1)}{k^1-k^l}\right)\omega^1 + \cdots + \left(\sum_{l\neq n}\frac{k_n^l - k_n^n/(n-1)}{k^n-k^l}\right)\omega^n. \tag{39}$$

In the generic case $k_i^i \neq 0$, the objects (36), (37), (38) and (39) can be expressed through the forms (4) and (5). However, they are important in the nongeneric situations, when some (or all) of k_i^i vanish so that (4) and (5) become identically zero. In particular, the cross-ratios of principal curvatures play an essential role in the study of Dupin hypersurfaces [8], [29]. In this respect it seems interesting to understand the role of the conformal class (38) and the 2-form $d\Omega$ in the modern Lie-geometric approach to Dupin hypersurfaces.

For hypersurfaces with nonholonomic net of curvature lines Theorem 2 leads to a nice geometric corollary, which we will discuss in the simplest nontrivial 3-dimensional case. Let us consider the structure equations (34) of a 3-dimensional hypersurface M^3. There are essentially two possibilities to distinguish:

1. Holonomic case: all three coefficients $c_{23}^1, c_{31}^2, c_{12}^3$ are zero. Such hypersurfaces possess parametrization by the coordinates of curvature lines.

2. Nonholonomic case: all three coefficients $c_{23}^1, c_{31}^2, c_{12}^3$ are nonzero.

It immediately follows from the Gauss-Codazzi equations, that for $n = 3$ intermediate cases are forbidden. In the nonholonomic case we can normalize covectors $\omega^1, \omega^2, \omega^3$ in such a way that the structure equations assume the form

$$d\omega^1 = a\omega^1\wedge\omega^2 + b\omega^1\wedge\omega^3 + \omega^2\wedge\omega^3,$$

$$d\omega^2 = p\omega^2 \wedge \omega^1 + q\omega^2 \wedge \omega^3 + \omega^3 \wedge \omega^1 , \tag{40}$$
$$d\omega^3 = r\omega^3 \wedge \omega^1 + s\omega^3 \wedge \omega^2 + \omega^1 \wedge \omega^2 .$$

This normalization fixes ω^i uniquely. As follows from the results of [15], in the 3-dimensional nonholonomic situation the Gauss-Codazzi equations completely determine the quadratic form (4) and the cubic form (5) through the coefficients a, b, p, q, r, s in the structure equations (40). Hence we can formulate the following result.

THEOREM 3. *A nonholonomic* 3*-dimensional hypersurface* M^3 *is defined by its structure equations* (40) *uniquely up to Lie sphere equivalence.*

We can reformulate this result as follows: two 3-dimensional nonholonomic hypersurfaces are Lie-equivalent if and only if there exists a point correspondence between them, mapping the lines of curvature of one of them onto the lines of curvature of the other. Hence 3-dimensional nonholonomic hypersurface is uniquely determined by geometry of its curvature lines.

This theorem should remain valid for multidimensional hypersurfaces if we generalize the notion of being nonholonomic in a proper way (e.g., $c^i_{jk} \neq 0$ for all $i \neq j \neq k$, which probably can be weakend).

7. Systems of hydrodynamic type. Reciprocal transformations and reciprocal invariants. In this section we consider 2-component systems of hydrodynamic type

$$u^i_t = v^i_j(u)u^j_x , \quad i, j = 1, 2 , \tag{41}$$

which naturally arise in polytropic gas dynamics, chromatography, plasticity, etc. and describe wide variety of models of continuous media. The main advantage of the 2-component case is the existence of the so-called Riemann invariants: coordinates, where the equations (41) assume the diagonal form

$$R^1_t = \lambda^1(R)R^1_x , \quad R^2_t = \lambda^2(R)R^2_x , \tag{42}$$

considerably simplifying their investigation. And system (42) possesses infinitely many conservation laws

$$h(R)dx + g(R)dt \tag{43}$$

with the densities $h(R)$ and the fluxes $g(R)$ governed by the equations

$$\partial_i g = \lambda^i \partial_i h , \quad i = 1, 2 , \tag{44}$$

$\partial_i = \partial/\partial R^i$, which are completely equivalent to the condition $h_t = g_x$, manifesting the closedness of the 1-form (43). Cross-differentiaiton of (44) results in the second-order equation

$$\partial_1\partial_2 h = \frac{\partial_2\lambda^1}{\lambda^2 - \lambda^1}\partial_1 h + \frac{\partial_1\lambda^2}{\lambda^1 - \lambda^2}\partial_2 h \tag{45}$$

for the conserved densities of system (42). Thus, conservation laws of the system (42) depend on two arbitrary functions of one variable. Let us choose two particular conservation laws

$B(R)dx + A(R)dt$, $N(R)dx + M(R)dt$ and introduce new independent variables X, T by the formulae

$$dX = Bdx + Adt\,, \quad dT = Ndx + Mdt\,, \tag{46}$$

which are correct since the right hand sides are closed. Changing from x, t to X, T in (42), we arrive at the transformed system

$$R^1_T = \Lambda^1(R)R^1_X\,, \quad R^2_T = \Lambda^2(R)R^2_X\,, \tag{47}$$

where the new characteristic velocities Λ^i are given by the formulae

$$\Lambda^i = \frac{\lambda^i B - A}{M - \lambda^i N}\,, \quad i = 1, 2\,. \tag{48}$$

REMARK. In principle one can apply the transformation (46) directly to the system (41) without rewriting it in Riemannian invariants. In this case the transformed equations assume the form

$$u^i_T = V^i_j(u)u^j_X$$

with the new matrix V given by

$$V = (Bv - AE)(ME - Nv)^{-1}\,, \quad E = \mathrm{id}.$$

Transformations of the type (46) are known as reciprocal and have been extensively investigated in [32], [33] (see also [36] and [10]–[12] for further discussion). Following [10], [11] we introduce the reciprocal invariants:

the symmetric 2-form

$$\frac{\partial_1\lambda^1\partial_2\lambda^2}{(\lambda^1 - \lambda^2)^2}dR^1dR^2 \tag{49}$$

and the differential

$$d\Omega \tag{50}$$

of the 1-form

$$\Omega = \left(\frac{\partial_1\partial_2\lambda^2}{\partial_2\lambda^2} + \frac{\partial_1\lambda^1}{\lambda^1 - \lambda^2}\right)dR^1 + \left(\frac{\partial_1\partial_2\lambda^1}{\partial_1\lambda^1} + \frac{\partial_2\lambda^2}{\lambda^2 - \lambda^1}\right)dR^2 \tag{51}$$

(Ω itself is not reciprocally invariant). Note that both objects (49) and (50) do not change under the reparametrization of Riemannian invariants $R^1 \to \varphi^1(R^1)$, $R^2 \to \varphi^2(R^2)$.

REMARK. In order to check the invariance of (49) and (50) under arbitrary reciprocal transformations it is sufficient to check their invariance under the following elementary ones:

$$dX = Bdx + Adt\,, \quad dT = dt\,,$$

which change only x and preserve t (under this transformation λ^i goes to $\Lambda^i = \lambda^i B - A$) and

$$dX = dt\,, \quad dT = dx\,,$$

which transforms λ^i into $\Lambda^i = 1/\lambda^i$. The invariance of (49) and (50) under these elementary transformations can be checked by direct calculation. Since any reciprocal transformation is a composite of elementary ones, we arrive at the required invariance.

It is quite remarkable that the invariants (49) and (50) form a complete set in the following sense: if the invariants of one system can be mapped onto the invariants of the other one by an appropriate change of coordinates R^i, then both systems are reciprocally related and the corresponding reciprocal transformation (46) can be constructed effectively (see [10], [11] for the discussion).

8. Hamiltonian systems. The system (41) is called Hamiltonian if it can be represented in the form

$$u^i_t = \varepsilon^i \delta^{ij} \frac{d}{dx}\left(\frac{\delta H}{\delta u^j}\right), \quad \varepsilon^i = \pm 1,$$

with the Hamiltonian operator $\varepsilon^i \delta^{ij} d/dx$ and the Hamiltonian $H = \int h(u)dx$. In this case the matrix v^i_j is just the Hessian of the density h (for definiteness, we choose $\varepsilon^i = 1$), so that the system (41) assumes the form

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x . \tag{52}$$

Here h_{ij} means $\partial^2 h/\partial u^i \partial u^j$. For the systems (42) in Riemannian invariants a necessary and sufficient condition for the existence of the Hamiltonian representation (52) is given by the following:

LEMMA [39]. *The system* (42) *is Hamiltonian if and only if there exists a flat diagonal metric* $ds^2 = g_{11}(dR^1)^2 + g_{22}(dR^2)^2$ *such that*

$$\partial_2 \ln \sqrt{g_{11}} = \frac{\partial_2 \lambda^1}{\lambda^2 - \lambda^1}, \quad \partial_1 \ln \sqrt{g_{22}} = \frac{\partial_1 \lambda^2}{\lambda^1 - \lambda^2}. \tag{53}$$

Introducing the Lame coefficients $H_1 = \sqrt{g_{11}}$, $H_2 = \sqrt{g_{22}}$ and the rotation coefficients β_{12}, β_{21} by the formulae

$$\partial_1 H_2 = \beta_{12} H_1, \quad \partial_2 H_1 = \beta_{21} H_2, \tag{54}$$

we can rewrite the flatness condition of the metric ds^2 in a simple form

$$\partial_1 \beta_{12} + \partial_2 \beta_{21} = 0. \tag{55}$$

Coordinates u^1, u^2 in (52) are just flat coordinates of the metric ds^2, where it assumes the standard Euclidean form $(du^1)^2 + (du^2)^2$.

REMARK. One can easily check that the quantities $\tilde{H}_1 = \lambda^1 H_1$, $\tilde{H}_2 = \lambda^2 H_2$ also satisfy (54). Thus, the characteristic velocities of any Hamiltonian system are the ratios of two different solutions of the Dirac equation (54).

Under reciprocal transformations (46) the metric coefficients g_{ii} transform according to the formulae

$$g_{ii} \to g_{ii}\frac{(M-\lambda^i N)^2}{(BM-AN)^2}, \quad i=1,2 \tag{56}$$

(see [36], [13]), so that the transformed metric coefficients and the transformed characteristic velocities Λ^i satisfy the same equations (53). It is important to emphasize that reciprocal transformations do not preserve in general the flatness condition of the metric ds^2 and hence destroy the Hamiltonian structure. However, for any Hamiltonian system there always exist sufficiently many canonical reciprocal transformations preserving the flatness condition [12], [13].

Let us introduce the cubic form

$$\partial_1\lambda^1 g_{11}(dR^1)^3 + \partial_2\lambda^2 g_{22}(dR^2)^3 . \tag{57}$$

Using the formulae (48) and (56) one can immediately check that this cubic form is conformally invariant under reciprocal trnasformations: it acquires the multiple $1/(BM-AN)$, so that the zero curves of (57) are reciprocally invariant. Hence, with any Hamiltonian system we can associate, besides the invariants (49) and (50), the reciprocally invariant 3-web of curves formed by coordinate lines $R^1=$ const, $R^2=$ const and the zero curves of the cubic form (57) which are defined by the equation

$$(\partial_1\lambda^1 g_{11})^{1/3}dR^1 + (\partial_2\lambda^2 g_{22})^{1/3}dR^2 = 0 .$$

A calculation similar to that in Section 4 shows that the invariant (50) is just the curvature form of this 3-web.

REMARK. It will be interesting to obtain explicit formulae for the reciprocal invariants (49), (50) and (57) in the flat coordinates u^i in terms of the Hamiltonian density h.

As we already know, the objects similar to (49), (50) and (57) arise in Lie sphere geometry. To clarify this point we recall the construction of [14], [12] relating Hamiltonian systems and surfaces in the Euclidean space. For definiteness, we restrict ourselves to the 2-component case.

9. Hamiltonian systems and surfaces in E^3. Let us consider a 2-component Hamiltonian system (52)

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x$$

and apply the reciprocal transformation

$$dX = Bdx + Adt, \quad dT = dt,$$

where

$$B = \frac{(u^1)^2 + (u^2)^2 + 1}{2}, \quad A = h_1 u^1 + h_2 u^2 - h$$

(this is indeed an integral of (52)). The transformed system assumes the form

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_T = \begin{pmatrix} h_{11}B - A & h_{12}B \\ h_{12}B & h_{22}B - A \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_X . \tag{58}$$

To reveal geometric meaning of system (58) we introduce a surface M^2 in the Euclidean space $E^3(x^1, x^2, x^3)$ with the radius-vector

$$\mathbf{r} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} h_1 - u^1 A/B \\ h_2 - u^2 A/B \\ -A/B \end{pmatrix} . \tag{59}$$

As one can verify by straightforward calculation, the unit normal of the surface M^2 is given by

$$\mathbf{n} = \begin{pmatrix} u^1/B \\ u^2/B \\ 1/B - 1 \end{pmatrix} .$$

Let us define the matrix w^i_j by the formula

$$\frac{\partial \mathbf{r}}{\partial u^j} = \sum_{i=1}^{2} w^i_j \frac{\partial \mathbf{n}}{\partial u^i} .$$

Geometrically, w^i_j is just the inverse of the Weingarten operator (shape operator) of the surface M^2. Using the formulae for $\mathbf{r}$ and $\mathbf{n}$ we arrive at the following expression for the matrix w^i_j:

$$\begin{pmatrix} h_{11}B - A & h_{12}B \\ h_{12}B & h_{22}B - A \end{pmatrix} ,$$

which coincides with that in (58). Hence the matrix of the system (58) is just the inverse of the Weingarten operator of the associated surface M^2. The characteristic velocities w^i of the system (58) are related to that of the system (52) by the formula

$$w^i = \lambda^i B - A , \tag{60}$$

and have geometric meaning of the radii of principal curvature of the surface M^2. Moreover, the Riemannian invariants of both systems (52) and (58) coincide and play the role of parameters of the lines of curvature. The equations (58) can be equivalently represented in the conservative from

$$\mathbf{n}_T = \mathbf{r}_X .$$

Some further properties of the correspondense (59) (in the general n-component case) have been discussed in [14], [12], in particular:

— commuting Hamiltonian systems correspond to surfaces with the same spherical image of the lines of curvature;

— multi-Hamiltonian systems correspond to surfaces, possessing nontrivial deformations preserving the Weingarten operator;

— canonical reciprocal transformations, preserving the Hamiltonian structure, correspond to Lie sphere transformations of the associated surfaces;

— the flat metric ds^2 defining Hamiltonian structure (see the lemma in Section 8) corresponds to the third fundamental form of the associated surface.

Since the correspondence between the systems (52) and (58) is reciprocal, the invariants (49), (50) and (57) coincide respectively with the symmetric 2-form

$$\frac{\partial_1 w^1 \partial_2 w^2}{(w^1 - w^2)^2} dR^1 dR^2,$$

the skew-symmetric 2-form $d\Omega$, where

$$\Omega = \left(\frac{\partial_1 \partial_2 w^2}{\partial_2 w^2} + \frac{\partial_1 w^1}{w^1 - w^2} \right) dR^1 + \left(\frac{\partial_1 \partial_2 w^1}{\partial_1 w^1} + \frac{\partial_2 w^2}{w^2 - w^1} \right) dR^2$$

and the conformal class of the cubic form

$$\partial_1 w^1 G_{11} (dR^1)^3 + \partial_2 w^2 G_{22} (dR^2)^3 .$$

Here R^1, R^2 are the parameters of curvature lines, w^1, w^2 are the radii of principal curvature and G_{11}, G_{22} are the components of the third fundamental form of the associated surface M^2. Since these objects do not change their form if we rewrite them in terms of principal curvatures k^i and the components of the metric g_{ii}, they concide with the Lie sphere invariants of the surface M^2. This provides remarkable differential-geometric interpretation of reciprocal invariants of hydrodynamic type systems.

10. Integrable evolutions of surfaces in Lie sphere geometry. Integrable evolutions of surfaces govrned by $(2+1)$-dimensional integrable equations have been introduced in [23]. The most interesting examples include evolution of surfaces in conformal geometry based on the generalized Weierstrass representation and evolution in affine geometry based on the Lelieuvre representation of surfaces in 3-space [23]–[25], [37]–[38]. The main idea is that linear systems used to construct a surface (the 2-dimensional Dirac operator in the case of Weierstrass representation and the Moutard equation in the Lelieuvre case) are viewed as the Lax operators of the intergrable $(2+1)$-dimensional hierarchies so that the corresponding time evolutions act on the induced surfaces. Here we sketch the construction of the third integrable evolution which is relevant to Lie sphere geometry (see also [19]).

As follows from Section 8, any 2-component Hamiltonian system

$$R_t^1 = \lambda^1(R) R_x^1 , \quad R_t^2 = \lambda^2(R) R_x^2$$

can be parametrized by a pair of solutions (H_1, H_2) and $(\tilde{H}_1, \tilde{H}_2)$ of the Dirac equation

$$\partial_1 H_2 = -\partial_2 \varphi H_1 , \quad \partial_2 H_1 = \partial_1 \varphi H_2 , \tag{61}$$

where $\beta_{12} = -\partial_2\varphi$, $\beta_{21} = \partial_1\varphi$ as a consequence of $\partial_1\beta_{12} + \partial_2\beta_{21} = 0$. Namely, one can put

$$\lambda^1 = \frac{\tilde{H}_1}{H_1}, \quad \lambda^2 = \frac{\tilde{H}_2}{H_2} .$$

Let us define the functions u^1, u^2 by the formulae

$$\begin{aligned} du^1 &= \cos\varphi\, H_1 dR^1 + \sin\varphi\, H_2 dR^2\,, \\ du^2 &= -\sin\varphi\, H_1 dR^1 + \cos\varphi\, H_2 dR^2\,, \end{aligned} \tag{62}$$

which are compatible in view of (61). As follows from the identity

$$(du^1)^2 + (du^2)^2 = H_1^2(dR^1)^2 + H_2^2(dR^2)^2\,,$$

the variables u^1, u^2 can be interpreted as the flat coordinates of the metric $H_1^2(dR^1)^2 + H_2^2(dR^2)^2$. Similarly, $\tilde{u}^1$, $\tilde{u}^2$ defined by

$$\begin{aligned} d\tilde{u}^1 &= \cos\varphi\, \tilde{H}_1 dR^1 + \sin\varphi\, \tilde{H}_2 dR^2\,, \\ d\tilde{u}^2 &= -\sin\varphi\, \tilde{H}_1 dR^1 + \cos\varphi\, \tilde{H}_2 dR^2 \end{aligned} \tag{63}$$

are the flat coordinates of the metric $\tilde{H}_1^2(dR^1)^2 + \tilde{H}_2^2(dR^2)^2$. The variables u, $\tilde{u}$ satisfy the identities

$$\partial_i \tilde{u}^1 = \lambda^i \partial_i u^1\,, \quad \partial_i \tilde{u}^2 = \lambda^i \partial_i u^2\,, \quad i = 1, 2$$

and hence define conservative representation of the system (42):

$$u_t^1 = \tilde{u}_x^1\,, \quad u_t^2 = \tilde{u}_x^2\,.$$

Moreover, since $d\tilde{u}^1 \wedge du^1 + d\tilde{u}^2 \wedge du^2 = 0$, we can introduce the function h by the formula

$$dh = \tilde{u}^1 du^1 + \tilde{u}^2 du^2 \tag{64}$$

(thus, $\tilde{u}^1 = h_1$, $\tilde{u}^2 = h_2$) so that the system under study assumes the Hamiltonian form

$$u_t^1 = (h_1)_x\,, \quad u_t^2 = (h_2)_x\,.$$

With this Hamiltonian system we now associate a surface following the construction of Section 9. Thus, any two solutions (H_1, H_2) and $(\tilde{H}_1, \tilde{H}_2)$ of the Dirac equation (61) define a surface in E^3 via the formulae (62), (63), (64) and (59). According to Section 9, the Lie-invariant functional of the corresponding surface coincides with

$$\iint \frac{\partial_1\lambda^1 \partial_2\lambda^2}{(\lambda^1 - \lambda^2)^2} dR^1 \wedge dR^2\,.$$

In view of the identities

$$\begin{aligned} \frac{\partial_1\lambda^1 \partial_2\lambda^2}{(\lambda^1 - \lambda^2)^2} dR^1 \wedge dR^2 &= \frac{\partial_2\lambda^1 \partial_1\lambda^2}{(\lambda^1 - \lambda^2)^2} dR^1 \wedge dR^2 - d\left(\frac{d\lambda^2}{\lambda^1 - \lambda^2}\right) \\ &= -\partial_1(\ln H_2)\partial_2(\ln H_1) dR^1 \wedge dR^2 - d\left(\frac{d\lambda^2}{\lambda^1 - \lambda^2}\right) \\ &= \partial_1\varphi\partial_2\varphi dR^1 \wedge dR^2 - d\left(\frac{d\lambda^2}{\lambda^1 - \lambda^2}\right), \end{aligned}$$

we conclude that for compact surfaces without umbilic points (for instance, immersed tori) the Lie sphere functional coincides with

$$\iint \partial_1\varphi\partial_2\varphi\, dR^1 \wedge dR^2 . \tag{65}$$

The functional (65) can be interpreted as the first nontrivial conserved quantity of the mVN hierarchy, associated with the Dirac operator (61). Indeed, imposing on H_1, H_2 the t-evolution

$$\begin{aligned} \partial_t H_1 &= \partial_2^3 H_1 - 3(\partial_1\partial_2\varphi)\partial_2 H_2 + 3p(\partial_1\varphi)H_2 , \\ \partial_t H_2 &= \partial_2^3 H_2 + 3\partial_2(pH_2) - 3(\partial_2^2\varphi)(\partial_2\varphi)H_2 , \end{aligned} \tag{66}$$

we arrive via the compatibility conditions of (66) with (61) at the following equation for φ:

$$\partial_t\varphi = \partial_2^3\varphi - (\partial_2\varphi)^3 + 3p\partial_2\varphi , \quad \partial_1 p = \partial_2(\partial_1\varphi\partial_2\varphi) . \tag{67}$$

Similarly, the τ evolution

$$\begin{aligned} \partial_\tau H_1 &= \partial_1^3 H_1 + 3\partial_1(qH_1) - 3(\partial_1^2\varphi)(\partial_1\varphi)H_1 , \\ \partial_\tau H_2 &= \partial_1^3 H_2 + 3(\partial_1\partial_2\varphi)\partial_1 H_1 - 3q(\partial_2\varphi)H_1 \end{aligned} \tag{68}$$

produces via the compatibility conditions with (61) the following equation:

$$\partial_\tau\varphi = \partial_1^3\varphi - (\partial_1\varphi)^3 + 3q\partial_1\varphi , \quad \partial_2 q = \partial_1(\partial_1\varphi\partial_2\varphi) . \tag{69}$$

Both these t- and τ-evolutions are compatible. Their linear combination

$$\begin{aligned} \partial_t\varphi = \partial_1^3\varphi + \partial_2^3\varphi - (\partial_1\varphi)^3 - (\partial_2\varphi)^3 + 3q\partial_1\varphi + 3p\partial_2\varphi , \\ \partial_1 p = \partial_2(\partial_1\varphi\partial_2\varphi) , \quad \partial_2 q = \partial_1(\partial_1\varphi\partial_2\varphi) \end{aligned} \tag{70}$$

is known as the $(2+1)$-dimensional potential mKdV equation, or the mVN equation [5]. Evolution of surfaces in E^3 governed by (70) has been discussed also in [35].

All these evolutions preserve the integral

$$\iint \partial_1\varphi\partial_2\varphi\, dR^1 dR^2 ,$$

which is the first conservation law in the mVN hierarchy.

The evolutions (66) and (68) induce geometric evolutions of surfaces, preserving the Lie-invariant functional (3). The stationary points of these evolutions can be shown to coincide with the diagonally cyclidic surfaces. We hope to report the details elsewhere.

11. Reciprocal transformations and reciprocal invariants of n-component systems. Let us consider an n-component system (6) of hydrodynamic type

$$u_t^i = v_j^i(u)u_x^j , \quad i, j = 1, \dots , n$$

with the characteristic velocities λ^i and the corresponding left eigenvectors $\mathbf{l}^i = (l_j^i)$ which satisfy the formulae

$$\sum_k l_k^i v_j^k = \lambda^i l_j^i .$$

Introducing the 1-forms $\omega^i = l^i_j du^j$ (note that $\mathbf{l}^i$ and ω^i are defined up to rescailing $\mathbf{l}^i \to p^i \mathbf{l}^i$, $\omega^i \to p^i \omega^i$), we can rewrite the equations (6) in the equivalent exterior form

$$\omega^i \wedge (dx + \lambda^i dt) = 0\,, \quad i = 1, \dots, n\,. \tag{71}$$

Differentiation of ω^i and λ^i results in the structure equations of system (6):

$$d\omega^i = c^i_{jk}\omega^j \wedge \omega^k\,, \quad d\lambda^i = \lambda^i_j \omega^j\,.$$

Systems in Riemannian invariants are specified by the conditions $c^i_{jk} = 0$ for any triple of indices $i \neq j \neq k$. Indeed, in this case the forms ω^i satisfy the equations $d\omega^i \wedge \omega^i = 0$ for any $i = 1, \dots, n$ and hence can be normalized so as to become just $\omega^i = dR^i$. In the coordinates R^i the equations (71) assume the familiar Riemannian invariant form

$$R^i_t = \lambda^i(R) R^i_x\,, \quad i = 1, \dots, n\,. \tag{72}$$

The exterior representation (71) is a natural analog of representation (72) which is applicable in the nondiagonalizable case as well. We emphasize that for $n \geq 3$ Riemannian invariants do not exist in general.

Applying to (6) the reciprocal transformation

$$dX = Bdx + Adt\,, \quad dT = Ndx + Mdt\,,$$

we arrive at the transformed equations

$$u^i_T = V^i_j(u) u^j_X$$

with the new matrix V given by

$$V = (Bv - AE)(ME - Nv)^{-1}\,, \quad E = \mathrm{id}$$

or, in the exterior form,

$$\omega^i \wedge (dX + \Lambda^i dT) = 0\,,$$

where

$$\Lambda^i = \frac{\lambda^i B - A}{M - \lambda^i N}\,.$$

Thus, the forms ω^i as well as the structure equations do not change, while λ^i transform as in the 2-component case—see formula (48).

REMARK. In the n-component case the equations (43) for the densities and fluxes of conservation laws $hdx + gdt$ assume the form

$$g_i = \lambda^i h_i\,, \quad i = 1, \dots, n\,,$$

where g_i and h_i are defined by the expansions

$$dg = g_i \omega^i\,, \quad dh = h_i \omega^i\,.$$

In [10], [11] we introduced the following reciprocally invariant objects:

1. The symmetric 2-form

$$\sum_{i\neq j} \frac{\lambda^i_i \lambda^j_j}{(\lambda^i - \lambda^j)^2} \omega^i \omega^j \,. \tag{73}$$

2. The skew-symmetric 2-form

$$d\Omega\,, \tag{74}$$

where

$$\Omega = \left(\sum_{k\neq 1} \frac{\lambda^k_1 - \lambda^1_1/(n-1)}{\lambda^1 - \lambda^k} \right) \omega^1 + \cdots + \left(\sum_{k\neq n} \frac{\lambda^k_n - \lambda^n_n/(n-1)}{\lambda^n - \lambda^k} \right) \omega^n \tag{75}$$

(Ω itself is not reciprocally invariant). Note that both objects (73) and (74) do not change if we reparametrize the 1-forms in the structure equations: $\omega^i \to p^i \omega^i$. The objects (73) and (74) are natural analogs of the corresponding invariants (49) and (50) in the 2-component case. However, for $n \geq 3$ the form Ω depends only on the first derivatives of the characteristic velocities λ^i.

In principle, for $n \geq 3$ there exist additional reciprocal invariants, namely, the invariant differentials

$$\frac{\lambda^i_i(\lambda^j - \lambda^l)}{(\lambda^i - \lambda^j)(\lambda^i - \lambda^l)} \omega^i \,, \quad i \neq j \neq l$$

as well as the cross-ratios

$$\frac{(\lambda^i - \lambda^j)(\lambda^k - \lambda^l)}{(\lambda^k - \lambda^j)(\lambda^i - \lambda^l)}$$

of any four characteristic velocities.

However, as follows from [10], [11], the structure equations and the invariants (73), (74) in fact define generic system of hydrodynamic type uniquely up to reciprocal equivalence (by being generic it is sufficient to understand genuinely nonlinear system, that is, a system with $\lambda^i_i \neq 0$ for any i).

12. Appendix. Surfaces in projective differential geometry. Based on [41] (see also [17]), let us briefly recall the standard way of defining surfaces M^2 in the projective space P^3 in terms of solutions of a linear system

$$\begin{aligned} \mathbf{r}_{xx} &= \beta \mathbf{r}_y + \frac{1}{2}(V - \beta_y)\mathbf{r}\,, \\ \mathbf{r}_{yy} &= \gamma \mathbf{r}_x + \frac{1}{2}(W - \gamma_x)\mathbf{r}\,, \end{aligned} \tag{76}$$

where β, γ, V, W are functions of x and y. If we cross-differentiate (76) and assume $\mathbf{r}$, $\mathbf{r}_x$, $\mathbf{r}_y$, $\mathbf{r}_{xy}$ to be independent, we arrive at the compatibility conditions [26, p. 120]

$$\begin{aligned} \beta_{yyy} - 2\beta_y W - \beta W_y &= \gamma_{xxx} - 2\gamma_x V - \gamma V_x\,, \\ W_x = 2\gamma\beta_y + \beta\gamma_y\,, &\quad V_y = 2\beta\gamma_x + \gamma\beta_x\,. \end{aligned} \tag{77}$$

For any fixed β, γ, V, W satisfying (77), the linear system (76) is compatible and possesses a solution $\mathbf{r} = (r^0, r^1, r^2, r^3)$, where $r^i(x, y)$ are linearly independent. They can be regarded as homogeneous coordinatres of a surface in the projective space P^3. For our purposes, one may think of M^2 as a surface in a three-dimensional Euclidean space with position vector $\mathbf{R} = (r^1/r^0, r^2/r^0, r^3/r^0)$. If we choose any other solution $\tilde{\mathbf{r}} = (\tilde{r}^0, \tilde{r}^1, \tilde{r}^2, \tilde{r}^3)$ of the same system (76), then the corresponding surface $\tilde{M}^2$ with position vector $\tilde{\mathbf{R}} = (\tilde{r}^1/\tilde{r}^0, \tilde{r}^2/\tilde{r}^0, \tilde{r}^3/\tilde{r}^0)$ constitutes a projective transform of M^2 so that any fixed solution β, γ, V, W of the equations (77) defines a surface M^2 uniquely up to projective equivalence. Moreover, a simple calculation yields

$$\mathbf{R}_{xx} = \beta \mathbf{R}_y + a\mathbf{R}_x\,,$$
$$\mathbf{R}_{yy} = \gamma \mathbf{R}_x + b\mathbf{R}_y$$

($a = -2r^0_x/r^0, b = -2r^0_y/r^0$), which implies that x, y are asymptotic coordinates of the surface M^2. In what follows, we assume that our surfaces are hyperbolic and the corresponding asymptotic coordinates x, y are real. The elliptic case is dealt with in an analogous manner by regarding x, y as complex conjugates. Since the equations (77) specify a surface uniquely up to projective equivalence, they can be viewed as the 'Gauss-Codazzi' equations in projective geometry.

Even though the coefficients β, γ, V, W define a surface M^2 uniquely up to projective equivalence via (76), it is not entirely correct to regard β, γ, V, W as projective invariants. Indeed, the asymptotic coordinates x, y are only defined up to an arbitrary reparametrization of the form

(78) $$x^* = f(x)\,, \quad y^* = g(y)\,,$$

which induces a scaling of the surface vector according to

(79) $$\mathbf{r}^* = \sqrt{f'(x)g'(y)}\,\mathbf{r}\,.$$

Thus by [6, p. 1], the form of the equations (76) is preserved by the above transformation with the new coefficients $\beta^*, \gamma^*, V^*, W^*$ given by

(80) $$\begin{aligned} \beta^* &= \beta g'/(f')^2\,, & V^*(f')^2 &= V + S(f)\,,\\ \gamma^* &= \gamma f'/(g')^2\,, & W^*(g')^2 &= W + S(g)\,, \end{aligned}$$

where $S(\cdot)$ is the usual Schwarzian derivative, that is,

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

The transformation formulae (80) imply that the symmetric 2-form

$$\beta\gamma\, dx dy$$

and the conformal class of the cubic form

$$\beta dx^3 + \gamma dy^3$$

are absolute projective invariants. They are known as the projective metric and the Darboux cubic form, respectively, and play an important role in projective differential geometry, since, in particular, they define a 'generic' surface uniquely up to projective equivalence.

One can also define projectively invariant differentials

$$\omega^1 = (\gamma\beta^2)^{1/3}dx\,, \quad \omega^2 = (\beta\gamma^2)^{1/3}dy\,,$$

so that $\beta\gamma dxdy = \omega^1\omega^2$. With the help of ω^1, ω^2 one can define projectively invariant differentiation. However, we will not take advantage of it in what follows.

Using (78)–(80), one can easily verify that the four points

$$\begin{gathered}\mathbf{r}\,, \quad \mathbf{r}_1 = \mathbf{r}_x - \frac{1}{2}\frac{\gamma_x}{\gamma}\mathbf{r}\,, \quad \mathbf{r}_2 = \mathbf{r}_y - \frac{1}{2}\frac{\beta_y}{\beta}\mathbf{r}\,, \\ \boldsymbol{\eta} = \mathbf{r}_{xy} - \frac{1}{2}\frac{\gamma_x}{\gamma}\mathbf{r}_y - \frac{1}{2}\frac{\beta_y}{\beta}\mathbf{r}_x + \left(\frac{1}{4}\frac{\beta_y\gamma_x}{\beta\gamma} - \frac{1}{2}\beta\gamma\right)\mathbf{r}\end{gathered} \tag{81}$$

are defined in an invariant way, that is, under the transformation formulae (78)–(89) they acquire a nonzero multiple which does not change them as points in the projective space P^3. These points form the vertices of the so-called Wilczynski moving tetrahedral [6]. Since the lines passing through $\mathbf{r}$, $\mathbf{r}_1$ and $\mathbf{r}$, $\mathbf{r}_2$ are tangential to the x- and y-asymptotic curves, respectively, the three points $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2$ span the tangent plane of the surface M^2 at $\mathbf{r}$. The line through $\mathbf{r}_1$, $\mathbf{r}_2$ lying in the tangent plane is known as the directrix of Wilczynski of the second kind. The line through $\mathbf{r}$, $\boldsymbol{\eta}$ is transversal to M^2 and is known as the directrix of Wilczynski of the first kind. It plays the role of a projective 'normal'. We stress that in projective differential geometry there exists no unique choice of an invariant normal. This is in contrast with Euclidean and affine geometries in which the normal is canonically defined. Some of the best-known and most-investigated normals are those of Wilczynski, Fubini, Green, Darboux, Bompiani and Sullivan [6, p. 35] with the directrix of Wilczynski being the most commonly used. It is known that the directrix of Wilczynski intersects the tangent Lie quadric of the surface M^2 at exactly two points $\mathbf{r}$ and $\boldsymbol{\eta}$ so that both points lie on the Lie quadric and are canonically defined. The Wilczynski tetrahedral proves to be the most convenient tool in projective differential geometry.

Using (76) and (81), we easily derive for $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\boldsymbol{\eta}$ the linear equations [20, p. 42]

$$\begin{gathered}\begin{pmatrix}\mathbf{r}\\ \mathbf{r}_1\\ \mathbf{r}_2\\ \boldsymbol{\eta}\end{pmatrix}_x = \begin{pmatrix}\gamma_x/2\gamma & 1 & 0 & 0\\ b/2 & -\gamma_x/2\gamma & \beta & 0\\ k/2 & 0 & \gamma_x/2\gamma & 1\\ \beta a/2 & k/2 & b/2 & -\gamma_x/2\gamma\end{pmatrix}\begin{pmatrix}\mathbf{r}\\ \mathbf{r}_1\\ \mathbf{r}_2\\ \boldsymbol{\eta}\end{pmatrix}, \\ \begin{pmatrix}\mathbf{r}\\ \mathbf{r}_1\\ \mathbf{r}_2\\ \boldsymbol{\eta}\end{pmatrix}_y = \begin{pmatrix}\beta_y/2\beta & 0 & 1 & 0\\ l/2 & \beta_y/2\beta & 0 & 1\\ a/2 & \gamma & -\beta_y/2\beta & 0\\ \gamma b/2 & a/2 & l/2 & -\beta_y/2\beta\end{pmatrix}\begin{pmatrix}\mathbf{r}\\ \mathbf{r}_1\\ \mathbf{r}_2\\ \boldsymbol{\eta}\end{pmatrix},\end{gathered} \tag{82}$$

where we introduced the notation

$$k = \beta\gamma - (\ln\beta)_{xy}, \quad l = \beta\gamma - (\ln\gamma)_{xy},$$
$$a = W - (\ln\beta)_{yy} - \frac{1}{2}(\ln\beta)_y^2, \quad b = V - (\ln\gamma)_{xx} - \frac{1}{2}(\ln\gamma)_x^2. \tag{83}$$

Under the transformations (78)–(80) these quantities transform as follows

$$k^* = k/f'g', \quad l^* = l/f'g', \quad a^* = a/(g')^2, \quad b^* = b/(f')^2, \tag{84}$$

and give rise to the projectively invariant quadratic form

$$ady^2 + bdx^2$$

and the quartic form

$$a\beta^2dx^4 + b\gamma^2dy^4.$$

The compatibility conditions of the equations (82) imply

$$(\ln\beta)_{xy} = \beta\gamma - k, \quad (\ln\gamma)_{xy} = \beta\gamma - l,$$
$$a_x = k_y + \frac{\beta_y}{\beta}k, \quad b_y = l_x + \frac{\gamma_x}{\gamma}l, \tag{85}$$
$$\beta a_y + 2a\beta_y = \gamma b_x + 2b\gamma_x,$$

which is just the equivalent form of the projective Gauss-Codazzi equations (77).

Note that the Gauss-Codazzi equations (28) and (85) (as well as (29) and (77)) are related by a complex change of variables

$$p = i\beta, \quad q = -i\gamma,$$

which is just the analytic manifestation of Lie's famous line-sphere correspondence.

Different types of surfaces can be defined by imposing additional constraints on β, γ, V, W (respectively, β, γ, k, l, a, b) so that, in a sense, projective differential geometry is the theory of (integrable) reductions of the underdetermined system (77) (respectively, (85)).

EXAMPLE 1. *Isothermally asymptotic surfaces* are specified by the condition $\beta = \gamma$, in which case the equations (77) assume the form of the stationary modified Veselov-Novikov (mVN) equation

$$\beta_{yyy} - 2\beta_y W - \beta W_y = \beta_{xxx} - 2\beta_x V - \beta V_x,$$
$$W_x = \frac{3}{2}(\beta^2)_y, \quad V_y = \frac{3}{2}(\beta^2)_x.$$

This fact has been pointed out in [16]. Isothermally asymptotic surfaces have a number of important geometric properties, in particular:

— The 3-web, formed by asymptotic curves and Darboux's curves, is hexagonal. (Darboux's curves are the zero curves of the Darboux cubic form $\beta dx^3 + \gamma dy^3$).

— Isothermally asymptotic surfaces arise as the focal surfaces of special W-congruences, preserving Darboux's curves.

Examples of isothermally asymptotic surfaces include arbitrary quadrics and cubics, quartics of Kummer, projective transforms of affine spheres and rotation surfaces. We refer to [26], [6], [20], [21], [18] for further discussion.

EXAMPLE 2. *Projectively minimal surfaces* are the extremals of the projective area functional

$$\iint \beta\gamma dxdy . \tag{86}$$

The Euler-Lagrange equations for the functional (86) adopt the form

$$\beta a_y + 2a\beta_y = 0 , \quad \gamma b_x + 2b\gamma_x = 0 ,$$

and may be obtained by equating to zero both sides of the equation $(85)_5$. It is remarkable that projective Gauss-Codazzi equations for projectively minimal surfaces identically coincide with those governing Lie-minimal surfaces in Lie sphere geometry (see Section 5). In particular, they possess the same Lax representation (30) with the spectral parameters λ, μ. Evaluating (30) at $\lambda = 1$, $\mu = 1$, we recover the Lie sphere frame for Lie-minimal surfaces. Evaluating (30) at $\lambda = -1$, $\mu = 1$, we recover projective Plücker frame for projectively minimal surfaces (construction of the projective Plücker frame is described below: see the formula (91)). Thus, the inverse scattering transform allows simultaneous treatment of both the Lie-minimal and projectively minimal surfaces.

Setting

$$a = \frac{\varphi(x)}{\beta^2} , \quad b = \frac{\psi(y)}{\gamma^2} ,$$

we have three cases to distinguish:

Case I (General case). Both $\varphi(x)$ and $\psi(y)$ are nonzero. In this case, we can always normalize $\varphi(x)$, $\psi(y)$ to ± 1 by means of the transformations (80). Let us assume, for instance, that $\varphi(x) = \psi(y) = 1$. With this normalization the equations (85) assume the form

$$\begin{aligned} &(\ln\beta)_{xy} = \beta\gamma - k , \quad (\ln\gamma)_{xy} = \beta\gamma - l , \\ &(\beta k)_y + 2\frac{\beta_x}{\beta^2} = 0 , \quad (\gamma l)_x + 2\frac{\gamma_y}{\gamma^2} = 0 . \end{aligned} \tag{87}$$

Case II (Surfaces of Godeaux-Rozet [6, p. 318]). In this case, $\varphi = 0$, and hence $a = 0$, while ψ is nonzero and may be normalized to ± 1. Here we assume that $\psi = 1$. On inserting this ansatz in (85), we obtain

$$k = \frac{s(x)}{\beta} .$$

Hence, if $s(x)$ is nonzero, it may be reduced to -1 by means of (80) so that the resulting equations take the form

$$(\ln\beta)_{xy} = \beta\gamma + \frac{1}{\beta} , \quad (\ln\gamma)_{xy} = \beta\gamma - l , \quad (\gamma l)_x + 2\frac{\gamma_y}{\gamma^2} = 0 . \tag{88}$$

Case III (Surfaces of Demoulin). In this case, both φ and ψ are zero and hence $a = b = 0$, so that

$$k = \frac{s(x)}{\beta} , \quad l = \frac{t(y)}{\gamma} .$$

Once again, the analysis falls into three subcases depending on whether s, t are zero or not. In the generic situations $s \neq 0$, $t \neq 0$ both s and t may be normalized to -1 and the resulting equaitons assume the form

$$(\ln \beta)_{xy} = \beta\gamma + \frac{1}{\beta}, \quad (\ln \gamma)_{xy} = \beta\gamma + \frac{1}{\gamma}. \tag{89}$$

In this form, the equations governing Demoulin surfaces have been set down in [20, p. 51]. The same system has been presented in [28] as a reduction of the two-dimensional Toda lattice.

In [9] Demoulin established in a purely geometric manner the existence of Bäcklund transformations for Godeaux-Rozet and Demoulin surfaces and proved the corresponding permutability theorems. Apparently, Demoulin did not formulate his results in terms of analytic expressions. In [17], a Toda lattice connection is used to derive explicitly a Bäcklund transformation for Demoulin surfaces.

REMARK. The specialization $\beta = \gamma$ reduces (89) to the Tzitzeica equation

$$(\ln \beta)_{xy} = \beta^2 + \frac{1}{\beta},$$

which governs affine spheres in affine differential geometry [40]. Geometrically this means that affine spheres lie in the intersection of two different integrable classes of projective surfaces, namely isothermally asymptotic and projectively minimal surfaces.

Projectively minimal, Godeaux-Rozet and Demoulin surfaces also arise in the theory of envelopes of Lie quadrics associated with the surface M^2. For brevity, we only recall the necessary definitions. The details can be found in [6], [20], [26], etc. Thus, let us consider a point p^0 on the surface M^2 and the x-asymptotic line passing through p^0. Let us take three additional points p^i, $i = 1, 2, 3$ on this asymptotic line close to p^0 and draw three y-asymptotic lines γ^i passing through p^i. The three straight lines which are tangential to γ^i and pass through the points p^i uniquely define a quadric $\boldsymbol{Q}$ containing them as rectilinear generators. As p^i tend to p^0, the quadric $\boldsymbol{Q}$ tends to a limiting quadric, the so-called Lie quadric of the surface M^2 at the point p^0. Even though this construction depends on the initial choice of either the x- or the y-asymptotic line through p^0, the resulting quadric $\boldsymbol{Q}$ is independent of that choice. Thus, we arrive at a two-parameter family of quadrics associated with the surface M^2. In terms of the Wilczynski tetrahedral, the parametric equation for $\boldsymbol{Q}$ is of the form [6, p. 311]

$$\boldsymbol{Q} = \boldsymbol{\eta} + \mu \mathbf{r}_1 + \nu \mathbf{r}_2 + \mu\nu \mathbf{r},$$

where μ, ν are parameters.

The case of projectively minimal surfaces is characterized by the additional requirement that the asymptotic lines on all these sheets correspond to the asymptotic lines of the surface M^2 itself. Moreover, for projectively minimal surface all four sheets of the envelope will be projectively minimal as well. In a sense, it is natural to call the family of Lie quadrics with this property a W-congruence of quadrics.

Now, in a neighbourhood of a generic point p^0 on M^2, the envelopes of the family of Lie quadrics consist of the surface M^2 itself and four (in general) distinct sheets. Surfaces of Godeaux-Rozet are characterized by the degenerate case of two distinct sheets, while Demoulin surfaces are present if all four sheets coincide. Surfaces of Godeaux-Rozet and Demoulin have been investigated extensively in [9], [22], [34], see also [17].

The similarity between projective and Lie sphere geometries becomes ever more transparent if we rewrite the equations of motion of the Wilczynski tetrahedral (82) in the Plücker coordinates.

For convenience of the reader we briefly recall this construction. Let us consider a line l in P^3 passing through the points $\mathbf{a}$ and $\mathbf{b}$ with the homogeneous coordinates $\mathbf{a} = (a^0 : a^1 : a^2 : a^3)$ and $\mathbf{b} = (b^0 : b^1 : b^2 : b^3)$. With the line l we associate a point $\mathbf{a} \wedge \mathbf{b}$ in the projective space P^5 with the homogeneous coordinates

$$\mathbf{a} \wedge \mathbf{b} = (p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}),$$

where

$$p_{ij} = \det \begin{pmatrix} a^i & a^j \\ b^i & b^j \end{pmatrix}.$$

The coordinates p_{ij} satisfy the well-known quadratic Plücker relation

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0. \tag{90}$$

Instead of $\mathbf{a}$ and $\mathbf{b}$ we may consider arbitrary linear combinations thereof without changing $\mathbf{a} \wedge \mathbf{b}$ as a point in P^5. Hence, we arrive at the well-defined Plücker correspondence $l(\mathbf{a}, \mathbf{b}) \to \mathbf{a} \wedge \mathbf{b}$ between lines in P^3 and points on the Plücker quadric in P^5. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are points in P^3 and κ is a scalar, the following properties hold:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}, \quad \mathbf{a} \wedge \mathbf{a} = 0, \quad \kappa(\mathbf{a} \wedge \mathbf{b}) = (\kappa\mathbf{a}) \wedge \mathbf{b} = \mathbf{a} \wedge (\kappa\mathbf{b}),$$

$$(\mathbf{a} + \mathbf{c}) \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{b} + \mathbf{c} \wedge \mathbf{b}, \quad (\mathbf{a} \wedge \mathbf{b})' = \mathbf{a}' \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{b}'.$$

The Plücker correspondence plays an important role in the projective differential geometry of surfaces and often sheds some new light on those properties of surfaces which are not 'visible' in P^3 but acquire a precise geometric meaning only in P^5. Thus, let us consider a surface $M^2 \subset P^3$ with the Wilczynski tetrahedral $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\boldsymbol{\eta}$ satisfying the equations (82). Since the two pairs of points $\mathbf{r}$, $\mathbf{r}_1$ and $\mathbf{r}$, $\mathbf{r}_2$ generate two lines in P^3 which are tangential to the x- and y-asymptotic curves, respectively, the formulae

$$\mathcal{U} = \mathbf{r} \wedge \mathbf{r}_1, \quad \mathcal{V} = \mathbf{r} \wedge \mathbf{r}_2$$

define the images of these lines under the Plücker embedding. Hence, with any surface $M^2 \subset P^3$ there are canonically associated two surfaces $\mathcal{U}(x, y)$ and $\mathcal{V}(x, y)$ in P^5 lying on the Plücker quadric (90). In view of the formulae

$$\mathcal{U}_x = \beta\mathcal{V}, \quad \mathcal{V}_y = \gamma\mathcal{U},$$

we conclude that the line in P^5 passing through a pair of points $(\mathcal{U}, \mathcal{V})$ can also be generated by the pair of points $(\mathcal{U}, \mathcal{U}_x)$ (and hence is tangential to the x-coordinate line on the surface $\mathcal{U}$) or by a pair of points $(\mathcal{V}, \mathcal{V}_y)$ (and hence is tangential to the y-coordinate line on the surface

$\mathcal{V}$). Consequently, the surfaces $\mathcal{U}$ and $\mathcal{V}$ are two focal surfaces of the congruence of straight lines $(\mathcal{U}, \mathcal{V})$ or, equivalently, $\mathcal{V}$ is the Laplace transform of $\mathcal{U}$ with respect to x and $\mathcal{U}$ is the Laplace transform of $\mathcal{V}$ with respect to y. We emphasize that the x- and y-coordinate lines on the surfaces $\mathcal{U}$ and $\mathcal{V}$ are not asymptotic but conjugate. Continuation of the Laplace sequence in both directions, that is, taking the x-transform of $\mathcal{V}$, the y-transform of $\mathcal{U}$, etc., leads, in the generic case, to an infinite Laplace sequence in P^5 known as the Godeaux sequence of a surface M^2 [6, p. 344]. The surfaces of the Godeaux sequence carry important geometric information about the surface M^2 itself.

The case of a closed, that is, periodic Godeaux sequence is particularly interesting. It turns out that the only surfaces $M^2 \subset P^3$ for which the Godeaux sequence is of period 6 (the value 6 turns out to be the least possible) are the surfaces of Demoulin [6, p. 360]. This result may be regarded as an equivalent geometric description of Demoulin surfaces.

Introducing

$$\mathcal{A} = \mathbf{r}_2 \wedge \mathbf{r}_1 + \mathbf{r} \wedge \boldsymbol{\eta}, \quad \mathcal{B} = \mathbf{r}_1 \wedge \mathbf{r}_2 + \mathbf{r} \wedge \boldsymbol{\eta},$$
$$\mathcal{P} = 2\mathbf{r}_2 \wedge \boldsymbol{\eta}, \quad \mathcal{Q} = 2\mathbf{r}_1 \wedge \boldsymbol{\eta},$$

we arrive at the following equations for the Plücker coordinates:

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 & \beta & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -\beta a & 0 & 0 \\ 0 & 0 & 0 & \gamma_x/\gamma & 1 & 0 \\ 0 & 0 & 0 & b & 0 & 1 \\ -\beta a & 0 & \beta & 0 & b & -\gamma_x/\gamma \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix},$$

(91)

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_y = \begin{pmatrix} \beta_y/\beta & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 \\ 0 & a & -\beta_y/\beta & -\gamma b & 0 & \gamma \\ \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ -\gamma b & 0 & 0 & 0 & l & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}.$$

The equations (91) are consistent with the following table of scalar products:

$$(\mathcal{U}, \mathcal{P}) = -1, \quad (\mathcal{A}, \mathcal{A}) = 1, \quad (\mathcal{V}, \mathcal{Q}) = 1, \quad (\mathcal{B}, \mathcal{B}) = -1,$$

all other scalar products being equal to zero. We emphasize that up to (essential) changes of signs, the formulae (91) coincide with (27).

13. Concluding remarks. In our discussion of the Lie invariants of surfaces (reciprocal invariants of hydrodynamic type systems) the choice of coordinatres R^i plays a crucial role. In the case of surfaces these are coordinates of the lines of curvature (Riemannian invariants in the case of hydrodynamic type systems). This choice is not accidental, since the lines of curvature are preserved by the Lie sphere group, while Riemannian invariants are preserved under reciprocal transformations. In fact, only in these special coordinates our invariants assume particularly symmetric and simple form. However, from the point of view of applications it is desirable to have a kind of invariant tensor formula, which will allow computation of these objects in an arbitrary coordinate system, for instance, in conformal parametrization in the case of surfaces or in the flat coordinates in the case of Hamiltonian systems.

14. Acknowledgements. I would like to thank A. I. Bobenko, B. G. Konopelchenko and U. Pinkall for their interest and useful remarks. I am especially grateful to K. R. Khusnutdinova for performing a number of important calculations without which the paper would not have appeared in its present form.

This research was partially supportred by INTAS and the Aleksander von Humboldt Foundation.

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