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EXOTIC INVOLUTIONS OF LOW-DIMENSIONAL SPHERES AND THE ETA-INVARIANT

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Abstract. We give a transparent description of the one-fold smooth suspension of Fintushel-Stern's exotic involution on the 4-sphere. Moreover we prove that any two involutions of the 4-sphere are stably (i.e., after one-fold suspension) smoothly conjugated if and only if the corresponding quotient spaces (real homotopy projective spaces) are stably diffeomorphic. We use the Atiyah-Patodi-Singer eta-invariant to detect smooth structures on homotopy projective spaces and prove that any homotopy projective space is detected in this way in dimensions 5 and 6.

1. Introduction. In this paper, following an idea of Gilkey [13], we study exotic homotopy projective spaces in dimensions 4, 5 and 6 by means of the eta-invariant. In particular we prove that the eta-invariant of certain Dirac-type operators completely detects homotopy projective spaces (or, equivalently, smooth fixed point free involutions on homotopy spheres) in dimensions 5 and 6. Moreover we give a very explicit geometric construction of these involutions. This gives more insight in exotic involutions than standard methods of surgery and homotopy topology (compare [20]), which provides information about the set of homotopy projective spaces as a whole but rarely about a given member of the set. In particular, we are able to identify one of our involutions as a smooth suspension of the Fintushel-Stern and the Cappell-Shaneson exotic involutions on S^4. Thus we get a transparent description of the one-fold smooth suspension of both the Fintushel-Stern and the Cappell-Shaneson involutions, and prove that, after forming the one-fold smooth suspension of both of these involutions, we get equivalent involutions of S^5. Moreover those suspended involutions are equivalent to an involution obtained by gluing Z_2-equivariantly $S^2 \times D^3$ and $D^3 \times S^2$ (both equipped with the ordinary "antipodal" Z_2 action $(x, y) \leftrightarrow (-x, -y)$) along their common boundary, with the help of a Z_2-equivariant autodiffeomorphism h_3 of $S^2 \times S^2$. We give a simple and transparent description of the diffeomorphism h_3 (being a composition of three copies of some other diffeomorphism h of $S^2 \times S^2$), and prove that any involution of S^5 can be obtained (up to equivalence) by the same construction with h_3 replaced by the n-th power of h, $n = 0, 1, 2, 3$. As a byproduct we also get a similar (although more complicated) description of all (up to smooth conjugation) smooth involutions of S^6. An explicit form of our involutions enables us to compute the Atiyah-Patodi-Singer eta-invariant of certain Dirac-type operators on corresponding homotopy projective spaces.

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This paper is organised as follows. In Section 2 we formulate main theorems of the paper and give, for readers convenience, some basic material concerning involutions on manifolds and exotic involutions on S^4. This section is concluded by a sketch of the proof of main theorems. In Section 3 we gather some basic facts about Pin^+ and Pin^c operators and their eta-invariant. In Section 4 we study involutions on S^5 and S^6. We give here an explicit and simple description of all (up to smooth conjugation) smooth involutions on these spheres, and prove that all these involutions can be detected by the eta-invariant. This will prove Theorem A of Section 2. We also include here some auxiliary technical lemmas which explain how the eta-invariant is affected by doing surgery on a given manifold. Section 5, being the core of the whole paper, is devoted to a more profound study of Fintushel-Stern's exotic involution. We describe here a "stratified" surgery, which is the key tool for the proof of main theorems, and prepare some auxiliary topological propositions. This surgery provides an alternative method for constructing exotic involutions on S^4 and, possibly, for describing them by a transparent formula (see Remark 2 in Section 5). In the final section we apply the methods described in the preceding Sections, to prove main theorems.

Let us note that some results of this paper have far-reaching generalisations. In particular, any number of the form $\pm(2k+1)/2^{n+1}$ mod Z can be realised as the value of the eta-invariant of the Pin^c operator on certain homotopy projective space of dimension $2n$ for all $n \geq 3$, and analogous results are valid for $Spin^c$ operators and odd-dimensional projective spaces (compare [25]).

2. Main theorems. In this section we formulate our main theorems of the paper. We precede these theorems by some background material concerning smooth free involutions on manifolds and exotic involutions on S^4 of Cappell-Shaneson and Fintushel-Stern.

First let us establish notation which will be used throughout this paper. $A \sqcup B$ is the disjoint union of spaces A and B, and kA is the disjoint union of k copies of A. $A\#B$ is the connected sum of manifolds A and B, and $k_{\#}A$ is the connected sum of k copies of A. Given a manifold M^n, let M_i^n, $i = 0, 1, \ldots$, be copies of M^n and apply a similar convention to other objects (maps, subsets etc.). If $\tilde{M}^n$ is a manifold with a free involution T^n, then $M^n = \tilde{M}^n/T^n$. R^n, D^n and S^{n-1} denote the Euclidean space, the unit disc and the unit sphere, respectively. We denote by *ant* the usual antipodal map on R^n as well as on its subspaces. Thus $RP^n = S^n/ant$ and $RP^k \tilde{\times} D^m = S^k \times D^m/ant$, and $D^m \tilde{\times} RP^k$ has an analogous meaning. $I = [0, 1]$ and $\overline{ant} : S^n \times I \to S^n \times I$ is given by $(x, t) \to (-x, t)$. $S^n \times I$ will be viewed as an invariant collar neighbourhood of S^n in D^{n+1}. $\mathrm{Fix}(T)$ is the set of fixed points of a map $T : X \to X$.

Given an imbedding $\phi : S^k \times D^m \to M^n$, $k + m = n$, we denote by $M^n_{\partial\phi}$ the manifold obtained from M^n by doing surgery on ϕ. If additionally $M^n \subset \partial M^{n+1}$, then M_ϕ^{n+1} is the trace of the surgery on ϕ, i.e., $M^{n+1} \cup_\phi D^{n+1}$, and $M_\phi^{n+1'}$ is the manifold $M^{n+1} \cup_\phi S^n \times I$; thus $M_\phi^{n+1'} \subset M_\phi^{n+1}$ in a natural way, and $\partial M_\phi^{n+1'} = \partial M_\phi^{n+1} \cup S^n$. Given Z_2-manifolds $(\tilde{M}^n, T^n) \subset \partial(\tilde{M}^{n+1}, T^{n+1})$ (i.e., $\tilde{M}^n$ is an invariant submanifold of $\partial(\tilde{M}^{n+1})$ and

$T^{n+1}|\tilde{M}^n = T^n$), and given an equivariant imbedding $\Phi : S^k \times D^m \to \tilde{M}^n$ with the quotient imbedding $\phi : RP^k \tilde{\times} D^m \to M^n$, $\tilde{M}^n_{\partial\phi}$, $\tilde{M}^{n+1}_{\Phi}$ and $\tilde{M}^{n+1'}_{\Phi}$ will have an obvious "equivariant" meaning. Thus $\tilde{M}^n_{\partial\Phi}$ (resp. $\tilde{M}^{n+1'}_{\Phi}$) comes with a naturally defined involution denoted by $T^n_{\partial\Phi}$ (resp. T^{n+1}_{Φ}), which is free if T^{n+1} is free, and we write $M^n_{\partial\phi}$ (resp. M^{n+1}_{ϕ}) to denote the corresponding quotient manifold. Let us also note that $\partial M^{n+1}_{\phi} = (\partial M^{n+1} \backslash M^n) \cup M^n_{\partial\phi} \cup RP^n$, where $M^n_{\partial\phi}$ is obtained from M^n by deleting the interior of $\phi(RP^k \tilde{\times} D^m)$ and then attaching $D^{k+1} \tilde{\times} RP^{m-1}$ with the help of the quotient map $\phi : RP^k \tilde{\times} D^m \to M^n$. Moreover M^{n+1}_{ϕ} is obtained by attaching $RP^n \times I$ to M^{n+1} with the help of the map ϕ.

As this paper deals only with smooth manifolds and fixed point-free smooth involutions, we agree that "manifold" will mean smooth manifold and "involution" will mean fixed point-free smooth involution unless otherwise stated.

Let T^n be an involution on a manifold $\tilde{M}^n$ (possibly with non-vacuous boundary). A T^n-invariant submanifold $\tilde{M}^{n-1} \subset \tilde{M}^n$ is called a characteristic submanifold provided that it cuts $\tilde{M}^n$ into two connected components, say A and A', permuted by T^n; thus $\tilde{M} = A \cup A'$, where $A' = T^n(A)$, and $A \cap A' = \tilde{M}^{n-1}$. Such a characteristic submanifold always exists and one can find a connected characteristic submanifold for $n \geq 3$ ([20]). Let T^n be an involution of a homotopy sphere Σ^n. It is said to desuspend if it admits a characteristic submanifold which is a homotopy sphere. There is a "surgery type" obstruction (Browder-Livesay invariant) $\alpha(T^n, \Sigma^n)$ which, for involutions of spheres of dimension $n > 5$, vanishes if and only if the involution T^n desuspends ([20]). Later in this paper we generalize this theorem to all dimensions ≥ 5. Then the quotient manifold Σ^n/T^n is a homotopy real projective space, and any homotopy projective space FRP^n is of this form. Therefore classifying free involutions on homotopy spheres is equivalent to classifying homotopy projective spaces. Two involutions (T^n_i, Σ^n_i), $i = 1, 2$, are called equivalent if there is an equivariant diffeomorphism $g : \Sigma^n_1 \to \Sigma^n_2$; or, equivalently, if the quotient manifolds FRP^n_i are diffeomorphic.

We will also need the notion of the smooth suspension of smooth free involutions of spheres. Assume Σ^n to be diffeomorphic to the ordinary sphere S^n. Then form a smooth manifold Σ^{n+1} as follows: Fix a diffeomorphism $g : \Sigma^n \to S^n$ and glue two copies of the standard n-disc D^{n+1}, say D^{n+1}_a and D^{n+1}_z, with the help of the involution $gT^ng^{-1} : \partial D^{n+1}_a \to \partial D^{n+1}_z$. Then we define a (free smooth) involution ΣT^n of Σ^{n+1} by the formula $D^{n+1}_a \ni x \mapsto x \in D^{n+1}_z$.

The Z_2 manifold (depending on g) $(\Sigma^{n+1}, \Sigma T^n)$ is called a smooth suspension of the involution (Σ^n, T^n), and it is clear that (Σ^n, T^n) is a characteristic submanifold of $(\Sigma^{n+1}, \Sigma T^n)$. If Σ^{n+1} is diffeomorphic to the ordinary sphere S^{n+1}, we can repeat this procedure and form the double suspension $(\Sigma^{n+2}, \Sigma^2 T^n)$ of the involution T^n. The double smooth suspension depends strongly on the identifications $\Sigma^k \simeq S^k$ used in its construction, and Σ^{n+2} needs not to be the ordinary sphere S^{n+2}. However, in this article, we shall not deal with this problem, since we confine ourselves to involutions of low dimensional spheres. Namely, we have the following simple proposition (this justifies our notation $(\Sigma^{n+1}, \Sigma T^n)$ which does not take care of the diffeomorphism g):

PROPOSITION 2.1. *Let T^n be a smooth involution of a homotopy sphere Σ^n. There exists precisely one (up to smooth conjugation) smooth suspension $(\Sigma^{n+1} \simeq S^{n+1}, \Sigma T^n)$ provided that $n \leq 5$. Moreover there exists precisely one double suspension $(\Sigma^{n+2} \simeq S^{n+2}, \Sigma^2 T^n)$ provided that $n \leq 4$.*

This follows immediately from the well-known fact that any autodiffeomorphism of the sphere S^n extends to an autodiffeomorphism of D^{n+1} for $n \leq 5$ ([17], [9]).

Now let us recall some basic facts concerning exotic involutions on S^4. Let us start with the Fintushel-Stern involution ([10]). In [10] it has been proved that the Brieskorn sphere $\Sigma(3, 5, 19)$ bounds a contractible 4-manifold U whose double is S^4. Moreover the involution t^3 "contained" in the natural S^1 action on $\Sigma(3, 5, 19)$ extends to a smooth fixed point-free involution on $S^4 = U^4 \cup_\Sigma U^4$, which permutes the two copies of U^4. This is the Fintushel-Stern exotic involution T_{FS}, and $S^4/T_{FS} = FRP^4_{FS}$ is the Fintushel-Stern exotic projective space. Thus $(\Sigma(3, 5, 19), t^3)$ is a characteristic submanifold for T_{FS}.

Now let us turn to the Cappell-Shaneson involution on S^4 ([8]). Let us fix a matrix $A \in GL(3, Z)$ (a CS-matrix) subject to the relations $\det A = -1$, $\det(1 - A) = 1$ and $\det(1 - A^2) = \pm 1$, and consider A as an (orientation-reversing) diffeomorphism of the torus $T^3 = S^1 \times S^1 \times S^1 = R^3/Z^3$ which leaves the "origin" $[0] \in T^3$ fixed. Let $M^4_A = T^3 \times I/((x, 0) \simeq (Ax, 1))$. Thus M^4_A is a smooth non-orientable manifold (the mapping torus of A). Note that the normal neighbourhood N_α in M^4_A of the image α of the segment $[0] \times I \subset T^3 \times I$ is diffeomorphic to the normal neighbourhood N of RP^1 in RP^4. Fix a diffeomorphism $N_\alpha \to N$, and let h_{CS} be its restriction to the boundary of N_α. Then form a (smooth, closed) manifold $FRP^4_{CS}(A)$ by gluing $RP^4 \setminus \operatorname{int} N$ and $M^4_A \setminus \operatorname{int} N_\alpha$ with the help of h_{CS}. In [8] it has been proved that $FRP^4_{CS}(A)$ is a 4-dimensional homotopy real projective space which is never diffeomorphic to the ordinary projective space RP^4. It is not known if the universal covering space of $FRP^4_{CS}(A)$ is always diffeomorphic to the ordinary sphere S^4.

Now let us confine ourselves to the case of the matrix

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

A_0 is a CS-matrix and we denote by FRP^4_{CS} the Cappell-Shaneson exotic projective space associated to this matrix. In [14] Gompf proved that the universal covering space of FRP^4_{CS} is diffeomorphic to S^4, and therefore FRP^4_{CS} can be viewed as the quotient manifold of the form S^4/T_{CS}, where T_{CS} is a (smooth fixed point-free) involution on S^4. The author does not know if the Cappell-Shaneson exotic involution T_{CS} is equivalent to the Fintushel-Stern involution T_{FS}. However, we shall prove that the smooth suspensions of these two involutions are equivalent and that both of them are given by a simple formula.

In order to formulate main theorems of this paper we shall need also some knowledge of 5 and 6-dimensional homotopy projective spaces. For any non-zero vector $x \in R^{n+1}$, let $R_x \in O(n + 1)$ be the reflection through the hyperplane perpendicular to x. Let $e_1, \ldots, e_{n+1}$ be the standard orthonormal basis of R^{n+1} and $c_n : S^n \to SO(n + 1)$ be given by $c_n(x) =$

$R_x R_{e_1}$. Then c_n is the clutching map for the tangent bundle to S^{n+1} ([16]); hence c_n is null-homotopic for $n = 2, 6$. Observe that $c_n(-x) = c_n(x)$, and let $\bar{c}_2 : S^3 \to SO(3)$ be an extension of c_2 such that $\bar{c}_2(-x) = \bar{c}_2(x)$.

Now let us define some auxiliary maps which will play an important role in this paper. Let $G : S^2 \times D^3 \to S^2 \times D^3$ be the autodiffeomorphism given by $G(x, y) = (x, c_2(x)y)$. Then $\bar{G} : S^3 \times D^3 \to S^3 \times D^3$ defined by $\bar{G}(x, y) = (x, \bar{c}_2(x)y)$ is an extension of G. Let $\Gamma : D^3 \times S^2 \to D^3 \times S^2$ be the composition tGt, where t is the permutation $(x, y) \leftrightarrow (y, x)$. Then $\bar{\Gamma} : D^4 \times S^2 \to D^4 \times S^2$ defined by $\bar{\Gamma}((y_1, y_2, y_3, y_4), x) = ((c_2(x)(y_1, y_2, y_3), y_4), x)$ is an extension of Γ. Let $\partial G : S^2 \times S^2 \to S^2 \times S^2$ be the restriction of G, and apply similar notation to the other maps. All the diffeomorphisms G, Γ, $\bar{G}$ and $\bar{\Gamma}$ are Z_2-equivariant with respect to the usual "antipodal" Z_2-action $(x, y) \leftrightarrow (-x, -y)$ so that they descend to maps $g : RP^2 \tilde{\times} D^3 \to RP^2 \tilde{\times} D^3$, $\gamma : D^3 \tilde{\times} RP^2 \to D^3 \tilde{\times} RP^2$, $\bar{g} : RP^3 \tilde{\times} D^3 \to RP^3 \tilde{\times} D^3$, and $\bar{\gamma} : D^4 \tilde{\times} RP^2 \to D^4 \tilde{\times} RP^2$, respectively. Let us also note that the maps G, Γ and $\bar{\Gamma}$ (but not $\bar{G}$) are all isotopic to identity, since $c_2 : S^2 \to SO(3)$ is null-homotopic, but they are not Z_2-equivariantly isotopic to identity. This property of these maps enables us to construct Z_2-manifolds diffeomorphic to the ordinary spheres, but the quotient manifolds of which are not diffeomorphic to the ordinary projective spaces.

Let $h = \partial\Gamma \circ \partial G : S^2 \times S^2 \to S^2 \times S^2$ and $\bar{h} = \partial\bar{\Gamma} \circ \partial\bar{G} : S^3 \times S^2 \to S^3 \times S^2$. Let h_i be the i-th power of h and similarly for $\bar{h}$; in particular $h_0 = id$. Let $\Sigma_i^5 = S^2 \times D^3 \cup_{h_i} D^3 \times S^2$ and $\Sigma_i^6 = S^3 \times D^3 \cup_{\bar{h}_i} D^4 \times S^2$. Let T_i^5 be a (smooth, fixed point-free) involution on Σ_i^5 defined uniquely by the requirement that $T_i^5 | S^2 \times D^3 : S^2 \times D^3 \to S^2 \times D^3$ and $T_i^5 | D^3 \times S^2 : D^3 \times S^2 \to D^3 \times S^2$, and these restrictions are both given by $(x, y) \to (-x, -y)$. Let T_i^6 be an involution of Σ_i^6 given by an analogous construction. It is clear now that T_0^5 (resp. T_0^6) is just the ordinary antipodal map on $\Sigma_0^5 = S^5$ (resp. on $\Sigma_0^6 = S^6$). Let us denote $FRP_i^5 = \Sigma_i^5 / T_i^5$ and $FRP_i^6 = \Sigma_i^6 / T_i^6$. Natural imbeddings $FRP_i^5 \subset FRP_i^6$ are now apparent, and it is clear that (Σ_i^5, T_i^5) is a characteristic submanifold of (Σ_i^6, T_i^6).

Now we have gathered all topological facts that we shall need to formulate our main theorems. However, as we mentioned above, we are going to apply the analytical eta-invariant to detect homotopy projective spaces. So let us introduce the following notation (for more detailed information concerning the eta-invariant and generalized Dirac-type operators, see [3], [13] and also Section 3 of this paper). Let M^{2k} be a smooth closed Riemannian Pin^c (resp. Pin^+)-manifold, and let ϕ (resp. ψ) be a Pin^c (resp. Pin^+)-structure on M^{2k}. Then there exists a Dirac-type, first-order elliptic differential operator D^c (resp. D^+) on M^{2k} determined by the Riemannian metric and the Pin^c (resp. Pin^+)-structure on M^{2k}. Note that D ($= D^c$ or D^+) is self-adjoint and the Atiyah-Patodi-Singer eta-invariant of D is well-defined. The mod Z (resp. mod $2Z$) reduction of the eta-invariant of D^c (resp. D^+, provided that $k \equiv 2 \bmod 4$) is a Pin^c (resp. Pin^+)-bordism invariant, and we denote it simply by $\eta(M^{2k}, \phi) \bmod Z$ (resp. $\eta(M^{2k}, \psi) \bmod 2Z$).

Now we can formulate main theorems of this paper. We are forced to start with involutions of 5 and 6-dimensional spheres. Recall that, up to diffeomorphisms, there exist precisely 4 homotopy projective spaces in dimensions 4 and 5 ([20]).

THEOREM A. (a1) *Σ_i^5 is a homotopy 5-sphere for any $i = 0, 1, \dots$, and hence $FRP_i^5 = \Sigma_i^5/T_i^5$ is a homotopy projective space.*

(a2) *Any involution of a 5-dimensional homotopy sphere is equivalent (i.e., smoothly conjugated) to precisely one of the involutions $T_0^5 = ant$, T_1^5, T_2^5, T_3^5.*

(b1) *Σ_i^6 is a homotopy 6-sphere for any $i = 0, 1, \dots$, and hence $FRP_i^6 = \Sigma_i^6/T_i^6$ is a homotopy projective space.*

(b2) *Any involution of a 6-dimensional homotopy sphere is equivalent to precisely one of the involutions $T_0^6 = ant$, T_1^6, T_2^6, T_3^6. Moreover, $\eta(FRP_i^6, \phi) = \pm(2i+1)/16$ (mod Z) for any Pin^c-structure ϕ on FRP_i^6. (In fact, $[FRP_i^6] = \pm(2i+1)[RP^6]$ in the cobordism group $\Omega_6^{Pin^c}$). Thus the eta-invariant of the Pin^c-operator completely detects homotopy projective spaces in dimension 6.*

This theorem gives a complete and particularly simple classification of involutions of homotopy 5 and 6-spheres.

THEOREM B. *The one-fold smooth suspension ΣT_{FS}^4 of the exotic Fintushel-Stern's involution is equivalent (smoothly conjugated) to the involution T_3^5 constructed above.*

Thus the one-fold suspension of the Fintushel-Stern involution is given by a simple construction and can be described by a transparent formula.

COMMENTARY 2.1. We could have used the eta-invariant (mod Z) of the tangential operator of the *Spin^c*-complex with coefficients in the virtual representation $\rho_0 - \rho_1$ (where ρ_0 (resp. ρ_1) is the trivial (resp. non-trivial) 1-dimensional representation of Z_2) to detect some odd-dimensional homotopy projective spaces in dimension 5 (compare [5, Lemma 2.3]). However, the range of the eta-invariant of this operator (being $Z[1/8]/Z$) is too small to detect all homotopy projective spaces in dimension 5. Namely, suspending an involution on S^5 one sees easily that any 5-dimensional homotopy projective space FRP^5 is the image under the Smith homomorphism of a 6-dimensional homotopy projective space FRP^6. Therefore $\eta(FRP^5) = 2\eta(FRP^6)$ by [5, Lemma 3.3]. But any such FRP^6 is *Pin^c*-bordant to an odd number of copies of the ordinary RP^6 (see the proof of Theorem A below, but this follows also by a simple argument using characteristic numbers and $\Omega_6^{Pin^c} = Z_{16} \oplus Z_4$ generated by RP^6 and $RP^2 \times CP^2$). Thus $\eta(FRP^5) = (2i+1)/8$ mod Z for some integer i. Consequently, there are only four possible values for the eta-invariant of 5-dimensional homotopy projective spaces. However, to any homotopy projective space there correspond precisely two mutually inverse values of the eta-invariant corresponding to mutually inverse *Spin^c*-structures. Therefore one can detect at most two 5-dimensional homotopy projective spaces using the eta-invariant. Topologically, the image of $FRP_i^6 = (2i+1)RP^6$ in $\Omega_{Pin^c}^6$ under the Smith homomorphism lies in the subgroup of $\Omega_{Pin^c}^5(BZ_2)$ generated by RP^5, which is isomorphic to Z_8, and is too small to detect 4 exotic projective spaces of dimension 5 by an argument similar to that given above.

Thus, instead of applying the eta-invariant of the *Spin*^{c}-operator, we define, for a homotopy projective space M^5, another invariant (η^c-invariant) derived from a 6-dimensional projective space which contains M^5 (see Definition 3.1 below). This also makes the paper a bit more concise. Similar remarks apply to the suspension of an involution of S^4. We cannot apply directly the above-mentioned results of [5], concerning the Smith homomorphism and the eta-invariant, to establish a satisfactory relation between T^4_{FS}, ΣT^4_{FS} and $\Sigma^2 T^4_{FS}$, since the eta-invariant of the *Pin*^{c}-operator does not detect homotopy projective spaces of dimension 4. The method of proving Theorem B in this paper is based on constructing explicitly some "stratified" cobordisms, and provides a detailed picture of the involution T^4_{FS}, as well as of its suspension. In fact, this method enables us to establish a connection between the eta-invariant (mod Z) of certain *Pin*^{c}-manifolds of dimension 6 and the eta-invariant (mod $2Z$) of their image under the Smith homomorphism in $\Omega^4_{Pin^+}$, but this is done only for a very special kind of manifolds (compare also [26]).

In order to formulate the next theorem, let us recall that two smooth closed 4-manifolds, say M^4 and $M^{4'}$, are called stably diffeomorphic if they become diffeomorphic after forming the connected sum with sufficiently many copies of $S^2 \times S^2$.

THEOREM C. *Let T^4_1 and T^4_2 be any two involutions on the standard 4-sphere S^4. Then the projective spaces S^4/T^4_1 and S^4/T^4_2 are stably diffeomorphic if and only if the suspended involutions ΣT^4_1 and ΣT^4_2 are equivalent.*

As an immediate corollary of Theorems B and C we get the following

THEOREM D. *The one-fold suspension ΣT^4_{CS} of the Cappell-Shaneson exotic involution is equivalent to the one-fold suspension ΣT^4_{FS} of the Fintushel-Stern involution, and these suspended involutions are both equivalent to the involution T^5_3 described above.*

PROOF. It has been proved in [23] that the Cappell-Shaneson exotic projective space FRP^4_{CS} is stably diffeomorphic to the Fintushel-Stern projective space FRP^4_{FS}. Now we use Theorems B and C to complete the proof.

It is well-known that a smooth involution T^n of a homotopy sphere Σ^n desuspends if and only if the Browder-Livesay invariant $\alpha(T^n, \Sigma^n) = 0$, provided that $n > 5$ ([20]). As an immediate consequence of our theorems we get an extension of this theorem to all dimensions ≥ 5.

THEOREM E. *A smooth involution T^n of a homotopy sphere Σ^n, where $n \geq 5$, desuspends if and only if the Browder-Livesay invariant $\alpha(T^n, \Sigma^n)$ vanishes. In fact, the smooth suspension ΣT^4 of any smooth involution T^4 of S^4 is equivalent to precisely one of the involutions T^5_0 and T^5_3.*

PROOF. It is well-known [20] that the Browder-Livesay invariant vanishes for precisely two (up to equivalence) involutions on S^5, and hence at most two of them desuspend. But two of our involutions $T_0^5, T_1^5, T_2^5, T_3^5$ certainly desuspended, namely $T_0^5 = ant$ and T_3^5 which desuspends to Fintushel-Stern's involution on S^4. This proves Theorem E.

Here is a brief outline of the proof of main theorems of this paper. First we use the eta-invariant of the Pin^c-operator to classify involutions on S^5 and S^6. This is the first step of the proof of Theorem B. Then we form the double suspension $\Sigma^2 T_{FS}^4$ of T_{FS}^4 and regard it as an involution of the quadruple $(\Sigma(3,5,19), S^4, S^5, S^6)$. Next we perform a few series of stratified and equivariant surgeries on this quadruple of Z_2-manifolds. The first series of surgeries is intended to simplify $\Sigma(3,5,19)$, but at the cost of making the 4-dimensional member of the quadruple topologically more complicated. More precisely, the first series of surgeries provides us with a stratified cobordism from $(\Sigma(3,5,19), S^4, S^5, S^6)$ equipped with the involution $(t^3, T_{FS}^4, \Sigma T_{FS}^4, \Sigma^2 T_{FS}^4)$ to eight copies of (S^3, S^4, S^5, S^6), each of which is equipped with the standard antipodal Z_2-action, and a quadruple of manifolds of the form $(S^3, 8_\#(S^2 \times S^2), S^5, S^6)$ equipped with an involution of the form $(ant, I_a^4, I_a^5, I_a^6)$. Next we perform another sequence of surgeries which provides us with an equivariant cobordism from $(S^3, 8_\#(S^2 \times S^2), S^5, S^6)$ to a quadruple of the form (S^3, S^4, S^5, S^6) equipped with the standard antipodal involution. We use the eta-invariant to detect the position of the quotient manifolds $FRP^n = S^n/$involution in the cobordism group $\Omega_{Pin^c}^n$. This enables us to identify the suspended involution ΣT_{FS}^4 as the involution T_3^5.

REMARK. We could apply a different method for proving Theorem B. Namely we could compute the Browder-Livesay invariant of our involutions T_i^5 and then apply Theorem C (using the fact that FRP_{FS}^4 is not stably diffeomorphic to RP^4) to identify ΣT_{FS}^4 as T_3^5. However, we prefer to apply the method of building explicitely appropriate cobordisms, since this provides a much more detailed picture of the exotic involution on S^4 and its suspension.

3. Pin-structures, Dirac-type operators and the eta-invariant. For convenience of the reader we collect in this section some basic facts concerning *Pin*-structures on manifolds, Dirac-type operators and the eta-invariant. Since the material presented here is now standard and can be found in many papers (see [3], [5], [13]), we will omit proofs.

Let α denote one of the symbols $+$, $-$ or c and $\varepsilon = \pm 1$. Then a Lie group Pin^α is well-defined (see [16], [3], [5]). Let ξ be a n-dimensional vector bundle over a paracompact space X. ξ is said to admit a Pin^α-structure if and only if the classifying map ξ of this bundle (we identify here the vector bundle with its classifying map) fits into the following commutative diagram:

$$\begin{array}{ccc} & & BPin^\alpha(n) \\ & \overset{\xi^\alpha}{\nearrow} & \downarrow{\scriptstyle B\sigma^\alpha} \\ X & \xrightarrow{\ \xi\ } & BO(n) \end{array}$$

A Pin^α-structure on the vector bundle ξ is a fibre-homotopy class of a map $\xi^\alpha : X \to BPin^\alpha(n)$ as in the diagram above. A manifold M is called a Pin^α-manifold if and only if its tangent bundle TM admits a Pin^{α}-structure, and a Pin^{α}-structure on TM is called simply a Pin^{α}-structure on M. The following proposition gives a useful characterization of Pin^{α}-bundles and Pin^{α}-manifolds.

PROPOSITION 3.1 ([5], [13], [28]). *Let ξ be an n-dimensional vector bundle over a paracompact space X, and $w_n(\xi)$ be the n-th Whitney-Stiefel class of ξ.*

(a) *ξ has a Pin^+ (resp. Pin^-)-structure if and only if $w_2(\xi) = 0$ (resp. $w_2(\xi)+w_1^2(\xi) = 0$). ξ has a Pin^c-structure if and only if $w_2(\xi)$ is the modulo 2 reduction of an integral cohomology class.*

(b) *If ξ has a Pin^{ε}-structure (resp. Pin^c-structure), then Pin^{ε} (resp. Pin^c)-structures on ξ are in a one-to-one correspondence with cohomology classes in $H^1(X, Z_2)$ (resp. $H^2(X, Z)$).*

For example, even-dimensional projective space RP^{2n} is a Pin^c-manifold which admits precisely two mutually inverse Pin^c-structures, and it is a Pin^+-manifold (with two mutually inverse Pin^+-structures) for dimensions $2n = 8k + 4$. As an immediate consequence of homotopy invariance of Whitney-Stiefel classes of closed manifolds, it follows that any closed manifold homotopy equivalent to a closed Pin^{α}-manifold is also a Pin^{α}-manifold.

Now let us turn to Dirac-type Pin^+ and Pin^c-operators (see [3], [13], [28]). Let M^{2n} be a Riemannian Pin^{α}-manifold, and Φ be a fixed Pin^{α}-structure on M. Φ determines a Pin^{α}-vector bundle $\Phi(\Delta^{\alpha}(n))$ over M whose fibre is a suitably chosen irreducible $C^{\alpha}(R^{2n+1})$-module $\Delta^{\alpha}(n)$ (where $C^{\alpha}(R^{2n+1})$ is a suitable Clifford algebra). There is a Dirac-type operator $D_{\Phi} : C^{\infty}(\Phi(\Delta^{\alpha})) \to C^{\infty}(\Phi(\Delta^{\alpha}))$ determined by the Riemannian structure on M and the Pin^{α}-structure Φ; we call this the Pin^{α}-operator on M (corresponding to the Pin^{α}-structure Φ). This is a first-order elliptic self-adjoint differential operator given in a local orthonormal frame $(e_1, e_2, \ldots, e_{2n+1})$ on $M^{2n} \times [0, 1)$ (equipped with an obvious product Riemannian metric) by the formula

$$D_{\Phi} s = \sum_{i=1}^{n} e_{2n+1}^{-1} \cdot e_i \cdot \nabla_{e_i} s \, .$$

The dot in this formula denotes the Clifford multiplication, and ∇ is the covariant derivative on $\Phi(\Delta^{\alpha}(n))$ determined by the Levi-Civita connection on M. Thus the Atiyah-Patodi-Singer eta-invariant of D_{Φ} is well-defined; we denote this invariant by $\eta(M^{2n}, \Phi^{\alpha})$. The following proposition justifies this notation.

PROPOSITION 3.2 ([13], [28]). (a) *$\eta(M^{2n}, \Phi^c)$ mod Z is a Pin^c-bordism invariant. It takes values in $Z[1/2^{n+1}]$ for non-orientable M and in $Z[1/2]$ for orientable M. Similarly, $\eta(M^{8k+4}, \Phi^+)$ mod 2Z is a Pin^+-bordism invariant.*

(b) *$\eta(RP^{2n}, \Phi^c) = \pm 2^{-(n+1)}$ mod Z for any Pin^c-structure Φ^c. Similarly, $\eta(RP^{8k+4}, \Phi^+) = \pm 2^{-(4k+3)}$ mod 2Z for any Pin^+-structure Φ^+.*

We shall also need the following result concerning exotic 4-dimensional projective spaces ([23], [28]):

PROPOSITION 3.3. (a) *$\eta(FRP^4_{CS}, \Phi^+) = \pm 7/8$ mod $2Z$ for any Pin^+-structure Φ^+ on FRP^4_{CS}. Similarly, $\eta(FRP^4_{FS}, \Phi^+) = \pm 7/8$ mod $2Z$ for any Pin^+-structure Φ^+ on FRP^4_{FS}.*

(b) *FRP^4_{CS} is stably diffeomorphic to FRP^4_{FS}, but not to the ordinary projective space RP^4.*

The eta-invariant (mod Z) of the Pin^c-operator proves to be a useful tool for detecting exotic projective spaces (or, equivalently, exotic involutions of spheres) in dimension 6. However, it is completely useless in the case of odd-dimensional (orientable) projective spaces due to its limited range (see also Commentary 2.1). The following definition introduces an invariant, which will extricate us from this unpleasant situation.

Let $FRP^5 = S^5/T^5$ be a homotopy projective space. Let $\Sigma T^5 : \Sigma^6 \to \Sigma^6$ be the smooth suspension of T^5, and let $\tilde{\Sigma} FRP^5 = \Sigma^6/\Sigma T^5$. $\tilde{\Sigma} FRP^5$ is a homotopy projective space which contains FRP^5 and has precisely two mutually inverse Pin^c-structures, say Φ^c_1 and Φ^c_2. Therefore $\eta(\tilde{\Sigma} FRP^5, \Phi^c_1) = -\eta(\tilde{\Sigma} FRP^5, \Phi^c_2)$ mod Z by Proposition 2a.

DEFINITION 3.1. With the notation above, define an invariant (the eta^c-invariant) $\eta^c(FRP^5)$ to be the (unordered) pair of numbers $\eta^c(FRP^5) = \{\eta(\tilde{\Sigma} FRP^5, \Phi^c_1) \bmod Z, \eta(\tilde{\Sigma} FRP^5, \Phi^c_2) \bmod Z\}$.

We will prove that the η^c-invariant completely detects homotopy projective spaces of dimension 5.

4. Homotopy projective spaces of dimensions 5 and 6 versus the eta-invariant. In this section we apply the eta-invariant of the Pin^c-operator to classify 5 and 6-dimensional smooth projective spaces. In particular, we prove Theorem A of Section 1. We also prove some auxiliary propositions which explain how doing surgery affects the eta-invariant.

Let us start with a version of Z_2-equivariant plumbing and surgery, which are the main tools in this paper.

Equivariant plumbing. For i even (resp. i odd) let ξ_i be an oriented k (resp. n)-dimensional Z_2-vector bundle over S^n (resp. S^k) with a Z_2-action covered by the Z_2-action on ξ_i, where $i = 0, 1, \ldots, l$. Let $D\xi_i$ be the unit disc bundle of ξ_i with respect to a Z_2-invariant fibre metric. Then $D\xi_i$ is a Z_2-manifold in a natural way. Let I_i be the involution of $D\xi_i$ given by the action of the non-trivial element of Z_2. Identify S^m, where $m = k$ or n depending on the parity of i, with the zero section of $D\xi_i$ as its invariant submanifold. Let $p_i, q_i \in S^m \subset D\xi_i$ be two distinct isolated fixed points of I_i. For i even (resp. i odd), fix orientation-preserving equivariant imbeddings with disjoint images $k_{p_i}, k_{q_i} : D^n \times D^k$ (resp. $(D^k \times D^n)$) $\to D\xi_i$ (where Z_2 acts on $D^n \times D^k$ by $(x, y) \mapsto (-x, -y)$) such that $k_{p_i}(0) = p_i$, $k_{q_i}(0) = q_i$, and for any $x \in D^n$ (resp. $x \in D^k$), $k_{p_i}(x \times D^k)$ (resp. $k_{p_i}(x \times D^n)$) is precisely a fibre of the bundle $D\xi_i \to S^n$ (resp. S^k), and similarly for k_{q_i}. For i even, let $D^n_{p_i} \times D^k_{p_i}$ (resp. $D^n_{q_i} \times D^k_{q_i}$) be the image of k_{p_i} (resp. k_{q_i}), and similarly for i odd.

Then we define the Z_2-equivariant plumbing of $D\xi_0, \ldots, D\xi_l$ to be a manifold $D\xi_0 \square \cdots \square D\xi_l$ obtained by identifying $D^n_{q_i} \times D^k_{q_i} \ni (x, y) \simeq (y, x) \in D^k_{p_{i+1}} \times D^n_{p_{i+1}}$. It is clear that

$D\xi_0\Box\cdots\Box D\xi_l$ comes with a naturally defined involution $I_0\Box\cdots\Box I_l$ extending all the involutions I_i and $\mathrm{Fix}(I_0\Box\cdots\Box I_l)\supset\{p_0,q_0=p_1,\ldots,q_{l-1}=p_l,q_l\}$. $D\xi_0\Box\cdots\Box D\xi_l$ is not a smooth Z_2-manifold, but this can be easily fixed by applying a Z_2-equivariant smoothing corners process.

Let $\alpha_i : S^{n-1}\to SO(k)$ (resp. $\alpha_i : S^{k-1}\to SO(n)$) be the clutching map for the bundle ξ_i for i even (resp. i odd). Let $\beta_i(x,y)=(x,\alpha_i(x)y)$ for $(x,y)\in S^{n-1}\times D^k$ (or $S^{k-1}\times D^n$ depending on the parity of i), and $T : D^n\times D^k\to D^k\times D^n$ be the permutation $(x,y)\to(y,x)$. It is not hard to see that $\partial(D\xi_0\Box\cdots\Box D\xi_l)=(\partial D\xi_0)_{\partial_\chi}$, where $\chi=k_{q_0}Tk_{p_1}^{-1}\beta_1k_{q_1}\ldots k_{q_{l-1}}Tk_{p_l}^{-1}\beta_lk_{q_l} : S^{n-1}\times D^k\to\partial D\xi_0$ or : $S^{k-1}\times D^n\to\partial D\xi_0$ (depending on the parity of l).

Now we can give two alternative descriptions of the involutions T_i^5 of Σ_i^5 and T_i^6 of Σ_i^6 described in Section 1. An advantage of both of these constructions over the one given in Section 2 is that they provide us with a suitable stratified equivariant cobordism from $((\Sigma_i^5,T_i^5),(\Sigma_i^6,T_i^6))$ to $2i+1$ copies of $((S^5,\mathit{ant}),(S^6,\mathit{ant}))$. This cobordism will play an essential role in our computation of the eta-invariant and determination of ΣT^4_{FS}.

Let us fix concordant decompositions $S^3=S^1\times D^2\cup D^2\times S^1$, $S^4=S^1\times D^3\cup D^2\times S^2$, $S^5=S^2\times D^3\cup D^3\times S^2$ and $S^6=S^3\times D^3\cup D^4\times S^2$, which are all invariant with respect to the usual "antipodal" Z_2-action. Let DS^n be the disc bundle of the tangent bundle TS^n, and let Z_2 act on DS^n by the differential of the map $R^{n+1}\ni(x_1,x_2,\ldots,x_{n+1})\mapsto(x_1,-x_2,\ldots,-x_{n+1})$. Denote by DT^n the involution on DS^n given by the action of the non-trivial element of Z_2. Then DT^n has precisely two fixed points $p=((1,0,\ldots,0),(0,\ldots,0))$ and $q=((-1,0,\ldots,0),(0,\ldots,0))$. Let $\bar{D}S^n$ be the disc bundle of the stable tangent bundle $TS^n\oplus 1$, and $\bar{D}T^n$ be an involution on $\bar{D}S^n$ given by $\bar{D}T^n(x,t)=(DT(x),-t)$. A natural equivariant imbedding $DS^n\subset\bar{D}S^n$ is now apparent, and $\mathrm{Fix}(\bar{D}T^n)=\mathrm{Fix}(DT^n)=\{p,q\}$. We will need also another 3-dimensional disc bundle over S^4, namely $\Delta S^3=D^4\times D^3\cup_{\bar{G}}D^4\times D^3$ (see Section 2 for the definition of $\bar{G}$). Then ΔS^3 comes with a naturally defined involution ΔT^3 which, when restricted to any copy of $D^4\times D^3\subset\Delta S^3$, is just the antipodal map $(x,y)\mapsto(-x,-y)$. A natural equivariant imbedding $(DS^3,DT^3)\hookrightarrow(\Delta S^3,\Delta T^3)$ is now apparent and $\mathrm{Fix}(\Delta T^3)=\mathrm{Fix}(DT^3)=\{p,q\}$.

A. This construction uses the equivariant plumbing as described above.

For $i=1,2,\ldots$, we define

$$\tilde{W}_i^6=DS_1^3\Box\cdots\Box DS_i^3\,,\quad J_i^6=DT_1^3\Box\cdots\Box DT_i^3\,.$$

For i odd, define

$$\tilde{W}_i^7=\Delta S_1^3\Box\bar{D}S_2^3\Box\cdots\Box\Delta S_i^3\,,\quad J_i^7=\Delta T_1^3\Box\bar{D}T_2^3\Box\cdots\Box\Delta T_i^3\,.$$

For i even, define

$$\tilde{W}_i^7=\Delta S_1^3\Box\bar{D}S_2^3\Box\cdots\Box\bar{D}S_i^3\,,\quad J_i^7=\Delta T_1^3\Box\bar{D}T_2^3\Box\cdots\Box\bar{D}T_i^3\,.$$

Observe that $(\tilde{W}_i^6,J_i^6)$ is a characteristic submanifold of $(\tilde{W}_i^7,J_i^7)$ in a natural way, and both of these Z_2-manifolds can be assumed to be smooth. Note that $\mathrm{Fix}(J_k^6)=\mathrm{Fix}(J_k^7)$

consists of precisely $k+1$ isolated points, namely $p_1, q_1 = p_2, \dots, q_{k-1} = p_k, q_k$. Let

$$(\tilde{V}_k^5, I_k^5) = \partial(\tilde{W}_k^6, J_k^6), \quad (\tilde{V}_k^6, I_k^6) = \partial(\tilde{W}_k^7, J_k^7).$$

It is not hard to see (compare Proposition 1 below) that for even k $\tilde{V}_k^5$ (resp. $\tilde{V}_k^6$) is a homotopy sphere. We will prove that the involution I_{2i}^5 of $\tilde{V}_{2i}^5$ is equivalent to the involution T_i^5 constructed in Section 2. An analogous statement is valid for $\tilde{V}_{2i}^6$ and I_{2i}^6.

Now we turn to yet another construction of the involutions T_i^5 and T_i^6.

B. This construction uses surgery in place of the plumbing.

Let $G_{i+1} : S^2 \times D_{i+1}^3 \subset S_{i+1}^5 \to S^2 \times D_i^3 \subset S_i^5$ be a copy of the map G, and let $\Gamma_{i+1} : D^3 \times S_{i+1}^2 \to D^3 \times S_i^2$ have an analogous meaning. Then

$$(\tilde{X}_0^6, P_0^6) = (D_0^6, ant_0),$$

and for even $i > 0$

$$(\tilde{X}_{i+1}^6, P_{i+1}^6) = (\tilde{X}_{iG}^6, P_{iG}^6),$$

while for odd i

$$(\tilde{X}_{i+1}^6, P_{i+1}^6) = (\tilde{X}_{i\Gamma}^6, P_{i\Gamma}^6),$$

where G (resp. Γ) stands for G_{i+1} (resp. Γ_{i+1}).

Similarly, let us put

$$(\tilde{X}_0^{6'}, P_0^{6'}) = (S^5 \times I_0, \overline{ant}_0),$$

and define

$$(\tilde{X}_{i+1}^{6'}, P_{i+1}^{6'}) = (\tilde{X}_{iG}^{6'}, P_{iG}^{6'}) \quad \text{or} \quad (\tilde{X}_{i\Gamma}^{6'}, P_{i\Gamma}^{6'})$$

depending on the parity of i.

Z_2-manifolds $(\tilde{X}_i^7, P_i^7)$ and $(\tilde{X}_{i+1}^{7'}, P_{i+1}^{7'})$ are defined analogously with G (resp. Γ) replaced by $\bar{G}$ (resp. $\bar{\Gamma}$). Natural imbeddings $(\tilde{X}_{i+1}^{n'}, P_{i+1}^{7'}) \subset (\tilde{X}_{i+1}^n, P_{i+1}^n)$, $n = 6, 7$ are apparent, and it is clear that $\tilde{X}_i^{n'}$ is obtained from $\tilde{X}_i^n$ by deleting small invariant discs around all fixed points of the involution P. Note that $(\tilde{X}_{i+1}^{6'}, P_{i+1}^{6'})$ (resp. $(\tilde{X}_{i+1}^6, P_{i+1}^6)$) is a characteristic submanifold of $(\tilde{X}_{i+1}^{7'}, P_{i+1}^{7'})$ (resp. $(\tilde{X}_{i+1}^7, P_{i+1}^7)$) and

$$\partial(\tilde{X}_i^{k'}, P_i^{k'}) = \partial(\tilde{X}_i^k, P_i^k) \sqcup (i+1)(S^{k-1}, ant).$$

Denote $(\tilde{Y}_i^{k-1}, Q_i^{k-1}) = \partial(\tilde{X}_i^{k'}, P_i^{k'})$, $k = 6, 7$. Using the notation in Section 2, one sees that $Y_{i+1}^5 = Y_{i\partial g}^5$ or $Y_{i\partial\gamma}^5$, and $Y_{i+1}^6 = Y_{i\partial\bar{g}}^6$ or $Y_{i\partial\bar{\gamma}}^6$, depending on the parity of i.

The following Proposition is elementary and its statements (a) and (b) are essentially well-known (compare [6]).

PROPOSITION 4.1. (a) *$(\tilde{W}_k^n, J_k^n)$ is diffeomorphic to $(\tilde{X}_k^n, P_k^n)$, $k = 0, 1, \dots, n = 6, 7$. Therefore, $(\tilde{V}_k^{n-1}, I_k^{n-1})$ is diffeomorphic to $(\tilde{Y}_k^{n-1}, Q_k^{n-1})$.*

(b) *$\tilde{W}_{2k}^6 \simeq \tilde{X}_{2k}^6$ is a stably framed 2-connected manifold, and $\tilde{V}_{2k}^5$ is a homotopy sphere.*

(c) *$\tilde{W}_{2k}^7 \simeq \tilde{X}_{2k}^7$ is 2-connected, and $\tilde{V}_{2k}^6$ is a homotopy sphere.*

(d) *The involution I_{2k}^5 of $\tilde{V}_{2k}^5$ is equivalent to the involution T_k^5 of Σ_k^5.*

(e) *The involution I_{2k}^6 of $\tilde{V}_{2k}^6$ is equivalent to the involution T_k^6 of Σ_k^6.*

PROOF. Observe that DS^3 is obtained by gluing two discs, say $D^3 \times D_1^3$ and $D^3 \times D_2^3$, with the help of the map $G : S^2 \times D_1^3 \to S^2 \times D_2^3$. Both the disc bundles $\bar{D}S^3$ and ΔS^3 are obtained by a similar construction—we use the maps $\bar{G}$ and $\bar{\Gamma}$, respectively. This observation immediately yields the assertion (a).

It follows from the very construction of $\tilde{X}_{2k}^6$ that $\tilde{Y}_{2k}^5 = \partial \tilde{X}_{2k}^6$ is obtained by gluing $D^3 \times S_0^2$ and $S^2 \times D_{2k}^3$ with the help of the map $\Gamma_{2k} \circ G_{2k-1} \circ \cdots \circ \Gamma_1 \circ G_1 | S^2 \times S^2$, which is nothing but the map h_k defined in Section 2. It is also clear that $\tilde{Y}_{2k}^6 = \partial \tilde{X}_{2k}^7$ is obtained by gluing $D^4 \times S_0^2$ and $S^3 \times D_{2k}^3$ with the help of the map $\bar{\Gamma}_{2k} \circ \bar{G}_{2k-1} \circ \cdots \circ \bar{\Gamma}_1 \circ \bar{G}_1 | S^3 \times S^2 = \bar{h}_k$. Moreover, Z_2 acts on the above-indicated two components of $\tilde{Y}_{2k}^5$ (resp. $\tilde{Y}_{2k}^6$) by the ordinary "antipodal" action. Now the assertions (d) and (e) follow (with the exception of the statement concerning the homotopy type of the manifolds $\tilde{V}$).

It is a standard fact that $\tilde{V}_{2k}^5$ is a homotopy sphere ([6]). In order to prove that $\tilde{V}_{2k}^6$ is a homotopy sphere, let us note that $\tilde{H}_l(\tilde{W}_{2k}^7; Z) = 0$ for $l \neq 3, 4$, while $\tilde{H}_3(\tilde{W}_{2k}^7; Z)$ (resp. $\tilde{H}_4(\tilde{W}_{2k}^7; Z)$) is free abelian of rank k generated by elements $v_1, v_2, \ldots, v_k$ (resp. $w_1, w_2, \ldots, w_k$), where v_i (resp. w_i) is represented by the zero-section S_i^3 (resp. S_i^4) of the bundle $\bar{D}S_i^3 \to S_i^3$ (resp. $\Delta S_i^3 \to S_i^4$) contained in $\tilde{W}_{2k}^7$. Moreover, $v_i \cdot w_i = \pm 1$ and $v_i \cdot w_j = 0$ for $|i - j| > 0$. Therefore any homology class v_i is primitive and can be killed by surgery. Therefore $\partial \tilde{W}_{2k}^7 = \tilde{V}_{2k}^6$ is a homotopy sphere, as claimed. This concludes the proof of Proposition 1 as well as of that of the statements (a1) and (b1) of Theorem A in Section 2.

Thus $((\tilde{X}_{2k}^{6'}, P_{2k}^{6'}), (\tilde{X}_{2k}^{7'}, P_{2k}^{7'}))$ is an equivariant cobordism from $((\Sigma_k^5, T_k^5), (\Sigma_k^6, T_k^6))$ to $2k+1$ copies of $((S^5, ant), (S^6, ant))$. Hence $(X_{2k}^{6'}, X_{2k}^{7'})$ is a cobordism from (FRP_k^5, FRP_k^6) to $2k + 1$ copies of (RP^5, RP^6). We are going to use this cobordism to compute the eta-invariant of the Pin^c-operator, so we shall need the following

PROPOSITION 4.2. *$X_{2k}^{7'}$ is a Pin^c-cobordism.*

Proof of this proposition is a simple calculation in cohomology and hence is omitted.

Thus $X_{2k}^{7'}$ is a Pin^c-cobordism from FRP_k^6 to $2k+1$ copies of RP^6. In order to compute the eta-invariant of FRP_k^6, we have to detect the Pin^c-structure inherited by any copy of $RP^6 \subset \partial X_{2k}^{7'}$ from a given Pin^c-structure on $X_{2k}^{7'}$. The next proposition provides us with an appropriate tool for doing this.

Let us from two auxiliary manifolds which are elementary pieces of the manifold $X_{2k}^{7'}$, namely $A^7 = (RP^6 \times I)'_{\bar{g}}$ and $B^7 = (RP^6 \times I)'_{\bar{\gamma}}$. To be more precise, A^7 (resp. B^7) is obtained by gluing two copies of $RP^6 \times I$ with the help of the map $\bar{g} : RP^3 \tilde{\times} D^3 \times 0_2 \subset RP^6 \times I_2 \to RP^3 \tilde{\times} D^3 \times 0_1 \subset RP^6 \times I_1$ (resp. $\bar{\gamma} : D^4 \tilde{\times} RP^2 \times 0_2 \subset RP^6 \times I_2 \to D^4 \tilde{\times} RP^2 \times 0_1 \subset RP^6 \times I_1$). Note that both the manifolds A^7 and B^7 are Pin^c-manifolds by Proposition 2 above. Let us also note that ∂A^7 (resp. ∂B^7) contains two copies of RP^6, namely $RP^6 \times 1_1$ and $RP^6 \times 1_2$, and denote these two copies of RP^6 by RP_1^6 and RP_2^6, respectively.

PROPOSITION 4.3. *Let Φ_A (resp. Φ_B) be a Pin^c-structure on A^7 (resp. B^7). Let Φ_{Ai} (resp. Φ_{Bi}), $i = 1, 2$, be the Pin^c-structure on $RP_i^6 \subset \partial A^7$ (resp. $RP_i^6 \subset \partial B^7$) induced by Φ_A (resp. Φ_B). Then the Pin^c-structures Φ_{A1} and Φ_{A2} (resp. Φ_{B1} and Φ_{B2}) coincide.*

PROOF. We prove the proposition for the manifold B^7 and the Pin^c-structures Φ_{B1} and Φ_{B2} on RP^6. The proof for A^7 is completely analogous. So let us assume, on the contrary, that the Pin^c-structures Φ_{B1} and Φ_{B2} on RP^6 are mutually inverse. Then they extend simultaneously to a Pin^c-structure Φ on the "tube" $RP^6 \times [0, 1]$. Consequently, the manifold $\bar{B}^7$ obtained by attaching to B^7 the "tube" $RP^6 \times [0, 1]$ by a map which identifies in an obvious way the "ends" $RP^6 \times 0$ and $RP^6 \times 1$ with the two copies of $RP^6 \subset \partial B^7$ is a Pin^c-manifold. We will show that $\bar{B}^7$ is not a Pin^c-manifold, thus arriving at a contradiction and proving the proposition.

The following simplified description of $\bar{B}^7$ will be useful. Take the "tube" $RP^6 \times [0, 1]$ and glue $RP^6 \times 0$ and $RP^6 \times 1$ with the help of the map $\bar{\gamma} : D^4 \tilde{\times} RP^2 \times 0 \subset RP^6 \times 0 \to D^4 \tilde{\times} RP^2 \times 1 \subset RP^6 \times 1$. This manifold is easily seen to be diffeomorphic to $\bar{B}^7$ and we denote it by the same symbol. In order to prove that $\bar{B}^7$ is not a Pin^c-manifold it sufficies to indicate a 7-dimensional submanifold of $\bar{B}^7$ which is not Pin^c. Let $E^7 = D^4 \tilde{\times} RP^2 \times [0, 1]/(x, 0) \simeq (\bar{\gamma}(x), 1)$. Thus E^7 is the mapping torus of the map $\bar{\gamma}$ and the natural inclusion $E^7 \subset \bar{B}^7$ is apparent. We will prove that E^7 is not a Pin^c-manifold.

Recall that $\bar{\gamma}$ is the quotient of the map $\bar{\Gamma} : D^4 \times S^2 \to D^4 \times S^2$ given by $((y_1, y_2, y_3, y_4); x) \mapsto ((c_2(x)(y_1, y_2, y_3), y_4); x)$. Therefore, $\bar{\gamma} : D^4 \tilde{\times} RP^2 \to D^4 \tilde{\times} RP^2$ is given by $([(y_1, y_2, y_3, y_4)]; [x]) \mapsto (d_2([x])[(y_1, y_2, y_3, y_4)]; [x])$, where $d_2 : RP^2 \to SO(4)$ is the composition of the quotient map of c_2 and the natural imbedding $SO(3) \hookrightarrow SO(4)$. In particular, $\bar{\gamma}$ is an authomorphism of the bundle $D^4 \tilde{\times} RP^2 \to RP^2$, and hence E^7 is a disc bundle of a 4-dimensional vector bundle $\xi^4 = RP^2 \times [0, 1] \times R^4/(x, 0, v) \simeq (x, 1, d_2(x)v)$ over $RP^2 \times S^1$. Let ν_i (resp. μ_2) be the canonical generator of $H^i(RP^2; Z_2) = Z_2$ for $i = 1, 2$ (resp. of $H^2(RP^2; Z) = Z_2$), and $\bar{\nu}_1$ (resp. $\bar{\mu}_1$) be the canonical generator of $H^1(S^1; Z_2) = Z_2$ (resp. of $H^1(S^1; Z) = Z$). Note that $H^2(E^7; Z_2) = H^2(RP^2 \times S^1; Z_2) = H^2(RP^2; Z_2) \oplus H^1(RP^2; Z_2) \otimes H^1(S^1; Z_2) = Z_2 \oplus Z_2$ generated by ν_2 and $\nu_1 \otimes \bar{\nu}_1$, and $H^2(E^7; Z) = H^2(RP^2 \times S^1; Z) = Z_2$ generated by μ_2. It is clear that ν_2 is the mod 2-reduction of μ_2, while $\nu_1 \otimes \bar{\nu}_1$ is not in the range of the mod 2-reduction operation.

We will show that $w_2(E^7) = \nu_2 + \nu_1 \otimes \bar{\nu}_1$, thus proving that E^7 is not a Pin^c-manifold. It sufficies to compute $w_2(T(RP^2 \times S^1) \oplus \xi) = \nu_2 + w_1(T(RP^2 \times S^1))w_1(\xi) + w_2(\xi) = \nu_2 + w_2(\xi)$, since $w_1(\xi) = 0$ because d_2 takes values in $SO(4)$. Let us put $w_2(\xi) = a\nu_2 + b\nu_1 \otimes \bar{\nu}_1$. It is clear that $a = 0$, since $\xi | RP^2 \times t$ is trivial; thus $w_2(\xi) = b\nu_1 \otimes \bar{\nu}_1$ and we must show that $b = 1$. Let $\zeta = \xi | RP^1 \times S^1$. Observe that for $[x] \in RP^1$, $d_2([x])$ is the rotation in the plane $\{e_1, x\}$ by the angle α between the vectors e_1 and x. Identify RP^1 with $S^1 = [0, 2\pi]$ with identified ends. It is clear now that $\zeta = \zeta_1 \oplus 2$, where 2 stands for the trivial 2-dimensional bundle and $\zeta_1 = S^1 \times [0, 1] \times R^2/(\alpha, 0, (v_1, v_2)) \simeq (\alpha, 1, (v_1 \cos\alpha - v_2 \sin\alpha, v_1 \sin\alpha + v_2 \cos\alpha))$. Thus the Euler class of ζ_1 is the generator of $H^2(RP^1 \times S^1; Z) \simeq Z$, and $w_2(\zeta_1) = w_2(\zeta) = w_2(\xi | RP^1 \times S^1)$ is the non-zero element of

$H^2(RP^1 \times S^1; Z_2) \simeq Z_2$. Consequently, $w_2(\xi) = \nu_1 \otimes \bar{\nu}_1$ and $w_2(D^7) = \nu_2 + \nu_1 \otimes \bar{\nu}_1$. This concludes the proof of Proposition 3 for the manifold B^7.

As mentioned above, the proof of the corresponding statement for the manifold A^7 is similar and hence is omitted.

As an immediate consequence of this proposition we get the following

COROLLARY 4.1. (a) *Let Φ^c be a Pin^c-structure on $X^{7'}_{2k}$, and Φ^c_i, $i = 1, 2, \dots, 2k+1$, be the Pin^c-structure induced by Φ^c on the i-th copy of RP^6 contained in $\partial X^{7'}_{2k}$. Let Φ^c_0 be the Pin^c-structure induced by Φ^c on $FRP^6_k \subset \partial X^{7'}_{2k}$. Then all the Pin^c-structures Φ^c_i, $i = 1, 2, \dots, 2k+1$, coincide. Therefore, $[(FRP^6_k, \Phi^c_0)] = (2k+1)[(RP^6, \Phi^c_1)]$ in the cobordism group $\Omega^6_{Pin^c}$. Consequently, $\eta(FRP^6_k, \Phi^c_0) = \pm(2k+1)/16$ mod Z and $\eta^c(FRP^5_k) = \pm(2k+1)/16$ mod Z.*

(b) *The η^c-invariant (resp. the eta-invariant of the Pin^c-operator) completely detects homotopy projective spaces in dimension* 5 (*resp.* 6).

In order to prove this corollary and to finish the proof of Theorem A of Section 2, it sufficies now to recall that there exist, up to equivalence, 4 involutions of homotopy spheres in dimensions 5 and 6 ([20]). Therefore they must be our involutions T^5_i (resp. T^6_i), $i = 0, 1, 2, 3$. This concludes our study of 5 and 6-dimensional homotopy projective spaces.

5. Fintushel-Stern's exotic involution on S^4.

In this section we give a detailed study of the Fintushel-Stern exotic involution on S^4. In particular, we prove a fundamental technical proposition (Proposition 4 below), which gives a link between the Fintushel-Stern exotic involution on S^4 and involutions on higher-dimensional spheres. Roughly speaking, this proposition states that there exists an equivariant cobordism from (S^4, T^4_{FS}) to eight copies of (S^4, ant) and some explicitely described Z_2-manifold $(\tilde{M}^4_8 \simeq 8_\# S^2 \times S^2, T^4_8)$, which imbeds appropriately into an equivariant cobordism from $(S^5, \Sigma T^4_{FS})$ to eight copies of (S^5, ant) and some Z_2-manifold of the form (Σ^5, I^5_9). In a further section of this paper, we identify (Σ^5, I^5_9) as (S^5, ant), and this, together with an explicit form of the cobordism, enables us to identify $(S^5, \Sigma T^4_{FS})$ as our involution (S^5, T^5_3). Some of the results given in this section can be found in [23], [25].

Let us introduce the following conventions and notation which are more convenient for our present purposes. Let $\beta : S^l \times S^m \to S^m \times S^l$ be the permutation $(x, y) \mapsto (y, x)$. Using β we fix concordant and Z_2-equivariant decompositions $S^n = D^{l+1} \times S^m \cup_\beta D^{m+1} \times S^l$ (abbreviated to $S^3 = T_1 \cup_\beta T_2$ for $n = 3$ and $l = m = 1$). Let us note that all the "surgery" constructions performed in Sections 2 and 3, with the help of the maps G, $\bar{G}$, Γ and $\bar{\Gamma}$, can be translated into the language of these new decompositions by replacing these maps by suitable compositions $\bar{\beta}^{-1} F \bar{\beta}$, where $\bar{\beta} : D^{l+1} \times S^m \to S^m \times D^{l+1}$ is an obvious extension of β and F stands for one of the maps G, $\bar{G}$, Γ or $\bar{\Gamma}$. In particular, we use the same symbol F instead of $\bar{\beta}^{-1} F \bar{\beta}$, and again we talk about the surgery on F and keep the same notation $M_{\partial F}$, M_F and so on. Let N^n be a smooth submanifold of a manifold V^v, and let $\phi : D^{l+1} \times S^m \to N^n$

and $\psi : D^{t+1} \times S^s \to V^v$ be smooth imbeddings, where $m + l + 1 = n$, $s + t + 1 = v$ and $s \geq m$, $t \geq l$.

DEFINITION 5.1. *With the notation above, we say that ψ essentially extends ϕ (and write $\psi \sim \phi$) provided that there exists, for the first, an autodiffeomorphism $\bar{g}$ of the disc $D^{s+t+2} \supset S^{s+t+1} = D^{t+1} \times S^s \cup_\beta D^{s+1} \times S^t$ such that the restriction $g : S^{s+t+1} \to S^{s+t+1}$ of $\bar{g}$ maps $D^{t+1} \times S^s$ onto itself; and, for the second, an autodiffeotopy h_t of V^v from $h_0 = id$ to h_1, such that $h_1 \circ \psi \circ g : D^{t+1} \times S^s \to V^v$ conicides with ϕ on $D^{l+1} \times S^m$, and $h_1 \circ \psi \circ g(D^{t+1} \times S^s) \cap N^n = \phi(D^{l+1} \times S^m)$.*

Let us note that if $\Phi : D^{w+1} \times S^u \to U^{u+w+1} \supset V^v$ essentially extends ψ, then Φ essentially extends ϕ. We will also use an obvious equivariant version of this notion (and write $\psi \sim_{Z_2} \phi$) in the case of Z_2-manifolds N^n and V^v and equivariant maps (as usual, we take the ordinary "antipodal" Z_2-action on S^n and $D^{l+1} \times S^m$, and use equivariant maps g and h_t). In this case we say also that the surgery on ψ essentially extends the surgery on ϕ. The following proposition is a simple consequence of the definition above.

PROPOSITION 5.1. *With the notation above, if an imbedding ψ essentially extends an imbedding ϕ, then there exists a manifold $nV^v_{\partial\psi}$ diffeomorphic to the manifold $V^v_{\partial\psi}$, which contains $N^n_{\partial\phi}$ as a smooth submanifold. Moreover, if $N^n \subset \partial K^{n+1}$ and $V^v \subset \partial W^{n+1}$, where K^{n+1} is a proper submanifold of a manifold W^{v+1}, there exists a manifold nW^{v+1}_ψ diffeomorphic to W^{v+1}_ψ, which contains K^{n+1}_ϕ as a smooth submanifold. In fact, it sufficies to take $nV^v_{\partial\psi} = V^v_{\partial\Phi}$ and $nW^{v+1}_\psi = W^{v+1}_\Phi$ for $\Phi = h_1 \circ \psi \circ g$. An analogous statement is clearly true for the "punctured" bordism W'_ψ. Moreover, all these statements have obvious Z_2-equivariant analogues.*

REMARK 5.1. The following notational and terminological convention will be applied in forthcoming sections of this paper. An imbedding $\Phi = h_1 \circ \psi \circ g$ as described above will be said to be *ϕ-good* and equivalent to ψ. Assume we are given two increasing n-tuples $(M_1, \dots, M_n) \subset (\partial N_1, \dots, \partial N_n)$ of smooth manifolds and a sequence $\phi = \{\phi_i : D^{l_i+1} \times S^{m_i} \to M_i\}$ of smooth imbeddings such that ϕ_{i+1} essentially extends ϕ_i. Form a sequence $\Phi = \{\Phi_i : D^{l_i+1} \times S^{m_i} \to M_i\}$ of smooth imbeddings such that $\Phi_1 = \phi_1$ and Φ_{i+1} is Φ_i-good and equivalent to ϕ_{i+1}, and form the manifolds $M_{i\partial\Phi_i}$ and $N_{i\Phi_i}$ (denoted shortly by $M_{i\partial\Phi}$ and $N_{i\Phi}$ respectively). Then $M_{i\partial\Phi} \subset M_{i+1,\partial\Phi}$ and $N_{i\Phi} \subset N_{i+1,\Phi}$ in a natural way.

We will usually neglect the replacement of ϕ by Φ and write $(M_1, \dots, M_n)_{\partial\phi}$ to denote the n-tuple of manifolds $(M_{1\partial\Phi}, \dots, M_{n\partial\Phi})$, and analogously $(N_1, \dots, N_n)_\phi$ to denote $(N_{1\Phi}, \dots, N_{n\Phi})$. Thus $N_{i\phi}$ is obtained from N_i by attaching a handle of index m_i, and we will say that $(N_1, \dots, N_N)_\phi$ is the trace of a surgery on $(N_1, \dots, N_n)$ of the type $(m_1, \dots, m_n)$. An analogous notation will be used in the case of Z_2-manifolds and their quotient manifolds and punctured manifolds $N'_{i\phi}$. Of course the manifold $N_{i\Phi}$ itself depends on the choice of Φ, but its diffeomorphism type does not. In fact, any two such manifolds are diffeomorphic in a natural way, and we can find a diffeomorphism between them, which is diffeotopic to identity while restricted to $N_i \subset N_{i\Phi}$.

If $\chi^3 : D^2 \times S^1 \to S^3$ is an orientation-preserving imbedding such that $\chi^3(0 \times S^1)$ is a trivial knot, then the isotopy class of χ^3 is determined by the linking number of $\chi^3(0 \times S^1)$ and $\chi^3(x \times S^1)$ for any $D^2 \ni x \neq 0$. We denote by $\chi(k)$ such an imbedding with the corresponding linking number k. The surgery on χ^3 will be called *Dehn's surgery*.

Now let us formulate two simple technical lemmas, which will be needed later.

LEMMA 5.1. (a) *Let $\chi^3(\pm 2) : D^2 \times S^1 \to D^2 \times S^1 \subset S^3$ be equivariant. Let an equivariant orientation-preserving imbedding $\chi^4 : D^3 \times S^1 \to S^4$ satisfy $\chi^4 \sim_{Z_2} \chi^3(\pm 2)$. Then χ^4 is (non-equivariantly) isotopic to the standard imbedding $D^3 \times S^1 \to D^3 \times S^1 \cup_\beta D^2 \times S^2 = S^4$, and both of the maps G and $\Gamma : D^3 \times S^2 \to S^5$ essentially extend χ^4. Therefore the surgery on G (resp. Γ) essentially extends the surgery on χ^4 and hence the surgery on χ^3.*

(b) *Let $\phi^3 : D^2 \times S^1 \to D^2 \times S^1 \subset S^3 = D^2 \times S^1 \cup_\beta D^2 \times S^1$ be the natural imbedding. Let $\phi^4 : D^3 \times S^1 \to D^2 \times S^2 \subset S^4 = D^2 \times S^2 \cup_\beta D^3 \times S^1$ satisfy $\phi^4 \sim_{Z_2} \phi^3$. Let $\phi^5 : D^3 \times S^2 \to D^3 \times S^2 \subset S^5 = D^3 \times S^2 \cup_\beta D^3 \times S^2$ be the natural imbedding. Then $\phi^5 \sim_{Z_2} \phi^4$.*

The assertion (a) follows from the fact that $G : D^3 \times S^2 \to D^3 \times S^2$ is a bundle-morphism and $G|D^2 \times S^1 : D^2 \times S^1 \to D^2 \times S^1$ is given by $(v, x) \to (c_1(x)v, x)$, which is easily seen to be of the form $\chi^3(2)$. Similar observations apply to prove this assertion for Γ. (b) follows by an easy "isotopy" argument.

Let $\tilde{\chi}^3(2k) : D^2 \times S^1 \to D^2 \times S^1 \subset S^3$ be equivariant. Let $\tilde{\chi}^4 : D^3 \times S^1 \to D^3 \times S^1 \subset S^4$ satisfy $\tilde{\chi}^4 \sim_{Z_2} \tilde{\chi}^3$. Let $(\tilde{M}^5, T^5) = (S^4 \times I_1, \overline{ant}) \cup_{\tilde{\chi}^4} (S^4 \times I_2, \overline{ant})$, where $\tilde{\chi}^4$ is understood as a map: $(D^3 \times S^1 \times 1)_2 \to (D^3 \times S^1 \times 1)_1$. Then ∂M^5 contains two copies of RP^4, namely $RP^4_1 = S^4 \times 0_1/ant$ and $RP^4_2 = S^4 \times 0_2/ant$. Moreover, M^5 is obtained by gluing two copies of $RP^4 \times [0, 1]$ by the quotient imbedding $\chi^4 : D^3 \tilde{\times} RP^1 \times 1_2 \subset RP^4 \times I_2 \to D^3 \tilde{\times} RP^1 \times 1_1 \subset RP^4 \times I_1$.

LEMMA 5.2 (see [23]). *With the notation above, we have the following*:

(a) *M^5 is a Pin$^+$-manifold.*

(b) *Let Φ^+ be a Pin$^+$-structure on M^5, and Φ^+_i be the Pin$^+$-structure on $RP^4_i \subset \partial M^5$ induced by Φ^+. Then Φ^+_1 coincides with (resp. is inverse to) Φ^+_2 if and only if k is odd (resp. even). Consequently, $\eta(RP^4, \Phi^+_1) = \eta(RP^4, \Phi^+_2) \bmod 2Z$ if and only if k is odd.*

(c) *The isotopy class of an imbedding $\gamma : D^3 \tilde{\times} RP^1 \to RP^4$ representing the non-trivial element of $\pi_1(RP^4)$ is detected by the eta-invariant of the "source" RP^4 equipped with the Pin$^+$-structure transfered by γ from the standard Pin$^+$-structure on the "target" RP^4.*

PROOF. Proof of the assertion (a) is similar to that of Proposition 4.2. The assertion (b) is nothing but Lemma 6 in Section 4 of [23] and its proof is similar to that Proposition 4.3. The assertion (c) follows from (b) and the well-known fact that there exist precisely two isotopy classes of imbeddings $\gamma : D^3 \tilde{\times} RP^1 \to RP^4$ representing the generator of $\pi_1(RP^4)$, which differs by the (unique) non-trivial automorphism of the bundle $D^3 \tilde{\times} RP^1 \to RP^1$.

Now, let us recall that the Brieskorn sphere $\Sigma(3, 5, 19)$ is a characteristic submanifold for the Fintushel-Stern involution on S^4 ([10], [23]). It is a Seifert manifold over S^2 with associated unnormalized Seifert invariants ((1, 1); (3, −1); (5, −2); (19, −5)). Let P^3 be the involution on $\Sigma(3, 5, 19)$ "contained" in the natural S^1-action. In Proposition 5.2 below we will apply the following convention. A diffeomorphism $\Theta_i^3 : D^2 \times S^1 \to D^2 \times S^1$, $i = 2, 3, \dots$, will be considered also as an imbedding $\Theta_i^3 : D^2 \times S^1 \times 1_i \subset S^3 \times I_i \to D^2 \times S^1 \times 1_{i-1} \subset S^3 \times I_{i-1}$; for $\Theta_i^3 \sim_{Z_2} \Theta_i^4 : D^3 \times S^1 \subset S^4 \to D^2 \times S^2 \subset S^4$, $i = 2, 3, \dots$, we apply a similar convention. Moreover, an imbedding $(\Theta_1^3, \Theta_1^4) : (D^2 \times S^1, D^3 \times S^1) \to (\Sigma(3, 5, 19), S^4)$ will be identified with its copy: $(D^2 \times S^1 \times 1_1, D^3 \times S^1 \times 1_1) \subset (S^3 \times I_1, S^4 \times I_1) \to (\Sigma(3, 5, 19) \times 1, S^4 \times 1) \subset (\Sigma(3, 5, 19) \times I, S^4 \times I)$.

PROPOSITION 5.2. *There exist a sequence of equivariant maps $\Theta^3 = (\Theta_1^3, \dots, \Theta_8^3)$, where $\Theta_1^3 : (T_1, ant) \subset (S^3, ant) \to (\Sigma(3, 5, 19), P^3)$ and $\Theta_i^3 : (T_1, ant) \subset (S^3, ant) \to (T_2, ant) \subset (S^3$, ant) for $i > 1$, and the associated sequence of equivariant maps $\Theta^4 = (\Theta_1^4 \sim_{Z_2} \Theta_1^3, \dots, \Theta_8^4 \sim_{Z_2} \Theta_8^3)$, where $\Theta_1^4 : (D^3 \times S^1, ant) \to (S^4, T_{FS}^4)$ and $\Theta_i^4 : (D^3 \times S^1, ant) \subset (S^4, ant) \to (D^2 \times S^2, ant) \subset (S^4, ant)$ for $i > 1$, such that the following conditions are satisfied:*

Let two sequences of Z_2-manifolds defined to be

$$(\tilde{M}_0^5, T_0^5) = (S^4 \times I_0, T_{FS}^4 \times id),$$

and for $1 \le i \le 8$

$$(\tilde{M}_i^5, T_i^5) = (\tilde{M}^{5'}_{i-1,\Theta_i^4}, T^5_{i-1,\Theta_i^4}),$$

$$(\tilde{N}_0^4, P_0^4) = (S^4 \times 1_0, T_{FS}^4),$$

and for $1 \le i \le 8$

$$(\tilde{N}_i^4, P_i^4) = (\tilde{N}^4_{i-1,\partial\Theta_i^4}, P^4_{i-1,\partial\Theta_i^4}).$$

Then the following hold.

(a) *Any map Θ_i^3, $i = 2, 3, \dots, 8$ satisfies $\Theta^3 = \Theta^3(\pm 2)$.*

(b) *$\tilde{N}_i^4$ is the connected sum of i copies of $S^2 \times S^2$.*

(c) *$(\tilde{N}_8^4, P_8^4)$ contains, as a characteristic submanifold, a Z_2-manifold $(\tilde{N}_8^3, P_8^3)$ which is diffeomorphic to (S^3, ant).*

It is to be stressed that $\tilde{N}_8^3$ need not to be obtained from $\Sigma(3, 5, 19)$ by doing the Dehn surgeries on the maps Θ_i^3.

PROOF. We have to define appropriate maps Θ_i^3, Θ_i^4. In [23, pages 19–21] we proved that there exist two sequences of equivariant imbeddings $(\tilde{\chi}_1^3, \dots, \tilde{\chi}_5^3)$ and $(\tilde{\chi}_6^3, \dots, \tilde{\chi}_8^3)$ (resp. $(\tilde{\chi}_1^4, \dots, \tilde{\chi}_5^4)$ and $(\tilde{\chi}_6^4, \dots, \tilde{\chi}_8^4)$), such that the following conditions are satisfied:

(1) $\tilde{\chi}_1^3$ (resp. $\tilde{\chi}_6^3$): $(D^2 \times S^1, ant) \to (\Sigma(3, 5, 19), P^3) \subset (S^4, T_{FS})$ is a diffeomorphism onto $T(19, -5)$ (resp. $T(3, -1)$) $\subset \Sigma(3, 5, 19)$, an invariant normal neighbourhood of the singular orbit corresponding to the Seifert invariants (19, −5) (resp. (3, −1)).

(2) For any $i = 2, 3, 4, 5, 7, 8$, $\tilde{\chi}_i^3 : (D^2 \times S^1, ant) \to (D^2 \times S^1, ant) \subset (S^3, ant)$ is a diffeomorphism of the form $\chi^3(\pm 2)$.

(3) $\tilde{\chi}_1^4 : (D^3 \times S^1, ant) \to (S^4, T_{FS})$ satisfies $\tilde{\chi}_1^4 \sim_{Z_2} \tilde{\chi}_1^3$, and similarly for $\tilde{\chi}_6^4$.

(4) For any $i = 2, 3, 4, 5, 7, 8$, $\tilde{\chi}_i^4 : (D^3 \times S^1, ant) \to (D^2 \times S^2, ant) \subset (S^4, ant)$ satisfies $\tilde{\chi}_i^4 \sim_{Z_2} \tilde{\chi}_i^3$.

(5) Consider the map $\tilde{\chi}_1^3$ (resp. $\tilde{\chi}_6^3$) as a map from $T_1 \times 1_1 \subset S^3 \times I_1$ (resp. $T_1 \times 1_6 \subset S^3 \times I_6$) onto $T(19, -5) \subset \Sigma(3, 5, 19) \times 1_0 \subset \Sigma(3, 5, 19) \times [0, 1]_0 \subset S^4 \times [0, 1]_0$ (resp. $T(3, -1) \subset \Sigma(3, 5, 19) \times 1_0$), and let $\tilde{\chi}_1^4 \sim_{Z_2} \tilde{\chi}_1^3$, $\tilde{\chi}_6^4 \sim_{Z_2} \tilde{\chi}_6^3$. Similarly, consider the map $\tilde{\chi}_i^3$, $i = 2, 3, 4, 5, 7, 8$, as a map $T_1 \times 1_i \subset S^3 \times 1_i \to T_2 \times 1_{i-1} \subset S^3 \times I_{i-1}$, and let $\tilde{\chi}_i^4 \sim_{Z_2} \tilde{\chi}_i^3 : D^3 \times S^1 \times 1_i \to D^2 \times S^2 \times 1_{i-1}$. Let

$$(\tilde{K}_0^5, I_0^5) = (S^4 \times I_0, T_{FS}^4 \times id),$$

and for $1 \leq i \leq 8$

$$(\tilde{K}_i^5, I_i^5) = (\tilde{K}'_{i-1,\tilde{\chi}_i^4}, I_{i-1,\tilde{\chi}_i^4}).$$

Also, let

$$(\tilde{L}_0^4, J_0^4) = (S^4 \times 1_0, T_{FS}^4),$$

and for $1 \leq i \leq 8$

$$(\tilde{L}_i^4, J_i^4) = (\tilde{L}_{i-1,\partial\tilde{\chi}_i^4}, J_{i-1,\partial\tilde{\chi}_i^4}).$$

Then K_8^5 is a Pin^+-bordism from $FRP_{FS}^4 = S^4 \times 0_0 / T_{FS}^4$ to eight copies of RP^4 (namely $S^4 \times 0_i / ant$) and some manifold L_8^4. Moreover, $(\tilde{L}_8^4, J_8^4)$ contains, as a characteristic submanifold, a Seifert manifold $\tilde{M}_8^3$ which is Z_2-equivariantly diffeomorphic to (S^3, ant).

Note that any K_i^5 is a Pin^+-manifold by Lemma 5.2(a). Therefore, any $\tilde{L}_i^4$ is the connected sum of i copies of $S^2 \times S^2$ by standard arguments. Thus $(\tilde{L}_8^4, J_8^4)$ satisfies the conditions (b) and (c), and all the maps $(\tilde{\chi}_i^4, \tilde{\chi}_i^3)$ but $(\tilde{\chi}_6^4, \tilde{\chi}_6^3)$ satisfy the condition (a). Define $\Theta_i^3 = \tilde{\chi}_i^3$ and let $\Theta_i^4 \sim \Theta_i^3$ for $i \neq 6$. In order to define $(\tilde{\Theta}_6^4, \tilde{\Theta}_6^3)$, let us observe that there exists an equivariant diffeotopy $h_t : \tilde{L}_5^4 \to \tilde{L}_5^4$ such that $h_0 = id$, and Θ_6^4 (=by definition $h_1 \circ \tilde{\chi}_6^4$) maps $D^3 \times S^1 \times 1_6$ into $D^2 \times S^2 \times 1_5 \subset \tilde{L}_5^4$ and satisfies the condition that $\Theta_6^4 \sim_{Z_2} \Theta_6^3$ for some $\Theta_6^3(\pm 2k) : T_1 \times 1_6 \to T_2 \times 1_5 \subset S^3 \times 1_5$, where $k = 0$ or 1. (In fact, use 1-connectivity of $\tilde{L}_5^4$ to build an equivariant diffeotopy which "moves" $\tilde{\chi}_6^4(D^3 \times S^1 \times 1_6)$ from its initial position to $D^2 \times S^2 \times 1_5 \subset \tilde{L}_5^4$; next observe that $\Theta_6^4 \sim_{Z_2} \Theta^3(l) : T_1 \times 1_6 \to T_2 \times 1_5 \subset S^3 \times 1_5$ for some l, which must be even since Θ_6^4 is equivariant; then use the fact that there exist precisely two equivariant isotopy classes of automorphisms of the bundle $D^3 \times S^1 \to S^1$). It is clear now that $(\tilde{M}_8^5, \tilde{N}_8^4)$ is diffeomorphic to $(\tilde{K}_8^5, \tilde{L}_8^4)$ and therefore $(\tilde{N}_8^4, P_8^4)$ contains, as a characteristic submanifold, a manifold $(\tilde{N}_8^3, P_8^3) \simeq (S^3, ant)$, since $\tilde{L}_8^4$ enjoys an analogous property. This proves the assertion (c).

In order to complete the proof of the assertion (a), it sufficies now to show $k = \pm 1$. But

$$M_8^5 = FRP_{FS}^4 \times I \cup_{D^3 \tilde{\times} RP^1 \times 1} RP^4 \times I_1 \cup \cdots \cup_{D^3 \tilde{\times} RP^1} RP^4 \times I_8 .$$

Therefore, by Lemma 2(b), (c) above, it sufficies to prove that the Pin^+-structures $\Phi_i^+ = \Phi^+ | RP^4 \times 0_i \subset \partial M_8^5$, $i = 5, 6$, induced by any Pin^+-structure Φ^+ on M_8^5 coincide; or, equivalently, that for a fixed Pin^+-structure Φ^+ on K_8^5 the Pin^+-structures $\Phi_i^+ = \Phi^+ | RP^4 \times$

$0_i \subset \partial K_8^5$, $i = 5, 6$, coincide. In order to prove this assertion, let us note that L_8^4 is the connected sum of RP^4 and some 1-connected *Spin*-manifold L_z^4 and $\tilde{L}_8^4 = L_z^4\# - L_z^4$ (the connected sum of L_z^4 with some fixed orientation and L_z^4 with reversed orientation). We claim that the manifold L_z^4 is a *Spin*-boundary. In fact, $\tilde{L}_8^4$ is obtained from S^4 by doing eight surgeries on some imbeddings $D^3 \times S^1 \to S^4$, therefore $\tilde{L}_8^4$ is the connected sum of eight copies of $S^2 \times S^2$ (the possibility of appearing here $S^2 \tilde{\times} S^2$, the S^2-bundle over S^2 with $w_2(S^2 \tilde{\times} S^2) \neq 0$, is excluded, since $\tilde{L}_8^4$ is a *Spin*-manifold). Thus L_z^4 is a *Spin*-manifold with the second Betti number $= 8$, and hence sign $(L_z^4) = 0$ by Rohlin's theorem. Therefore L_z^4 is a *Spin*-boundary, as claimed.

Consequently, $\eta(L_8^4, \Phi_L^+) = \eta(RP^4, \Phi_z^+) \bmod 2Z$, where Φ_L^+ is the Pin^+-structure on L_8^4 determined by Φ^+ and Φ_z^+ is the Pin^+-structure on RP^4 determined by the restriction of Φ^+ to the RP^4-part of L_8^4. Now, using Lemma 2(b) above, one sees easily that

$$\eta(RP^4, \Phi_1^+) = \eta(RP^4, \Phi_2^+) = \cdots = \eta(RP^4, \Phi_5^+) \bmod 2Z$$

and

$$\eta(RP^4, \Phi_6^+) = \eta(RP^4, \Phi_7^+) = \eta(RP^4, \Phi_8^+) \bmod 2Z$$

(see the definition of $\tilde{\chi}_i^3$ and θ_i^3). In [23] we proved that $\eta(FRP_{FS}^4, \Phi) = \pm 7/8 \bmod 2Z$ for any Pin^+-structure Φ. Without loss of generality we can assume that $\eta(FRP_{FS}^4, \Phi_0^+) = 7/8 \bmod 2Z$, where $\Phi_0^+ = \Phi^+ | FRP_{FS}^4$. Thus

$$7/8 = \eta(FRP_{FS}^4, \Phi_0^+) = 5\eta(RP^4, \Phi_5^+) + 3\eta(RP^4, \Phi_6^+) + \eta(RP^4, \Phi_z^+) .$$

Since $\eta(RP^4, \Phi) = \pm 1/8 \bmod 2Z$, it follows that $\eta(RP^4, \Phi_6^+) = \eta(RP^4, \Phi_5^+) \bmod 2Z$. Thus $\Phi_6^+ = \Phi_5^+$ and the assertion (a) is proved.

The argument for (b) is the same as the analogous statement for $\tilde{L}_8^4$ given above. Now Proposition 2 is proved.

REMARK 5.2. Proposition 3 above shows that the Fintushel-Stern involution on S^4 can be obtained from an involution on $4_\# S^2 \times S^2 \# 4_\# S^2 \times S^2$, which permutes the two copies of $4_\# S^2 \times S^2$ and "desuspends" to the ordinary antipodal involution on S^3 by killing $H_2(4_\# S^2 \times S^2 \# 4_\# S^2 \times S^2)$ by an equivariant surgery. We suspect that a bit more detailed description of this surgery could provide a transparent formula for T_{FS}.

Now we can formulate the main technical proposition of this section, which provides us with a suitable tool (an equivariant stratified cobordism), that will enable us later to identify ΣT_{FS}^4. Let

$$\Theta_i^5 = G : D^3 \times S^2 \times 1_i \subset S^5 \times I_i \to D^3 \times S^2 \times 1_{i-1} \subset S^5 \times I_{i-1} ,$$

and

$$\Theta_i^6 = \bar{G} : D^3 \times S^3 \times 1_i \subset S^6 \times I_i \to D^3 \times S^3 \times 1_{i-1} \subset S^6 \times I_{i-1}$$

for $i = 2, 4, 6, 8$. Similarly,

$$\Theta_i^5 = \Gamma : D^3 \times S^2 \times 1_i \subset S^5 \times I_i \to D^3 \times S^2 \times 1_{i-1} \subset S^5 \times I_{i-1} ,$$

and

$$\Theta_i^6 = \bar{\Gamma} : D^4 \times S^2 \times 1_i \subset S^6 \times I_i \to D^4 \times S^2 \times 1_{i-1} \subset S^6 \times I_{i-1}$$

for $i = 3, 5, 7$. In order to define $\Theta_1^5 : D^3 \times S^2 \times 1_1 \to D^3 \times S^2 \times 1_0$ and $\Theta_1^6 =: D^4 \times S^2 \times 1_1 \to D^4 \times S^2 \times 1_0$, we proceed as follows. There exist precisely two isotopy classes of smooth imbeddings $h : D^3 \times S^1 \to S^4$ detected by the trace of the surgery on h, and h extends to an imbedding $D^3 \times S^2 \to S^5$ if and only if the trace of the surgery on h is $S^2 \times S^2$. This is precisely the case of the map Θ_1^4 of Proposition 2, and we define $\Theta_1^5 : D^3 \times S^2 \times 1_1 \subset S^5 \times 1_1 \to D^3 \times S^2 \times 1_0 \subset S^5 \times 1_0$ to be an equivariant extension of Θ_1^4 such that $\Theta_1^5(D^3 \times S^2) \cap S^4 = \Theta_1^4(D^3 \times S^1)$. Next we define $\Theta_1^6 : D^4 \times S^2 \times 1_1 \subset S^6 \times 1_1 \to D^4 \times S^2 \times 1_0 \subset S^6 \times 1_0$ to be an equivariant extension of Θ_1^5 such that $\Theta_1^6(D^4 \times S^2) \cap S^5 = \Theta_1^5(D^3 \times S^2)$. It follows from Proposition 2 and Lemma 1 together with the definition of the maps $G, \Gamma, \bar{G}, \bar{\Gamma}$ that Θ_i^n essentially extends Θ_i^{n-1} for $i = 1, \dots, 8$ and $n = 5, 6$.

Thus, using Propositions 1 and 2 and Remark 1, we can form a triple of Z_2-manifolds

$$((\tilde{M}_i^5, T_i^5), (\tilde{M}_i^6, T_i^6), (\tilde{M}_i^7, T_i^7)), \quad i = 0, \dots, 8$$

such that

$$(\tilde{M}_0^{n+1}, T_0^{n+1}) = (S^n \times I_0, \Sigma^{n-4} T_{FS}^4 \times id),$$

where $n = 4, \dots, 6$, and $\Sigma^0 T_{FS}^4 = T_{FS}^4$, and

$$(\tilde{M}_i^{n+1}, T_i^{n+1}) = (\tilde{M}_{i-1,\Theta_i^n}^{n+1'}, T_{i-1,\Theta_i^n}^{n+1}).$$

Similarly, a triple of Z_2-manifolds

$$((\tilde{N}_i^4, P_i^4), (\tilde{N}_i^5, P_i^5), (\tilde{N}_i^6, P_i^6)), \quad i = 0, \dots, 8$$

such that

$$(\tilde{N}_0^n, P_0^n) = (S^n \times 1_0, \Sigma^{n-4} T_{FS}^4),$$

and

$$(\tilde{N}_i^n, P_i^n) = (\tilde{N}_{i-1,\partial\Theta_i^n}^n, P_{i-1,\partial\Theta_i^n}^n).$$

It is clear that $(\tilde{M}_i^n, T_i^n)$ (resp. $(\tilde{N}_i^{n-1}, P_i^{n-1})$) is a characteristic submanifold of $(\tilde{M}_i^{n+1}, T_i^{n+1})$ (resp. $(\tilde{N}_i^n, P_i^n)$), $n = 5, 6, 7$, $i = 0, \dots, 8$. An argument completely analogous to the one used in the proof of Proposition 1 in Section 4 shows that $\tilde{N}_i^5$ is a homotopy sphere for any even integer i. Moreover, an easy homological argument shows that $\tilde{N}_i^6$ is a homotopy sphere for even i. Thus we get the following

PROPOSITION 5.3. *There exists a smooth Z_2-manifold $(\tilde{M}^7, T^7)$ (where T^7 is fixed point-free) such that the following conditions are satisfied:*

(a) $\partial(\tilde{M}^7, T^7) = (S^6, \Sigma^2 T_{FS}^4) \sqcup 8(S^6, ant) \sqcup (\tilde{N}^6, P^6)$, *where P^6 is an involution of a homotopy sphere $\tilde{N}^6$.*

(b) *P^6 desuspends to an involution P^5 of a homotopy sphere $\tilde{N}^5 \subset \tilde{N}^6$.*

(c) $(\tilde{N}^5, P^5)$ *has a characteristic submanifold* $(\tilde{N}^4, P^4)$ *such that* $\tilde{N}^4$ *is diffeomorphic to the connected sum of eight copies of* $S^2 \times S^2$. *Moreover,* $(\tilde{N}^4, P^4)$ *has a characteristic submanifold* $(\tilde{N}^3, P^3) \simeq (S^3, ant)$. *Thus* N^3, N^5 *and* N^6 *are homotopy projective spaces.*

(d) $\partial M^7 = FRP^6_{FS} \sqcup 8RP^6 \sqcup N^6$, *where* $FRP^6_{FS} = S^6/\Sigma^2 T^4_{FS}$. *Moreover,* M^7 *is a* Pin^c*-manifold and any* Pin^c*-structure* Φ^c *on* M^7 *induces the same* Pin^c*-structure on any of two copies of* $RP^6 \subset \partial M^7$. *Consequently,* $\eta(FRP^6_{FS}, \Phi^c_{FS}) = 8\eta(RP^6, \Phi^c_1) + \eta(N^6, \Phi^c_N)$ *mod* Z, *where* Φ^c_1 *(resp.* Φ^c_N, *resp.* Φ^c_{FS}*) is the* Pin^c*-structure on* RP^6 *(resp.* N^6, *resp.* FRP^6_{FS}*) induced by* Φ^c.

PROOF. We define $(\tilde{M}^7, T^7)$ to be $(\tilde{M}^7_8, T^7_8)$ of Proposition 2 and apply Propositions 1 and 2 to prove (a), (b) and (c). The assertion (d) follows immediately by Proposition 3 of Section 4.

6. Proof of main theorems. In this section we prove Theorems B and C in Section 1, thus establishing all the results of this paper. Let us start with two technical lemmas which will be needed later.

LEMMA 6.1. *Let* T^n *be an involution on* S^n *which desuspends to an involution* T^{n-1} *on a homotopy sphere* $\Sigma^{n-1} \subset S^n$, *where* $n = 4, 5, 6$. *If* $n = 4$, *assume additionally that* Σ^{n-1} *cuts* S^n *into two submanifolds diffeomorphic to the ordinary disc* D^4 *(note that an analogous condition for* $n > 4$ *is always satisfied). If* (Σ^{n-1}, T^{n-1}) *is equivalent to* (S^{n-1}, ant), *then* (Σ^n, T^n) *is equivalent to* (S^n, ant).

This follows from the fact that any autodiffeomorphism of S^{n-1} extends to an autodiffeomorphism of D^n for $n = 4, 5, 6$ ([9],[17]). Now we can formulate a technical lemma which enables us to detect certain involutions of 5-dimensional spheres basing on some data about their characteristic submanifolds.

LEMMA 6.2. *Let* P^5 *be a smooth fixed point-free involution of* S^5 *such that the following two conditions are satisfied:*

(a) *There exists a characteristic submanifold* $(\tilde{X}^4, P^4) \subset (S^5, P^5)$ *such that* $\tilde{X}^4$ *is diffeomorphic to the connected sum of* $2k$*-copies of* $S^2 \times S^2$ *for some integer* $k < 8$.

(b) *There exists a characteristic submanifold* $(\tilde{X}^3, P^3) \subset (\tilde{X}^4, P^4)$ *such that* $(\tilde{X}^3, P^3)$ *is diffeomorphic to* (S^3, ant).

Then P^5 *is smoothly conjugated to the ordinary antipodal map ant:* $S^5 \to S^5$.

PROOF. We give only a brief outline of the proof. The details, which are standard and rather dull, could be provided by the reader without undue difficulty.

First, let us note that $\tilde{X}^4 = L_0 \cup_{S^3} L_0'$ for some 1-connected *Spin*-manifold L_0 and $L_0' = P^4(L_0)$. Thus $\tilde{X}^4$ is the equivariant connected sum of two copies of some 1-connected *Spin*-manifold L. Moreover, the second Betti number of L is < 16 by the assumption (a). Consequently, $|\text{sign}(L)| < 16$, and therefore $\text{sign}(L) = 0$ by the Rohlin theorem. Thus L is stably diffeomorphic to the connected sum of k copies of $S^2 \times S^2$ by a well-known theorem of Wall ([30]).

Now we build an equivariant cobordism from $C_0 = ((\tilde{X}^4, P^4), (S^5, P^5), (S^6, \Sigma P^5))$ to $((S^4, ant), (S^5, ant), (S^6, ant))$ as follows. First we do on C_0 a sequence of l stratified equivariant surgeries of type $(1, 2, 2)$, away from $S^3 \subset \tilde{X}^4$, so as to get a Z_2-cobordism from C_0 to $C_1 = ((k+l)_\# S^2 \times S^2 \cup_{S^3} (k+l)_\# S^2 \times S^2, P_1^4), (\tilde{M}_1^5, P_1^5), (\tilde{M}_1^6, P_1^6))$. Next, we kill $\pi_2((k+l)_\# S^2 \times S^2 \cup_{S^3} (k+l)_\# S^2 \times S^2)$ by a sequence of stratified equivariant surgeries of type $(2, 2, 2)$, away from S^3, and we get a new triple $((S^4, P_2^4), (\tilde{M}_2^5, P_2^5), (\tilde{M}_2^6, P_2^6))$. Now, we kill $\pi_2(\tilde{M}_2^5)$ by a sequence of equivariant surgeries of type $(2, 2)$ on $(\tilde{M}_2^5, \tilde{M}_2^6)$, away from S^4, and we get $((S^4, P_2^4), (S^5, P_3^5), (\tilde{M}_3^6, P_3^6))$. Finally, we kill $\pi_2(\tilde{M}_3^6)$ and $\pi_3(\tilde{M}_3^6)$ by equivariant surgery, away from S^5, and we get a quadruple of Z_2-manifolds $((S^3, ant), (S^4, P_2^4), (S^5, P_3^5), (S^6, P_4^6))$. Using Lemma 6.1 we see that all these involutions are equivalent to the standard "antipodal" involution. Passing to quotient manifolds, we get a Pin^c-cobordism from $(FRP^5 = S^5/P^5, FRP^6 = S^6/\Sigma P^5)$ to (RP^5, RP^6). Thus $\eta_c(FRP^5) = \pm 1/16 \bmod Z$, and Lemma 6.2 follows from Corollary 4.1(b).

Essentially the same arguments apply to prove the following generalization of Lemma 6.2.

LEMMA 6.3. *Let P^5 be an involution of S^5 such that the following two conditions are satisfied:*

(a) *There exists a 1-connected characteristic submanifold $(\tilde{X}^4, P^4) \subset (S^5, P^5)$ and a characteristic submanifold $(\tilde{X}^3, P^3) \subset (\tilde{X}^4, P^4)$ such that $(\tilde{X}^3, P^3)$ is diffeomorphic to (S^3, ant).*

(b) *Let $\tilde{X}^4 = L_0 \cup_{\tilde{X}^3} L_0'$, where $L_0' = P^4(L_0)$.*

Assume that $\operatorname{sign}(L_0) = 0$. *Then P^5 is smoothly conjugated to the ordinary antipodal map ant: $S^5 \to S^5$.*

REMARK 6.1. Lemma 6.2 is no longer true if we relax the assumption that $k \leq 8$. However, it remains valid also for $k = 8$ by a theorem of Donaldson. In fact, $\operatorname{sign}(L_0) = 0$ also in this case, since L_0 is a smooth 1-connected 4-manifold with $b_2(L_0) = 16$, the boundary of which is S^3, and which has even intersection form, and we get the required equation combining theorems of Rohlin and Donaldson.

Now the proof of Theorem B is immediate. We apply Lemmas 6.1 and 6.2 to the objects described in Proposition 3 of the previous section, and we see that $(\tilde{N}^5, P^5) \simeq (S^5, ant)$ and $(\tilde{N}^6, P^6) \simeq (S^6, ant)$, and therefore $\eta(N^6, \Phi_N^c) = \pm\eta(RP^6, \Phi^c) = \pm 1/16 \bmod Z$. Thus

$$\eta(FRP_{FS}^6, \Phi_{FS}^c) = \pm 7/16 \bmod Z$$

and

$$\eta^c(FRP_{FS}^5) = \pm 7/16 \bmod Z\,.$$

Consequently, $FRP_{FS}^6 = S^6/\Sigma^2 T_{FS}^4$ (resp. $FRP_{FS}^5 = S^5/\Sigma T_{FS}^4$) is diffeomorphic to FRP_3^6 (resp. FRP_3^5) by Corollary 1 in Section 4, thus proving Theorem B.

Now let us turn to the proof of Theorem C.

We start with the "if"-implication. So let us assume that the involutions ΣT_1^4 and ΣT_2^4 are smoothly conjugated. Then we can regard the Z_2-manifold $(\tilde{Z}^6, T^6) = (S^5 \times I, \Sigma T_1^4)$ as an equivariant cobordism from $(S^5, \Sigma T_1^4)$ to $(S^5, \Sigma T_2^4)$ having a fixed identification $(S^5, \Sigma T_2^4) \simeq (S^5 \times 1, \Sigma T_1^4)$. Thus $(S^5 \times 0, \Sigma T_1^4)$ contains (S^4, T_1^4) as a characteristic submanifold, and (S^4, T_2^4) is a characteristic submanifold of $(S^5 \times 1, \Sigma T_1^4)$. Fix a classifying map $f_1 : FRP_1^5 = S^5 \times 0/\Sigma T_1^4 \to RP^N$, N large, for the Z_2-bundle $S^5 \to FRP_1^5$ such that f_1 is transversal to RP^{N-1} and $f_1^{-1}(RP^{N-1}) = FRP_1^4 = S^4 \times 0/T_1^4$. Similarly, fix a classifying map $f_2 : FRP_1^5 \simeq FRP_2^5 = S^5 \times 1/\Sigma T_2^4 \to RP^N$ for the Z_2-bundle $S^5 \to FRP_2^5$ such that f_2 is transversal to RP^{N-1} and $f_2^{-1}(RP^{N-1}) = FRP_2^4 = S^4 \times 0/T_2^4$ (the existence of such maps follows by an easy obstruction-theoretic argument). Since both f_1 and f_2 are classifying maps for the unique non-trivial Z_2-bundle over FRP_1^5 and $FRP_2^5 \simeq S^5 \times 1/\Sigma T_1^4$ respectively, they extend simultaneously to the classifying map $F : Z^6 = FRP_1^5 \times I \to RP^N$ for the non-trivial Z_2-bundle $S^5 \times I \to FRP_1^5 \times I$. Make F transversal to RP^{N-1} so as $F^{-1}(RP^{N-1}) \cap \partial Z^6 = FRP_1^4 \sqcup FRP_2^4$, and let Z^5 be the connected component of the manifold $F^{-1}(RP^{N-1})$ which contains FRP_1^4. An easy point-set argument then shows that Z^5 contains also FRP_2^4, and therefore $Z^5 \subset Z^6$ is a cobordism from FRP_1^4 to FRP_2^4. Let $\tilde{Z}^5 \subset \tilde{Z}^6$ be the obvious cover of Z^5, and observe that $(\tilde{Z}^5, T^5 = T^6|\tilde{Z}^5)$ is a characteristic submanifold of $(\tilde{Z}^6, T^6)$. Convert $\tilde{Z}^5$ into a 1-connected characteristic submanifold using appropriate equivariant intrinsic surgery (attaching to the original $\tilde{Z}^5$, away from its boundary, some pairs of 2-handles inside $\tilde{Z}^6$). Denote this new characteristic submanifold again by $\tilde{Z}^5$ and its quotient by Z^5.

We will show that Z^5 is now a Pin^+-cobordism from FRP_1^4 to FRP_2^4. In order to prove this assertion, let us observe that T^6 preserves orientation of $\tilde{Z}^6$, while T^5 reverses orientation of $\tilde{Z}^5$, and therefore any circle S in Z^5 reverses orientation of Z^5 if and only if it reverses orientation of the normal bundle ξ to Z^5 in Z^6. Thus $w_1(TZ^5) = w_1(\xi)$ and $w_2(TZ^6)|Z^5 = w_2(TZ^5 \oplus \xi) = w_2(TZ^5) + w_1^2(\xi)$. But $\pi_1(Z^5) = \pi_1(Z^6) = Z_2$ and one sees easily that $w_1(\xi) = w|Z^5$, where w is the generator of $H^*(Z^6; Z_2) \simeq H^*(FRP_1^5; Z_2)$, which is the truncated polynomial algebra over Z_2 generated by $w \in H^1(FRP_1^5; Z_2)$. Moreover $w_2(TZ^6) = w^2$. Thus, using the expression for $w_2(TZ^6)|Z^5$ given above, we get $w^2|Z^5 = w_2(Z^5) + w^2|Z^5$ and $w_2(Z^5) = 0$, as claimed. Therefore FRP_1^4 is Pin^+-cobordant to FRP_2^4. But FRP_1^4 is homeomorphic to FRP_2^4 by the Freedman's topological s-cobordism theorem, and hence FRP_1^4 is stably diffeomorphic to FRP_2^4 or to $FRP_2^4 \# K$ (where K is the Kummer surface) by [15]. But $[K] \neq 0$ in $\Omega_4^{Pin^+}$. Consequently, FRP_1^4 is stably diffeomorphic to FRP_2^4. This proves the "if" part of Theorem C.

In order to prove the "only if" implication of Theorem C, let us recall that there exist precisely two stable diffeomorphism classes of 4-dimensional homotopy projective spaces represented by RP^4 and FRP_{FS}^4 (in order to prove this use the above-mentioned result of [15] and the fact that the eta-invariant of the Pin^+-operator completely detects elements of $\Omega_4^{Pin^+} = Z_{16}$). Thus it sufficies to prove that, given an involution T_0^4 on S^4, ΣT_0^4 is smoothly conjugated to the usual antipodal involution (resp. to ΣT_{FS}^4) provided that $FRP_0^4 = S^4/T_0^4$

is stably diffeomorphic to RP^4 (resp. to FRP^4_{FS}). So let us assume that FRP^4_0 is stably diffeomorphic to FRP^4_{FS}, and fix an even integer k so as $FRP^4_0\#k_\#S^2\times S^2 \simeq FRP^4_{FS}\#k_\#S^2\times S^2$. Let $C_0 = (X^5, S^5\times I, S^6\times I)$ be a standard cobordism from $(k_\#S^2\times S^2, S^5, S^6)$ to (S^4, S^5, S^6) (we use the fact that k is even to construct such a cobordism). Fix an arc $\gamma\subset X^5$, which meets $k_\#S^2\times S^2$ (resp. S^4) $\subset \partial X^5$ transversally, precisely at $\gamma(0)$ (resp. $\gamma(1)$). Similarly, fix an analogous arc $\delta\subset FRP^4_{FS}\times I$. Next, form the connected sum along γ and δ of C_0 and $C_1 = (FRP^4_{FS}\times I, FRP^5_{FS}\times I, FRP^6_{FS}\times I)$. In this way, we get a Pin^c-cobordism C_2 from $(FRP^4_{FS}\#k_\#S^2\times S^2 \simeq FRP^4_0\#k_\#S^2\times S^2, FRP^5_{FS}, FRP^6_{FS})$ to $(FRP^4_{FS}, FRP^5_{FS}, FRP^6_{FS})$.

Now, proceeding analogously as in the proof of Lemma 6.2, we kill $k_\#S^2\times S^2\subset FRP^4_0\#k_\#S^2\times S^2$ by stratified surgery, and subsequently kill π_2 and π_3 of 5 and 6-dimensional members of the new triple of manifolds, leaving the 4-dimensional member unchanged. In this way, we get a Pin^c-cobordism from $(FRP^4_{FS}, FRP^5_{FS}, FRP^6_{FS})$ to (FRP^4_0, X^5, X^6), where X^5 and X^6 are certain homotopy projective spaces. But $(\tilde{X}^5, T^5_0)$ (resp. $(\tilde{X}^6, T^6_0)$), where $(\tilde{X}^n, T^n_0)$ is the universal cover of X^n, is the smooth suspension ΣT^4_0 (resp. the double smooth suspension $\Sigma^2 T^4_0$) of T^4_0. Thus $\eta_c(X^5) = \pm 7/8 \bmod Z$, and therefore ΣT^4_0 is equivalent to ΣT^4_{FS}, as claimed.

The proof of the corresponding assertion for FRP^4_0 stably diffeomorphic to RP^4 is completely analogous, and hence is omitted. This concludes the proof of Theorem C. Now all the assertions in this paper are proved.

REFERENCES

[1] D. W. ANDERSON, E. H. BROWN AND F. P. PETERSON, SU-cobordism, KO-characteristic numbers, and the Kervaire invariant, Ann. Math. 83 (1966), 54–67.

[2] D. W. ANDERSON, E. H. BROWN AND F. P. PETERSON, The structure of the Spin-cobordism ring, Ann. Math. 86 (1967), 271–298.

[3] M. F. ATIYAH, V. K. PATODI AND I. M. SINGER, Spectral assymetry and Riemannian geometry I, II, III, Math. Proc. Cambridge Philos. Soc. 77 (1975), 43–69, 78 (1975), 405–432, 79 (1976), 71–99.

[4] A. BAHRI AND P. B. GILKEY, Pin-cobordism and equivariant Spin-cobordism of cyclic 2-groups, Proc. Amer. Math. Soc. 99 (1987), 380–382.

[5] A. BAHRI AND P. B. GILKEY, The eta-invariant, Pin-bordism, and equivariant Spin-bordism for cyclic 2-groups, Pacific J. Math. 128 (1987), 1–24.

[6] W. BROWDER, Surgery on simply-connected manifolds, Moscow (1984) (Russian edition).

[7] W. BROWDER AND G. R. LIVESAY, Fixed point free involutions on homotopy spheres, Bull. Amer. Math. Soc. 73 (1967), 242–245.

[8] S. E. CAPPELL AND J. L. SHANESON, Some new 4-manifolds, Ann. of Math. 104 (1976), 61–72.

[9] J. CERF, Sur les diffeomorphismes de la sphere de dimension trois, Lecture Notes in Math. 53, Springer-Verlag, Berlin-New York, 1968.

[10] R. FINTUSHEL AND R. J. STERN, An exotic free involution on S^4, Ann. of Math. 113 (1981), 357–365.

[11] C. H. GIFFEN, Smooth homotopy projective spaces, Bull. Amer. Math. Soc. 75 (1969), 509–513.

[12] C. H. GIFFEN, Weakly complex involutions and cobordism of projective spaces, Ann. of Math. 90 (1969), 418–432.

[13] P. B. GILKEY, The eta-invariant for even dimensional Pin-manifolds, Adv. Math. 58 (1985), 243–284.

[14] R. GOMPF, Killing the Akbulut-Kirby sphere, Topology 30 (1991), 97–115.

[15] I. HAMBLETON, M. KRECK AND P. TEICHNER, Nonorientable 4-manifolds with fundamental group of order 2, Trans. Amer. Math. Soc. 344 (1994), 649–665.

[16] D. HUSEMOLLER, Fibre bundles, McGraw-Hill Book Co., New York, 1966.
[17] M. A. KERVAIRE AND J. W. MILNOR, Groups of homotopy spheres I, Ann. of Math. 77 (1963), 504–537.
[18] R. KIRBY, Problems in low-dimensional manifold theory, Proc. Sympos. Pure. Math. 32 (1978), 273–312.
[19] R. KIRBY, 4-Manifold problems, Contemp. Math. 35, 513–528, Amer. Math. Soc., Providence, RI, 1984.
[20] S. LOPEZ DE MEDRANO, Involutions on manifolds, Ergebnisse der Mathematik 59, Springer, 1971.
[21] R. MANDELBAUM, Four dimensional topology, Moscow, 1981 (russian edition).
[22] W. D. NEUMANN AND F. RAYMOND, Seifert manifolds, plumbing, μ-invariant and orientation-reversing maps, Algebraic and geometric topology, 163–196, Lecture Notes in Math. 664, Springer-Verlag, Berlin, 1978.
[23] W. J. OLEDZKI, On the eta-invariant of Pin^+-operator on some exotic 4-dimensional projective space, Compositio. Math. 78 (1991), 1–27.
[24] W. J. OLEDZKI, Some exotic projective spaces detected by spectral invariants, to appear.
[25] W. J. OLEDZKI, Exotic involutions of spheres detected by spectral invariants, Preprint Inst. of Math. of Polish Acad. Sci. 534 (1995), 1–308.
[26] W. J. OLEDZKI, Some topological formulas for the eta-invariant of Dirac-type operators, in preparation.
[27] P. ORLIK AND C. P. ROURKE, Free involutions on homotopy $(4k+3)$-spheres, Bull. Amer. Math. Soc. 74 (1968), 949–953.
[28] S. STOLZ, Exotic structures on 4-manifolds detected by spectral invariants, Invent. Math. 94 (1988), 147–162.
[29] R. E. STONG, Notes on cobordism theory, Princeton University Press, Princeton, New Jersey, 1968.
[30] C. T. C. WALL, Surgery on Compact Manifolds, Academic Press, London-New York, 1970.

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