Tohoku Math. J.
52 (2000), 555–577

CONFIGURATIONS OF CONICS WITH MANY TACNODES

GÁBOR MEGYESI

(Received April 14, 1999)

Abstract. We investigate configurations of conics in the projective plane which have the property that the number of tacnodes is equal or close to the upper bound obtained from the Miyaoka-Yau inequality. We show that for 5 conics there are exactly 3 configurations, including 2 new ones, achieving the maximum 17 tacnodes, and for 6 conics the maximum number of tacnodes is 22, which together with previous results implies that the Miyaoka-Yau bound can never be achieved.

1. Introduction. Let us consider sets of k smooth conics in the complex projective plane such that their union has only nodes and tacnodes (A_1 and A_3 singularities), but no other types of singularities, in particular, no three conics pass through one point. Let $t(k)$ be the maximal number of tacnodes for a given k. Obviously, $t(k) \leq k(k-1)$. If $k \geq 3$, we can consider the double cover of the plane branched along the union of the conics and apply the Miyaoka-Yau inequality [4, 1.1] to it, or take the boundary divisor B consisting of the union of conics with coefficient 1/2 and apply [2, Theorem 4.3] to the pair $(\boldsymbol{P}^2, B)$, and we obtain the inequality [1]

$$t(k) \leq \frac{4}{9}k^2 + \frac{4}{3}k\,. \tag{1}$$

If equality held, the double cover X of $\boldsymbol{P}^2$ branched along the union of the conics would be a surface for which equality holds in the Miyaoka-Yau inequality for singular surfaces, and if Y were a smooth surface with a covering $Y \to X$ étale outside the singularities of X, we would have $c_1(Y)^2 = 3c_2(Y)$, this is why this problem is interesting in algebraic geometry. We shall, however, prove in Theorem 18, that equality cannot be achieved for any k.

Smooth conics in $\boldsymbol{P}^2$ are parametrized by an open subset of $(\boldsymbol{P}^5)^*$, each tacnode imposes one condition and $\dim \mathrm{Aut}(\boldsymbol{P}^2) = 8$, so by a naïve dimension count, one would expect a $5k - t - 8$ dimensional family of configurations modulo projective equivalence for k conics with t tacnodes. The examples in [1] with $k = 14$, $t = 98$ and $k = 12$, $t = 72$ show that there exist configurations with negative expected dimension, and we shall see in Section 8 that certain combinatorial types of configurations with positive expected dimension do not exist.

It was proved in [3, Theorem 1, 6] that the inequality (1) is not sharp in the sense that $t(k)$ is less than the integer part of the right-hand side for $k = 8, 9, 12$ and for $k \geq 15$, and in fact, $t(k) \leq ck^{2-1/7633}$ for a suitable constant c. It was also proved in [3, Theorem 6] that in any configuration of six conics with 24 tacnodes there must be exactly 8 tacnodes and 4 nodes on each conic, which restricts the combinatorial possibilities we have to consider.

2000 *Mathematics Subject Classification*. Primary 14N05.

For $k = 5$ the upper bound is 17, the expected dimension for 5 conics and 17 tacnodes is 0, and we shall show that up to projective equivalence there are exactly three configurations realising it, including the one constructed by Naruki [5].

We shall prove that the Miyaoka-Yau bound of 24 for $k = 6$ cannot be achieved either, and in fact, $t(6) = 22$, where the expected dimension is 0. As a corollary, we obtain that $t(7) \leq 30$.

We shall proceed by investigating systematically configurations of 2, 3 and 4 conics with many tacnodes and partly for their use later and partly for their own interest. During this we shall also recover some known results.

We shall work over $\boldsymbol{C}$ and from now on the word conic, unless specified otherwise, will mean a smooth conic.

DEFINITION. A *configuration* of conics will mean a set of smooth conics in the projective plane whose union has no singularities other than nodes and tacnodes. $(PQRS)$ will denote the cross ratio of the points P, Q, R, S if they lie on a line, we shall use $(PQRS)_C$ to denote their cross ratio on the conic C, if they do not lie on a line. The conics in a configuration will be denoted by $C_1, C_2, \ldots, C_n$. If C_i and C_j are tangent to each other at two points, e_{ij} will denote the line connecting the two contact points, if C_i and C_j are tangent to each other at one point, e_{ij} will denote their common tangent line at the contact point and e'_{ij} the line connecting the two points of transversal intersection.

To each configuration of conics we can associate a graph with possible double edges but no loops as in [5]. The vertices correspond to the conics. Under the conditions on the singularities, two conics may meet transversally at four points, then the two vertices are connected by two edges, the two conics may meet transversally at two points and be tangent to each other at one point, then the two vertices are connected by one edge, or they may be tangent at two points, then the two vertices are not connected at all.

The first problem is to determine when two conics are tangent to each other at one point or two points. By taking a point $(X_0 : Y_0 : Z_0)$ on one of the curves and considering the second point of intersection of the line $Y - Y_0 = t(Z_0X - X_0Z)$ with it, we can parametrize it in the form $X = p_1(t)$, $Y = p_2(t)$, $Z = p_3(t)$, where $p_i(t) \in \boldsymbol{C}[t]$ is of degree at most 2. By substituting $p_1(t)$, $p_2(t)$ and $p_3(t)$ for X, Y and Z into the equation of the other conic, we obtain a polynomial $q(t)$ of degree at most 4. The two conics are tangent to each other if and only if $q(t)$ has a double root or it has degree 2 corresponding to a double root at ∞, which happens if and only if the discriminant of $q(t)$, considered as a quartic, is 0. If t_0 is the double root, the point of contact is $(p_1(t_0) : p_2(t_0) : p_3(t_0))$.

The two conics are tangent to each other at two points if and only if $q(t)$ is the square of another polynomial, but is not a fourth power.

DEFINITION. Let $g(t) = \sum_{i=0}^{4} g_i t^i \in \boldsymbol{C}[t]$. We define the following functions of $g(t)$:

$$S_1(g) = g_4 g_1^2 - g_0 g_3^2 ,$$
$$S_2(g) = g_3^3 + 8 g_1 g_4^2 - 4 g_2 g_3 g_4 ,$$

$$S_3(g) = g_1^3 + 8g_3g_0^2 - 4g_2g_1g_0\,,$$
$$S_4(g) = g_2^2 - 4g_0g_4\,,$$
$$S_5(g) = g_1^2 - 4g_0g_2\,,$$
$$S_6(g) = g_3^2 - 4g_2g_4\,.$$

LEMMA 1. *$g(t) = g_4t^4 + g_3t^3 + g_2t^2 + g_1t + g_0$ ($g_i \in \boldsymbol{C}$) is the square of a polynomial if and only if $S_1(g) = S_2(g) = S_3(g) = 0$ and one of the following conditions holds*:

(a) $g_4g_3g_1g_0 \neq 0$,
(b) $g_1 = g_3 = 0$ *and* $S_4(g) = 0$,
(c) $g_4 = g_3 = 0$ *and* $S_5(g) = 0$,
(d) $g_1 = g_0 = 0$ *and* $S_6(g) = 0$.

PROOF. Let us first assume that $g_4g_3g_1g_0 \neq 0$. If $g(t)$ has a square root, it must be $r(t) = \pm\sqrt{g_4}(t^2 + g_3t/(2g_4) + g_1/g_3)$, because these are the only coefficients which make the t^4, t^3 or t terms in $g(t) - (r(t))^2$ vanish. By equating the constant terms, we obtain $S_1(g) = 0$, while from the coefficient of t^2 we get $S_2(g) = 0$. Conversely, if $S_1(g) = S_2(g) = 0$, then $r(t)$ is the square root of $g(t)$.

By writing the square root of $g(t)$ in the form $\pm\sqrt{g_0}(1 + g_1t/(2g_0) + g_3t^2/g_1)$, we similarly obtain $S_3(g) = 0$. It is also true that if $g_4g_3g_1g_0 \neq 0$ and $S_1(g) = 0$, then $S_2(g) = 0$ and $S_3(g) = 0$ are equivalent.

Let us now assume that $g(t)$ is a square, but some of the coefficients g_4, g_3, g_1 and g_0 is 0. It is easy to check that the only possibilities are those listed in (b), (c) and (d). $S_1(t) = S_2(t) = S_3(t) = 0$ is automatically satisfied in these cases, and the additional conditions are obviously necessary and sufficient for $g(t)$ to be a square. □

There is another method for determining when two conics are tangent to each other. We can write the equations of the conics in the form $\mathbf{x}^T A\mathbf{x} = 0$ and $\mathbf{x}^T B\mathbf{x} = 0$, where $\mathbf{x} = (X, Y, Z)$ and A, B are 3×3 symmetric matrices, then the two conics are tangent to each other if and only if $\det(A - tB)$, a cubic polynomial in t, has a multiple root, which can be determined by calculating its discriminant. This method is simpler when we only want to know whether two conics are tangent to each other, but less suitable for determining when two conics are tangent to each other at two points.

LEMMA 2. (Cf. [5, Section 5]). *Let C_1, C_2 be conics given by the equations $Q_1(X, Y, Z) = 0$ and $Q_2(X, Y, Z) = 0$, respectively. If C_3 is a conic which is tangent to both C_1 and C_2 at two points, its equation can be written in the form*

$$(2)\qquad Q_3(X, Y, Z) = Q_1(X, Y, Z) + L_1^2(X, Y, Z) = \lambda Q_2(X, Y, Z) + L_2^2(X, Y, Z)\,,$$

where $\lambda \in \boldsymbol{C}$ and $L_1 = 0$, $L_2 = 0$ define lines connecting the two points where C_3 is tangent to C_1 or C_2, respectively. Furthermore, $Q_1 - \lambda Q_2 = L_2^2 - L_1^2 = 0$ defines a degenerate conic, and L_1 and L_2 are linear combinations of the defining equations of the components of this conic (if it is a double line, $L_1 = 0$ and $L_2 = 0$ define this line with the reduced structure). λ is uniquely determined by C_3, while L_1 and L_2 are determined up to sign.

PROOF. We can multiply Q_3, L_1 and L_2 by suitable scalars so that (2) holds. Then the rank of the conic $Q_1 - \lambda Q_2 = L_2^2 - L_1^2 = 0$ is at most two, and writing it in the form $(L_2 - L_1)(L_1 + L_2) = 0$ makes it obvious that L_1 and L_2 are linear combinations of the equations of the components of this degenerate conic.

L_1 and L_2 are determined by C_3 up to multiplication by scalars. If they define the same line, then $Q_1 - \lambda Q_2$ is a multiple of L_1^2 and λ is obviously unique. If they define different lines, then let P be the point of intersection of these lines. The degenerate conic defined by $Q_1 - \lambda Q_2 = 0$ is the union of two lines meeting at P. These lines must either pass through two points of intersection of C_1 and C_2 or be tangent to C_1 and C_2 at a point where the two conics are tangent to each other. Given P, this determines the two lines, hence λ uniquely. The uniqueness of λ implies that L_1 and L_2 are determined up to sign by C_3. □

Let P, Q, R and S be the intersection points of C_1 and C_2 with appropriate multiplicity. By the above lemma, C_3 determines a singular element in the pencil spanned by C_1 and C_2. There exists a corresponding partition of P, Q, R, S into two pairs such that this singular conic is the union of the lines passing through the two points in each pair. If the two points in a pair coincide, we take the line to be the common tangent line to C_1 and C_2 at that point. Following Naruki, we shall call this partition a *reference*, and say that C_3 belongs to a given reference.

We shall apply the lemmas in this section to determine when two conics are tangent to each other at one or two points, or to construct conics tangent to given ones, using *Maple* to carry out the more complicated calculations. The other two main tools will be graphs to determine the combinatorial possibilities and projective geometry.

2. Two conics with two tacnodes. Let the two conics be C_1 and C_2. We can choose homogeneous coordinates on $\boldsymbol{P}^2$ such that C_1 has the equation $X^2 + Y^2 = Z^2$, and C_1, C_2 are tangent to each other at $(0 : \pm 1 : 1)$. C_2 then has an equation of the form $(X^2/r^2) + Y^2 = Z^2$ for some $r \in \boldsymbol{C}$, $r \neq 0$, $r \neq \pm 1$. Let G be the subgroup of $\mathrm{Aut}(\boldsymbol{P}^2) = PGL_3(\boldsymbol{C})$ fixing C_1 and the points $(0 : \pm 1 : 1)$. G consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \alpha \end{pmatrix} \quad \text{with } \alpha^2 - \beta^2 = 1\,,$$

which also fix C_2. To save space, instead of this matrix, we shall just write the pair (α, β). Then G can be described as the set of pairs $(\alpha, \beta) \in \boldsymbol{C}^2$ with $\alpha^2 - \beta^2 = 1$, and the multiplication is given by $(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma + \beta\delta, \alpha\delta + \beta\gamma)$. $G \cong \boldsymbol{C}^*$ via the isomorphism $w \mapsto ((w^2 + 1)/(2w), (w^2 - 1)/(2w))$. We have the following statement.

PROPOSITION 3. *Any two smooth conics C_1 and C_2 which are tangent to each other at two points are projectively equivalent to the pair defined by the equations $X^2 + Y^2 = Z^2$ and $(X^2/r^2) + Y^2 = Z^2$ for some $r \in \boldsymbol{C}$, $r \neq 0$, $r \neq \pm 1$. r^2 is an invariant of the ordered pair (C_1, C_2), which will be denoted by $[C_1/C_2]$ following Naruki* [5]. *Clearly,*

$[C_1/C_2][C_2/C_1] = 1$. *Given another such pair, C_3 and C_4, $C_1 \cup C_2$ and $C_3 \cup C_4$ are projectively equivalent if and only if $[C_1/C_2] = [C_3/C_4]$ or $[C_1/C_2][C_3/C_4] = 1$.*

$[C_1/C_2]$ can also be defined without the use of coordinates. The (possibly singular) conics passing through two given points and having given tangent lines at those points form a pencil, so they correspond to points on the projective line. If C_0 and C_∞ are the union of the two tangent lines and twice the line connecting the two given points, respectively, then $[C_1/C_2] = (C_0, C_\infty, C_1, C_2)$.

3. Three conics with six tacnodes. Let us assume that we have a configuration of three conics with six tacnodes. By the previous section we may assume that two of the conics, C_1 and C_2 are given by the equations $Q_1(X, Y, Z) = X^2 + Y^2 - Z^2 = 0$ and $Q_2(X, Y, Z) = (X^2/r^2) + Y^2 - Z^2 = 0$. Let L_1 and L_2 be as in Lemma 2. The singular conics with equations of the form $Q_1 - \lambda Q_2$ are $(1 - 1/r^2)X^2 = 0$ and $(1 - r^2)(Y^2 - Z^2) = 0$. In the first case C_3 would be tangent to C_1 and C_2 at their contract points, which is not allowed. In the second case L_1 and L_2 are of the form $\alpha Y + \beta Z$. By acting on C_3 by a suitable element of G, we may assume that $L_2 = \alpha Y$, then we get $L_1 = \pm\alpha Z$ and $\alpha^2 = (1 - r^2)$, so the equation of C_3 is $X^2 + Y^2 = r^2 Z^2$. C_3 is tangent to C_1 at $(1 : \pm i : 0)$ and to C_2 at $(\pm r : 0 : 1)$. The other possible choices for C_3 are its images under the group G. All such conics are invariant under $(X : Y : Z) \to (-X : Y : Z)$, in other words $(-1, 0) \in G$ acts trivially on them. Let H be the quotient of the G by the 2-element subgroup generated by $(-1, 0)$, this means identifying (α, β) and $(-\alpha, -\beta)$, H now acts freely and transitively on the conics that are tangent to both C_1 and C_2 at two points. We have following:

PROPOSITION 4. *If any two of the conics C_1, C_2 and C_3 are tangent to each other at two points and no three of them pass through one point, they are projectively equivalent to the three conics given by $X^2 + Y^2 = Z^2$, $(X^2/r^2) + Y^2 = Z^2$ and $X^2 + Y^2 = r^2 Z^2$ for some $r \in \mathbf{C}$, $r \neq 0$, $r \neq \pm 1$. We have $[C_1/C_2] = [C_2/C_3] = [C_3/C_1] = r^2$, so we can consider r^2 as the invariant of the three conics with a given cyclic ordering* ([5, *Proposition* 6.1]). *Given two such sets of three conics with invariants r^2 and r'^2, their unions are projectively equivalent if and only if $r^2 = r'^2$ or $r^2 = 1/r'^2$. If we fix C_1 and C_2, the possibly choices for C_3 are images of each other under the action of the group H.*

C_1, C_2 and C_3 above have some obvious parametrizations, for example, $(2t : t^2 - 1 : t^2 + 1)$, $(2rt : t^2 - 1 : t^2 + 1)$ and $(2rt : r(t^2 - 1) : t^2 + 1)$, respectively, which we shall use without writing them down again explicitly.

4. Three conics with five tacnodes. Let us number the conics so that C_2 and C_3 are tangent to C_1 at two points, and to each other at one point. By choosing suitable homogeneous coordinates we may assume that C_1 is given by the equation $X^2 + Y^2 = Z^2$, and that C_2 and C_3 are tangent to each other at $(0 : 0 : 1)$ and their common tangent there is the line $Y = 0$.

Let the equation of C_2 be $aX^2 + bY^2 + cZ^2 + dYZ + eXZ + fXY = 0$. The conditions that $(0 : 0 : 1) \in C_2$ and the tangent line to C_2 at $(0 : 0 : 1)$ is $Y = 0$ imply $c = e = 0$.

By substituting the standard parametrization of C_1 into the equation of C_2, we obtain the quartic

$$q(t) = (d+b)t^4 + 2ft^3 + (4a-2b)t^2 - 2ft + (b-d)\,.$$

$S_1(q) = 0$ gives $f^2 d = 0$. If $d = 0$, then C is not smooth, so $f = 0$.

C_2 must be tangent to C_1 at the points $(\pm\sqrt{1-l^2} : l : 1)$ for some $l \in \boldsymbol{C} \setminus \{0, \pm 1\}$. By comparing the equations of the tangent lines to C_1 and C_2 at these points, we obtain that the equation of C_2 must be $l^2X^2 + (l^2+1)Y^2 - 2lYZ = 0$. This conic can be parametrized as $(2lt : 2lt^2 : (1+l^2)t^2 + l^2)$.

PROPOSITION 5. (a) *If the conics C_1, C_2 and C_3 form a configuration such that C_2 and C_3 are both tangent to C_1 at two points and they are tangent to each other at one point, then the three conics are projectively equivalent to the three conics given by the equation $X^2+Y^2 = Z^2$, $l^2X^2 + (l^2+1)Y^2 - 2lYZ = 0$ and $m^2X^2 + (m^2+1)Y^2 - 2mYZ = 0$, $l, m \in \boldsymbol{C} \setminus \{0, \pm 1\}$, $l \neq m$, $lm \neq 1$.*

(b) *l/m is an invariant of the pair C_2, C_3, which will be denoted by $[C_2/C_3]$, and we have* $[C_1/C_2][C_3/C_1] = [C_2/C_3]^2$ ([5, Proposition 5.2]).

PROOF. (a) The only thing that needs proving is that $lm \neq 1$. C_2 meets the line $X = 0$ at the points $(0:0:1)$ and $(0 : 2l : l^2+1)$. If $(0 : 2l : l^2+1) = (0 : 2m : m^2+1)$, which is equivalent to $l = m$ or $lm = 1$, then C_2 and C_3 are tangent to each other at two points, which we excluded, therefore $lm \neq 1$.

(b) Let C_0 be the union of the common tangent line to C_2 and C_3 at their contact point and of the line connecting the two points of transversal intersection. Let C_∞ be the union of the two lines connecting the transversal intersection points with the contact point. Then l/m is the cross ratio $(C_0, C_\infty, C_2, C_3)$, so it is indeed an invariant. We have $[C_1/C_2] = l^2$, $[C_3/C_1] = 1/m^2$, so the identity holds. □

Given just C_2 and C_3, we can reconstruct a lot of information about C_1. Recall that the fixed points of any involution of $\boldsymbol{P}^2$ consist of a line l and a point $P \notin l$, and l and P determine the involution. Given any point A in the plane, $A \neq P$, $A \notin l$, let Q be the intersection of the line AP with l. The image of A is the unique point A' on the line AP such that the cross ratio $(PQAA') = -1$.

LEMMA 6. *Let C_2 and C_3 be two conics which meet transversally at the points P, Q and are tangent to each other at the point N. Let L be the point of intersection of e_{23} and e'_{23}. Let M be the point on e'_{23} such that $(PQLM) = -1$. Then any conic C_1 which is tangent to both C_2 and C_3 at two points other than N is invariant under the involution fixing L and the line MN, and e_{12}, e_{13} also pass through L.*

PROOF. In the special coordinates used above $N = (0:0:1)$, the tangent line at N is the line $Y = 0$ and $L = (1:0:0)$. M is a point on the line $X = 0$ different from N, so MN is the line $X = 0$, and C_1 is indeed invariant under the involution $(X:Y:Z) \to (-X:Y:Z)$. This statement is, however, purely projective and independent of the choice of coordinates. It follows from Lemma 2 that e_{12}, e_{13} pass through L. □

5. Four conics with 11 or 12 tacnodes.

PROPOSITION 7. (a) ([5, Proposition 6.1]). *Any configuration of four conics with* 12 *tacnodes is projectively equivalent to the four conics given by the four choices of signs in* $X^2 \pm Y^2 \pm Z^2 = 0$.

(b) *Any configuration of four conics with* 11 *tacnodes is projectively equivalent to the four conics given by the equations* $X^2 + Y^2 + Z^2$, $(X^2/r^2) + Y^2 = Z^2$, $X^2 + Y^2 = r^2Z^2$ *and*

$$(1 - r^2)X^2 + (3r^2 + 1)Y^2 + r^2(r^2 + 3)Z^2 - 4r(r^2 + 1)YZ = 0$$

for some $r \in \boldsymbol{C} \setminus \{0, \pm 1, \pm i\}$. *Alternatively, we can take the last two conics to be* $X^2 + (r^2 + 1)Y^2 \pm 2rYZ = 0$ *to make the whole setup more symmetric.*

PROOF. By Proposition 4 we may assume in both cases that the first three conics are in $C_1 : X^2 + Y^2 = Z^2$, $C_2 : (X^2/r^2) + Y^2 = Z^2$ and $C_3 : X^2 + Y^2 = r^2Z^2$ for some $r \in \boldsymbol{C} \setminus \{0, \pm 1\}$. C_4 must be the image of C_3 under some $(\alpha, \beta) \in H$, so its equation is $X^2 + (\alpha Y - \beta Z)^2 + r^2(-\beta Y + \alpha Z)^2 = 0$. It is tangent to C_3 if and only if $(\alpha, \beta) = ((r^2 + 1)/(r^2 - 1), \pm 2r/(r^2 - 1))$. We can take the $+$ sign for β, then the equation of C_4 is as claimed, and the contact point of C_4 with C_3 is $(0 : \pm r : 1)$.

If $r^2 = -1$, then we get the equations given in (a) and the configuration has 12 tacnodes. Conversely, if a configuration of four conics has 12 tacnodes, then by repeated application of Proposition 4, $[C_1/C_2] = [C_3/C_1] = [C_1/C_4] = [C_2/C_1] = 1/[C_1/C_2]$, so $r^2 = [C_1/C_2] = -1$. If $r^2 \neq -1$, then we get a configuration with 11 tacnodes.

The element $(1/\sqrt{1 - r^2}, -r/\sqrt{1 - r^2}) \in H$, which is the square root of $(\alpha, \beta)^{-1}$, transforms C_3, C_4 to $X^2 + (r^2 + 1)Y^2 \pm 2rYZ = 0$. □

REMARK. It is easy to see that we cannot have four conics with 11 or 12 tacnodes such that they are all defined over $\boldsymbol{R}$, and all their contact points are real, but 10 tacnodes are possible.

C_3 and C_4 are now in a more special position, $[C_3/C_4] = -1$, and we can derive more information about the conics tangent to both of them than in Lemma 6.

LEMMA 8. *Let* C_3 *and* C_4 *be two conics which meet transversally at the points* P, Q *and are tangent to each other at the point* N. *Assume that there exist two conics,* C_1 *and* C_2, *such that the four conics together form a configuration with* 11 *tacnodes. Let* L *and* M *be defined as in Lemma* 6. *Then the points of contact of* C_1 *and* C_2 *lie on the line* MN.

Let C_5 *be any conic tangent to both* C_3 *and* C_4 *at two points other than* N, *including the possibility* $C_5 = C_1$ *or* $C_5 = C_2$. *Then* C_5 *is invariant under the involution fixing the line* LM *and the point* N *and under the involution fixing* LN *and* M, *while* C_3 *and* C_4 *are exchanged by these two involutions. If* $C_5 \neq C_1$, $C_5 \neq C_2$, *and the five conics form a configuration, then* C_5 *is not tangent to either* C_1 *or* C_2.

PROOF. We can assume by Proposition 7(b) that C_3 and C_4 have the equations $X^2 + (r^2 + 1)Y^2 \pm 2rYZ = 0$. By Lemma 2 we obtain that C_5 has an equation of the form $X^2 + bY^2 + cZ^2 = 0$. We have $L = (1 : 0 : 0)$, $M = (0 : 1 : 0)$ and $N = (0 : 0 : 1)$, so the line MN is the line $X = 0$, and involutions mentioned are $(X : Y : Z) \mapsto (X : Y : -Z)$ and $(X : Y : Z) \mapsto (X : -Y : Z)$, which leave C_5 invariant and exchange C_3 and C_4.

If C_5 is tangent to C_1 or C_2 at two points, then these points must lie on the line MN and then the conics do not form a configuration. (In fact, there are only two conics, C_1 and C_2, which are tangent to both C_3 and C_4 at two points and pass through the contact points of C_1 and C_2.) If C_5 is tangent to C_1 or C_2 at one point, then the contact point must be invariant under both of the involutions described above. The only such points are L, M and N, but there exist no smooth conics invariant under the involutions which passes through any of these points.

The statements of the lemma are purely projective, therefore they hold independently of the choice of coordinates. □

6. **Four conics with 10 tacnodes.**

There are three possible graphs, , or .

PROPOSITION 9. *Any configuration of four conics with graph is projectively equivalent to the conics $X^2 + Y^2 = Z^2$, $(X^2/r^2) + Y^2 = Z^2$ and*

$$X^2 + (\alpha^2 - r^2\beta^2)Y^2 + (\beta^2 - r^2\alpha^2)Z^2 \pm 2\alpha\beta(r^2 - 1)YZ = 0$$

for some $r \in \mathbf{C} \setminus \{0, \pm 1\}$, $\alpha, \beta \in \mathbf{C}$, $\alpha^2 - \beta^2 = 1$.

PROOF. We may assume that C_1 and C_2 are the conics $X^2 + Y^2 = Z^2$ and $(X^2/r^2) + Y^2 = Z^2$, and that the double edge is between C_3 and C_4. By the results of the previous section, C_3 and C_4 are images of $X^2 + Y^2 = r^2Z^2$ under suitable elements $h_1, h_2 \in H$. By acting on them by a square root of $h_1^{-1}h_2^{-1}$, we can transport them into such a position that they are the images of $X^2 + Y^2 = r^2Z^2$ under h and h^{-1} for some $h \in H$. If $h = (\alpha, \beta)$, the equations of C_3 and C_4 are as stated. □

Given just C_3 and C_4 we can derive the following information about C_1 and C_2.

LEMMA 10. *Let C_3 and C_4 be two conics which meet transversally at four points P, Q, R and S. Assume that there exist two other conics C_1 and C_2 such that the four conics together form a configuration with graph . Then the following statements hold.*

(a) *C_1 and C_2 belong to the same reference, say, $\{P, Q\}$, $\{R, S\}$.*

(b) $(PQRS)_{C_3} \cdot (PQRS)_{C_4} = 1$.

(c) *If J is the point of intersection of the lines PS and QR, K that of PR and QS, and L the intersection of PQ and RS, then the two contact points of C_1 and C_2 lie on the line JK.*

Let C_5 be any conic tangent to both C_3 and C_4 belonging to the same reference as C_1 and C_2, including the possibility $C_5 = C_1$ or $C_5 = C_2$.

(d) *C_3, C_4 and C_5 are invariant under the involution of $\mathbf{P}^2$ fixing the line JK and the point L.*

(e) *Let M be the point of intersection of PQ and JK, N that of RS and JK. Then C_5 is invariant under the involution fixing the line PQ and the point N and the involution fixing the line RS and M, while C_3 and C_4 are exchanged by these two involutions.*

(f) *If $C_6 \neq C_5$ is a conic tangent to both C_3 and C_4 at two points belonging to the same reference as C_5, then C_5, C_6 are either not tangent to each other, or they are tangent at two points lying on the line JK.*

PROOF. Parts (a), (b) and (c) are obvious if we choose homogeneous coordinates such that the four conics are given by the equations in the previous proposition. We can choose P, Q to be the two points on the line $Z = 0$, and R, S to be the two points on $Y = 0$. JK is the line $X = 0$. From Lemma 2 we obtain that the equation of C_5 is of the form $X^2 + bY^2 + cZ^2 = 0$.

The coordinates of M are $(0 : 1 : 0)$, and those of N are $(0 : 0 : 1)$. The involution in (d) is $(X : Y : Z) \mapsto (-X : Y : Z)$, while those in (e) are $(X : Y : Z) \mapsto (X : Y : -Z)$ and $(X : Y : Z) \mapsto (X : -Y : Z)$, so (d) and (e) are clear, too.

If C_5 and C_6 were tangent to each other at one point, that point would have to be invariant under the three involutions in (d) and (e). The only such points are L, M and N, but there is no smooth conic passing through any of these points which is invariant under the three involutions.

If C_5 and C_6 are tangent at two points, then e_{56} is the line JK, since it is also the line e_{12} and it can be constructed from C_3, C_4 and the given reference.

All the statements of the lemma are purely projective, so they hold independently of the choice of coordinates. □

LEMMA 11. *Let C_3 and C_4 be two conics which meet transversally at four points. If there exists a labelling of the four intersection points such that $(PQRS)_{C_3} \cdot (PQRS)_{C_4} = 1$ and $(PQRS)_{C_3} \neq (1 \pm i\sqrt{3})/2$, then the partition of the four points into the sets $\{P, Q\}$ and $\{R, S\}$ with this property is unambiguous. If $(PQRS)_{C_3} \cdot (PQRS)_{C_4} = 1$ and $(PQRS)_{C_3} = (1 \pm i\sqrt{3})/2$, then these conditions hold for any labelling of the four points and there is no distinguished partition.*

PROOF. Let $(PQRS)_{C_3} = \gamma$. If we exchange the two pairs $\{P, Q\}$ and $\{R, S\}$ or change the order of the elements within a pair, the cross ratio on C_3 will be γ or $1/\gamma$. For any other permutation, the cross ratio on C_3 will be $f(\gamma)$ and that on C_4 will be $f(1/\gamma)$, where $f(t)$ is one of the functions $1 - t$, $1/(1 - t)$, $t/(t - 1)$ and $1 - 1/t$. $f(\gamma)f(1/\gamma) = 1$ holds for any of these functions if and only if $\gamma = (1 \pm i\sqrt{3})/2$. □

Let us now consider the graph . We may assume that C_1, C_2 and C_3 are as in Proposition 5. C_4 is tangent to C_2 and C_3 at two points, so by Lemma 6, its equation cannot contain XY or XZ terms. C_4 must be tangent to C_1 at one of the points $(0 : \pm 1 : 1)$ by changing the sign of Y (and of l, m), we may assume that it is $(0 : 1 : 1)$. The equation of C_4 must be of the form $X^2 + bY^2 + cZ^2 - (b + c)YZ = 0$.

By substituting a parametrization of C_2 into the equation of C_4 and applying Lemma 1, we obtain $((b - c)^2 + 4c)l^2 - 4(b + c)l + 4(c + 1) = 0$. By the same argument with C_5 instead of C_4, we obtain the same equation with m instead of l, so $u = l$ and $u = m$ are the two roots of

$$((b-c)^2+4c)u^2-4(b+c)u+4(c+1)=0\,. \tag{3}$$

$u=0$ is not a root, the leading coefficient cannot vanish, the roots must be distinct and their product cannot be 1. These give the conditions $c\neq -1$, $(b-c)^2+4c\neq 0$ and $b-c\neq \pm 2$.

By taking suitable linear combinations of the relations between the roots and coefficients of the above quadric, we obtain

$$(c+1)((2lm-l-m)^2c+(l-m)^2)=0$$

and

$$(lm-l-m)c+lmb-l-m=0\,.$$

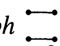
Since $c\neq -1$, the only solution is

$$c=-\frac{(l-m)^2}{(2lm-l-m)^2}$$

and

$$b=\frac{(l+m)(4lm-3l-3m+4)-4lm}{(2lm-l-m)^2}\,.$$

In addition to the conditions on l,m imposed in Proposition 5, we must also require that $l+m\neq 2lm$ to avoid division by 0, and $l+m\neq 2lm$, $l\neq -m$, to ensure that C_4 is not singular and $C_1\neq C_4$. Hence we have the following:

PROPOSITION 12. *Any configuration of four conics with graph*  *is projectively equivalent to the conics* $C_1 : X^2+Y^2=Z^2$, $C_2 : l^2X^2+(l^2+1)Y^2-2lYZ=0$, $C_3 : m^2X^2+(m^2+1)Y^2-2mYZ=0$ *and*

$$\begin{aligned} C_4 : (2lm-l-m)^2X^2 &+ ((l+m)(4lm-3l-3m+4)-4lm)Y^2 \\ &- (l-m)^2Z^2-4(l-1)(m-1)(l+m)YZ=0\,, \end{aligned}$$

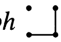
where $l,m\in \boldsymbol{C}\setminus\{0,\pm 1\}$, $l\neq \pm m$, $lm\neq 1$, $l+m\neq 2$, $l+m\neq 2lm$.

Let us now consider the graph . We may assume that C_1, C_2 and C_3 are as in Proposition 5, and C_4 tangent to C_1 at two points and to C_2 and C_3 at one point. If $[C_1/C_4]=s^2$, then the equation of C_4 can be written as

$$X^2+Y^2+Z^2+\frac{(1-s^2)(\alpha X+\beta Y+\gamma Z)^2}{s^2(\alpha^2+\beta^2-\gamma^2)}=0\,,$$

where $\alpha X+\beta Y+\gamma Z=0$ is the equation of e_{14}. α, β and γ are only determined up to scalars. From the condition that C_4 is tangent to C_2 and C_3, we get two equations for α, β and γ. After discarding the solutions corresponding to the cases when C_4 is tangent to C_2 or C_3 at two points or when it passes through the contact points of some of the other conics, the only solutions are $(\alpha:\beta:\gamma)=((1-rs):\pm 2\sqrt{rs}:\pm(r+s))$ and $(\alpha:\beta:\gamma)=((1+rs):\pm 2i\sqrt{rs}:\pm(r-s))$. These can all be obtained from one another by changing the sign of r,s or one of the coordinates. They can all be written in the form $(\alpha:\beta:\gamma)=((1-\rho^2\sigma^2):\pm 2\rho\sigma:\pm(\rho^2+\sigma^2))$, where ρ and σ are suitable fourth roots of r^2 and s^2, respectively. The pairs (ρ, σ) and $(-\rho, -\sigma)$ determine the same conics. The contact point of C_2 and C_4 is $(\rho^2(\sigma^2 - \rho^2) : -2\rho\sigma : \rho^2 + \sigma^2)$, while that of C_3 and C_4 is $(\rho^2(\rho^2\sigma^2 - 1) : -2\rho^3\sigma : \rho^2\sigma^2 + 1)$. If $r = \pm s$ or $r = \pm 1/s$, then one of these contact points is the contact point of C_1 with C_2 or C_3, which we have to exclude.

The lines e_{24} and e_{34} intersect at $(\rho^2 : -\rho/\sigma : 1)$. Thus we have the following proposition.

PROPOSITION 13. *Any configuration of 4 conics with graph*  *is projectively equivalent to the conics given by the equations* $C_1 : X^2 + Y^2 = Z^2$, $C_2 : X^2/\rho^4 + Y^2 = Z^2$, $C_3 : X^2 + Y^2 = \rho^4 Z^2$ *and*

$$C_4 : X^2 + Y^2 - Z^2 + \frac{((1 - \rho^2\sigma^2)X + 2\rho\sigma Y + (\rho^2 + \sigma^2)Z)^2}{\sigma^4(1 - \rho^4)} = 0$$

for some $\rho, \sigma \in \mathbf{C} \setminus \{0, \pm 1, \pm i\}$, $\rho^4 \neq \sigma^4$, $\rho^4\sigma^4 \neq 1$. *The intersection point of* e_{24} *and* e_{34} *lies on the common tangent line to* C_2 *and* C_3 *at one of their contact points.*

7. Five conics with 17 tacnodes. For five conics the Miyaoka-Yau bound is 17, and we shall show exactly how it can be achieved.

There are six possible graphs on 5 vertices with 3 edges, shown on Figure 1. The labelling of the vertices is chosen in each case to make the proof more convenient.

Let us consider the first graph. By Proposition 7(a) we may assume that C_1, C_2, C_3, C_4 are the conics $X^2 \pm Y^2 \pm Z^2 = 0$. But then the only conic which is tangent to C_1 and C_2 at two points and also tangent to C_3 is C_4, so this graph is impossible. The second and third graphs are impossible by Lemma 8.

Let us now consider the fourth graph. We may assume that C_1, C_2 and C_3 are as in Proposition 4, and then $C_4 = h(C_3)$ and $C_5 = h^{-1}(C_3)$, where $h = ((r^2 + 1)/(r^2 - 1), 2r/(r^2 - 1)) \in H$. In general, two conics which are both tangent to C_1 and C_2 at two points are tangent to each other if and only if one of them is the image of the other under h, so C_4 and C_5 are

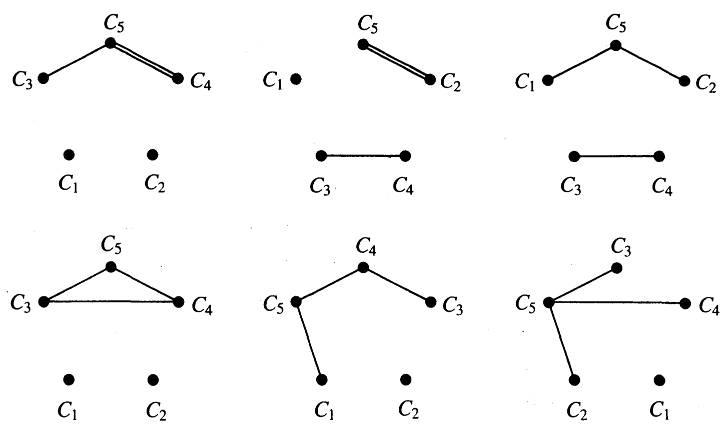


FIGURE 1. The six graphs on 5 vertices with 3 edges.

tangent to each other if and only if $h^3 = 1$, which happens if and only if $r^2 = -1/3$ or $r^2 = -3$. These are reciprocals of each other and give projectively equivalent configurations. If we take $r^2 = -1/3$, we obtain the conics $X^2 + Y^2 - Z^2 = 0$, $-3X^2 + Y^2 - Z^2 = 0$, $3X^2 + 3Y^2 + Z^2 = 0$ and $3X^2 - 2Z^2 \pm i\sqrt{3}YZ = 0$. This is indeed a configuration, the one obtained in [5, Section 7], and it is the only one with this graph up to projective equivalence.

Let us now consider the fifth graph. We may assume that C_1, C_2, C_3 and C_4 have equations as stated in the first alternative in Proposition 7(b). By applying the argument of the proof of Proposition 7 with the roles of C_1 and C_3 reversed, C_5 must be the image of C_1 under the action of an element of the subgroup of $PGL_3(\boldsymbol{C})$ consisting of the matrices of the form

$$\begin{pmatrix} \alpha & 0 & r^2\beta \\ 0 & 1 & 0 \\ \beta & 0 & \alpha \end{pmatrix} \quad \text{with } \alpha^2 - r^2\beta^2 = 1\,,$$

which fix C_2, C_3 and the points $(\pm r : 0 : 1)$. C_1 and C_5 must be tangent to each other at one of the points $(\pm 1 : 0 : 1)$, we may assume it is $(1 : 0 : 1)$, then C_5 is the image of C_1 under a group element which maps $(-1 : 0 : 1)$ to $(1 : 0 : 1)$. Hence the equation of C_5 is

$$(r^2 + 3)X^2 + (r^2 - 1)Y^2 + (3r^2 + 1)Z^2 - 4(r^2 + 1)XZ = 0\,. \tag{4}$$

The discriminant expressing condition that C_4 and C_5 are tangent to each other is

$$2^{18}(r^2 + 1)^6(r^2 - 1)^{10}r^2(r^4 - 6r^2 + 1)^2\,.$$

The only feasible solutions are the roots of $r^4 - 6r^2 + 1$, $r = \pm 1 \pm \sqrt{2}$, but then C_2, C_4 and C_5 are tangent to each other at the same point. For example, if $r = \sqrt{2} - 1$, then this common point is $(\sqrt{2} - 1 : 1 : \sqrt{2})$.

Let us now consider the sixth graph. By Proposition 7(a), we may assume that the first four conics are $C_1 : X^2 + Y^2 + Z^2 = 0$, $C_2 : X^2 + Y^2 - Z^2 = 0$, $C_3 : X^2 - Y^2 + Z^2 = 0$ and $C_4 : -X^2 + Y^2 + Z^2 = 0$. Let $\alpha X + \beta Y + \gamma Z = 0$ be the equation of the line e_{15}, then the equation of C_5 is $\lambda(X^2 + Y^2 + Z^2) + (\alpha X + \beta Y + \gamma Z)^2 = 0$ for some suitable $\lambda \in \boldsymbol{C}$.

From the condition that C_2 and C_5 are tangent to each other we obtain the vanishing of the discriminant

$$2^{14}\lambda^2(\alpha^2 + \beta^2)^2(4\lambda^2 + 4(\alpha^2 + \beta^2 + \gamma^2)\lambda + (\alpha^2 + \beta^2 - \gamma^2)^2)\,,$$

so $\alpha^2 + \beta^2 = 0$ or $\lambda = -(\alpha^2 + \beta^2 + \gamma^2)/2 \pm \gamma\sqrt{\alpha^2 + \beta^2}$.

If $\alpha^2 + \beta^2 = 0$, then C_5 passes through one of the contact points of C_1 and C_2, $(1 : \pm i : 0)$, so we must have the second possibility. By doing the same calculations with C_3 and C_4 and comparing the three expressions for λ, we obtain that $\alpha^2(\beta^2 + \gamma^2) = \beta^2(\alpha^2 + \gamma^2) = \gamma^2(\alpha^2 + \beta^2)$, so α, β, γ can only differ from each other by a sign, and then by changing the sign of some of the coordinates if necessary, we may assume that $\alpha = \beta = \gamma = 1$. Then $\lambda = -3/2 \pm \sqrt{2}$, and the equation of C_5 is

$$(\pm 2\sqrt{2} - 1)(X^2 + Y^2 + Z^2) + 4(XY + YZ + ZX) = 0\,. \tag{5}$$

It is easy to check that the five conics form a configuration with either choice of sign, so we obtain two configurations of five conics with 17 tacnodes.

We claim these two configurations are not projectively equivalent. Let C_5^+ and C_5^- be the conics obtained by choosing the $+$ and $-$ sign in (5), respectively. Let us assume that there exists $\phi \in \mathrm{Aut}(\boldsymbol{P}^2)$ transforming $C_1, C_2, C_3, C_4, C_5^+$ into $C_1, C_2, C_3, C_4, C_5^-$. C_1 must be invariant, C_2, C_3, C_4 may be permuted among each other, and C_5^+ must be mapped to C_5^-. The contact points of C_1 and C_5^+ or of C_1 and C_5^- lie on the line $X + Y + Z = 0$, so this line must be invariant under ϕ. Similarly, the lines e_{12}, e_{13}, e_{14} are the lines $X = 0$, $Y = 0$ and $Z = 0$, so they must be permuted among themselves. These conditions imply that ϕ simply permutes X, Y, Z, so it leaves C_5^+ and C_5^- invariant.

Thus we have the following theorem.

THEOREM 14. *There exist exactly three configurations of five conics with* 17 *tacnodes up to projective equivalence. One consists of the conics* $X^2+Y^2-Z^2 = 0$, $-3X^2+Y^2-Z^2 = 0$, $3X^2+3Y^2+Z^2 = 0$ *and* $3X^2-2Z^2 \pm i\sqrt{3}YZ = 0$, *the other two contain the four conics with equations* $X^2 \pm Y^2 \pm Z^2 = 0$ *and one of the conics* $(\pm 2\sqrt{2}-1)(X^2+Y^2+Z^2) + 4(XY+YZ+ZX) = 0$.

8. Five conics with 15 or 16 tacnodes. The aim of this section is to prove that certain graphs cannot occur as graphs of configuration of five conics with 15 or 16 tacnodes. The expected dimensions of families of such configuration is 2 and 1, respectively, but if one tries to construct examples, there are always three conics which pass through the same point.

LEMMA 15. *None of the graphs shown in Figure* 2 *can occur as the graph of a configuration of five conics.*

REMARK. The impossibility of the first five graphs is stated but not proved in [5, Section 12].

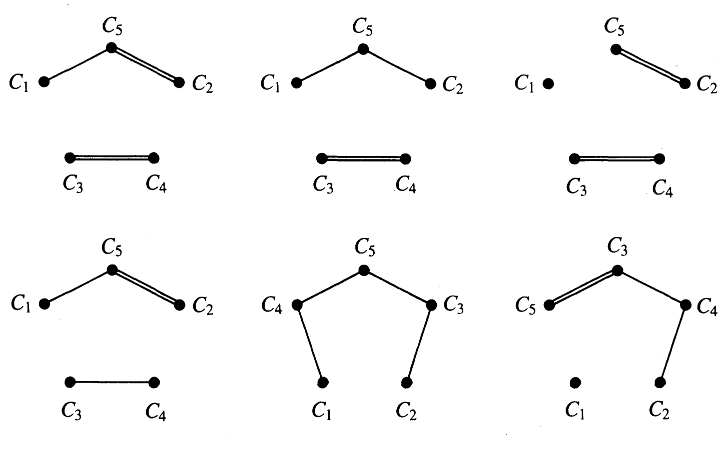


FIGURE 2. Graphs that cannot occur.

PROOF. Let us consider the first and second graphs. By Lemma 6, e_{13}, e_{35}, e_{15} and e'_{15} are concurrent, and so are e_{14}, e_{45}, e_{15} and e'_{15}. As e_{15} and e'_{15} are distinct, all six lines must be concurrent, but then C_1 and C_5 belong to the same reference with respect to C_3 and C_4, which contradicts Lemma 10.

Let us now consider the third graph. By Lemma 10, C_1, C_1 and C_5 belong to the same reference with respect to C_3 and C_4, and all three of them would have to be tangent to each other at the same two points, this is a contradiction.

The proof given for the third graph in Figure 1 also works for the fourth graph in Figure 2.

Let us now consider the fifth graph. We may assume that C_1, C_2, C_3 and C_4 are as in Proposition 12. By Lemma 2, C_5 must have equation

$$X^2 + Y^2 - Z^2 + \frac{1}{4}\left(\left(\alpha - \frac{1}{\alpha}\right)\sqrt{1-m^2}X + \left(\alpha + \frac{1}{\alpha}\right)(mY - Z)\right)^2 = 0$$

for some $\alpha \in \boldsymbol{C} \setminus \{0, \pm 1\}$.

Let us substitute the parametrization of C_2 into the equation of C_5. By factorising the discriminant of the resulting quartic and discarding the obviously non-zero factors, we obtain that one of the following must hold:

$$((l-m)\alpha^2 + 2(l+m)\alpha + (l-m))((l-m)\alpha^2 - 2(l+m)\alpha + (l-m)) = 0$$

or

$$(l-m)^2\alpha^4 + 2(2(lm-1)^2 - (l-m)^2)\alpha^2 + (l-m)^2 = 0\,.$$

The latter equation would imply that the point of intersection of e_{12} and e_{15},

$$\left(\frac{(\alpha^2+1)(lm-1)}{(1-\alpha^2)\sqrt{1-m^2}} : l : 1\right),$$

lies on C_1, so we must have $(l-m)\alpha^2 \pm 2(l+m)a + (l-m) = 0$ for a suitable choice of sign.

The point of intersection of e_{34} and e_{35} is

$$\left(\frac{(\alpha^2-1)(m-1)^2(l+m)}{(\alpha^2+1)\sqrt{1-m^2}} : (l-m)m : (2lm^2 + (1-2m)(l+m))\right).$$

This point cannot lie on C_3, which implies that

$$(lm + m^2 + l - 3m)(m^2 - 3lm + l + m) \neq 0$$

irrespective of which sign we choose in $(l-m)\alpha^2 \pm 2(l+m)\alpha + (l-m) = 0$.

Let us assume that $(l-m)\alpha^2 - 2(l+m)\alpha + (l-m) = 0$. Let us substitute the parametrization of C_4 into the equation of C_5. By factorising the discriminant of the resulting quartic in the ring

$$\boldsymbol{C}(l, m)[\alpha]/((l-m)\alpha^2 - 2(l+m)\alpha + (l-m))$$

(it is necessary to work in this ring because otherwise the expression is too big for the computer), and discarding the obviously non-zero factors, we obtain that either $l^2m + lm^2 - 4lm +$

$l+m=0$ or $g_{11}(l,m,\alpha)=0$ where $g_{11}(l,m,\alpha)$ is a polynomial, which is homogeneous of degree 11 in l,m and linear in α.

If we solve g_{11} for α and substitute it into $(l-m)\alpha^2-2(l+m)a+(l-m)=0$, we obtain $l=m$, so $g_{11}\neq 0$, therefore we must have $lm(l+m)-4lm+l+m=0$. We reach the same conclusion if we assume $(l-m)\alpha^2+2(l+m)\alpha+(l-m)=0$.

Now $l+m=4lm/(lm+1)$, so if we choose β such that $lm=-\beta^2$, then l,m are the roots of the equation

$$u^2+\frac{4\beta^2 u}{1-\beta^2}-\beta^2=0\,.$$

The two roots are $(1-\beta)\beta/(\beta+1)$ and $(\beta+1)\beta/(\beta-1)$, they are interchanged if we change the sign of β, so we may take $l=(1-\beta)\beta/(\beta+1)$ and $m=(\beta+1)\beta/(\beta-1)$. $\beta\neq\pm i$ because then we would have $l=m$.

The two points of contact of C_2 and C_4 are

$$\left(\pm\sqrt{\beta^4+2\beta^3+2\beta-1}:\beta(\beta-1):\beta^2-\beta+2\right),$$

and the four possibilities for α are $(\pm 2\beta\pm i(\beta^2-1))/(\beta^2+1)$. For each of these choices, one of the contact points of C_2 and C_4 lies on C_5, in fact, the three conics are tangent to each other there. This shows that the fourth graph cannot be the graph of a configuration either.

Let us consider now the sixth graph. By repeated application of Proposition 4, $[C_3/C_1]=[C_1/C_2]=[C_5/C_1]=[C_1/C_4]=r^2$, but this contradicts Proposition 13. □

9. Six conics.

THEOREM 16. *There does not exist a configuration of six conics with* 23 *or* 24 *tacnodes.*

PROOF. First we show that there does not exist a configuration of 6 conics with 24 tacnodes. If such a configuration existed, its graph would have six edges and each vertex would have degree 2 by [3, Theorem 6(ii)]. This gives the four possible graphs shown below.

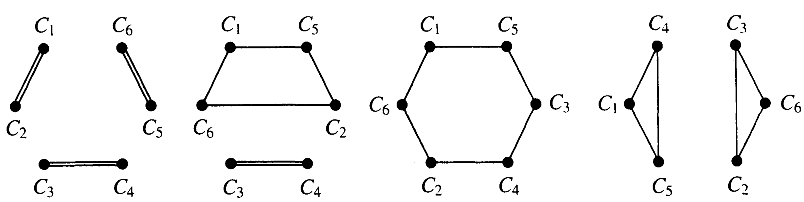


FIGURE 3. Regular graphs of degree 2 on 6 vertices.

The first, second and third graphs contain one of the forbidden subgraphs from Proposition 15.

Let us now consider the fourth graph. We may assume that C_1, C_2, C_3 and C_4 have the equations described in Proposition 12. C_5, which is tangent to C_1 at $(0:-1:1)$, has the equation

$$(2lm + l + m)^2X^2 - ((l + m)(4lm + 3l + 3m + 4) + 4lm)Y^2 \\ - (l - m)^2Z^2 - 4(l + 1)(m + 1)(l + m)YZ = 0\,,$$

where $l, m \in \boldsymbol{C} \setminus \{0, \pm 1\}$, $l \neq \pm m$, $lm \neq 1$, $l + m \neq \pm 2$, $l + m \neq \pm 2lm$.

As C_6 is tangent to C_4 and C_5 at two points, it must be invariant under the involution $(X : Y : Z) \mapsto (-X : Y : Z)$ by Lemma 6. Therefore C_6 is tangent to C_2 at $(0 : 2l : (l^2 + 1))$ and to C_3 at $(0 : 2m : (m^2 + 1))$, so its equation must be of the form $aX^2 + ((l^2 + 1)Y - 2lZ)((m^2 + 1)Y - 2mZ) = 0$ for some a. By substituting the parametrization of C_1 into the equation of C_6, and calculating the discriminant of the resulting quartic, we can determine that C_1 and C_6 are tangent to each other at two points if and only if $a = (l - m)^2$ or $a = (lm - 1)^2$.

Let us now consider the case $a = (l - m)^2$. We can parametrize C_4 by taking its second point of intersection with the lines $Y = tX + Z$ passing through $(0 : 1 : 1)$. If we substitute this parametrization into the equation of C_6, we obtain a quartic which must be a square, since C_6 is tangent to C_4. This quartic only contains even degree terms, so by Lemma 1(b) we obtain

$$2^6lm(m - 1)^2(l - 1)^2(l - m)^2(l + m - 2)^2(2lm - l - m)^4(lm + l + m - 3) = 0\,.$$

The only feasible solution is $lm + l + m - 3 = 0$. By an analogous argument with C_5 in place of C_4, we obtain $lm - l - m - 3 = 0$. Hence $l + m = 0$, which we have already excluded. If we take $a = (lm - 1)^2$ in the equation of C_6, the above argument yields $3lm \pm (l + m) - 1 = 0$, so again $l + m = 0$. This shows that the fourth graph is also impossible.

We never used here that C_4 and C_5 are tangent to each other, so even if we change the edge between them to a double edge, that graph still cannot be realized. If we add the edge (C_1, C_6) to this graph, then C_1 and C_6 are invariant under $(X : Y : Z) \mapsto (-X : Y : Z)$ by Lemma 6, so their point of contact with C_1 would have to lie on the line $X = 0$. These points are also the contact points of C_1 with C_4 and C_5, so this graph is impossible, too.

Now we show that none of the remaining graphs for configurations of six conics with 23 tacnodes can be realized either. Such a graph must have 6 vertices and 7 edges. All vertices must have degree 4 or less. If there is a vertex of degree 4, the graph obtained by removing it must be one of the two possible graphs on 5 vertices with 3 edges. If there is a vertex of degree 3, the graph obtained removing it cannot be any of the forbidden graphs listed in Lemma 15. We shall consider six different cases.

Case 1. C_6 has degree 4, and by removing it we get the fourth graph in Figure 1. We may assume that C_i, $1 \leq i \leq 5$, are given by the equations in Theorem 14. In addition to the forbidden subgraphs, we also have to consider that there exist automorphisms of $\boldsymbol{P}^2$ inducing any permutation of C_3, C_4 and C_5, that if C_6 is tangent to two of C_3, C_4 and C_5 at two points, then it cannot be tangent to C_1 and C_2 by Lemma 8, and the only two conics tangent to each of C_3, C_4 and C_5 at two points are C_1 and C_2. This leaves three graphs we have to consider.

Subcase 1.1. There are four simple edges from C_6 to C_1, C_2, C_4 and C_5. The equation of C_6 is

$$3X^2 + 3Y^2 + Z^2 + (\alpha X + \beta Y + \gamma Z)^2 = 0,$$

where $\alpha X + \beta Y + \gamma Z = 0$ is the equation of the line connecting the two contact points of C_3 and C_6. We must have

$$\alpha\beta\gamma(\alpha^2 + \beta^2)(\alpha^2 + 3\gamma^2)(\beta^2 + 3\gamma^2) \neq 0$$

to ensure that C_6 does not pass through any of the contact points of C_3 with the other curves, and that C_6 is tangent to C_1 and C_2 at a point other than one of their contact points with each other or with C_3.

Let us substitute into the parametrization of C_1, C_2, C_4 and C_5 into the equation of C_6. Let the respective discriminants of these quartics be Δ_1, Δ_2, Δ_4 and Δ_5, each of these must vanish. $\Delta_1/(\alpha^2 + \beta^2)^2$, $\Delta_2/(\alpha^2 + 3\gamma^2)^2$, $\Delta_4 + \Delta_5$ and $i\sqrt{3}(\Delta_4 - \Delta_5)$ are all polynomials in $\boldsymbol{Q}[\alpha, \beta, \gamma]$. By using the `gbasis` command in *Maple*, we can see that α^2 is in the ideal generated by the above polynomials, in fact, it is an element of the Gröbner basis, but $\alpha \neq 0$, this is a contradiction.

Subcase 1.2. There are simple edges from C_6 to C_1, C_3, C_4 and C_5. The equation of C_6 is

$$-3X^2 + Y^2 - Z^2 + (\alpha X + \beta Y + \gamma Z)^2 = 0.$$

By the same method as above we now obtain from the vanishing of the discriminants that $3\beta^2 + \gamma^2 = 0$. This is, however, not possible, because then C_6 would have to be tangent to C_1 at one of contact points of C_1 with C_4 or C_5. Perhaps the easiest way to see this is if we act by a suitable one of the elements $(1/2, \pm i\sqrt{3}/2) \in H$, which permute C_3, C_4 and C_5 cyclically while fixing C_1 and C_2, and then the image of C_6 has equation

$$-3X^2 + Y^2 - Z^2 + (\alpha X + 2\beta Y)^2 = 0.$$

Similarly, there is no solution when there is an edge in the graph from C_6 to C_2 instead of C_1.

Case 2. C_6 has degree 4, and by removing it we get the sixth graph in Figure 1. Let us use the notation introduced in the proof of Theorem 14. We shall assume that $C_5 = C_5^+$. C_1, C_2, C_3 and C_4 are defined over $\boldsymbol{Q}$, while C_5^+ and C_5^- are conjugate to each other under the action of $\mathrm{Gal}(\boldsymbol{Q}(\sqrt{2})/\boldsymbol{Q})$, so if there is no suitable configuration with one of them, there does not exist any with the other either. In addition to the forbidden subgraphs, we also have to consider that any conic tangent to two of C_1, C_2, C_3 and C_4 at two points, and tangent to another one of them, must be one of the four conics, and that there exist projective transformations of the plane fixing C_1 and C_5, and permuting C_2, C_3 and C_4 arbitrarily. This leaves us with five graphs to consider. First we shall consider the two cases where there is an edge between C_5 and C_6, then the three where there is not.

Subcase 2.1. There are four simple edges from C_6 are to C_2, C_3, C_4 and C_5. The conics which are tangent to C_1 at two points and to C_2, C_3 and C_4 at one point are C_5^+, C_5^-, and the conics obtained from them by changing the sign of one of the coordinates. The contact points of C_5^+, C_5^- lie on C_1, so we cannot have $C_6 = C_5^-$. Let $C_5^{+\prime}$, $C_5^{-\prime}$ be the images of C_5^+, C_5^- under the map $(X : Y : Z) \mapsto (X : Y : -Z)$. C_5^+ and $C_5^{+\prime}$ are not tangent to each other, while C_5^+ and $C_5^{-\prime}$ are, but their contact point lies on C_2, so we do not obtain a configuration.

Subcase 2.2. There are four simple edges from C_6 are to C_1, C_2, C_3 and C_5. C_6 must be the image of C_5^+ or C_5^- under a map of the form $(X : Y : Z) \mapsto (\pm X : \pm Y : iZ)$, but none of these conics is tangent to C_5^+.

Subcase 2.3. There is a double edge from C_6 to C_1 and simple edges to C_3 and C_4. By Lemma 2, the equation of C_6 is

$$\begin{aligned}(2\sqrt{2}-1)(X^2+Y^2+Z^2)+4(XY+YZ+ZX)\\ +\left(\alpha(X+Y+Z\sqrt{2})-\frac{2}{\alpha}(X+Y+(2-\sqrt{2})Z)\right)^2=0\end{aligned}$$

for some $\alpha \neq 0$.

C_6 is tangent to C_3 if and only if $\alpha^2 = -2$, $\alpha^2 = 2$, $\alpha^2 = 10 - 8\sqrt{2}$ or $\alpha = \pm(2-\sqrt{2})$. In the first case $C_6 = C_1$, in the other cases C_6 passes through the points $(-1 : 0 : 1)$, $(-1 : 0 : 1)$ or $(1 : -\sqrt{2} : 1)$, respectively, each of which is the contact point of C_i and C_j for some i, j, $1 \leq i < j \leq 5$, so we do not obtain a configuration.

Subcase 2.4. There is a double edge to C_2 and simple edges to C_3 and C_4. Then by Lemma 2, the equation of C_6 is

$$\begin{aligned}(2\sqrt{2}-1)(X^2+Y^2+Z^2)+4(XY+YZ+ZX)\\ +(\alpha(X+\omega Y+\omega^2 Z)+(X+\omega^2 Y+\omega Z)/\alpha)^2=0\end{aligned}$$

for some $\alpha \neq 0$, where ω is a primitive cube root of 1. The polynomials in α expressing the conditions that C_6 is tangent to C_3 and that C_6 is tangent to C_4 are coprime, so there is no configuration with this graph.

Subcase 2.5. There is a simple edge from C_6 to each of C_1, C_2, C_3 and C_4. The equation of C_6 is

$$(2\sqrt{2}-1)(X^2+Y^2+Z^2)+4(XY+YZ+ZX)+(\alpha X+\beta Y+\gamma Z)^2=0,$$

where $\alpha X + \beta Y + \gamma Z = 0$ is the equation of the line connecting the two contact points of C_5 and C_6. The condition that C_6 is tangent to C_i, $1 \leq i \leq 4$, is expressed by a discriminant. Let Δ_i be the polynomial in α, β and γ obtained by dividing this discriminant by the factors corresponding to C_6 passing through one of the contact points of C_i and C_5, and let us normalize them so that Δ_2, Δ_3 and Δ_4 have the same constant terms. Δ_1 is symmetric in α, β, γ and so are $\Phi_1 = \Delta_2 + \Delta_3 + \Delta_4$, $\Phi_2 = \Delta_2\Delta_3 + \Delta_3\Delta_4 + \Delta_4\Delta_2$, $\Phi_3 = \Delta_2\Delta_3\Delta_4$. Therefore they can be expressed in terms of σ_1, σ_2 and σ_3, the elementary symmetric polynomials in α, β and γ. For example,

$$\begin{aligned}\Delta_1 &= (\alpha^2+\beta^2+\gamma^2)^2-4(\alpha^2+\beta^2+\gamma^2-4\alpha\beta-4\beta\gamma-4\gamma\alpha)+36\\ &= (\sigma_1^2-2\sigma_2)^2-4(\sigma_1^2-6\sigma_2)+36,\end{aligned}$$

Φ_1, Φ_2, Φ_3 are rather longer. By using Δ_1 and Φ_1 successively to eliminate σ_2 and σ_3, we obtain two polynomials in σ_1 whose greatest common divisor is $(\sigma_1^2+18)^6$, so this polynomial is in the ideal generated by Δ_1, Φ_1, Φ_2 and Φ_3 in $\boldsymbol{Q}(\sqrt{2})[\sigma_1, \sigma_2, \sigma_3]$. Therefore the only solutions of $\Delta_1 = \Phi_1 = \Phi_2 = \Phi_3 = 0$ are $\sigma_1 = \pm 3i\sqrt{2}$, $\sigma_2 = -6$, $\sigma_3 = -2\sigma_1/3$, which give $\alpha = \beta = \gamma = \pm i\sqrt{2}$, but then $C_6 = C_1$.

In the remaining cases the maximal degree of the vertices of the graph is 3. These graphs can be systematically enumerated in the following way. The sum of the degrees is 14, so there are four possibilities for the degree sequence, (3, 3, 2, 2, 2, 2), (3, 3, 3, 2, 2, 1), (3, 3, 3, 3, 1, 1) and (3, 3, 3, 3, 2, 0). If the graph has a vertex of degree 2, we can remove it and connect its two neighbours by an edge, possibly a loop. If the vertex is the vertex on a loop, we simply remove it together with the loop. By repeating this process, we obtain a graph whose degree sequence is the same as that of the original with the 2's omitted. These graphs are easier to enumerate, and the original graph can be recovered by a suitable subdivision and by adding cycles. Eliminating the graphs containing a forbidden subgraph gives the graphs shown in Figure 4.

Case 3. The degree sequence is (3, 3, 2, 2, 2, 2). There is only one such graph we have to investigate, the first graph in Figure 4. We may assume that C_1, C_2, C_3 and C_4 are as in Proposition 13, and C_5 is given by (4), where $r^2 = \rho^4$. C_5 is the image of C_1 under

$$\begin{pmatrix} \dfrac{r^2+1}{r^2-1} & 0 & \dfrac{2r^2}{r^2-1} \\ 0 & 1 & 0 \\ \dfrac{2}{r^2-1} & 0 & \dfrac{r^2+1}{r^2-1} \end{pmatrix} \in PGL_3(\boldsymbol{C}),$$

which fixes C_2 and C_3, so C_6 is the image under this transformation of a conic given by an

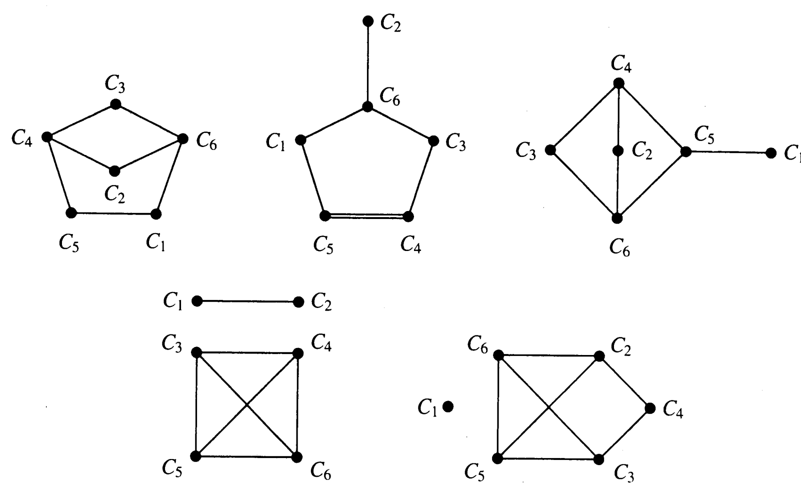


FIGURE 4. Some graphs on 6 vertices with 7 edges.

equation like that of C_4, but with some ρ', σ' instead of ρ, σ. We have $\rho^4 = \rho'^4 = [C_2/C_3] = r^2$.

The condition that C_4 is tangent to C_5 at a point other than its contact point with C_1, C_2 or C_3 gives

$$(\rho^2 - 1)^2\sigma^4 + 6(1 - \rho^4)\sigma^2 + (\rho^2 + 1)^2 = 0\,, \tag{6}$$

therefore

$$\sigma^2 = \frac{(3 \pm 2\sqrt{2})(\rho^2 + 1)}{\rho^2 - 1}\,, \tag{7}$$

while the similar condition on C_1 and C_6 gives

$$\sigma'^2 = \frac{(3 \pm 2\sqrt{2})(1 - \rho'^2)}{\rho'^2 + 1}\,.$$

We may take the $+$ sign in (7). Taking into account that (ρ', σ') and $(-\rho', -\sigma')$ define the same conic, we only need to consider the following cases: $\rho' = \rho$, $\sigma' = \pm i/\sigma$ or $\sigma' = \pm i(3 + 2\sqrt{2})/\sigma$, and $\rho' = i\rho$, $\sigma' = \pm i\sigma$ or $\sigma' = \pm i(3 - 2\sqrt{2})\sigma$. In each case we will have three equations for ρ and σ. The first one is (6).

By Lemma 5, e_{45}, e'_{45}, e_{46} are concurrent, and so are e_{16}, e'_{16}, e_{46}. e_{45} and e'_{45} intersect at $P = (\rho((-2\rho^4+\rho^2-1)\sigma^2+\rho^2(\rho^2+1)) : (\rho^4+1)(\rho^2+1)\sigma : \rho((-\rho^4+\rho^2-2)\sigma^2-\rho^2-1))$, while e_{16} and e'_{16} at $Q = (\rho'(\rho'^2(\rho'^2 - 1)^2 - (\rho'^2 + 1)^2\sigma'^2) : \sigma'(\rho'^8 - 1) : -\rho'(\rho'^2(\rho'^2 + 1)^2\sigma'^2 + (\rho'^2 - 1)^2))$.

P and Q cannot coincide, because then the involution associated to the triples C_1, C_4, C_6 and C_4, C_5, C_6 would be the same. Therefore $P = Q$ would have to lie on the line $X = Z$, the common tangent line to C_1 and C_5 at their contact point, which would imply $\rho^2 = 0$ or -1, which are excluded. Thus P and Q span the line e_{46}, and the line can be parametrized as $\lambda P + \mu Q$, $(\lambda : \mu) \in \boldsymbol{P}^1$. The restrictions of C_4 and C_6 to e_{46} must be multiples of each other, so the ratios of the coefficients of λ^2 and $\lambda\mu$ must agree. This gives us the second equation.

We can substitute into the equation of C_6 the parametrization of C_4 obtained by taking the residual intersection with lines through the contact point of C_2 and C_4, S_1 of the resulting quartic is the third equation.

For any of the possible values of ρ' and σ', if we take the resultants of the first equation with the second and third equations with respect to σ, the resultants are coprime, or only have common factors corresponding to excluded values of ρ or to cases where three conics would pass through one point. This shows that this graph cannot be realized.

Case 4. The degree sequence is $(3, 3, 3, 2, 2, 1)$. There are two such graphs without forbidden subgraphs.

Subcase 4.1. The graph is the second graph in Figure 4. We may assume that the C_i, $1 \le i \le 5$, have the equations used in the proof of Theorem 14, when we dealt with the fifth graph in Figure 1. We saw there that $r \ne \pm\sqrt{2} \pm 1$, and we also have to exclude $r = \pm i$, because then C_4 and C_5 would coincide.

By applying Lemma 6 to the triples C_2, C_4, C_6 and C_2, C_5, C_6, we see that e_{26}, e'_{26}, e_{46} and e_{56} are concurrent. Their common point is $(2r^2 : 2r : r^2+1)$, which is also the singular point of one of the singular elements of the pencil generated by C_4 and C_5. By Lemma 2, if $Q_4 = 0$ is the equation of C_4, then C_6 has equation

$$Q_4 + \frac{1}{4}\left(\alpha((r+i)X + (1-ir)Y - 2rZ) + \frac{r^2+1}{\alpha}(r-i)X + (1+ir)Y - 2rZ)\right)^2 = 0$$

for some $\alpha \neq 0$. The discriminant expressing the condition that C_1 and C_6 are tangent to each other can be factorized in $\mathbf{Z}[i, r, \alpha]$, and each factor implies that C_6 passes through a contact point of C_i and C_j for some i, j, $1 \leq i < j \leq 5$, so there is no configuration with this graph.

Subcase 4.2. The graph is the third graph in Figure 4. We may assume that C_1, C_2, C_3 and C_4 are as in Proposition 13 and that the equation of C_5 is given by (4) with $r = \rho^2$. We have $[C_1/C_6] = 1/[C_1/C_4] = 1/\sigma^4$, so by Proposition 13 applied to C_1, C_2, C_3 and C_6, the equation of C_6 must be of the form

$$X^2 + Y^2 - Z^2 + \frac{((1-\rho'^2\sigma'^2)X + 2\rho'\sigma'Y + (\rho'^2+\sigma'^2)Z)^2}{\sigma'^4(1-\rho'^4)} = 0, \tag{8}$$

where $\rho' = \rho$ or $i\rho$ and $\sigma' = \pm 1/\sigma$ or $\pm i/\sigma$.

By Lemma 2, the line $((1-\rho'^2\sigma'^2)X + 2\rho'\sigma'Y + (\rho'^2+\sigma'^2)Z) = 0$ must pass through $((1-\rho^2\sigma^2) : 2\rho\sigma : -(\rho^2+\sigma^2))$, the intersection point of the common tangent lines to C_1 and C_4 at their contact points. This rules out 6 of the 8 possibilities for ρ' and σ', the only two remaining are $\rho = \rho'$, and $\sigma' = \pm i/\sigma$, and then we have the equation

$$(\rho^2+\sigma^2)(1-\rho^2\sigma^2) \pm 2i\rho^2\sigma^2 = 0, \tag{9}$$

where we take the $+$ sign if $\sigma' = i/\sigma$, and the $-$ sign if $\sigma' = -i/\sigma$.

After discarding the factors corresponding to C_6 passing through the contact points of some of the other conics, the conditions that C_5 is tangent to C_4 and C_6 give the equations (6) and

$$(\rho^2+1)^2\sigma^4 - 6(1-\rho^4)\sigma^2 + (\rho^2-1) = 0.$$

The calculation of the resultant of the above equation with (6) and (9) with respect to σ shows that the three equations have no common solutions.

Case 5. The degree sequence is $(3, 3, 3, 3, 1, 1)$. There is only one such graph without forbidden subgraphs, the fourth one in Figure 4. We can choose coordinates such that C_1, C_2, are invariant under $(X : Y : Z) \mapsto (-X : Y : Z)$, then all the others must also be invariant by Lemma 6. The contact point of C_i and C_j for $3 \leq i < j \leq 6$ must lie on the line $X = 0$, so some of them must coincide, since each conic has only two points of intersection with this line.

Case 6. The degree sequence is $(3, 3, 3, 3, 2, 0)$. There is only one such graph without forbidden subgraphs, the fifth one in Figure 4.

We may assume that C_1, C_2, C_3 and C_4 are as in Proposition 13. By the argument in Subcase 4.2, the equations of C_5 and C_6 must be given by (8) with $\rho' = \rho$ and $\sigma' = \pm i/\sigma$.

Then the equation (9) has to be satisfied simultaneously with both $+$ and $-$ signs, so $\rho = 0$ or $\sigma = 0$, which we have excluded. $\square$

10. Conclusions.

THEOREM 17. *We have* $t(6) = 22$ *and* $27 \leq t(7) \leq 30$.

PROOF. $t(6) \leq 22$ by the previous theorem, so it is sufficient to exhibit one configuration of six conics with 22 tacnodes. Our example is similar in spirit to the configuration of five conics with 17 tacnodes whose graph is a triangle. Let that C_1, C_2, C_3 have equations as in Proposition 4. Then all other conics tangent to C_1 and C_2 at two points other than $(0 : \pm 1 : 1)$ are the images of C_3 under a suitable element of the group H, and two such conics are tangent to each other if and only if one of them is the image of the other under $h = ((r^2+1)/(r^2-1), 2r/(r^2-1)) \in H$. If we choose r such that h has order 4 in H, which happens if and only if $r^2 = -(3 \pm 2\sqrt{2})$, then C_1, C_2, C_3, $h(C_3)$, $h^2(C_3)$ and $h^3(C_3)$ form a configuration with 22 tacnodes.

If we choose r such that h has order 5 in H, then we can obtain a configuration of 5 conics with 27 tacnodes. Hence $t(7) \geq 27$. The inequality (1) gives $t(7) \leq 31$, but $t(6) = 22$ implies

$$t(7) \leq \frac{42 \cdot t(6)}{30} = \frac{154}{5} < 31 .$$

$\square$

REMARK. (i) The expected number of configurations of 6 conics with 22 tacnodes is finite, but the possibilities are probably too numerous to list.

(ii) The same idea can be used to produce a configuration of k conics with $5k - 8$ tacnodes for any $k \geq 4$, and $5k - 8$ is exactly the number for which the expected dimension is 0. For $k \geq 8$ there is a better method. The example in [1] with $k = 14$, $t = 98$ and consists of two sets of 7 conics, such that two conics from different sets are tangent to each other at 2 points. For $k \leq 14$, if we choose $[k/2]$ conics from one set and $[(k+1)/2]$ from the other, we get k conics with $[k^2/2]$ tacnodes. For $k > 14$, we can take copies of the whole configuration of 14 conics and transform them by suitable elements of $PGL_3(\boldsymbol{C})$, this gives $\sim 7k$ tacnodes. There is also a better than linear asymptotic bound of the form $t(k) \geq Ak^{1+B/\log\log k}$ [3, Theorem 5].

THEOREM 18. *For every* k,

$$t(k) < \frac{4k^2}{9} + \frac{4k}{3} .$$

PROOF. The right-hand side is not an integer unless $3 | k$. It was shown in [3, Theorem 6] that equality cannot hold for $k = 9$, $k = 12$ or $k \geq 15$. $t(3) = 6$ obviously and we have shown that $t(6) = 22$. $\square$

Acknowledgments. I am grateful to Professor F. Hirzebruch, who pointed out his paper [1] to me during my stay at the Max-Planck-Institut für Mathematik in 1993–94, which provided the inspiration for the current paper.

REFERENCES

[1] F. HIRZEBRUCH, Singularities of algebraic surfaces and characteristic numbers, Contemp. Math. 58 (1986), 141–155.

[2] G. MEGYESI, Generalization of the Bogomolov-Miyaoka-Yau inequality to singular surfaces, Proc. London Math. Soc. 78 (1999), 241–282.

[3] G. MEGYESI AND E. SZABÓ, On the tacnodes of configurations of conics in the projective plane, Math. Ann. 305 (1996), 693–703.

[4] Y. MIYAOKA, The maximal number of quotient singularities on surfaces with given numerical invariants, Math. Ann. 268 (1984), 159–171.

[5] I. NARUKI, Some invariants for conics and their applications, Publ. Res. Inst. Math. Sci. 19 (1983), 1139–1152.

DEPARTMENT OF MATHEMATICS
UMIST, P. O. BOX 88
MANCHESTER, M60 1QD
ENGLAND

E-mail address: gmegyesi@am.ma.umist.ac.uk