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NON-ISOTROPIC HARMONIC TORI IN COMPLEX PROJECTIVE SPACES AND CONFIGURATIONS OF POINTS ON RATIONAL OR ELLIPTIC CURVES

TETSUYA TANIGUCHI

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Abstract. Recently, McIntosh develops a method of constructing all non-isotropic harmonic tori in a complex projective space in terms of their spectral data. In this paper, we classify all spectral data whose spectral curves are smooth rational or elliptic curves. We also construct explicitly corresponding harmonic maps.

1. Introduction. Harmonic maps of a two-sphere in complex Grassmann manifolds have been extensively studied and classified in [3], [20] and [21]. On the other hand, until recently not much has been known for harmonic maps of two-tori in these manifolds. However, concerning harmonic two-tori in a compact symmetric space of rank one, it has been known that any non-conformal harmonic torus can be obtained by integrating certain commuting Hamiltonian flows (cf. [2]). Also, it was proved by Burstall [1] that any non-superminimal harmonic torus in a sphere or a complex projective space is covered by a primitive harmonic map of finite type into a certain generalized flag manifold. Furthermore, Udagawa [19] generalized Burstall's result to those harmonic tori into a complex Grassmann manifold $G_2(\boldsymbol{C}^4)$ of 2-dimensional complex linear subspaces in $\boldsymbol{C}^4$ and, by using a Symes formula, constructed weakly conformal non-superminimal harmonic maps from the complex line to $G_2(\boldsymbol{C}^4)$. Employing these facts, as well as algebro-geometric methods, McIntosh has recently constructed a significant correspondence between the following spaces: the space of non-isotropic, linearly full harmonic maps into a complex projective n-space, $\psi : \boldsymbol{R}^2 \to \boldsymbol{C}P^n$, of finite type, up to isometries, and that of spectral data, that is, triplets $(X, \pi, \mathcal{L})$ consisting of a real, complete, connected algebraic curve X (called the spectral curve for ψ), a rational function π on X and a line bundle $\mathcal{L}$ over X, which satisfy certain conditions (cf. [12] and [13]).

The purpose of this paper is to determine all spectral data $(X, \pi, \mathcal{L})$ for which the spectral curve X is a smooth rational or elliptic curve (Theorems 3.1 and 3.5). Corresponding to them, we construct non-trivial examples of harmonic maps of two-tori into complex projective spaces. Moreover, we prove a criterion on the periodicity of these harmonic maps (Theorems 3.3 and 3.7).

This paper is organized as follows. In Section 3, we recall the definition of spectral data introduced by McIntosh. All spectral data with smooth rational or elliptic spectral curves

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are classified (Theorems 3.1 and 3.5), and corresponding harmonic maps are explicitly constructed (Theorems 3.2 and 3.6). Moreover, we prove a necessary and sufficient condition for a constructed harmonic map to be doubly periodic (Theorems 3.3 and 3.7). We also construct some examples of harmonic tori by using the method developed in this section. In Sections 4 and 5, the proofs of Theorems 3.1 and 3.5 are given respectively. Section 6 is devoted to proving Theorems 3.2 and 3.6. Finally, in Section 7, we introduce certain homomorphisms into generalized Jacobians of spectral curves and prove Theorem 3.7.

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2. Jacobi's theta functions and Weierstrass' zeta functions. C. G. J. Jacobi introduced four functions θ_1, θ_2, θ_3 and θ_4 of variables $p(u) = \exp(\pi\sqrt{-1}u)$ and $q = \exp(\pi\sqrt{-1}\tau)$, where u is the usual covering coordinate of an elliptic curve $X = \boldsymbol{C}/\boldsymbol{L}$ and τ stands for its period ratio with familiar standardization that the imaginary part $\mathrm{Im}\tau$ of τ is positive. If we take $\boldsymbol{L}$ to be $\boldsymbol{Z} \oplus \tau\boldsymbol{Z}$ for simplicity, then these Jacobi's theta functions are given as follows:

$$\begin{aligned}
\theta_1(u) &= \theta_1(u \mid \tau) = \sqrt{-1}\sum(-1)^n p^{2n-1} q^{(n-1/2)^2}\,, \\
\theta_2(u) &= \theta_2(u \mid \tau) = \sum p^{2n-1} q^{(n-1/2)^2}\,, \\
\theta_3(u) &= \theta_3(u \mid \tau) = \sum p^{2n} q^{n^2}\,, \\
\theta_4(u) &= \theta_4(u \mid \tau) = \sum(-1)^n p^{2n} q^{n^2}\,.
\end{aligned}$$

Here the sums are taken over $n \in \boldsymbol{Z}$. Under the addition of half-periods, these functions transform according to the following table.

	$u+1/2$	$u+\tau/2$	$u+1/2+\tau/2$	$u+1$	$u+\tau$	$u+1+\tau$
θ_1	θ_2	$-\sqrt{-1}a\theta_4$	$-a\theta_3$	$-\theta_1$	$-b\theta_1$	$b\theta_1$
θ_2	$-\theta_1$	$-a\theta_3$	$\sqrt{-1}a\theta_4$	$-\theta_2$	$b\theta_2$	$-b\theta_2$
θ_3	θ_4	$a\theta_2$	$\sqrt{-1}a\theta_1$	θ_3	$b\theta_3$	$b\theta_3$
θ_4	θ_3	$\sqrt{-1}a\theta_1$	$a\theta_2$	θ_4	$-b\theta_4$	$-b\theta_4$

For example, we have the transformation rules

$$\theta_1(u+\tau) = -b(u)\theta_1(u)\,, \tag{2.1}$$

$$\theta_1(u+1/2) = \theta_2(u)\,, \tag{2.2}$$

$$\theta_1(u+\tau/2) = -\sqrt{-1}a(u)\theta_4(u)\,, \tag{2.3}$$

$$\theta_3(u+\tau/2) = a(u)\theta_2(u)\,, \tag{2.4}$$

$$\theta_4(u+1/2) = \theta_3(u)\,, \tag{2.5}$$

where $a(u) = p(u)^{-1}q^{-1/4}$ and $b(u) = p(u)^{-2}q^{-1}$. Special values of these functions are obtained as follows:

$$\begin{gathered}\lim_{t\to\infty} q^{-1/4}\frac{\partial\theta_1}{\partial u}(0\,|\,\sqrt{-1}t) = 2\pi\,, \quad \lim_{t\to\infty} q^{-1/4}\theta_2(0\,|\,\sqrt{-1}t) = 2\,, \\ \lim_{t\to\infty}\theta_3(0\,|\,\sqrt{-1}t) = 1\,, \quad \lim_{t\to\infty}\theta_4(0\,|\,\sqrt{-1}t) = 1\,.\end{gathered} \tag{2.6}$$

On the other hand, Weierstrass' zeta function ζ_w is defined by

$$\zeta_w(u) = \zeta_{w,\tau}(u) = \frac{1}{u} + \sum_{\omega\in\boldsymbol{L}\setminus(0,0)}\left\{\frac{1}{(u-\omega)} + \frac{u}{\omega^2} + \frac{1}{\omega}\right\}. \tag{2.7}$$

Note that these functions have the following properties. θ_1 is an odd function. θ_2, θ_3 and θ_4 are even functions. Concerning ζ_w, there exist complex numbers $A = A_\tau$ and $B = B_\tau$ depending only on τ such that

$$\zeta_w(u+1) - \zeta_w(u) = A\,, \quad \zeta_w(u+\tau) - \zeta_w(u) = B\,, \quad A\tau - B = 2\pi\sqrt{-1}\,. \tag{2.8}$$

Moreover, if τ is pure imaginary, we have $\overline{\theta_1(u)} = \theta_1(\bar{u})$, $\overline{\zeta_w(u)} = \zeta_w(\bar{u})$, $\bar{A} = A$ and $\bar{B} = -B$.

For further details and formulas regarding these functions, we refer the reader to McKean and Moll [14, Chapter 3].

3. Main results. Let $\boldsymbol{P}^1$ be the smooth rational curve and λ an affine coordinate on it. Let ρ be an anti-holomorphic involution on $\boldsymbol{P}^1$ defined by $\lambda \mapsto 1/\bar{\lambda}$. Then the fixed point set of ρ consists of the equator S^1 defined by $\{\lambda \in \boldsymbol{P}^1 \mid |\lambda| = 1\}$.

First we recall the definition of spectral data introduced by McIntosh (cf. Section 2.1 in [13]).

DEFINITION 3.1. A spectral data is a triplet $(X, \pi, \mathcal{L})$ of isomorphism classes which satisfies the following conditions:

(1) X is a complete, connected, algebraic curve of arithmetic genus p, with a real involution ρ_X.

(2) π is a meromorphic function on X of degree $N = n + 1$ satisfying $\pi \circ \rho_X = 1/\bar{\pi}$, with a distinguished zero P_0 of degree $m + 1$ ($m \geqq 1$) and a pole $P_\infty = \rho_X(P_0)$. We regard X as a covering of degree $n + 1$ of the rational curve $\boldsymbol{P}^1$ via π.

(3) $\mathcal{L}$ is a line bundle over X of degree $p + n$ satisfying

$$\mathcal{L} \otimes \overline{\rho_{X*}\mathcal{L}} \cong \mathcal{O}_X(R)\,,$$

where R is the ramification divisor for π. By identifying $\mathcal{L}$ with a divisor line bundle $\mathcal{O}_X(D)$, we can find a meromorphic function f on X which satisfies the following conditions:

(a) The divisor (f) of f is given by $D + \rho_{X*}D - R$ and $\overline{\rho_X^* f} = f$.

(b) Let $X_{\boldsymbol{R}}$ be the preimage of S^1 by π. Then f is non-negative on $X_{\boldsymbol{R}}$.

(4) π has no branch points on S^1 and ρ_X fixed every point of $X_{\boldsymbol{R}}$.

Two triplets are the same if there exists a biholomorphic map between spectral curves which carries the real structure, the meromorphic function and the isomorphism class of the line bundle each other.

Our main theorems which refine the correspondence proved by McIntosh may be stated as follows. (See Section 6.1 for the detail of this correspondence.)

THEOREM 3.1. *Let X be the smooth rational curve. Then* $(X, \pi, \mathcal{L})$ *is a spectral data if and only if the following conditions are satisfied*:

(1) (X, ρ_X) *is real isomorphic to* $(\boldsymbol{P}^1, \rho)$. *By the affine coordinate* λ, π *is expressed as*

$$\pi(\lambda) = \alpha_0 \lambda^{m+1} \frac{\prod_{j=1}^{n-m} (\lambda - P_j)}{\prod_{j=1}^{n-m} (\lambda - Q_j)}, \quad P_0 = 0, \quad \alpha_0 = \frac{\prod_{j=1}^{n-m} (1 - Q_j)}{\prod_{j=1}^{n-m} (1 - P_j)}$$

for some m and n with $1 \leqq m \leqq n-1$. *Here* $P_j \in X^S = \{\lambda \in X \mid 0 < |\lambda| < 1\}$ *and* $Q_j = 1/\bar{P}_j$ *for any* $1 \leqq j \leqq n-m$.

(2) $\mathcal{L}$ *is a line bundle of degree n.*

THEOREM 3.2. *Choosing a complex corrdinate on the source suitably, the harmonic map* $\Psi : \boldsymbol{R}^2 \to \boldsymbol{C}P^n$ *corresponding to the spectral data* $(X, \pi, \mathcal{L} = \mathcal{O}_X(D))$ *in Theorem* 3.1 *is given by*

$$z = x + \sqrt{-1}y \mapsto [\Psi_0(z) : \Psi_1(z) : \cdots : \Psi_n(z)],$$

where $\Psi_i(z)$ *is a function defined by*

$$\Psi_i(z) = \exp(\eta_i^{-1} z - \eta_i \bar{z}) \cdot \frac{\prod_{j=1}^{n-m} (\eta_i - P_j)}{\prod_{j=1}^{n-m} (\eta_i - R_j)}. \tag{3.2}$$

Here $\{\eta_0, \dots, \eta_n\}$ *is the inverse image* $\pi^{-1}(1)$ *of* 1 *by* π *and* $R_+ = \sum_{j=1}^{n-m} R_j$ *is a divisor given by the intersection of* X^S *with R, that is,* $R_+ = X^S \cap R$.

Furthermore we obtain the following

THEOREM 3.3. Ψ *is doubly periodic with periods* $v_1, v_2 \in \boldsymbol{C}$ *if and only if the set*

$$V = \bigcap_{1 \leqq i \leqq n} \frac{\pi}{\beta_i} (\boldsymbol{R} \oplus \sqrt{-1}\boldsymbol{Z}) \tag{3.3}$$

contains the 2*-dimensional lattice* $M = \boldsymbol{Z}v_1 \oplus \boldsymbol{Z}v_2$, *where* $\beta_1, \dots, \beta_n$ *are complex numbers defined by* $\beta_i = \eta_i^{-1} - \eta_0^{-1}$.

COROLLARY 3.4. *Let* $(X, \pi, \mathcal{L})$ *be a spectral data in Theorem* 3.1 *such that the degree of* π *is* 3. *Then the corresponding harmonic map* $\Psi : \boldsymbol{R}^2 \to \boldsymbol{C}P^2$ *in Theorem* 3.2 *is always doubly periodic with periods* v_1, v_2, *where* v_1 *and* v_2 *are complex numbers in the set*

$$\boldsymbol{Z}v_+ \oplus \boldsymbol{Z}v_- = \boldsymbol{Z}\pi(\beta_1 \operatorname{Im}(\beta_2/\beta_1))^{-1} \oplus \boldsymbol{Z}\pi(\beta_2 \operatorname{Im}(\beta_1/\beta_2))^{-1}.$$

PROOF. In this case, the set V in Theorem 3.3 reduces to $\boldsymbol{Z}v_+ \oplus \boldsymbol{Z}v_-$. Hence Corollary 3.4 follows from Theorem 3.3. $\square$

Now we turn to the case of a smooth elliptic spectral curve X. Let us denote by $\mathrm{Pic}^d(X)$ and $J(X)$ the set of line bundles on X of degree d and the Jacobian of X, respectively. Note that $J(X)$ can be identified with $X = \boldsymbol{C}/(\boldsymbol{Z} \oplus \boldsymbol{Z}\tau)$. We then define a biholomorphic map $J : \mathrm{Pic}^0(X) \to J(X)$ by $J(L) = \sum_{j=1}^{k}(P_j - Q_j) \pmod{\boldsymbol{Z} \oplus \boldsymbol{Z}\tau}$, provided that $L \in \mathrm{Pic}^0(X)$ is expressed as a divisor line bundle $\mathcal{O}_X(\sum_{j=1}^{k}(P_j - Q_j))$.

THEOREM 3.5. *Let X be a smooth elliptic curve. Then $(X, \pi, \mathcal{L})$ is a spectral data if and only if the following conditions are satisfied*:

(1) *X is an elliptic curve $X_\tau = \boldsymbol{C}/(\boldsymbol{Z} \oplus \boldsymbol{Z}\tau)$, where τ is a pure imaginary number $\sqrt{-1}t$ with $t > 0$. ρ_X is an anti-holomorhic involution induced by the usual conjugation of $\boldsymbol{C}$. Regarded as a doubly periodic meromorphic function on $\boldsymbol{C}$, π is expressed as*

$$\pi(u) = C\frac{\theta_1(u - P_0)^{m+1}\prod_{j=1}^{n-m-1}\theta_1(u - P_j)\cdot\theta_1(u - P_{n-m} + W)}{\theta_1(u - Q_0)^{m+1}\prod_{j=1}^{n-m}\theta_1(u - Q_j)}$$

for some m and n with $1 \leqq m \leqq n - 1$. Here $P_i \in X^S = \{x \in X \mid 0 < \mathrm{Im}\,x < \mathrm{Im}\,\tau/2 \pmod{\mathrm{Im}\,\tau\boldsymbol{Z}}\}$ and $Q_i = \bar{P_i} \pmod{\boldsymbol{Z} \oplus \boldsymbol{Z}\tau}$ for any $0 \leqq i \leqq n - m$; $W = (m+1)P_0 + \sum_{i=1}^{n-m} P_i - (m+1)Q_0 - \sum_{i=1}^{n-m} Q_i$; $P_0 \neq P_i$ for $i \neq 0$; W belongs to $\boldsymbol{Z} \oplus \boldsymbol{Z}\tau$; and C is the unique constant such that $\pi(0) = 1$.

(2) *Let $r : \mathrm{Pic}^{n+1}(X) \to \mathrm{Pic}^0(X)$ be a map defined by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}_X(-R_+)$, where $R_+ = \sum_{j=0}^{n} R_j$ is a divisor of degree $n+1$ given by the intersection of X^S with R, that is, $R_+ = X^S \cap R$. Then, $\mathcal{L}$ is an element of the inverse image of $(\boldsymbol{Z} \oplus \sqrt{-1}\boldsymbol{R})/(\boldsymbol{Z} \oplus \tau\boldsymbol{Z})$ by the composition $J \circ r$.*

THEOREM 3.6. *Choosing a complex coordinate on the source suitably, the harmonic map $\Psi : \boldsymbol{R}^2 \to \boldsymbol{CP}^n$ corresponding to the spectral data $(X_\tau, \pi, \mathcal{L} = \mathcal{O}_X(D))$ in Theorem 3.5 is given by*

$$z = x + \sqrt{-1}y \mapsto [\Psi_0(z) : \Psi_1(z) : \cdots : \Psi_n(z)],$$

where $\Psi_i(z)$ is a function defined by

$$\begin{aligned}\Psi_i(z) &= \mu_i^{-1}\exp(z[\zeta_w(\eta_i - P_0) - A\eta_i] - \bar{z}[\zeta_w(\eta_i - Q_0) - A\eta_i]) \\ &\cdot\frac{\theta_1(\eta_i - P_0)^m\prod_{j=1}^{n-m}\theta_1(\eta_i - P_j)\theta_1(\eta_i + mP_0 + \sum_{j=1}^{n-m}P_j - D - z + \bar{z})}{\prod_{j=0}^{n}\theta_1(\eta_i - R_j)}.\end{aligned} \tag{3.4}$$

Here $\{\eta_0, \ldots, \eta_n\}$ is the inverse image $\pi^{-1}(1)$ of 1 by π, μ_i is a constant given by $\mu_i = \exp(2\pi\sqrt{-1}(D - R_+)\mathrm{Im}\eta_i/t)$, and A is a constant given in the equation (2.8).

Moreover we prove the following

THEOREM 3.7. *The harmonic map $\Psi : \boldsymbol{R}^2 \to \boldsymbol{CP}^n$ in Theorem 3.6 is doubly periodic with periods $v_1, v_2 \in \boldsymbol{C}$ if and only if the set $V = \bigcap_{0 \leqq i \leqq n} V_i$ contains the 2-dimensional lattice $M = \boldsymbol{Z}v_1 \oplus \boldsymbol{Z}v_2$, where $V_0, \ldots, V_n$ are the sets defined by*

$$V_i = \begin{cases}\pi\beta_i^{-1}(\boldsymbol{R} \oplus \sqrt{-1}\boldsymbol{Z}), & \text{if } \beta_i \neq 0, \\ \boldsymbol{C}, & \text{otherwise.}\end{cases}$$

Here $\beta_0, \beta_1, \dots, \beta_n$ *are complex numbers defined by*

$$\beta_0 = 2\pi/t\,, \quad \beta_i = [\zeta_w(\eta_0 - P_0) - \zeta_w(\eta_i - P_0) - B(\eta_0 - \eta_i)\tau^{-1}] \quad (1 \leqq i \leqq n)\,.$$

COROLLARY 3.8. *Let* $(X, \pi, \mathcal{L})$ *be a spectral data in Theorem* 3.5 *such that the degree of* π *is* 2 *and* $\mathrm{Im}\,\beta_1 \neq 0$. *Then the corresponding harmonic map* $\Psi : \boldsymbol{R}^2 \to \boldsymbol{C}P^1$ *in Theorem* 3.6 *is always doubly periodic with periods* v_1, v_2, *where* v_1 *and* v_2 *are complex numbers in the set*

$$\boldsymbol{Z}v_+ \oplus \boldsymbol{Z}v_- = \boldsymbol{Z}\pi(\mathrm{Im}\beta_1)^{-1} \oplus \boldsymbol{Z}\overline{\beta_1}(\mathrm{Im}\beta_1)^{-1}t/2\,.$$

PROOF. In this case, the set V in Theorem 3.7 reduces to $\boldsymbol{Z}v_+ \oplus \boldsymbol{Z}v_-$. Hence Corollary 3.8 follows from Theorem 3.7. □

We now give some explicit examples of harmonic maps by applying the theorems above.

EXAMPLE 3.9. Let $(X = \boldsymbol{P}^1, \pi, \mathcal{L})$ be a spectral data defined as follows. The map $\pi : \boldsymbol{P}^1 \to \boldsymbol{P}^1$ is given by $\lambda \mapsto \lambda^{n+1}$. $\mathcal{L}$ is the divisor line bundle

$$\mathcal{L} = O_X(n0)\,,$$

and $P_0 = 0$, a point as in Condition (2) of Definition 3.1. Then we choose the constant function $f = 1$ as a meromorphic function in Condition (3) of Definition 3.1. Setting $\omega = \exp(2\pi\sqrt{-1}/(n+1))$, we see that $\pi^{-1}(1)$ is given by $\{1, \omega, \omega^2, \dots, \omega^n\}$. Then the corresponding harmonic map $\Psi : \boldsymbol{R}^2 \to \boldsymbol{C}P^n$ is given by

$$z = x + \sqrt{-1}y \mapsto [\Psi_0(z) : \cdots : \Psi_n(z)]\,,$$

where $\Psi_i = \exp(\omega^{-j}z - \omega^j\bar{z})$. Note that Ψ is a superconformal map. Moreover, if $n = 1, 2$, 3 or 5, then ψ is doubly periodic.

EXAMPLE 3.10. Let $(X = \boldsymbol{P}^1, \pi, \mathcal{L})$ be a spectral data defined as follows. The map $\pi : \boldsymbol{P}^1 \to \boldsymbol{P}^1$ is now given by

$$\lambda \mapsto \frac{1-\beta}{1-\alpha}\lambda^2\left(\frac{\lambda-\alpha}{\lambda-\beta}\right),$$

where α is a real number such that $0 < |\alpha| < 1$ and $\beta = 1/\alpha$. The ramification divisor R of π is given by $R = (R_1) + (0) + (\rho_X(R_1)) + (\infty)$, where $R_1 = (\alpha^2 + 3 - \sqrt{\alpha^4 - 10\alpha^2 + 9})/4\alpha$. $\mathcal{L}$ is the divisor line bundle given by

$$\mathcal{L} = O_X(R_1 + \infty)\,,$$

and $P_0 = 0$. Moreover, $\pi^{-1}(-1) = \{\eta_0, \eta_1, \eta_2\}$ is given by

$$\eta_0 = 1\,, \quad \eta_1 = \frac{\alpha - 1 + \sqrt{4 - (\alpha-1)^2}\sqrt{-1}}{2}\,, \quad \eta_2 = \frac{\alpha - 1 - \sqrt{4 - (\alpha-1)^2}\sqrt{-1}}{2}\,.$$

Then the corresponding harmonic map $\Psi : \boldsymbol{R}^2 \to \boldsymbol{C}P^2$ is given by

$$z = x + \sqrt{-1}y \mapsto [\Psi_0(z) : \Psi_1(z) : \Psi_2(z)]\,,$$

where

$$\Psi_i(z) = \exp(\eta_i^{-1}z - \eta_i\bar{z}) \cdot \frac{(\eta_i - \alpha)}{(\eta_i - R_1)}\,.$$

Note that Ψ is a harmonic map of isotropy order 1 and is nowhere conformal. Moreover, by Corollary 3.4, Ψ has two complex periods v_1 and v_2, which are in the lattice $\boldsymbol{Z}v_+ \oplus \boldsymbol{Z}v_-$ defined by

$$v_+ = \left(-\frac{1}{\sqrt{4-(\alpha-1)^2}} + \frac{\sqrt{-1}}{\alpha-3}\right)\pi\,, \quad v_- = \left(\frac{1}{\sqrt{4-(\alpha-1)^2}} + \frac{\sqrt{-1}}{\alpha-3}\right)\pi\,.$$

EXAMPLE 3.11. Let $(X_\tau = X_{\sqrt{-1}}, \pi, \mathcal{L})$ be a spectral data defined as follows. We define the map $\pi : X_\tau \to \boldsymbol{P}^1$ by $u \mapsto \lambda = g(u)/g(1/2)$, where $g(u)$ is a meromorphic function on X given by

$$g(u) = \frac{\theta_1(u-R_0)^2\theta_1(u-R_0-2\sqrt{-1})}{\theta_1(u-R_3)^3}$$

with $R_0 = 1/2+\sqrt{-1}/6$ and $R_3 = 1/2+5\sqrt{-1}/6$. In this case, there exists a point $R_2 \in X^S$ such that the ramification divisor R is expressed as $2R_0 + R_2 + \rho_X(2R_0 + R_2)$. We define the divisor line bundle $\mathcal{L}$ by

$$\mathcal{L} = O_X(2R_0 + R_2)\,.$$

Set $P_0 = R_0$ as a distinguished zero of π as in Condition (2) of Definition 3.1. We choose the constant function $f = 1$ as a meromorphic function in Condition (3) of Definition 3.1. In this case, $\zeta_w(\sqrt{-1}r) = -\sqrt{-1}\zeta_w(r)$ for $r \in \boldsymbol{R}$. From this, together with (2.8), we get $A = \pi$. Since $\pi^{-1}(1)$ is $\{0, 1/2, \sqrt{-1}/2\}$, the corresponding harmonic map $\Psi : \boldsymbol{R}^2 \to \boldsymbol{C}P^2$ is given by

$$z = x + \sqrt{-1}y \mapsto [\psi(0,z) : \psi(1/2,z) : \psi(\sqrt{-1}/2, z)]\,,$$

where

$$\psi(u,z) = \exp[z\{\zeta_{w,\tau}(u-R_0) - \pi u\} - \bar{z}\{\zeta_{w,\tau}(u-R_3) - \pi u\}]\frac{\theta_1(u-R_2-z+\bar{z})}{\theta_1(u-R_2)}\,.$$

Note that Ψ is a superconformal map into $\boldsymbol{C}P^2$.

EXAMPLE 3.12. Let $(X_\tau = X_{\sqrt{-1}}, \pi, \mathcal{L})$ be a spectral data defined as follows. We now define the map $\pi : X_\tau \to \boldsymbol{P}^1$ by $u \mapsto \lambda = \mathfrak{p}(u-R_2)/\mathfrak{p}(3\sqrt{-1}/4)$, where $R_2 = 3\sqrt{-1}/4$ and $\mathfrak{p}$ is Weierstrass' $\mathfrak{p}$ function defined by

$$\mathfrak{p}(u) = \frac{1}{u^2} + \sum_{(m,n)\neq(0,0)} \left\{\frac{1}{(u-(m+\sqrt{-1}n))^2} - \frac{1}{(m+\sqrt{-1}n)^2}\right\}\,.$$

The ramification divisor R of π is given by $R = R_0 + R_1 + R_2 + R_3$, where $R_0 = \sqrt{-1}/4$, $R_1 = (2+\sqrt{-1})/4$ and $R_3 = (2+3\sqrt{-1})/4$. Define the divisor line bundle $\mathcal{L}$ by

$$\mathcal{L} = O_X(R_0 + R_1)\,.$$

Set $P_0 = R_0$ as a distinguished zero of π as in Condition (2) of Definition 3.1. The constant function $f = 1$ can be taken as a meromorphic function in Condition (3) of Definition 3.1. Since $\pi^{-1}(1)$ is $\{0, \sqrt{-1}/2\}$, the corresponding harmonic map $\Psi : \boldsymbol{R}^2 \to \boldsymbol{C}P^1$ is given by

$$z = x + \sqrt{-1}y \mapsto [\psi(0,z) : \psi(\sqrt{-1}/2, z)]\,,$$

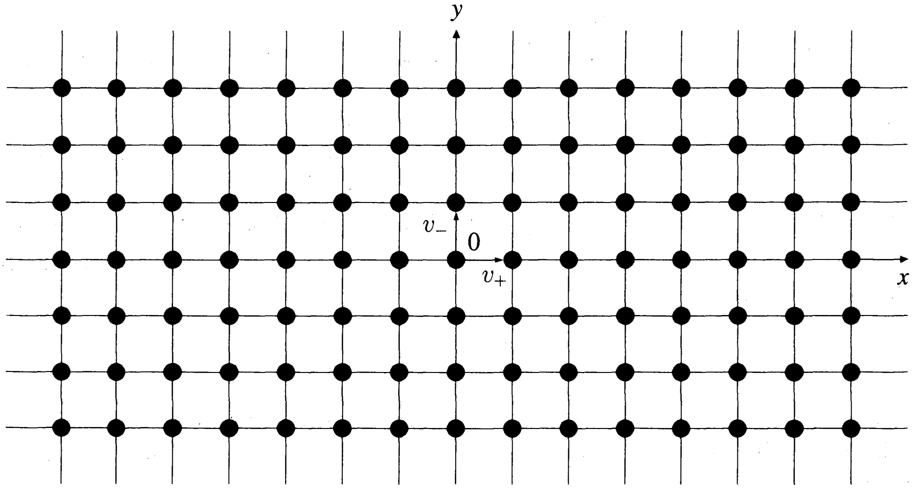

FIGURE 1.

where

$$\psi(u,z)=\exp[z\{\zeta_{w,\tau}(u-R_0)-\pi u\}-\bar{z}\{\zeta_{w,\tau}(u-R_2)-\pi u\}]\frac{\theta_1(u-R_1-z+\bar{z})}{\theta_1(u-R_1)}\,.$$

Note that Ψ is a harmonic map of isotropy order 1 and is nowhere conformal.

Concerning the periodicity of Ψ, the corresponding set V in Theorem 3.7 then consists of the lattice points in Figure 1.

From Corollary 3.8, we see that Ψ has two periods v_+ and v_- defined by

$$v_+ = 2\pi/(4\zeta_w(1/4)-\pi) \fallingdotseq 0.4962\ldots\,, \qquad v_- = \sqrt{-1}/2\,,$$

that is, $\Psi(v_- + z) = \Psi(v_+ + z) = \Psi(z)$. Moreover, Ψ maps the torus $T = \boldsymbol{C}/(\boldsymbol{Z}v_+ \oplus \boldsymbol{Z}v_-)$ to an annulus in the Riemann sphere $\boldsymbol{CP}^1$.

4. Classification of spectral data with the smooth rational spectral curve. This section is devoted to the proof of Theorem 3.1. First, we shall describe the real structures of the smooth rational curve $\boldsymbol{P}^1$.

We first note that there are two real structures on $\boldsymbol{P}^1$ (cf. Section 2.1 in [5]). One is $(\boldsymbol{P}^1, \rho)$. The other is $(\boldsymbol{P}^1, \sigma)$, where σ is the anti-holomorphic involution defined by

$$\lambda \mapsto -1/\bar{\lambda}\,.$$

However, it is not suitable to choose the latter as the involution of the spectral curve $X = \boldsymbol{P}^1$, since it has no fixed points on $\boldsymbol{P}^1$ and does not satisfy Condition (4) in Definition 3.1.

Throughout this section, we shall always assume that $X = \boldsymbol{P}^1$ and $\rho_X = \rho$.

PROPOSITION 4.1. *Let π be a non-constant holomorphic map from X to $\boldsymbol{P}^1$ satisfying the following conditions:*

(1) $\pi \circ \rho_X = \rho \circ \pi$,

(2) *ρ_X fixes every point of $X_{\boldsymbol{R}}$,*

(3) *π has no branch points on S^1.*
Then π is either (A) χ *or* (B) $1/\chi$, *where χ is the meromorphic function defined by*

$$\chi(\lambda) = \alpha_0 \lambda^k \frac{\prod_{j=1}^{l}(\lambda - \alpha_j)}{\prod_{j=1}^{l}(\lambda - \beta_j)} .$$

Here k and l are some non-negative integers with $k + l \neq 0$; $\alpha_0 \in \boldsymbol{C}^* = \boldsymbol{C} \setminus 0$; $\alpha_1, \dots, \alpha_l$ are non zero complex numbers satisfying $|\alpha_i| < 1$ and $|\alpha_0 \alpha_1 \cdots \alpha_l| = 1$; and $\beta_i = 1/\bar{\alpha}_i$.

Conversely, any map π expressed as above satisfies Conditions (1), (2) *and* (3).

We devide the proof of Proposition 4.1 into several lemmas.

LEMMA 4.2. *The map π satisfies Condition* (1) *in Proposition* 4.1 *if and only if it is of the following form*:

$$\pi(\lambda) = \alpha_0 \lambda^k \frac{\prod_{j=1}^{l}(\lambda - \alpha_j)}{\prod_{j=1}^{l}(\lambda - \beta_j)} ,$$

where k is an integer and $\alpha_0 \in \boldsymbol{C}^*$, and $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l$ are complex numbers belonging to $\boldsymbol{C}^* \setminus S^1$ which satisfy $\beta_i = 1/\bar{\alpha}_i$ $(1 \leqq i \leqq l)$ and $|\alpha_0 \alpha_1 \cdots \alpha_l| = 1$.

PROOF. The map π intertwines the involutions ρ_X on X and ρ on $\boldsymbol{P}^1$, precisely when

$$\pi(\lambda)\overline{\pi(1/\bar{\lambda})} = 1 . \tag{4.1}$$

From this it follows that if π has a pole (resp. zeto) of order k at p, then $\rho_X(p)$ is the zero (resp. pole) of π of order k. Since ρ_X fixes every point of S^1, there exist no zeros and poles on S^1. Thus π must be of the following form

$$\pi(\lambda) = \alpha_0 \lambda^k \frac{\prod_{j=1}^{l}(\lambda - \alpha_j)}{\prod_{j=1}^{l}(\lambda - \beta_j)} , \tag{4.2}$$

where $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l$ are all complex numbers contained in $\boldsymbol{C}^* \setminus S^1$ and $\alpha_0 \in \boldsymbol{C}^*$. Using (4.1), we also get

$$|\alpha_0 \alpha_1 \cdots \alpha_l| = 1 . \tag{4.3}$$

Conversely, let π be the map defined as in (4.2) with $|\alpha_0 \alpha_1 \cdots \alpha_l| = 1$. Then clearly π satisfies the equation (4.1). □

LEMMA 4.3. *Let π be a map as in Lemma* 4.2 *and suppose that π satisfies Condition* (3) *in Proposition* 4.1. *Then π satisfies Condition* (2) *in Proposition* 4.1 *if and only if π is either* (A) χ *or* (B) $1/\chi$, *where χ is the meromorphic function as in Proposition* 4.1.

PROOF. Let $\pi|_S$ denote the restriction of π to S^1. It is easy to see that π maps S^1 into S^1. This together with Condition (3) in Proposition 4.1 implies that Condition (2) in Proposition 4.1 is equivalent to that $|d|$, the absolute value of the mapping degree d of $\pi|_S$, is equal to $|k| + l$, the degree of π. Note that d is given by the integral

$$\frac{1}{2\pi\sqrt{-1}} \int_{S^1} \frac{1}{\pi(\lambda)} d\pi(\lambda) .$$

Then, by a straightforward calculation, we get

$$d = k + \frac{1}{2\pi\sqrt{-1}} \int_{S^1} \sum_{i=1}^{l} \frac{\alpha_i - \beta_i}{(\lambda - \alpha_i)(\lambda - \beta_i)} d\lambda\,,$$

where the second term in the right-hand side is equal to

$$\#\{\alpha_i;\, |\alpha_i| < 1\} - \#\{\beta_i;\, |\beta_i| < 1\}\,.$$

Thus $|d| = |k| + l$, precisely when k and $\alpha_1, \dots, \alpha_l$ satisfy

$$\begin{cases} \text{(A)} \quad k \geqq 0 \text{ and } |\alpha_i| < 1 \ (1 \leqq i \leqq l) \text{ or} \\ \text{(B)} \quad k \leqq 0 \text{ and } |\alpha_i| > 1 \ (1 \leqq i \leqq l)\,. \end{cases}$$

This completes the proof of Lemma 4.3. □

LEMMA 4.4. *Let π be a map as in Proposition* 4.1. *Then the ramification divisor of π does not intersect S^1 in X.*

PROOF. Let π be a map as in (A). Differentiating π by λ, we get

$$\frac{d\pi}{d\lambda} = \alpha_0 \lambda^{k-1} \left(\prod_{i=1}^{l} \frac{\lambda - \alpha_i}{\lambda - \beta_i} \right) \left[k + \lambda \sum_{i=1}^{l} \left(\frac{\alpha_i - \beta_i}{(\lambda - \alpha_i)(\lambda - \beta_i)} \right) \right].$$

Suppose that the ramification divisor of π intersects S^1, that is, there exists a point λ on S^1 such that

$$k + \lambda \sum_{i=1}^{l} \left(\frac{\alpha_i - \beta_i}{(\lambda - \alpha_i)(\lambda - \beta_i)} \right) = 0\,.$$

Then, setting $\tilde{\lambda} = \exp(\sqrt{-1}\theta)\lambda$, $\tilde{\alpha}_i = \exp(\sqrt{-1}\theta)\alpha_i$ and $\tilde{\beta}_i = \exp(\sqrt{-1}\theta)\beta_i$, we have

$$k + \tilde{\lambda} \sum_{i=1}^{l} \left(\frac{\tilde{\alpha}_i - \tilde{\beta}_i}{(\tilde{\lambda} - \tilde{\alpha}_i)(\tilde{\lambda} - \tilde{\beta}_i)} \right) = 0\,. \tag{4.4}$$

Choose $\theta \in \boldsymbol{R}$ such that $\tilde{\lambda} = 1$. Then the left-hand side of (4.4) becomes

$$k + \sum_{i=1}^{l} \frac{1 - |\tilde{\alpha}_i|^2}{|1 - \tilde{\alpha}_i|^2}\,,$$

which is positive because $|\tilde{\alpha_k}| = |\alpha_k| < 1$. This is a contradiction. Thus the ramification divisor does not intersect S^1.

The proof for the case when π is a map as in (B) proceeds in a similar way. □

By Lemma 4.2, Lemma 4.3 and Lemma 4.4, Proposition 4.1 has been proved.

PROPOSITION 4.5. *Let π be a meromorphic function on $X = \boldsymbol{P}^1$ and $\mathcal{L}$ a line bundle over X. Then $(X, \pi, \mathcal{L})$ is a spectral data if and only if it satisfies the following conditions*:

(1) *π is a meromorphic function as in Proposition* 4.1.

(2) *The degree of $\mathcal{L}$ is $N - 1$, where N is the degree of π.*

(3) *π has a zero P_0 of order $\geqq 2$.*

PROOF. We first prove the "only if" part. Conditions (2) and (4) in Definition 3.1 require that π is a meromorphic function as in Proposition 4.1. By Condition (3) in Definition 3.1, the degree of $\mathcal{L}$ must be $N-1$. Condition (3) in Proposition 4.5 assures Condition (2) in Definition 3.1. Hence the "only if" part is clear.

To prove the "if" part, we only have to show the existence of a divisor D and a meromorphic function f satisfying Conditions (3-a) and (3-b) in Definition 3.1. Let D be any divisor of degree $N-1$. First, the existence of a meromorphic function f whose divisor (f) satisfies

$$(f) = -R + D + \rho_{X*}D$$

is obvious, since $X = \boldsymbol{P}^1$. It has been shown in [Section 3.1, 12] that there exists a meromorphic function f such that

$$\overline{\rho_X^* f} = f\,.$$

So it suffices to show that f_S, the restriction of f to S^1, is non-negative on S^1. Let S_{zp} be the intersection of S^1 with the set of zeros and poles of f. Restricting f_S to $S^1 \setminus S_{zp}$, we get a real function f^*. Considering the restriction of $(-R + D + \rho_{X*}D)$ to S^1, we see that f_S has only zeros and poles whose orders are all even. So the sign of f^* remains invariant at each point of S_{zp}. Changing the sign of f if necessary, we get the desired function. Hence Proposition 4.5 is proved. □

Now let us prove Theorem 3.1.

PROOF OF THEOREM 3.1. To prove this theorem, it suffices to show that for every spectral data $(X, \pi, \mathcal{L})$ with P_0 as in Proposition 4.5, there exists a real automorphism ϕ on (X, ρ_X) such that the value of λ at $\phi^{-1}(P_0)$ is equal to 0 and the pull-back of π by ϕ is of a form in Condition (1) of Theorem 3.1. But this is quite straightforward. □

5. Classification of spectral data with smooth elliptic spectral curves. This section is devoted to the proof of Theorem 3.5. First, we describe all smooth real elliptic curves which can be spectral curves. Second, meromorphic functions on these spectral curves, which satisfy Conditions (2) and (4) in Definition 3.1, are determined (Proposition 5.2). Finally, after preparing a device (Proposition 5.7) useful to select line bundles satisfying Condition (3) in Definition 3.1, we prove Theorem 3.5.

Let S^+ (resp. S^-) be the northern (resp. southern) hemisphere defined by $S^+ = \{\lambda \in \boldsymbol{P}^1 \mid |\lambda| > 1\}$ (resp. $S^- = \{\lambda \in \boldsymbol{P}^1 \mid |\lambda| < 1\}$). Let $X = X_\tau = \boldsymbol{C}/(\boldsymbol{Z} \oplus \tau\boldsymbol{Z})$ be an elliptic curve, where τ belongs to the upper half plane $\mathfrak{H} := \{\operatorname{Im}\tau > 0\}$. Let ρ_X be an anti-holomorphic involution of X and X^ρ the fixed point set of ρ_X.

It should be remarked that a real elliptic curve (X, ρ_X) with $X^\rho = \emptyset$ is not suitable for our purpose, since ρ_X has no fixed points on X and hence violates Condition (4) in Definition 3.1.

THEOREM 5.1 ([5]). *Let (X, ρ_X) be as above and $X^\rho \neq \emptyset$. Then (X, ρ_X) is isomorphic to $(\boldsymbol{C}/(\boldsymbol{Z} \oplus \tau\boldsymbol{Z}), \sigma)$, where τ belongs to* (F0) $\{\sqrt{-1}t \mid t \in \boldsymbol{R}, t > 0\}$ *or* (F1) $\{1/2 + \sqrt{-1}t \mid t \in \boldsymbol{R}, t > 0\}$, *and σ is the anti-holomorphic involution on $\boldsymbol{C}/(\boldsymbol{Z} \oplus \tau\boldsymbol{Z})$*

induced by the usual conjugation of $\boldsymbol{C}$. *Moreover, if* X *is an elliptic curve of type* (F0), *then* X^ρ *consists of two circles* S_A^1 *and* S_B^1 *defined by*

$$S_A^1 = (\boldsymbol{R} \oplus \tau\boldsymbol{Z})/(\boldsymbol{Z} \oplus \boldsymbol{Z}\tau)\,, \quad S_B^1 = (\boldsymbol{R} \oplus \tau(1/2 + \boldsymbol{Z}))/(\boldsymbol{Z} \oplus \boldsymbol{Z}\tau)\,,$$

and $X \setminus X^\rho$ *consists of two tubes* X^N *and* X^S *defined by*

$$X^N = (\{x \in \boldsymbol{C} \mid \operatorname{Im}\tau/2 < \operatorname{Im} x < \operatorname{Im}\tau\} \oplus \boldsymbol{Z}\tau)/(\boldsymbol{Z} \oplus \boldsymbol{Z}\tau)\,,$$
$$X^S = (\{x \in \boldsymbol{C} \mid 0 < \operatorname{Im} x < \operatorname{Im}\tau/2\} \oplus \boldsymbol{Z}\tau)/(\boldsymbol{Z} \oplus \boldsymbol{Z}\tau)\,.$$

On the other hand, if X *is an elliptic curve of type* (F1), *then* X^ρ *consists of a circle* S_A^1 *defined by*

$$S_A^1 = (\boldsymbol{R} \oplus \tau\boldsymbol{Z})/(\boldsymbol{Z} \oplus \boldsymbol{Z}\tau)\,,$$

and $X \setminus X^\rho$ *is connected.*

PROPOSITION 5.2. *Let* X_τ *be an elliptic curve and* ρ_X *an anti-holomorphic involution on* X_τ *with* $X^\rho \neq \emptyset$. *Let* π *be a non-constant holomorphic map from* X_τ *to* $\boldsymbol{P}^1$ *satisfying the following conditions*:

(1) $\pi \circ \rho_X = \rho \circ \pi$,
(2) ρ_X *fixes every point of* $\pi^{-1}(S^1)$,
(3) π *has no branch points on* S^1.

Then X_τ *is an elliptic curve of type* (F0). *Moreover, regarded as a doubly periodic meromorphic function on* $\boldsymbol{C}$, π *is either* (A) χ *or* (B) $1/\chi$, *where* χ *is a meromorphic function defined by*

$$\chi(u) = C\exp(-2\pi\sqrt{-1}qu)\prod_{i=1}^{n+1}\frac{\theta_1(u-\alpha_i)}{\theta_1(u-\beta_i)}\,.$$

Here θ_1 *is Jacobi's theta function as in Section* 2; n *is a positive integer*; $q, \alpha_1, \dots, \alpha_{n+1}, \beta_1, \dots, \beta_{n+1}$, *and* C *are constants satisfying the following conditions*:

(1) $\alpha_i \in X^S$ *and* $\sum_i(\alpha_i - \beta_i)$ *is expressed as* $p + q\tau \in \boldsymbol{Z} \oplus \boldsymbol{Z}\tau$.
(2) $\beta_i = \rho_X(\alpha_i)$, *that is,* $\alpha_i + \beta_i$ *is expressed as* $r_i + s_i\tau \in \boldsymbol{R} \oplus \boldsymbol{Z}\tau$.
(3) $|C| = \exp(\pi\sqrt{-1}\sum_i s_i(\alpha_i - \beta_i))$.

Conversely, any map π *expressed as above satisfies Conditions* (1), (2) *and* (3).

The proof of Proposition 5.2 is divided into several lemmas.

LEMMA 5.3. *There exist no non-constant holomorphic maps from an elliptic curve* X_τ *of type* (F1) *to* $\boldsymbol{P}^1$ *satisfying Condition* (2) *in Proposition* 5.2.

PROOF. Suppose that such a map π exists. Let $X^* = X \setminus X^\rho$, $X^+ = \{x \in X^* \mid \pi(x) \in S^+\}$, and $X^- = \{x \in X^* \mid \pi(x) \in S^-\}$. Then X^+ and X^- are open and $X^* = X^+ \cup X^-$. Since X^* is connected by Theorem 5.1, X^* coincides with either X^+ or X^-. In particular, π is not surjective, which is a contradiction. □

Owing to Lemma 5.3, we may assume that X_τ is an elliptic curve of type (F0).

LEMMA 5.4. *Let* X_τ *be an elliptic curve of type* (F0) *and* π *a non-constant holomorphic map from* X_τ *to* $\boldsymbol{P}^1$. *Then* π *satisfies Condition* (1) *of Proposition* 5.2 *if and only if it is* *of the following form*:

$$\pi(u) = C\exp(-2\pi\sqrt{-1}qu)\prod_{i=1}^{k}\frac{\theta_1(u-\alpha_i)}{\theta_1(u-\beta_i)}$$

for some $k \geqq 2$. *Here* θ_1 *is Jacobi's theta function as in Proposition* 5.2; q, α_i, β_i $(1 \leqq i \leqq k)$ *are constants satisfying*

$$\sum_i(\alpha_i - \beta_i) = p + q\tau \in \boldsymbol{Z} \oplus \boldsymbol{Z}\tau\,, \quad \alpha_i + \beta_i = r_i + s_i\tau \in \boldsymbol{R} \oplus \boldsymbol{Z}\tau \ \ (1 \leqq i \leqq k);$$

and C *is a constant such that* $|C| = \exp(\pi\sqrt{-1}\sum_i s_i(\alpha_i - \beta_i))$.

PROOF. The map π intertwines the involutions ρ_X on X_τ and ρ on $\boldsymbol{P}^1$ if and only if

$$\pi(u)\overline{\pi(\rho_X(u))} = 1\,. \tag{5.1}$$

It follows from this that if π has a pole (resp. zero) of order k at p, then $\rho_X(p)$ is the zero (resp. pole) of π of order k. Since ρ_X fixes X^ρ pointwise, there exist no zeros and poles of π on X^ρ.

Suppose that $\pi : X_\tau \to \boldsymbol{P}^1$ satisfies Condition (1) in Proposition 5.2. Then the divisor of π must be of the form

$$(\pi) = (\alpha_1) + \cdots + (\alpha_k) - (\beta_1) - \cdots - (\beta_k)\,, \tag{5.2}$$

where α_i, β_i are points on $X_\tau \setminus X^\rho$ satisfying $\beta_i = \rho_X(\alpha_i)$, that is, $\alpha_i + \beta_i$ is expressed as $r_i + s_i\tau \in \boldsymbol{R} \oplus \boldsymbol{Z}\tau$ $(1 \leqq i \leqq k)$. Then it follows from Abel's theorem that $\sum_i \alpha_i - \sum_i \beta_i$ belongs to $\boldsymbol{Z} \oplus \tau\boldsymbol{Z}$, and hence there exist integers p and q such that $\sum_i \alpha_i - \sum_i \beta_i = p + q\tau$. The meromorphic function π is now determined, up to multiplication by a constant C, and is expressed as follows:

$$\pi(u) = C\exp(-2\pi\sqrt{-1}qu)\frac{\theta_1(u-\alpha_1)\cdots\theta_1(u-\alpha_k)}{\theta_1(u-\beta_1)\cdots\theta_1(u-\beta_k)}\,. \tag{5.3}$$

Using (5.1), we get $\pi(0)\overline{\pi(0)} = 1$, that is, $|C| = \exp(\pi\sqrt{-1}\sum_i s_i(\alpha_i - \beta_i))$.

Conversely, let π be a map defined by (5.3) with $|C| = \exp(\pi\sqrt{-1}\sum_i s_i(\alpha_i - \beta_i))$. Then, clearly π satisfies the identity (5.1). □

LEMMA 5.5. *Let* π *be a map as in Lemma* 5.4. *Suppose that* π *satisfies Condition* (3) *in Proposition* 5.2. *Then* π *is either* (A) χ *or* (B) $1/\chi$, *where* χ *is the meromorphic function in Proposition* 5.2, *if and only if* π *satisfies Condition* (2) *in Proposition* 5.2.

PROOF. Since $X = X_\tau$ is an elliptic curve of type (F0), $X^* = X \setminus X^\rho$ consists of two connected components. More precisely, $X^* = X^N \cup X^S$. Let $X^{N,+}$ and $X^{N,-}$ be the subsets of X defined by $X^{N,\pm} = \{x \in X^N \mid \pi(x) \in S^\pm\}$, respectively. Similarly, define $X^{S,\pm} = \{x \in X^S \mid \pi(x) \in S^\pm\}$. Suppose that π satisfies Condition (2) in Proposition 5.2. Then we see that $\pi(X^*) \cap S^1 = \emptyset$. It then follows that $X^N = X^{N,+} \cup X^{N,-}$ and $X^S = X^{S,+} \cup X^{S,-}$. Since X^N and X^S are connected, we see that (a) $X^N = X^{N,+}$, $X^S = X^{S,-}$ or (b) $X^N = X^{N,-}$, $X^S = X^{S,+}$. In the case (a) (resp. (b)), π must be a function of type (A) (resp. (B)) as in Proposition 5.2.

Conversely, if π is either (A) χ or (B) $1/\chi$, then it is easy to see that π maps S_A^1 and S_B^1 into S^1. Let π_A denotes the restriction of π to S_A^1 and d_A be the degree of the map $\pi_A : S_A^1 \to S^1$. Similarly, define $\pi_B : S_B^1 \to S^1$ by the restriction of π to S_B, and denote by d_B its degree. Since $|d_A| + |d_B|$ coincides with the degree of π by the residue theorem, we see that for any point $p \in S^1$, $\pi^{-1}(p)$ is contained in $X^\rho = S_A^1 \cup S_B^1$. This implies that π satisfies Condition (2) in Proposition 5.2. □

LEMMA 5.6. *Let π be a map as in Proposition* 5.2. *Then the ramification divisor of π does not intersect $X^\rho = S_A^1 \cup S_B^1$.*

PROOF. Let π be a meromorphic function of type (A) as in Proposition 5.2. Note that the number of zeros of π on X^S is given by the integral

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial X^S} \frac{1}{\pi(u)} d\pi(u)\,,$$

which is equal to $k = n+1$ from Proposition 5.2. Since π maps S_A^1 and S_B^1 into S^1, for every point $p \in S^1$ we see that $\pi^{-1}(p)$ contains at least k distinct points. Recall that the degree of π is k. It implies that

$$\#\{\pi^{-1}(p)\} = k\,. \tag{5.4}$$

Suppose that there exists a point x such that $x \in R \cap (S_A^1 \cup S_B^1)$, where R is the ramification divisor of π. Setting $q = \pi(x)$, we see that $\#\{\pi^{-1}(q)\} = k$ by the identity (5.4).

Let $\pi^{-1}(q) = \{P_1, \dots, P_k\}$ and U_i a neighbourhood of P_i such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Let $V(q)$ be the neighbourhood of q defined by $V(q) = \bigcap_i \pi(U_i)$. Denote by e the degree of π at x. It then follows from the assumption $e \geqq 2$ that there exists a neighbourhood $W(x)$ of x such that $\pi(W(x)) \subset V(q)$ and the degree of $\pi\,|_{W(x)\setminus\{x\}}$, the restriction of π to $W(x) \setminus \{x\}$, is e. Take a point $y \in \pi(W(x)) \setminus \{q\}$. Then, there exist a point $Y_i \in U_i$ for each $i \neq 1$ and points $Z_1, \dots, Z_e \in U_1$ such that π maps all of these points to y. Also, we see that $\#\{\pi^{-1}(y)\} \geqq k - 1 + e \geqq k + 1$. This contradicts that the degree of π is k. Hence R does not intersect $S_A^1 \cup S_B^1$.

The proof for a meromorphic function of type (B) as in Proposition 5.2 proceeds in a similar manner. □

By Lemma 5.4, Lemma 5.5 and Lemma 5.6, Proposition 5.2 has been proved.

PROPOSITION 5.7. *Let $(X = \boldsymbol{C}/(\boldsymbol{Z}\oplus\sqrt{-1}t\boldsymbol{Z}), \rho_X)$ be a real curve of type* (F0), *which is identified with its Jacobian $J(X)$. Let E and F be divisors on X satisfying*

$$E + \rho_X(E) \cong F + \rho_X(F)\,, \tag{5.5}$$

where $\cong$ means linear equivalence. Let f be a non-constant meromorphic function such that

$$(f) = E + \rho_X(E) - (F + \rho_X(F))\,, \quad \overline{\rho_X^* f} = f\,, \tag{5.6}$$

where (f) is the divisor of f. Then f^ρ, the restriction of f to $X^\rho = S_A^1 \cup S_B^1$, is a non-negative or a non-positive real function if and only if

$$J(E - F) \in (\boldsymbol{Z} \oplus \sqrt{-1}\boldsymbol{R})/(\boldsymbol{Z} \oplus \sqrt{-1}t\boldsymbol{Z})\,, \tag{5.7}$$

where $J(E-F)$ is defined by

$$\sum_i (P_i - Q_i) \quad \operatorname{mod} \boldsymbol{Z} \oplus \boldsymbol{Z}\sqrt{-1}t\,,$$

provided $E-F$ is expressed as $E-F=\sum_i (P_i - Q_i)$.

PROOF. Let S_{zp} be the intersection of $S^1_A \cup S^1_B$ with the set of zeros and poles of f^ρ. Restricting f^ρ to $(S^1_A \cup S^1_B) \setminus S_{zp}$, we get a real function f^*. Considering the restriction of $(E+\rho_X(E)-F-\rho_X(F))$ to $S^1_A \cup S^1_B$, we see that f^ρ has only zeros and poles with even order. So the sign of f^* remains invariant at each point of S_{zp}, and hence f^ρ is non-negative or non-positive on each connected component of $S^1_A \cup S^1_B$. Consequently, f^ρ is a non-negative or a non-positive real function on $S^1_A \cup S^1_B$ if and only if there exist points $\alpha \in S^1_A \setminus S_{zp}$ and $\beta \in S^1_B \setminus S_{zp}$ such that $f(\beta)/f(\alpha) > 0$.

Note that the divisors E and F satisgy the equivalence (5.5) precisely when $J(E-F)$ belongs to $L(0)$ or $L(1/2)$, where $L(s)$ $(0 \leqq s < 1)$ is defined by $L(s) = ((\boldsymbol{Z}+s) \oplus \sqrt{-1}\boldsymbol{R})/(\boldsymbol{Z} \oplus \sqrt{-1}t\boldsymbol{Z})$. Then the following lemma completes the proof of Proposition 5.7. □

LEMMA 5.8. *In the case $J(E-F) \in L(0)$, there exist $\alpha \in S^1_A$ and $\beta \in S^1_B$ such that $f(\beta)/f(\alpha) > 0$. In the case $J(E-F) \in L(1/2)$, there exist $\alpha \in S^1_A$ and $\beta \in S^1_B$ such that $f(\beta)/f(\alpha) < 0$.*

PROOF. The divisor $E+\rho_X(E)-(F+\rho_X(F))$ is expressed as $\sum_{i=1}^{2k}(P_i - Q_i)$ with $P_i \neq Q_j$ $(1 \leqq i, j \leqq 2k)$. By Abel's theorem, there exist integers p and q such that

$$p + q\tau = \sum_{i=1}^{2k} (P_i - Q_i)\,. \tag{5.8}$$

Then the meromorphic function g having this divisor is determined up to a non-zero constant and is expressed as follows:

$$g(u) = \gamma \exp(-2\pi\sqrt{-1}qu) \frac{\theta_1(u-P_1)\cdots\theta_1(u-P_{2k})}{\theta_1(u-Q_1)\cdots\theta_1(u-Q_{2k})}\,, \tag{5.9}$$

where γ is a non-zero complex number and q is the integer given in (5.8).

It is not hand to see by moving the points $P_1, \dots, P_{2k}, Q_1, \dots, Q_{2k}$ appropriately that we can construct a 1-parameter family g_s of meromorphic functions on X which satisfies the following conditions:

(1) If $J(E-F) \in L(0)$, then $g_0 = g$ and $g_1 = \begin{cases} \gamma G_k^{(0)} & \text{for } k \geqq 2\,, \\ \gamma G_k^{(0)} \text{ or } \gamma/G_k^{(0)} & \text{for } k = 1\,. \end{cases}$

If $J(E-F) \in L(1/2)$, then $g_0 = g$ and $g_1 = \gamma G_k^{(1/2)}$. Here $G_k^{(0)}$ and $G_k^{(1/2)}$ are meromorphic functions on X_τ defined by

$$G_k^{(0)}(u) = \exp(-2\pi\sqrt{-1}ku)\left(\frac{\theta_1(u-1/2-\tau/2)}{\theta_1(u-1/2)}\right)^{2k},$$

$$G_k^{(1/2)}(u) = \left(\frac{\theta_1(u-1/2-\tau/2)}{\theta_1(u-\tau/2)}\right)^2 G_{k-1}^{(0)}(u)\,.$$

(2) g_s depends smoothly on the parameter s for $0 \leqq s \leqq 1$. If we denote the divisors consisting of poles and zeros of g_s by $\sum_i P_i^s$ and $\sum_i Q_i^s$ respectively, then they are invariant under ρ_X and $P_i^s \neq Q_j^s$ for $1 \leqq i, j \leqq 2k$.

Also, we can construct 1-parameter families of points $\alpha_s \in S_A^1$ and $\beta_s \in S_B^1$ satisfying the following conditions:

(1) For each $0 \leqq s \leqq 1$, α_s and β_s do not belong to $\{P_1^s, \dots, P_{2k}^s, Q_1^s, \dots, Q_{2k}^s\}$.

(2) $\alpha_1 = \varepsilon + 1/2$ and $\beta_1 = \varepsilon + 1/2 + \tau/2 = \varepsilon + 1/2 + \sqrt{-1}t/2$, where ε is a small positive constant.

We see that the sign of $g_s(\beta_s)/g_s(\alpha_s)$ does not depend on the choice of s, and hence $f(\beta_0)/f(\alpha_0) = g_0(\beta_0)/g_0(\alpha_0)$ and $g_1(\beta_1)/g_1(\alpha_1)$ have the same sign.

Assume that $J(E-F) \in L(0)$ and $k \geqq 2$. Let us determine the sign of $g_1(\beta_1)/g_1(\alpha_1) = G_k^{(0)}(\varepsilon + 1/2 + \tau/2)/G_k^{(0)}(\varepsilon + 1/2)$. Using the identities (2.1) and (2.3), we see that

$$
\begin{aligned}
&G_k^{(0)}(\varepsilon + 1/2 + \tau/2)/G_k^{(0)}(\varepsilon + 1/2) \\
&= \exp(-2\pi\sqrt{-1}k(\tau/2)) \left(\frac{\theta_1(\varepsilon)^2}{\theta_1(\varepsilon - \tau/2)\theta_1(\varepsilon + \tau/2)} \right)^{2k} \\
&= \exp(-2\pi\sqrt{-1}k(\tau/2)) \left(\frac{\theta_1(\varepsilon)^2}{(\sqrt{-1}a(-\varepsilon)\theta_4(\varepsilon))(-\sqrt{-1}a(\varepsilon)\theta_4(\varepsilon))} \right)^{2k} \\
&= \left(\frac{\theta_1(\varepsilon)}{\theta_4(\varepsilon)} \right)^{4k} = \left(\frac{\theta_1(\varepsilon \mid \sqrt{-1}t)}{\theta_4(\varepsilon \mid \sqrt{-1}t)} \right)^{4k}
\end{aligned}
$$

If we fix ε, we get a nowhere vanishing real function ϕ defined by

$$
\phi(t) = \left(\frac{\theta_1(\varepsilon \mid \sqrt{-1}t)}{\theta_4(\varepsilon \mid \sqrt{-1}t)} \right)^{4k} \quad (t > 0)\,.
$$

By (2.6), we get the following Taylor expansion:

$$
\lim_{t \to \infty} q^{-k}\phi(t) = (2\pi)^{4k}\varepsilon^{4k} + O(\varepsilon^{4k+1})\,, \tag{5.10}
$$

from which we see that for a small positive ε, this is positive. If $k = 1$, then we can see that the sign of $f(\beta_0)/f(\alpha_0)$ is positive in a similar fashion. Thus Lemma 5.8 is verified in the case that $J(E-F) \in L(0)$.

In the case $J(E-F) \in L(1/2)$, the sign of $g_1(\beta_1)/g_1(\alpha_1) = G_k^{(1/2)}(\varepsilon + 1/2 + \tau/2)/G_k^{(1/2)}(\varepsilon + 1/2)$ is similarly determined as follows. Using the identities (2.1), (2.2), (2.3), (2.4) and (2.5), we obtain

$$
\begin{aligned}
&G_k^{(1/2)}(\varepsilon+1/2+\tau/2)/G_k^{(1/2)}(\varepsilon+1/2)\\
&=\left(\frac{\theta_1(\varepsilon)}{\theta_1(\varepsilon+1/2)}\right)^2\left(\frac{\theta_1(\varepsilon-\tau/2)}{\theta_1(\varepsilon+1/2-\tau/2)}\right)^{-2}\frac{G_{k-1}^{(0)}(\varepsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\varepsilon+1/2)}\\
&=\left(\frac{\theta_1(\varepsilon)}{\theta_2(\varepsilon)}\right)^2\left(\frac{\theta_1(\varepsilon+\tau/2)/b(\varepsilon-\tau/2)}{\theta_1(\varepsilon+1/2+\tau/2)/b(\varepsilon+1/2-\tau/2)}\right)^{-2}\frac{G_{k-1}^{(0)}(\varepsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\varepsilon+1/2)}\\
&=\left(\frac{b(\varepsilon+1/2-\tau/2)}{b(\varepsilon-\tau/2)}\right)^{-2}\left(\frac{\theta_1(\varepsilon)}{\theta_2(\varepsilon)}\right)^2\left(\frac{\theta_1(\varepsilon+\tau/2)}{\theta_1(\varepsilon+1/2+\tau/2)}\right)^{-2}\frac{G_{k-1}^{(0)}(\varepsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\varepsilon+1/2)}\\
&=\left(\frac{b(\varepsilon+1/2-\tau/2)}{b(\varepsilon-\tau/2)}\right)^{-2}\left(\frac{\theta_1(\varepsilon)}{\theta_2(\varepsilon)}\right)^2\left(\frac{\sqrt{-1}a(\varepsilon)\theta_4(\varepsilon)}{\sqrt{-1}a(\varepsilon+1/2)\theta_4(\varepsilon+1/2)}\right)^{-2}\\
&\qquad\times\frac{G_{k-1}^{(0)}(\varepsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\varepsilon+1/2)}\\
&=\left(\frac{b(\varepsilon+1/2-\tau/2)a(\varepsilon)}{b(\varepsilon-\tau/2)a(\varepsilon+1/2)}\right)^{-2}\left(\frac{\theta_1(\varepsilon)}{\theta_2(\varepsilon)}\right)^2\left(\frac{\theta_4(\varepsilon)}{\theta_4(\varepsilon+1/2)}\right)^{-2}\frac{G_{k-1}^{(0)}(\varepsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\varepsilon+1/2)}\\
&=\left(\frac{b(\varepsilon+1/2-\tau/2)a(\varepsilon)}{b(\varepsilon-\tau/2)a(\varepsilon+1/2)}\right)^{-2}\left(\frac{\theta_1(\varepsilon)}{\theta_2(\varepsilon)}\right)^2\left(\frac{\theta_4(\varepsilon)}{\theta_3(\varepsilon)}\right)^{-2}\frac{G_{k-1}^{(0)}(\varepsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\varepsilon+1/2)}\\
&=-\left(\frac{\theta_3(\varepsilon)}{\theta_2(\varepsilon)\theta_4(\varepsilon)}\right)^2\theta_1(\varepsilon)^2\frac{G_{k-1}^{(0)}(\varepsilon+1/2+\tau/2)}{G_{k-1}^{(0)}(\varepsilon+1/2)}.
\end{aligned}
$$

From (5.10), together with (2.6), we get the following Taylor expansion:

$$\lim_{t\to\infty} q^{-(k-1)}G_k^{(1/2)}(\varepsilon+1/2+\tau/2)/G_k^{(1/2)}(\varepsilon+1/2)=-2^{4(k-1)}\pi^{4k-2}\varepsilon^{4k-2}+O(\varepsilon^{4k-1}).$$

If we take a small positive ε, this is negative. Thus Lemma 5.8 also holds in the case $J(E-F)\in L(1/2)$. □

Now we are in a position to prove Theorem 3.5.

PROOF OF THEOREM 3.5. Conditions (2) and (4) in Definition 3.1 are equivalent to the following assertions:

(1) π is a meromorphic function as in Proposition 5.2.

(2) π has a zero P_0 of order $m+1\geqq 2$.

It is clear that $R=R_+ + \rho_{X*}(R_+)$. Applying Proposition 5.7 to $E=D$ and $F=R_+$, we see that Condition (3) in Definition 3.1 is equivalent to Condition (2) in Theorem 3.5.

Take any spectral data, that is, a triplet $(X,\pi,\mathcal{L})$ with P_0, which satisfies the above assertions and Condition (2) in Theorem 3.5. Consider the following real automorphism ϕ_a on (X,ρ_X) defined by $u\mapsto u+a$, where a is a real number. Them, by using ϕ_a and ρ_X, we can construct a real automorphism ϕ on (X,ρ_X) such that $(X,\phi^*\pi,\phi^*\mathcal{L})$ is a triplet in

Theorem 3.5, where $\phi^*\pi$ and $\phi^*\mathcal{L}$ denote the pull-backs by ϕ of π and $\mathcal{L}$, respectively. Hence Theorem 3.5 follows. □

6. Construction of harmonic maps into complex projective spaces. By applying McIntosh's method of constructing harmonic maps in terms of spectral data, we shall construct harmonic maps corresponding to spectral data having smooth rational or elliptic spectral curves. We also prove Theorems 3.2 and 3.6. From now on, for a Riemann surface X and a sheaf $\mathcal{F}$ on X, we denote by $H^0(X, \mathcal{F})$ and $H^0(Y, \mathcal{F})$ the spaces of holomorphic global sections of $\mathcal{F}$ and its restriction to an open subset Y of X, respectively.

6.1. Construction of Harmonic maps corresponding to spectral data. Let $(X, \pi, \mathcal{L})$ be a spectral data as in Definition 3.1. By identifying $\mathcal{L}$ with a divisor line bundle $\mathcal{O}_X(D)$, we equip $H^0(X, \mathcal{L})$ with a positive definite Hermitian form h as follows.

For given $u, v \in H^0(X, \mathcal{L})$, we define a rational function $h(u, v)$ on $\boldsymbol{P}^1$ by

$$h(u, v)(p) = \sum_{x \in \pi^{-1}(p)} f(x)u(x)\overline{(v \circ \rho_X)(x)}\,, \tag{6.1}$$

where p is a point on $\boldsymbol{P}^1$. Then it is known that $h(u, v)$ is a constant function and the following holds.

THEOREM 6.1 ([13]). *The Hermitian form h is positive definite on $H^0(X, \mathcal{L})$.*

Let $\pi^{-1}(1) = \{\eta_0, \dots, \eta_n\}$, the inverse image of 1 by π, and θ_i $(0 \leqq i \leqq n)$ a local trivialization for $\mathcal{L}$ over a neighbourhood of η_i. Using these local trivializations, the Hermitian form h in (6.1) has also the following expression. For $u \in H^0(X, \mathcal{L})$, let $u_0, \dots, u_n$ be the complex numbers defined by $u(\eta_i) = u_i\theta_i(\eta_i)$. For $v \in H^0(X, \mathcal{L})$, we define the complex numbers $v_0, \dots, v_n$ in a similar way. Then (6.1) becomes

$$h(u, v) = \sum_{i=0}^{n} a_i u_i \overline{v_i}\,, \tag{6.2}$$

where $a_0, \dots, a_n$ are positive real numbers depending only on the choice of $\theta_0, \dots, \theta_n$.

Next we construct a line bundle $L(z)$ with a complex parameter z. Let $U(P_0)$ be a neighbourhood of P_0 and $U(P_\infty)$ a neighbourhood of P_∞ defined by $U(P_\infty) = \rho_X(U(P_0))$. Let ζ be a meromorphic function on $U(P_0) \cup U(P_\infty)$ satisfying $\pi = \zeta^{m+1}$ and $\zeta \circ \rho_X = 1/\bar{\zeta}$. We fix an open cover $X_A \cup X_I$ of X, where $X_A = X \setminus \{P_0, P_\infty\}$ and $X_I = U(P_0) \cup U(P_\infty)$. Let $L(z)$ be the unique line bundle with local trivializations θ_A^z and θ_I^z over X_A and X_I respectively, such that

$$\theta_I^z = \exp(z\zeta^{-1} - \bar{z}\zeta)\theta_A^z \quad \text{on } X_A \cap X_I\,. \tag{6.3}$$

Let $\mathcal{L}_0$ be an ideal sheaf of $\mathcal{L}$ defined by $\mathcal{L}_0 = \mathcal{L}(-mP_0 - E_0)$, where E_0 is the restriction of the zero divisor of π to X_A, that is, $E_0 = P_1 + P_2 + \cdots + P_{n-m}$. Then it is known that $H^0(X, \mathcal{L}_0 \otimes L(z))$ is a 1-dimensional complex vector space. For each $z \in \boldsymbol{C}$, fix a non-zero global section τ of $\mathcal{L}_0 \otimes L(z)$. Then $\tau \otimes \theta_A^{z\,-1}$ belongs to $H^0(X_A, \mathcal{L})$ and we can find holomorphic functions $\psi_0^z, \dots, \psi_n^z$ over $\boldsymbol{P}^1 \setminus \{0, \infty\}$ such that

$$\tau_A \otimes \theta_A^{z-1} = (\psi_0^z \circ \pi)\sigma_0 + \cdots + (\psi_n^z \circ \pi)\sigma_n \,, \tag{6.4}$$

where $\{\sigma_0, \dots, \sigma_n\}$ is an orthonormal basis of $H^0(X, \mathcal{L})$ with respect to the Hermitian form h.

Now we are going to construct a harmonic map corresponding to the spectral data $(X, \pi, \mathcal{L})$. Let $\psi : \boldsymbol{R}^2 \to \boldsymbol{C}P^n$ be a map defined by

$$z = x + \sqrt{-1}y \mapsto [\psi_0^z(1) : \cdots : \psi_n^z(1)] \,.$$

Then it is known that ψ is a harmonic map corresponding to the spectral data $(X, \pi, \mathcal{L})$. This construction is due to McIntosh, which is described in detail in [12] and [13]. However, it seems difficult to compute $\psi_0^z, \dots, \psi_n^z$ in general.

We shall now present a method which determines the values of $\psi_0^z(\lambda), \dots, \psi_n^z(\lambda)$ at $\lambda = 1$. We define a complex $(n+1) \times (n+1)$ matrix $M = (M_{ij})$ by

$$M_{ij}\theta_i(\eta_i) = \sigma_j(\eta_i) \,. \tag{6.5}$$

Let t_j^z be complex numbers defined by

$$\tau \otimes \theta_A^{z-1}(\eta_j) = t_j^z \theta_j(\eta_j) \,. \tag{6.6}$$

Substituting (6.5) and (6.6) to (6.4), we obtain

$${}^t(t_0^z, \dots, t_n^z) = M\,{}^t(\psi_0^z(1), \dots, \psi_n^z(1)) \,. \tag{6.7}$$

LEMMA 6.2. *The determinant of M does not vanish.*

PROOF. Since $\{\sigma_0, \dots, \sigma_n\}$ is an orthonormal basis with respect to h, we have $h(\sigma_i, \sigma_j) = \delta_{ij}$. From this and the identity (6.2), it is easy to see that the following identity holds:

$$M \operatorname{diag}(a_0, \dots, a_n) M^* = I_{n+1} \,,$$

where $\operatorname{diag}(a_0, \dots, a_n)$ denotes the diagonal matrix with diagonal components $a_0, \dots, a_n$, and I_{n+1} is the unit matrix of degree $n+1$. In particular, we see that the determinant of M does not vanish. □

Hence the inverse matrix M^{-1} of M exists, and $\psi_0^z(1), \dots, \psi_n^z(1)$ are determined as

$${}^t(\psi_0^z(1), \dots, \psi_n^z(1)) = M^{-1}\,{}^t(t_0^z, \dots, t_n^z) \,. \tag{6.8}$$

Moreover, it is known that the components of the matrix M and $t_0^z, \dots, t_n^z$ can be expressed by using theta functions and Baker-Akhizer functions (cf. [11]).

Constructing a special orthonormal basis, the above formula becomes much simpler. For $0 \leqq i \leqq n$, take a non-zero element $\sigma_i \in H^0(X, \mathcal{L}(-\eta_0 - \cdots - \eta_{i-1} - \eta_{i+1} - \cdots - \eta_n))$. Rescaling σ_i, we obtain an orthonormal basis $\{\sigma_i\}$ of $\mathcal{L}$, that is, $h(\sigma_i, \sigma_j) = \delta_{ij}$. Then the matrix M is diagonal and M_{ii} is given by

$$M_{ii} = \left.\frac{\sigma_i}{\theta_i}\right|_{\eta_i} .$$

Therefore the right hand side of the equation (6.8) becomes

$$ {}^{t}\left(\left.\frac{\tau\otimes\theta_A(z)^{-1}}{\sigma_0}\right|_{u=\eta_0}, \left.\frac{\tau\otimes\theta_A(z)^{-1}}{\sigma_1}\right|_{u=\eta_1}, \dots, \left.\frac{\tau\otimes\theta_A(z)^{-1}}{\sigma_n}\right|_{u=\eta_n} \right). \tag{6.9} $$

Let $\psi(z,\bar{z},u)$ be a function on X such that $\psi(z,\bar{z},u)\theta_A(z)$ is an element of $H^0(X,\mathcal{L}_0\otimes L(z))$. Setting $\tau=\psi(z,\bar{z},u)\theta_A(z)$ and substituting τ into (6.9), we get

$$ \psi_i^z(1) = \left.\frac{\psi(z,\bar{z},u)}{\sigma_i}\right|_{u=\eta_i} \quad \text{for } 0\leqq i\leqq n. \tag{6.10} $$

Before closing this subsection, we prove the following lemma for later use.

LEMMA 6.3. *Given a function $\phi(z,\bar{z},u)$ on X with the parameter z, let U and V be neighbourhoods of the set of the points $\{P_0, P_\infty\}$ which satisfy the following conditions*:

(1) $\{P_0, P_\infty\}\subset U\subset V\subset X_I$.

(2) *$\phi(z,\bar{z},u)$ is a holomorphic section of $\mathcal{O}_X(M)$ on $X\setminus U$ for any $z\in\boldsymbol{C}$, where M is a divisor on $X\setminus V$.*

(3) *$\phi(z,\bar{z},u)\exp(-z\zeta^{-1}+\bar{z}\zeta)$ is a holomorphic section of $\mathcal{O}_X(N)$ on V for any $z\in\boldsymbol{C}$, where N is a divisor on U.*

Then $\phi(z,\bar{z},u)\theta_A(z)$ belong to $H^0(X,\mathcal{F}\otimes L(z))$ for any $z\in\boldsymbol{C}$, where $\mathcal{F}\cong\mathcal{O}_X(M+N)$.

PROOF. From the condition (2), $\phi(z,\bar{z},u)\otimes\theta_A(z)$ clearly belongs to $H^0(X\setminus U,\mathcal{O}_X(M)\otimes L(z)) = H^0(X\setminus U,\mathcal{F}\otimes L(z))$. It suffices to show that $\phi(z,\bar{z},u)\otimes\theta_A(z)$ belongs to $H^0(V,\mathcal{O}_X(N)\otimes L(z)) = H^0(V,\mathcal{F}\otimes L(z))$. By using (6.3), we see that $\phi(z,\bar{z},u)\otimes\theta_A(z) = \phi(z,\bar{z},u)\exp(-z\zeta^{-1}+\bar{z}\zeta)\otimes\theta_I(z)$ on $V(\subset X_I)$. On the other hand, from the condition (3) it follows that $\phi(z,\bar{z},u)\exp(-z\zeta^{-1}+\bar{z}\zeta)$ is an element of $H^0(V,\mathcal{F})$ and hence $\phi(z,\bar{z},u)\otimes\theta_A(z)$ belongs to $H^0(V,\mathcal{F}\otimes L(z))$. Thus $\phi(z,\bar{z},u)\theta_A(z)$ is a global holomorphic section of $\mathcal{F}\otimes L(z)$ on X. □

6.2. Proof of Theorem 3.2. Using the results in Section 6.1, let us now construct harmonic maps corresponding to spectral data whose spectral curves are smooth rational curves, and prove Theorem 3.2.

Let $(X,\pi,\mathcal{L})$ be a spectral data as in Theorem 3.1. We may assume that π, R and $\mathcal{L}$ are of the following form:

$$ \pi(\lambda)=\alpha_0\lambda^{m+1}\frac{\prod_{j=1}^{n-m}(\lambda-P_j)}{\prod_{j=1}^{n-m}(\lambda-Q_j)}, \quad P_0=0, \quad R=D+\rho_X(D), \quad \mathcal{L}=\mathcal{O}_X(D), $$

where α_0 is a constant as in Theorem 3.1 and D is a divisor defined by $D=mP_0+\sum_{i=1}^{n-m}R_i$. First we prove the following

LEMMA 6.4. *Let $(X,\pi,\mathcal{L})$ be a spectral data as above. Define a function $\psi(z,\bar{z},\lambda)$ on X with parameter z by*

$$ \psi(z,\bar{z},\lambda)=\exp\left(\frac{z}{\kappa}\lambda^{-1}-\overline{\left(\frac{z}{\kappa}\right)}\lambda\right)\cdot\frac{\prod_{j=1}^{n-m}(\lambda-P_j)}{\prod_{j=1}^{n-m}(\lambda-R_j)}. \tag{6.11} $$

Here $\kappa = (\partial\zeta/\partial\lambda)\,|_{\lambda=P_0}$ *is the value of the differential of the meromorphic function* ζ *as in* (6.3) *at* $\lambda = P_0$. *Then* $\psi(z, \bar{z}, u)\theta_A(z)$ *is an element of* $H^0(X, \mathcal{L}_0 \otimes L(z))$ *for any* $z \in \boldsymbol{C}$.

PROOF. Denote by $D\,|_{P_0 \cup Q_0}$ the restriction of the divisor $D = mP_0 + \sum_{i=1}^{n-m} R_i$ to $P_0 \cup Q_0$. Then, applying Lemma 6.3 to $M = D - D\,|_{P_0\cup Q_0} - E_0$, $N = D\,|_{P_0\cup Q_0} - mP_0$, and $\phi = \psi$, we get the assertion. $\square$

Next we construct a special orthonormal basis of global sections of $\mathcal{L} = \mathcal{O}_X(mP_0 + \sum_{i=1}^{n-m} R_i)$ following the method explained above. Here we choose $f = 1$ as a meromorphic function on X in Condition (3) of Definition 3.1. For $0 \leqq i \leqq n$, let us denote by σ_i the following element

$$\sigma_i = \frac{\eta_i^m \prod_{j=1}^{i-1}(\eta_i - R_j)}{\prod_{j=0}^{i-1}(\eta_i - \eta_j)\cdot\prod_{j=i+1}^{n}(\eta_i - \eta_j)} \frac{\prod_{j=0}^{i-1}(\lambda - \eta_j)\cdot\prod_{j=i+1}^{n}(\lambda - \eta_j)}{\lambda^m \prod_{j=1}^{i-1}(\lambda - R_j)}.$$

Then we see that $\sigma_i \in H^0(X, \mathcal{L}(-\eta_0 - \cdots - \eta_{i-1} - \eta_{i+1} - \cdots - \eta_n))$ and $h(\sigma_i, \sigma_i) = 1$ for $0 \leqq i \leqq n$. Thus we get an orthonormal basis $\{\sigma_i\}_{0\leqq i\leqq n}$ of $H^0(X, \mathcal{L})$, that is, $h(\sigma_i, \sigma_j) = \delta_{ij}$.

Owing to (6.10), the corresponding harmonic map: $\boldsymbol{R}^2 \to \boldsymbol{CP}^n$ is given by

$$z = x + \sqrt{-1}y \mapsto [\psi_0^z(1) : \psi_1^z(1) : \cdots : \psi_n^z(1)],$$

where each $\psi_i^z(1)$ is a function defined by

$$\psi_i^z(1) = \exp\left(\frac{z}{\kappa}\eta_i^{-1} - \overline{\left(\frac{z}{\kappa}\right)}\eta_i\right) \cdot \frac{\prod_{j=1}^{n-m}(\eta_i - P_j)}{\prod_{j=1}^{n-m}(\eta_i - R_j)}. \tag{6.12}$$

Define a map $F : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ by $z = x + \sqrt{-1}y \mapsto \kappa z$. Then the composition $\psi \circ F$ gives rise to the harmonic map given in (3.2). This completes the proof of Theorem 3.2.

6.3. Proof of Theorem 3.6. By an argument similar to that in Section 6.2, we now construct harmonic maps corresponding to spectral data whose spectral curves are smooth elliptic curves, and prove Theorem 3.6.

LEMMA 6.5. *Let* $(X = X_{\sqrt{-1}t}, \pi, \mathcal{L} = \mathcal{O}_X(\sum_{i=1}^{k+n+1} E_i - \sum_{i=1}^{k} F_i))$ *be a spectral data as in Theorem* 3.5. *Define a function* $\psi(z, \bar{z}, u)$ *on* X *with parameter* z *by*

$$\begin{aligned}\psi(z, \bar{z}, u) =& \exp\left(\frac{z}{\kappa}[\zeta_w(u - P_0) - Au] - \overline{\left(\frac{z}{\kappa}\right)}[\zeta_w(u - Q_0) - Au]\right)\\ &\cdot\frac{\prod_{j=1}^{k}\theta_1(u - F_j)\cdot\theta_1(u - P_0)^m\cdot\prod_{j=1}^{n-m}\theta_1(u - P_j)\cdot\theta_1(u - G - H)}{\prod_{j=1}^{k+n+1}\theta_1(u - E_j)}\end{aligned} \tag{6.13}$$

Here ζ_w *is Weierstrass' zeta function as in* (2.7),

$$G = \sum_{i=1}^{k+n+1} E_i - \sum_{i=1}^{k} F_i - mP_0 - \sum_{i=1}^{n-m} P_i, \quad H = H(z, \bar{z}) = \frac{z}{\kappa} - \overline{\left(\frac{z}{\kappa}\right)},$$

A is the constant as in (2.8), *and* $\kappa = (\partial\zeta/\partial u)\,|_{u=P_0}$ *is the value of the differential of the meromorphic function* ζ *in* (6.3) *at* $u = P_0$. *Then* $\psi(z, \bar{z}, u)\theta_A(z)$ *is an element of* $H^0(X, \mathcal{L}_0 \otimes L(z))$ *for any* $z \in \boldsymbol{C}$.

PROOF. The proof of this lemma is similar to that of Lemma 6.4. □

Next we construct, following the method used in Section 6.1, a special orthonormal basis of global sections of $\mathcal{L} = \mathcal{O}_X(\sum_{i=1}^{k+n+1} E_i - \sum_{i=1}^{k} F_i)$. Here we choose

$$f = \frac{\prod_{j=1}^{k+n+1} \theta_1(u - E_j)}{\prod_{j=1}^{k} \theta_1(u - F_j) \prod_{j=0}^{n} \theta_1(u - R_j)} \cdot \frac{\prod_{j=1}^{k+n+1} \theta_1(u - \overline{E_j})}{\prod_{j=1}^{k} \theta_1(u - \overline{F_j}) \prod_{j=0}^{n} \theta_1(u - \overline{R_j})}$$

as a meromorphic function on X in Condition (3) of Definition 3.1. Let μ_i be the constant in Theorem 3.6 and set $\hat{\eta}_i = \sum_{i=1}^{k+n+1} E_i - \sum_{i=1}^{k} F_i - (\eta_0 + \cdots + \eta_{i-1} + \eta_{i+1} + \cdots + \eta_n)$. Denoting by σ_i the element

$$\mu_i \frac{\prod_{j=0}^{n} \theta_1(\eta_i - R_j) \cdot \prod_{j=1}^{k} \theta_1(u - F_j) \cdot \prod_{j=0}^{i-1} \theta_1(u - \eta_j) \cdot \theta_1(u - \hat{\eta}_i) \cdot \prod_{j=i+1}^{n} \theta_1(u - \eta_j)}{\prod_{j=1}^{i-1} \theta_1(\eta_i - \eta_j) \cdot \theta_1(\eta_i - \hat{\eta}_i) \cdot \prod_{j=i+1}^{n} \theta_1(\eta_i - \eta_j) \cdot \prod_{j=1}^{k+n+1} \theta_1(u - E_j)},$$

we see that $\sigma_i \in H^0(X, \mathcal{L}(-\eta_0 - \cdots - \eta_{i-1} - \eta_{i+1} - \cdots - \eta_n))$ and $h(\sigma_i, \sigma_i) = 1$ for $0 \leqq i \leqq n$. Thus we get an orthonormal basis $\{\sigma_i\}_{0 \leqq i \leqq n}$ of $H^0(X, \mathcal{L})$, that is, $h(\sigma_i, \sigma_j) = \delta_{ij}$. These are well-defined by the following lemma.

LEMMA 6.6. *The above constants $\hat{\eta}_i$ are not equal to η_i* $(\operatorname{mod} \boldsymbol{Z} \oplus \boldsymbol{Z}\tau)$.

PROOF. If $\hat{\eta}_i = \eta_i \bmod \boldsymbol{Z} \oplus \boldsymbol{Z}\tau$, then $h(\sigma_i, \sigma_i) = 0$, which is a contradiction because h is positive definite. □

On account of (6.10), the corresponding harmonic map: $\boldsymbol{R}^2 \to \boldsymbol{C}P^n$ is given by

$$z = x + \sqrt{-1}y \mapsto [\psi_0^z(a) : \psi_1^z(1) : \cdots : \psi_n^z(1)],$$

where each $\psi_i^z(1)$ is a function defined by

$$\begin{aligned}
\psi_i^z(1) &= \mu_i^{-1} \exp\left(\frac{z}{\kappa}[\zeta_w(\eta_i - P_0) - A\eta_i] - \overline{\left(\frac{z}{\kappa}\right)[\zeta_w(\eta_i - Q_0) - A\eta_i]}\right) \\
&\quad \cdot \frac{\theta_1(\eta_i - P_0)^m \prod_{j=1}^{n-m} \theta_1(\eta_i - P_j) \cdot \theta_1(\eta_i - G - H(z, \bar{z}))}{\prod_{j=0}^{n} \theta_1(\eta_i - R_j)}.
\end{aligned} \tag{6.14}$$

Define a map $F : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ by $z = x + \sqrt{-1}y \mapsto \kappa z$. Then the composition $\psi \circ F$ gives rise to the harmonic map given in (3.4). This completes the proof of Theorem 3.6.

7. Periodicity conditions of harmonic maps in terms of generalized Jacobians. McIntosh studied periodicity conditions of the corresponding harmonic maps by introducing certain homomorphisms into generalized Jacobians. In this section, when X is a smooth elliptic curve, we reformulate McIntosh's periodicity conditions by introducing certain families of lines on the complex plane $\boldsymbol{C}$, and prove Theorem 3.7.

Let $(X, \pi, \mathcal{L})$ be a spectral data as in Definition 3.1. Let $L(z)$ be the line bundle as in Section 6 and $\theta_A(z)$ the local trivialization of $L(z)$ over X_A as in (6.3). Let $J(X_o)$ be a generalized Jacobian defined by

$$J(X_o) = \bigcup_{L \in J(X)} \{(\operatorname{Hom}(L|_{\eta_1}, L|_{\eta_0}) \setminus \{0\}) \times \cdots \times (\operatorname{Hom}(L|_{\eta_n}, L|_{\eta_0}) \setminus \{0\})\}.$$

We define a map $\hat{L} : \boldsymbol{R}^2 \to J(X_o)$ by $z = x + \sqrt{-1}y \mapsto (L(z), h_1(z), \dots, h_n(z))$, where $h_i(z)$ is an element of $\mathrm{Hom}(L(z)\,|_{\eta_i}, L(z)\,|_{\eta_0}) \setminus \{0\} (\cong \boldsymbol{C}^*)$ defined by the condition that $h_i(z)$ maps $\theta_A(z)\,|_{\eta_i}$ to $\theta_A(z)\,|_{\eta_0}$. Then McIntosh proved the following

THEOREM 7.1 ([13]). *The harmonic map $\psi : \boldsymbol{R}^2 \to \boldsymbol{C}P^n$ corresponding to the above spectral data is doubly periodic if and only if $\hat{L} : \boldsymbol{R}^2 \to J(X_o)$ is doubly periodic.*

In the case of the smooth rational curve X, the maps Φ in the proof of Theorem 3.3 and $\hat{L}$ are essentially the same.

Let us determine the map $\hat{L}$ when $(X, \pi, \mathcal{L})$ is a spectral data with a smooth elliptic curve as its spectral curve. First, we compute the map $L : \boldsymbol{R}^2 \to J(X)$ defined by $z = x+\sqrt{-1}y \mapsto L(z)$. Let T_z be a divisor defined by

$$T_z = (D) - m(P_0) - (S) - E_0\,, \tag{7.1}$$

where S is a point on X defined by $S = G + H$ and E_0 is the divisor given in Section 6.1. Then $\psi(z, \bar{z}, u) \otimes \theta_A(z)$ belongs to $H^0(X, \mathcal{O}_X(T_z) \otimes L(z))$ $(\cong H^0(X, \mathcal{L}_0(-S) \otimes L(z)))$ by Lemma 6.5. Moreover, we see that $\psi(z, \bar{z}, u) \otimes \theta_A(z)$ is a non-vanishing global holomorphic section of $\mathcal{O}_X(T_z) \otimes L(z)$. In particular, the line bundle $L(z) \otimes \mathcal{O}_X(T_z)$ is tribial, that is, $L(z) \otimes \mathcal{O}_X(T_z) \cong \mathcal{O}_X$, and hence $L(z) \cong \mathcal{O}_X(-T_z)$. Using (7.1) and identifying Jacobian $J(X)$ with $X \cong \boldsymbol{C}/(\boldsymbol{Z} \oplus \sqrt{-1}t\boldsymbol{Z})$, we see that $L : \boldsymbol{R}^2 \to J(X)$ is given by

$$z = x + \sqrt{-1}y \mapsto -D + mP_0 + S + E_0 = H(z, \bar{z}) = z/\kappa - \overline{(z/\kappa)} \mod \boldsymbol{Z} \oplus \boldsymbol{Z}\sqrt{-1}t\,,$$

where κ is the complex number in Lemma 6.5.

Second, we determine $\theta_A(z)$. Let Θ be a meromorphic function on $\boldsymbol{C}^2$ defined by

$$\Theta(w, u) = \frac{\prod_{j=1}^{k+n+1} \theta_1(u - E_j)}{\prod_{j=1}^{k} \theta_1(u - F_j) \cdot \theta_1(u - P_0)^m \prod_{j=1}^{n-m} \theta_1(u - P_j) \cdot \theta_1(u - G - w)}\,.$$

Using $\psi(z, \bar{z}, u) \otimes \theta_A(z) \in H^0(X, L(z) \otimes \mathcal{O}_X(T_z)) = H^0(X, \mathcal{O}_X) \cong \boldsymbol{C}$, we see that

$$\theta_A(z) = C \exp\left(-\frac{z}{\kappa}[\zeta_w(u - P_0) - Au] + \overline{\left(\frac{z}{\kappa}\right)}[\zeta_w(u - Q_0) - Au]\right) \Theta(H(z, \bar{z}), u)\,,$$

where C is a non-zero constant.

Now we give an explicit description of $\hat{L}$. Let $v : S^1_J = \{e^{\sqrt{-1}\theta} \,|\, 0 \leqq \theta < 2\pi\} \to J(X)$ be a map defined by $e^{\sqrt{-1}\theta} \mapsto \sqrt{-1}t\theta/2\pi \mod \boldsymbol{Z} \oplus \boldsymbol{Z}\sqrt{-1}t$. Let $J_S \to S^1_J$ be the pull-back of $J(X_o)$ by v. For $0 \leqq i \leqq n$, we define $B_i : e^{\sqrt{-1}\theta} \in S^1_J \mapsto B_i(e^{\sqrt{-1}\theta}) \in \mathrm{Hom}(v(e^{\sqrt{-1}\theta})\,|_{\eta_i}, v(e^{\sqrt{-1}\theta})\,|_{\eta_0})$, sections of $J_S \to S^1_J$, by the condition that each $B_i(e^{\sqrt{-1}\theta})$ maps the element $\exp(\sqrt{-1}\eta_i\theta)\Theta(\sqrt{-1}t\theta/(2\pi), \eta_i)$ of $\mathcal{O}_X(-T_z)\,|_{\eta_i}$ to the element $\exp(\sqrt{-1}\eta_0\theta)\Theta(\sqrt{-1}t\theta/(2\pi), \eta_0)$ of $\mathcal{O}_X(-T_z)\,|_{\eta_0}$. Since the image of $\boldsymbol{R}^2$ by L is contained in $\boldsymbol{Z} \oplus \boldsymbol{R}\tau \mod \boldsymbol{Z} \oplus \boldsymbol{Z}\tau (\subset J(X))$, we can regard $\hat{L} : \boldsymbol{R}^2 \to J(X_o)$ as a map $\boldsymbol{R}^2 \to J_S$. Using this identification, the map $\hat{L} : \boldsymbol{R}^2 \to J_S$ is given by

$$z = x + \sqrt{-1}y \mapsto (\exp(2\pi H(z, \bar{z})/t) \in S^1_J, h_1(z, \bar{z}), h_2(z, \bar{z}), \dots, h_n(z, \bar{z}))\,,$$

where $h_i(z,\bar z)$ is an element of $\mathrm{Hom}(v(\exp(2\pi H(z,\bar z)/t))|_{\eta_i}, v(\exp(2\pi H(z,\bar z)/t))|_{\eta_0})$ being defined by $h_i(z,\bar z)=\exp(b_i(z,\bar z))B_i(\exp(2\pi H(z,\bar z)/t))$ with

$$\begin{aligned} b_i(z,\bar z) = &-\frac{z}{\kappa}\left[\zeta_w(\eta_0-P_0)-\zeta_w(\eta_i-P_0)-\frac{B}{\tau}(\eta_0-\eta_i)\right] \\ &+\overline{\left(\frac{z}{\kappa}\right)}\left[\zeta_w(\eta_0-Q_0)-\zeta_w(\eta_i-Q_0)-\frac{B}{\tau}(\eta_0-\eta_i)\right]. \end{aligned}$$

LEMMA 7.2. *For $1\leqq i\leqq n$, each $b_i(z,\bar z)$ is pure imaginary.*

PROOF. We may assume that $0\leqq \mathrm{Im}\,P_0, \mathrm{Im}\,Q_0, \mathrm{Im}\,\eta_0,\dots,\mathrm{Im}\,\eta_n < \mathrm{Im}\,\tau$. On this assumption, $Q_0=\overline{P_0}+\tau$. Using $\overline{\zeta_w(u)}=\zeta_w(\bar u)$ and $\bar B=-B$, we then get

$$\begin{aligned} &\overline{[\zeta_w(\eta_0-P_0)-\zeta_w(\eta_i-P_0)-B\tau^{-1}(\eta_0-\eta_i)]} \\ &\quad=[\zeta_w(\overline{\eta_0-P_0})-\zeta_w(\overline{\eta_i-P_0})]-B\tau^{-1}(\overline{\eta_0-\eta_i}) \qquad (7.2)\\ &\quad=[\zeta_w(\overline{\eta_0}-Q_0+\tau)-\zeta_w(\overline{\eta_i}-Q_0+\tau)]-B\tau^{-1}\overline{(\eta_0-\eta_i)}. \end{aligned}$$

In the case that $\eta_0\in S^1_A$ and $\eta_i\in S^1_B$, it follows from $\zeta_w(u+\tau)=\zeta_w(u)+B$ that the right hand side of (7.2) is equal to

$$\begin{aligned} &[\zeta_w(\eta_0-Q_0+\tau)-\zeta_w(\eta_i-\tau-Q_0+\tau)]-B\tau^{-1}(\eta_0-\eta_i+\tau) \\ &\quad=[\zeta_w(\eta_0-Q_0)-\zeta_w(\eta_i-Q_0)]-B\tau^{-1}(\eta_0-\eta_i), \end{aligned}$$

which implies that b_i is pure imaginary. Similarly, we can also see that b_i is pure imaginary in other cases. □

Thus we can consider $\hat L:\boldsymbol{R}^2\to J_S$ to be a map $L_T:\boldsymbol{R}^2\to T^{n+1}=S^1_J\times S^1\times\cdots\times S^1$ defined by

$$z=x+\sqrt{-1}y\mapsto(\exp(2\pi H(z,\bar z)/t),\exp(b_1(z,\bar z)),\dots,\exp(b_n(z,\bar z))).$$

Evidently, $\hat L$ is doubly periodic if and only if L_T is doubly periodic. Then we have the following

PROPOSITION 7.3. *The harmonic map $\psi:\boldsymbol{R}^2\to \boldsymbol{C}P^n$, defined by* (6.14), *corresponding to a spectral data $(X,\pi,\mathcal{L})$ is doubly periodic with periods $v_1, v_2\in\boldsymbol{C}$ if and only if the set $V=\bigcap_{0\leqq i\leqq n}V_i$ contains the 2-dimensional lattice $M=\boldsymbol{Z}v_1\oplus\boldsymbol{Z}v_2$, where $V_0,\dots,V_n$ are the sets defined by*

$$V_i=\begin{cases}\pi\beta_i^{-1}(\boldsymbol{R}\oplus\sqrt{-1}\boldsymbol{Z}), & \text{if } \beta_i\neq 0,\\ \boldsymbol{C}, & \text{otherwise.}\end{cases} \tag{7.3}$$

Here $\beta_0,\beta_1,\dots,\beta_n$ are complex constants defined by

$$\beta_0=2\pi/(\kappa t),\quad \beta_i=[\zeta_w(\eta_0-P_0)-\zeta_w(\eta_i-P_0)-B(\eta_0-\eta_i)\tau^{-1}]/\kappa \quad (1\leqq i\leqq n).$$

PROOF. Recall that ψ has two periods v_1, v_2 if and only if L_T has two periods v_1, v_2 by Theorem 7.1. If L_T has two periods v_1, v_2, then the set $\boldsymbol{Z}v_1\oplus\boldsymbol{Z}v_2$ is contained in V, since V is the set of all points on which the value of L_T is equal to the initial value $L_T(0)=(1,\dots,1)\in T^{n+1}$.

Conversely, if V contains a 2-dimensional lattice $M = \boldsymbol{Z}v_1 \oplus \boldsymbol{Z}v_2$, then clearly v_1 and v_2 are periods of L_T, since L_T is a homomorphism from the additive group $\boldsymbol{R}^2$ to T^{n+1}. Hence Condition (7.3) is a necessary and sufficient condition for L_T to be doubly periodic with periods v_1, v_2. $\square$

Now let us prove Theorem 3.7.

PROOF OF THEOREM 3.7. From the argument in the proof of Theorem 3.6, we see that the map given in Theorem 3.7 is a composition $\psi \circ F$, where ψ is the map in Proposition 7.3 and F is a map defined by $\boldsymbol{R}^2 \to \boldsymbol{R}^2$, $z = x + \sqrt{-1}y \mapsto \kappa z$. Thus Theorem 3.7 follows immediately from Proposition 7.3. $\square$

The proof of Theorem 3.3 in similar to that of Theorem 3.7.

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MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI 980–8578
JAPAN

E-mail address: tetsu@math.tohoku.ac.jp