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NORMAL COORDINATE SYSTEMS FROM A VIEWPOINT OF REAL ANALYSIS

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Abstract. Normal coordinate systems for pseudo-Riemannian metrics are investigated from a viewpoint of the theory of partial differential equations. Given a cartesian coordinate system x, a local metric for which x is a normal coordinate system is determined by a metric tensor at the origin and any one of certain three matrix functions. These are related one another by three partial differential equations. Solvability of these equations in C^∞ framework and power series expansion of solutions are discussed.

Introduction. In a pseudo-Riemannian manifold (M, g) of class C^∞, let $x = (x^1, \dots, x^n)$ be a normal coordinate system for g with origin at an arbitrary fixed point o of M. If we expand the covariant metric tensor g_{ab} into power series of x, every homogeneous degree part is an invariant polynomial. It depends neither on the dimension of M nor on the signature of g, and the coefficients are polynomial functions of the Riemannian curvature tensor and its covariant derivatives evaluated at o. Expansion was already written in Cartan's book [C, p. 243] up to the third degree. Herglotz [H] showed a principle of higher degree expansion. Sakai [Sa] obtained an expansion of the volume form $\det(g_{ab})^{1/2}$ up to the sixth degree and applied it to an iso-spectral problem (see also Berger-Gauduchon and Mazet [BGM], Gray [Gr], Willmore [Wi], Gilkey [Gi]). It was Günther [Gü1] who gave a definitive result for g_{ab} with explicit formula for general terms.

Günther was motivated by his own work on the Huygens principle for hyperbolic operators of the second order ([Gü2]). The minor premise of the Huygens principle is realized in a short time interval if and only if a particular Hadamard coefficient vanishes identically. Also, this in turn is equivalent to a set of an infinite number of equalities on coefficients of the operator. Hadamard coefficients which appear in elementary solutions are common to elliptic, parabolic and hyperbolic operators. So, several problems in analysis and in geometry are relied upon the study of the same objects. The 0-th Hadamard coefficient is essentially equal to $|\det(g_{ab})|^{-1/4}$ and higher order ones involve also the contravariant metric tensor g^{ab}, $|\det(g_{ab})|^{\pm 1/2}$ and their derivatives. So, we need a unified treatment of g_{ab}, g^{ab} and several powers of $|\det(g_{ab})|$.

In the present article, we define auxiliary $n \times n$ matrix functions $\mathcal{N}$, $\mathcal{R}$, $\mathcal{S}$ of class C^∞ from a given covariant metric tensor $G = (g_{ab})_{a,b=1}^n$ (Section 1). $\mathcal{N}$ comes from the Levi-Civita connection, $\mathcal{R}$ is a part of the curvature tensor and $\mathcal{S}$ consists of coefficients

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representing a standard basis of tangent vectors by means of basic Jacobi fields. They solve the following partial differential equations (see Section 2):

$$YS = -SN\,, \tag{1}$$

$$YYS + YS + SR = O\,, \tag{2}$$

$$YN + N = R + N^2\,, \tag{3}$$

where $Y = x^s\partial/\partial x^s$ is the Euler vector field and O is the zero matrix. Near the origin, $\|G-G(0)\|$, $\|\mathcal{N}\|$, $\|\mathcal{R}\|$, $\|\mathcal{S}-I\|$ are of $O(|x|^2)$ (I is the unit matrix). By the Prüfer transform (1), the Sturm-Liouville type equation (2) is translated into the Riccati type equation (3), and vice versa. Covariant metric tensor is represented as

$$G = \mathcal{S}G(0)\,{}^t\mathcal{S}\,. \tag{4}$$

To give a metric g is equivalent to give $G(0)$ and any one of $\mathcal{N}$, $\mathcal{R}$, $\mathcal{S}$. This is our main result (Theorem 4.1). For example, if $\mathcal{R}$ is given, we can find $\mathcal{S}$ by solving (2) and we have G from (4). If one of $\mathcal{N}$, $\mathcal{R}$, $\mathcal{S}$ is real analytic, then others are, too, and g is real analytic (Corollary 4.1). We can formulate the separation of variables for (1), (2), (3) (Corollary 4.2). Since differentiations are generated only by Y, (1), (2), (3) are reduced to ordinary differential equations

$$U'(t) = U(t)\Phi(t)\,, \quad U(0) = I\,; \tag{5}$$

$$tV''(t) + 2V'(t) = V(t)\Psi(t)\,, \quad V(0) = I\,; \tag{6}$$

$$tN'(t) + N(t) = P(t) + N(t)^2\,, \quad P(0) = N(0) = O\,, \tag{7}$$

respectively, where Φ, Ψ, P are given and U, V, N are unknown. Existence, uniqueness and smoothness of solutions to (1), (2), (3) are reduced to those of (5), (6), (7) with parameters x (Section A). If we know the Taylor series of $\Psi(t)$ at $t = 0$, we can write down the Taylor series of $V(t)$ (see (a.7)). Applied to $\Psi(t) = -(1/t)\mathcal{R}(tx)$, $V(t) = \mathcal{S}(tx)$, this is nothing but the Günther formula (6.2). We can also represent the Taylor series of $\mathcal{N}$ by means of $\mathcal{R}$ (see (a.12), (a.13), (6.5)). Real analyticity is guaranteed by simple majorant series (see Section A.4). $\mathcal{N}$ exploses eventually on the set $\{x;\, \det\mathcal{S}(x) = 0\}$ (see Remark A.1). Since

$$Y\log|\det(g_{ab})| = -2\,\mathrm{tr}\mathcal{N}\,, \tag{8}$$

we can expand any power of $|\det(g_{ab})|$ in a unified way (see (6.7)). Expansions in terms of $\mathcal{N}$ are much simpler than those in terms of $\mathcal{R}$. The author, however, does not yet understand the geometrical meaning of $\mathcal{N}$.

It should be noticed that Y and Y^2 are hyperbolic except at the origin and are degenerate in all directions at the origin (see Petrovsky [P]). They are not of the Fuchs type in the sense of Baouendi-Goulaouic [BG] but of the Euler type if they are to be labeled.

We are very much inspired by a work [KB] of Kowalski and Belger. They showed how to construct a real analytic metric when prescribed are values of the curvature tensor and all its covariant derivatives at o.

One of remaining questions in this article is to investigate the structure of the space V of matrix functions (see Sections 3, 7). Application of our framework to the Hadamard coefficients will be the subject of a forthcoming paper.

1. Notation and preliminaries. Given a pseudo-Riemannian manifold (M, g) of dimension n, let TM be the tangent bundle of M. The Levi-Civita connection on TM is denoted by ∇. The curvature tensor $R : TM \times TM \to \mathrm{End}(TM)$ is defined to be $R(\xi, \eta)\zeta = [\nabla_\xi, \nabla_\eta]\zeta - \nabla_{[\xi,\eta]}\zeta$. Given a local coordinate system $x = (x^1, \dots, x^n)$, we denote $\partial_j = \partial/\partial x^j$, $g_{jk} = g(\partial_j, \partial_k)$ and $\nabla_j = \nabla_{\partial_j}$. Then, components of ∇ and R are given as follows.

$$(1.1)_1 \qquad \nabla_s \partial_j = \Gamma_s{}^k{}_j \partial_k\,, \quad R(\partial_a, \partial_s)\partial_r = R^k{}_{ras}\partial_b\,, \quad R_{bras} = g_{bc}R^c{}_{ras}\,,$$

where

$$(1.1)_2 \qquad \begin{aligned} \Gamma_j{}^a{}_k &= (1/2)g^{ab}(\partial_j g_{kb} + \partial_k g_{jb} - \partial_b g_{jk})\,, \\ R^b{}_{ras} &= \partial_a \Gamma_s{}^b{}_r + \Gamma_s{}^h{}_r \Gamma_a{}^b{}_h - \partial_s \Gamma_a{}^b{}_r - \Gamma_a{}^h{}_r \Gamma_s{}^b{}_h\,, \end{aligned}$$

and the summation convention is used.

Let o be an arbitrary reference point of M. A local coordinate system x with origin at o ($x = 0$ at o) available in a neighborhood of o is said to be *normal* if and only if

$$(1.2)_1,\ (1.2)_2 \qquad x^k g_{jk}(x) = x^k g_{jk}(0) \quad \text{or} \quad x^j x^k \Gamma_j{}^a{}_k(x) = 0\,.$$

$(1.2)_1$ and $(1.2)_2$ are equivalent. In a normal coordinate system, the *Euler vector field* $Y = x^s\partial_s$ is of special importance. Every geodesic emanating from o is written as $x = tp$ with real parameters p^j satisfying $(p^1)^2 + \dots + (p^n)^2 = 1$. Then, $(Yf)(tp) = (t\partial/\partial t)\{f(tp)\}$ for any real-valued function f. So, Y is tangential to the geodesic from o to x. For a matrix function $U = (u_{ab})$ or $(u_a{}^b)$, we define YU to be $(x^s\partial_s u_{ab})$ or $(x^s\partial_s u_a{}^b)$, respectively. A smooth function f is a homogeneous polynomial of degree m if $Yf = mf$.

From now on, we fix a normal coordinate system x with origin at o. So, x is simply a cartesian coordinate system in $\boldsymbol{R}^n$ and we work in a neighborhood of $x = 0$. We introduce certain $n \times n$ matrix functions.

(1°) *Metric tensor.* Denote $G = (g_{ab}(x))^n_{a,b=1}$, $G^{-1} = (g^{ab}(x))^n_{a,b=1}$,

$$(1.3) \qquad \begin{aligned} \gamma = G(0) = (\gamma_{ab})^n_{a,b=1}\,, \quad \gamma^{-1} &= G(0)^{-1} = (\gamma^{ab})^n_{a,b=1}\,, \\ \mathcal{H} = G\gamma^{-1} &= (g_{as}(x)\gamma^{sb})^n_{a,b=1}\,, \end{aligned}$$

where $\gamma_{ab} = g_{ab}(0)$, $\gamma^{ab} = g^{ab}(0)$.

(2°) *Jacobi fields.* Let X_A be the parallel transport of $(\partial_A)_o$ along geodesics emanating from o in such a way that $\nabla_Y X_A = 0$ $(1 \le A \le n)$. If we set

$$X_A = \tau_A{}^j(x)\partial_j \quad \text{and} \quad \partial_j = \sigma_j{}^A(x)X_A\,,$$

then $\sigma_j{}^A\tau_A{}^k = \delta_j{}^k$, $\tau_A{}^j\sigma_j{}^B = \delta_A{}^B$, $\sigma_j{}^A(0) = \delta_j{}^A$, $\tau_A{}^j(0) = \delta_A{}^j$. We denote

$$(1.4) \qquad \mathcal{S} = (\sigma_j{}^A(x))^n_{j,A=1}\,, \quad \text{so} \quad \mathcal{S}^{-1} = (\tau_B{}^k(x))^n_{B,k=1}\,.$$

(3°) *A part of the Levi-Civita connection.* We set

$$\mathcal{N} = (\mathcal{N}_A{}^B)_{A,B=1}^n, \quad \mathcal{N}_A{}^B = -\tau_A{}^a(x)x^s\Gamma_s{}^b{}_a(x)\sigma_b{}^B(x), \tag{1.5}$$
$$\mathcal{N}' = (\mathcal{N}'_a{}^b)_{a,b=1}^n = \mathcal{S}\mathcal{N}\mathcal{S}^{-1}, \quad \mathcal{N}'_a{}^b = -x^s\Gamma_s{}^b{}_a(x).$$

$\mathcal{N}$ defines an endomorphism $\xi \to \xi\mathcal{N}$ ($\xi^A X_A \to \xi^B\mathcal{N}_B{}^A X_A$ or $\xi'^a\partial_a \to -\xi'^a\nabla_Y\partial_a$) of TM. (Restriction of $\mathcal{N}$ to the orthogonal complement of x, identified with a coordinate vector, may be related with the shape operator of a geodesic sphere because our (2.4) below is analogous to the equation (2.10) of Kowalski-Belger [KB]).

(4°) *A part of the curvature.* Let $\{X_*{}^A\}_{A=1}^n \subset T^*M$ be the dual basis to $\{X_A\}_{A=1}^n$, that is, $\langle X_A, X_*{}^B\rangle = \delta_A{}^B$, where $\langle\,,\,\rangle$ stands for the duality. Then, we have

$$\tau_A{}^a(x)x^r x^s R^b{}_{ras}(x)\sigma_b{}^B(x) = \langle R(X_A, Y)Y, X_*{}^B\rangle.$$

So, we set

$$\mathcal{R} = (\langle R(X_A, Y)Y, X_*{}^B\rangle)_{A,B=1}^n, \tag{1.6}$$

where $Y = x^s\partial_s$. $\mathcal{R}$ represents $R(\cdot, Y)Y \in \mathrm{End}(TM)$ ($\xi\mathcal{R} = R(\xi, Y)Y$).

REMARK 1.1. If x and $\bar{x}$ are two normal coordinate systems with the same origin for the same metric, the transformation $x \to \bar{x}$ is linear: $\bar{x} = xT$ ($\bar{x}^\alpha = x^s t_s{}^\alpha$), say, with a constant, non-singular real matrix $T = (t_s{}^\alpha)_{s,\alpha=1}^n$. (Coordinate vectors are always regarded as row vectors in this article). If $\bar{\gamma}$, $\bar{G}$, $\bar{\mathcal{H}}$, $\bar{\mathcal{N}}$, $\bar{\mathcal{R}}$, $\bar{\mathcal{S}}$ are defined as above with respect to the coordinate system $\bar{x}$, the law of transformation is as follows.

$$\gamma = T\bar{\gamma}\,{}^tT, \quad G = T\bar{G}\,{}^tT, \quad \mathcal{A} = T\bar{\mathcal{A}}T^{-1} \quad \text{for } \mathcal{A} = \mathcal{H}, \mathcal{N}, \mathcal{R} \text{ or } \mathcal{S}. \tag{1.7}$$

On the other hand, Y is invariant, $x^s\partial/\partial x^s = \bar{x}^\alpha\partial/\partial\bar{x}^\alpha$. Therefore, analytic or geometric considerations in what follows are the same in any normal coordinate system. In particular, partial differential equations derived in Section 2 are invariant under change of variables from a normal coordinate system to another normal coordinate system.

2. Relationship between two of $\mathcal{H}, \mathcal{N}, \mathcal{R}, \mathcal{S}$.

LEMMA 2.1.

$$Y\mathcal{S} = -\mathcal{S}\mathcal{N} = -\mathcal{N}'\mathcal{S}; \tag{2,1),(2.1'}$$

$$YG = -2\mathcal{N}'G \quad\text{and}\quad \mathcal{N}'G = G\,{}^t\mathcal{N}'; \tag{2.2}$$

$$G = \mathcal{S}\gamma\,{}^t\mathcal{S} \quad\text{or}\quad \mathcal{H} = \mathcal{S}\gamma\,{}^t\mathcal{S}\gamma^{-1}; \tag{2.3}$$

$$Y\mathcal{N} + \mathcal{N} = \mathcal{R} + \mathcal{N}^2; \tag{2.4}$$

$$YY\mathcal{S} + Y\mathcal{S} + \mathcal{S}\mathcal{R} = O. \tag{2.5}$$

PROOF. (2.1) Since $\nabla_s\partial_j = \nabla_s(\sigma_j{}^A X_A) = (\partial_s\sigma_j{}^A)X_A + \sigma_j{}^A\nabla_s X_A = \Gamma_s{}^k{}_j\partial_k = \Gamma_s{}^k{}_j\sigma_k{}^A X_A$, we have $(\partial_s\sigma_j{}^A - \Gamma_s{}^k{}_j\sigma_k{}^A)X_A + \sigma_j{}^A\nabla_s X_A = 0$. This implies $Y\sigma_j{}^A = x^s\Gamma_s{}^k{}_j\sigma_k{}^A$ because $\nabla_Y X_A = 0$, which proves (2.1′) and (2.1).

(2.2) We multiply x^s to both sides of $2\Gamma_s{}^c{}_a g_{cb} - \partial_s g_{ab} = \partial_a g_{sb} - \partial_b g_{sa}$. Since $x^s\partial_a g_{sb} = \gamma_{ab} - g_{ab} = x^s\partial_b g_{sa}$ in view of $(1.2)_1$, we have (2.2).

(2.3) Set $G' = G - \mathcal{S}\gamma\,{}^t\mathcal{S}$. Then, $YG' = -\mathcal{N}'G' - G'\,{}^t\mathcal{N}'$ and $G'(0) = O$. We apply Lemma 2.2 below with $\mathcal{A}_1 = \mathcal{A}_2 = O$ and $\kappa = r = 0$ to see that $G' = O$.

(2.4) Multiplying $x^r x^s$ to both sides of $(1.1)_2$, we have $Y\mathcal{N}' + \mathcal{N}' = \mathcal{R}' + \mathcal{N}'^2$, where $\mathcal{R}' = (x^r x^s R^b{}_{ras})^n_{a,b=1} = \mathcal{S}\mathcal{R}\mathcal{S}^{-1}$. Rewriting this with the aid of (2.1), we have (2.4).

(2.5) $Y(Y\mathcal{S} + \mathcal{S}\mathcal{N}) = O$ by (2.1). Applying (2.4) to this, we have (2.5). q.e.d.

We state the lemma assumed in the proof of (2.3). Let $\mathcal{M}_n$ be the ring of $n \times n$ real matrices endowed with the norm $\|A\| = \sup\{|vA|;\, v \in \boldsymbol{R}^n,\, |v| \le 1\}$, where $|\ |$ stands for the Euclidean norm. Zero and unit matrices are denoted by O and I, respectively.

LEMMA 2.2. *Let r be a non-negative integer and $\mathcal{F}$ be an $\mathcal{M}_n$-valued function of class C^{r+1} in a star-shaped neighborhood Ω of $x = 0$ satisfying a partial differential equation*

$$Y\mathcal{F} + \kappa\mathcal{F} = \mathcal{A}_1\mathcal{F}\mathcal{A}_2 + \mathcal{A}_3\mathcal{F} + \mathcal{F}\mathcal{A}_4\,, \tag{2.6}$$

where κ is a real number, $\mathcal{A}_1, \dots, \mathcal{A}_4$ are $\mathcal{M}_n$-valued functions of class C^1 in Ω such that $\mathcal{A}_3(0) = \mathcal{A}_4(0) = O$, and, $\mathcal{A}_1(0) = O$ or $\mathcal{A}_2(0) = O$. If $\mathcal{F}$ and all its partial derivatives up to order r vanish at $x = 0$ and if $\kappa + r + 1 > 0$, then $\mathcal{F}$ is identically equal to zero in Ω.

PROOF. Set $\kappa' = \kappa + r$, $F(t) = t^{-r}\mathcal{F}(tx)$ and $A_j(t) = \mathcal{A}_j(tx)$. Then,

$$F(t) = \int_0^1 \theta^{\kappa'-1}(A_1FA_2 + A_3F + FA_4)(\theta t)d\theta\,.$$

If we set $a(t) = (\|A_1(t)\|\|A_2(t)\| + \|A_3(t)\| + \|A_4(t)\|)/|t|$, then

$$\|F(t)\| \le |t|\int_0^1 \theta^{\kappa'}a(\theta t)\|F(\theta t)\|d\theta\,.$$

This implies $\|F(t)\| = 0$ as long as $tx \in \Omega$, so $\mathcal{F}$ is identically equal to O. q.e.d.

The reasoning fails if $\kappa + r$ is a negative integer. For example, $\mathcal{F}(x) = (x^1)^pI$ solves $Y\mathcal{F} - p\mathcal{F} = O$ $(p = 1, 2, 3, \dots, \kappa = -p, r = p - 1)$. Remark also that the lemma can be applied to functions with values in row vectors if we set $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = O$.

3. Construction of a metric from γ and one of $\mathcal{N}, \mathcal{R}, \mathcal{S}$. In what follows, we fix a constant, real, symmetric, non-singular matrix γ. Given a matrix function $\mathcal{A} = \mathcal{A}(x) = (a_r{}^s(x))^n_{r,s=1}$ of class C^∞, the homogeneous part of degree p in the Taylor series at $x = 0$ is denoted by $\mathcal{A}_{(p)}$.

DEFINITION 3.1. *Let V be the real vector space of $n \times n$ matrix functions $\mathcal{A}$ of class C^∞ in a neighborhood of $x = 0$ satisfying*

$$\mathcal{A}\gamma = \gamma\,{}^t\!\mathcal{A} \quad \text{and} \quad x\mathcal{A} = 0\,, \tag{3.1}$$

that is, $a_r{}^h(x)\gamma_{hs} = \gamma_{rh}a_s{}^h(x)$ and $x^h a_h{}^s(x) = 0$. Let $V_{(p)}$ be the linear subspace of V consisting of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_{(p)}$. If $\mathcal{A} \in V$, then $\mathcal{A}_{(p)} \in V_{(p)}$ $(p \ge 0)$.

We shall show in Lemma 7.1 below that $V_{(0)} = \{O\}$ and $V_{(1)} = \{O\}$.

3.1. We have four matrix functions $\mathcal{H}, \mathcal{N}, \mathcal{R}, \mathcal{S}$ of class C^∞ in a neighborhood of the origin. First of all, let us enumerate some properties of each one not involving others.

LEMMA 3.1. *Let $\gamma, \mathcal{H}, \mathcal{N}, \mathcal{R}, \mathcal{S}$ be matrices defined in* (1.3) *through* (1.6). *Then, they satisfy* (h), (n), (r), (s) *respectively as follows*:

(h), (n), (r) $$\mathcal{H} - I \in V ; \quad \mathcal{N} \in V ; \quad \mathcal{R} \in V ;$$

(s) $$\mathcal{S}_{(0)} = I , \quad \mathcal{S}_{(1)} = O , \quad (Y\mathcal{S})\gamma\, {}^t\mathcal{S} = \mathcal{S}\gamma\, {}^t(Y\mathcal{S}) \quad \text{and} \quad x\mathcal{S} = x .$$

PROOF. (1°) Obviously $\mathcal{H}_{(0)} = \mathcal{S}_{(0)} = I, \mathcal{N}_{(0)} = \mathcal{R}_{(0)} = \mathcal{R}_{(1)} = O$. $\mathcal{H}_{(1)} = \mathcal{N}_{(1)} = O$ because $\partial_j g_{ab}(0) = \Gamma_a{}^b{}_s(0) = 0$, so $\mathcal{S}_{(1)} = -\mathcal{N}_{(1)} = O$ from (2.1) and $\mathcal{H}\gamma = G = {}^tG = \gamma\, {}^t\mathcal{H}$.

(2°) We set $\tilde{\mathcal{S}} = (Y\mathcal{S})\gamma\, {}^t\mathcal{S} - \mathcal{S}\gamma\, {}^t(Y\mathcal{S}), \tilde{\mathcal{N}} = \gamma\, {}^t\mathcal{N} - \mathcal{N}\gamma, \tilde{\mathcal{R}} = \gamma\, {}^t\mathcal{R} - \mathcal{R}\gamma$. Then,

$(3.2)_1, (3.2)_2, (3.2)_3$ $$Y\tilde{\mathcal{S}} + \tilde{\mathcal{S}} = \mathcal{S}\tilde{\mathcal{R}}\, {}^t\mathcal{S}, \quad \tilde{\mathcal{S}} = \mathcal{S}\tilde{\mathcal{N}}\, {}^t\mathcal{S}, \quad Y\tilde{\mathcal{N}} + \tilde{\mathcal{N}} = \mathcal{N}\tilde{\mathcal{N}} + \tilde{\mathcal{N}}\, {}^t\mathcal{N} + \tilde{\mathcal{R}},$$

$(3.2)_4$ $$G\, {}^t\mathcal{R}' - \mathcal{R}'G = \mathcal{S}\tilde{\mathcal{R}}\, {}^t\mathcal{S},$$

where $\mathcal{R}' = \mathcal{S}\mathcal{R}\mathcal{S}^{-1}$. From $(3.2)_4$, we see that $\tilde{\mathcal{R}} = O$ because $\mathcal{R}'G = (x^r x^s R_{bras})^n_{a,b=1}$ is symmetric. Then, $\tilde{\mathcal{S}} = O$ by $(3.2)_1$ and Lemma 2.2 with $\mathcal{A}_j = O, \kappa = r = 1$. And $\tilde{\mathcal{N}} = O$ by $(3.2)_3$.

(3°) First, $x\mathcal{H} = x$ from $(1.2)_1$. Second, $x\mathcal{N}' = 0$ from $(1.2)_2$, so $x\mathcal{S}\mathcal{N} = 0$ from (1.5). If we set $\eta = x - x\mathcal{S}$, we have $Y\eta - \eta = x\mathcal{S}\mathcal{N} = 0$. We can apply Lemma 2.2 with $\mathcal{A}_j = O, \kappa = -1$ and $r = 1$ to have $\eta = 0$ or $x\mathcal{S} = x$. Third, $x\mathcal{N} = x\mathcal{S}\mathcal{N} = 0$. Fourth, $x\mathcal{R}' = 0$ because $R^b{}_{ras} = -R^b{}_{rsa}$. So, $x\mathcal{R} = x\mathcal{S}^{-1}\mathcal{R}'\mathcal{S} = x\mathcal{R}'\mathcal{S} = 0$. q.e.d.

REMARK 3.1. (s) implies also $x\gamma\, {}^t\mathcal{S} = x\gamma$ (In fact, if we set $\zeta = x\gamma\, {}^t\mathcal{S} - x\gamma$, then $x\tilde{\mathcal{S}} = 0$ reduces to $Y\zeta = \zeta$ because $x\mathcal{S} = x$. So $\zeta = 0$ by Lemma 2.2 as above).

3.2. Next, we discuss $\mathcal{N}, \mathcal{R}, \mathcal{S}$ apart from their geometrical meanings.

LEMMA 3.2. *If we give a γ and any one of $\mathcal{N}, \mathcal{R}, \mathcal{S}$ of class C^∞ in a neighborhood of $x = 0$ and satisfying* (n), (r), (s) *respectively, we can define, in a unique way, others which satisfy* (2.1), (2.4) *and* (2.5).

PROOF. (1°) Given an element $\mathcal{N}$ of V, we define $\mathcal{R}$ by (2.4). Then, $\mathcal{R}_{(0)} = O$, $\mathcal{R}_{(1)} = 2\mathcal{N}_{(1)} = O$ and $x\mathcal{R} = xY\mathcal{N} = Y(x\mathcal{N}) - (Yx)\mathcal{N} = Y0 - x\mathcal{N} = 0$. Also $\tilde{\mathcal{R}} = O$ in view of $(3.2)_3$, so $\mathcal{R} \in V$.

(2°) Given an element $\mathcal{R}$ of V, there exists a unique $\mathcal{S}$ satisfying (2.5) and $\mathcal{S}(0) = I$ (see Section A.2′). We have $\mathcal{S}_{(0)} = I, 2\mathcal{S}_{(1)} = -\mathcal{S}_{(0)}\mathcal{R}_{(1)} - \mathcal{S}_{(1)}\mathcal{R}_{(0)} = O$. Next, $\eta = x - x\mathcal{S}$ satisfies $YY\eta - Y\eta + \eta\mathcal{R} = 0$. This means that $z'' = zR(t)$ if we set $z(t) = \eta(tx)$ and $R(t) = -(1/t^2)\mathcal{R}(tx)$. $R(t)$ is continuous near $t = 0$ and $z(0) = z'(0) = 0$. Therefore, $z(t)$ is identically equal to zero, so η is zero. Finally $\tilde{\mathcal{S}} = O$ by $(3.2)_1$. Hence $\mathcal{S}$ satisfies (s).

(3°) Given an $\mathcal{S}$ satisfying (s), we set $\mathcal{N} = -\mathcal{S}^{-1}Y\mathcal{S}$. Then, $\mathcal{N}_{(0)} = O, \mathcal{N}_{(1)} = -\mathcal{S}_{(0)}\mathcal{S}_{(1)} = O$ and $x\mathcal{N} = -x\mathcal{S}^{-1}Y\mathcal{S} = -xY\mathcal{S} = (Yx)\mathcal{S} - Y(x\mathcal{S}) = x\mathcal{S} - Yx = x - x = 0$. Furthermore, $\tilde{\mathcal{N}} = O$ by $(3.2)_2$. Therefore, $\mathcal{N} \in V$. q.e.d.

REMARK 3.2. (i) Given an element $\mathcal{N}$ of V, we can find a unique $\mathcal{S}$ satisfying (2.1) and $\mathcal{S}(0) = I$ (see Section A.1′). This $\mathcal{S}$ satisfies also (s).

(ii) Given an $\mathcal{S}$ satisfying (s), we define $\mathcal{R}$ by (2.5). Then, $\mathcal{R}$ belongs to V.

(iii) Given an element $\mathcal{R}$ of V, there exists a unique solution $\mathcal{N}$ to (2.4) such that $\mathcal{N}(0) = O$. We make there use of the argument in Section A.3′. $\mathcal{N}$ belongs to V.

REMARK 3.3. The vector space V is also a module over the ring of real-valued functions of class C^∞. Furthermore, if $\mathcal{A}, \mathcal{B} \in V$, then $Y\mathcal{A} \in V$ and $\mathcal{AB} + \mathcal{BA} \in V$ as is easily verified.

4. Main result.

4.1. The following is the main result of this article.

THEOREM 4.1. *If we give a constant, real, symmetric, non-singular matrix γ and any one of the matrix functions $\mathcal{N}$, $\mathcal{R}$, $\mathcal{S}$ of class C^∞ satisfying* (n), (r), (s) *respectively, we can define a unique pseudo-Riemannian metric $g = g_{ab}dx^a dx^b$ of class C^∞ in a neighborhood of $x = 0$ related with $\mathcal{N}$, $\mathcal{R}$, $\mathcal{S}$ by* (2.1), (2.3), (2.4) *and* (2.5). *x is a normal coordinate system with respect to g and $g_{ab}(0) = \gamma_{ab}$.*

PROOF. If we give γ and one of $\mathcal{N}$, $\mathcal{R}$, we have $\mathcal{S}$ satisfying (s) as in Lemma 3.2, so we set $\mathcal{H} = \mathcal{S}\gamma\, {}^t\mathcal{S}\gamma^{-1}$. Then, $\mathcal{H}_{(0)} = I$, $\mathcal{H}_{(1)} = \mathcal{S}_{(1)} + \gamma\, {}^t\mathcal{S}_{(1)}\gamma^{-1} = O$, $x\mathcal{H} = x\mathcal{S}\gamma\, {}^t\mathcal{S}\gamma^{-1} = x\gamma\, {}^t\mathcal{S}\gamma^{-1} = x\gamma\gamma^{-1} = x$ (see Remark 3.1), so $\mathcal{H} - I \in V$. If we set $G = \mathcal{H}\gamma$, it is symmetric, $G(0) = \gamma$, and G satisfies $(1.2)_2$, so x is a normal coordinate system for this metric. q.e.d.

COROLLARY 4.1. *If any one of $\mathcal{N}$, $\mathcal{R}$, $\mathcal{S}$ is real analytic in a neighborhood Ω of $x = 0$, the metric g is real analytic in a neighborhood Ω' of $x = 0$ contained in Ω.*

PROOF. We refer to Proposition A.4 below. If $\mathcal{R}$ is real analytic, the solution $\mathcal{N}$ to (2.4), $\mathcal{N}(0) = O$, is real analytic. If $\mathcal{N}$ is real analytic, the solution $\mathcal{S}$ to (2.1), $\mathcal{S}(0) = I$, is real analytic. If $\mathcal{S}$ is real analytic, $G = \mathcal{S}\gamma\, {}^t\mathcal{S}$ is also real analytic. q.e.d.

4.2. Let us remark a functorial property. A constant real matrix $\pi = (\pi_a{}^b)^n_{a,b=1}$ is said to be a *γ-projection* if $\pi^2 = \pi$ and $\pi\gamma = \gamma\, {}^t\pi$ ($\pi_a{}^c\pi_c{}^b = \pi_a{}^b$, $\pi_a{}^c\gamma_{cb} = \gamma_{ac}\pi_b{}^c$). If π is a γ-projection, $I - \pi$ is, too. We suppose that π is proper, that is, $O \neq \pi \neq I$. Given also a metric g satisfying $G(0) = \gamma$ as above, we define G_ρ to be $G_\rho(x) = \rho G(x\rho)\, {}^t\rho$ for $\rho = \pi$ and for $\rho = I - \pi$. G_ρ defines a metric g_ρ on $\operatorname{Im}\rho = \rho\boldsymbol{R}^n$, whose metric tensor is independent of $x - x\rho$. A γ-projection π is said to *decompose* g if $g = g_\pi + g_{I-\pi}$ or $G = G_\pi + G_{I-\pi}$.

Given also a matrix function $\mathcal{A} = (\mathcal{A}_a{}^b(x))^n_{a,b=1}$, we define $\mathcal{A}_\rho$ to be $\mathcal{A}_\rho(x) = \rho\mathcal{A}(x\rho)\rho$ for $\rho = \pi, I - \pi$. A γ-projection π is said to *decompose* $\mathcal{A}$ if $\mathcal{A} = \mathcal{A}_\pi + \mathcal{A}_{I-\pi}$. Let us show that the separation of variables works also in the non-linear equation (2.4).

COROLLARY 4.2. *If π is a γ-projection, then the following three conditions are equivalent*:

(a) *π decomposes any one of $\mathcal{N}$, $\mathcal{R}$, $\mathcal{S}$.*

(b) *π decomposes $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{S}$.*

(c) *π decomposes g.*

PROOF. Let ρ be π or $I-\pi$. The Euler vector field splits as $Y = Y_\pi + Y_{I-\pi}$, so $Y\mathcal{A}_\rho = (Y\mathcal{A})_\rho = Y_\rho \mathcal{A}_\rho$. If π decomposes $\mathcal{R}$, then $\mathcal{A} = \mathcal{N}_\rho$ is a solution to $Y\mathcal{A} + \mathcal{A} = \mathcal{R}_\rho + \mathcal{A}^2$, $\mathcal{A}(0) = O$, which is unique by Lemma 2.2 and π decomposes $\mathcal{N}$. If π decomposes $\mathcal{N}$, then $\mathcal{B} = \mathcal{S}_\rho$ is a unique solution to $Y\mathcal{B} = -\mathcal{B}\mathcal{N}_\rho$, $\mathcal{B}(0) = \rho$, and π decomposes $\mathcal{S}$. If π decomposes $\mathcal{S}$, then $G_\rho = \mathcal{S}_\rho \gamma\, {}^t\mathcal{S}_\rho$ and π decomposes G. If π decomposes G, then π decomposes $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{S}$, too. q.e.d.

There are many γ-projections. Note that the space of γ-projections is non-compact if γ is indefinite.

5. Examples. If γ and one of $\mathcal{N}$, $\mathcal{R}$, $\mathcal{S}$ are given, Theorem 4.1 allows us to construct local metrics satisfying $g_{ab}(0) = \gamma_{ab}$ with respect to which x is a normal coordinate system.

5.1. If one of $\mathcal{H} - I$, $\mathcal{N}$, $\mathcal{R}$, $\mathcal{S} - I$ belongs to a $V_{(p)}$ ($p \geq 2$. see Definition 3.1), others are obtained by a simple power series calculus.

If $\mathcal{H} - I \in V_{(p)}$, then

$$(5.1) \qquad \mathcal{S} = \mathcal{H}^{1/2}, \quad \mathcal{N} = \frac{p}{2}(\mathcal{H}^{-1} - I), \quad \mathcal{R} = \left(\frac{p}{2}\mathcal{H}^{-1} + \frac{1}{2}I\right)^2 - \left(\frac{p+1}{2}\right)^2 I\,.$$

If $\mathcal{S} - I \in V_{(p)}$, then

$$(5.2) \qquad \mathcal{N} = p(\mathcal{S}^{-1} - I)\,, \quad \mathcal{R} = (p^2 + p)(\mathcal{S}^{-1} - I)\,, \quad G = \mathcal{S}^2\gamma\,.$$

If $\mathcal{N} \in V_{(p)}$, then

$$(5.3) \qquad \mathcal{S} = e^{-(1/p)\mathcal{N}}\,, \quad \mathcal{R} = (p+1)\mathcal{N} - \mathcal{N}^2\,, \quad G = e^{-(2/p)\mathcal{N}}\gamma\,.$$

If $\mathcal{R} \in V_{(p)}$, then

$$(5.4)_1 \qquad \mathcal{S} = \sigma(\mathcal{R})\,, \quad \mathcal{N} = \nu(\mathcal{R})\,, \quad G = \sigma(\mathcal{R})^2\gamma\,,$$

where

$$(5.4)_2 \qquad \begin{aligned} \sigma(z) &= \Gamma\left(\frac{p+1}{p}\right)\left(\frac{p}{\sqrt{z}}\right)^{1/p} J_{1/p}\left(\frac{2\sqrt{z}}{p}\right), \\ \nu(z) &= \sqrt{z}J_{(p+1)/p}\left(\frac{2\sqrt{z}}{p}\right)\Big/ J_{1/p}\left(\frac{2\sqrt{z}}{p}\right), \end{aligned}$$

and $J_\nu(z)$ is the Bessel function of order ν (see Watson [Wa]).

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!\Gamma(\nu + k + 1)}\,.$$

If, in particular, $p = 2$, then $\sigma(z) = (\sin\sqrt{z})/\sqrt{z}$. Also, if $\mathcal{R}$ is a constant multiple of P defined in (5.5) below, then g is of constant curvature.

Note that $\mathcal{S} - I$ does not belong to V in general. For example, $\mathcal{S}_{(5)}\gamma \neq \gamma\, {}^t\mathcal{S}_{(5)}$ if $\mathcal{R}_{(2)}\mathcal{R}_{(3)} \neq \mathcal{R}_{(3)}\mathcal{R}_{(2)}$, so $\mathcal{S} - I \notin V$ (see (6.2), $(a.7)$ below in which $a_{23} \neq a_{32}$).

5.2. In this subsection, we denote

(5.5) $$x_a = \gamma_{aj} x^j\,, \quad u = x^a x_a\,, \quad \pi_a{}^b = u\delta_a{}^b - x_a x^b\,, \quad P = (\pi_a{}^b)^n_{a,b=1}\,.$$

Then, $P \in V_{(2)}$ and $P^2 = uP$. Let φ be a real-valued function of class C^∞ in a star-shaped neighborhood Ω of $x = 0$ such that $u\varphi(x) < 1$ in Ω. We are interested in the following metric

(5.6) $$g_{ab} = \{1 - u\varphi(x)\}\gamma_{ab} + \varphi(x)\gamma_{ac}\gamma_{bd}x^c x^d \quad \text{or} \quad \mathcal{H} = I - \varphi(x)P\,.$$

We obtain $\mathcal{N}'$ from (2.2), $\mathcal{S}$ by solving (2.1′) and $\mathcal{R}$ from (2.4):

$(5.7)_1, (5.7)_2$ $$\mathcal{S} = I - \frac{\varphi}{1 + \sqrt{1-\psi}}P\,, \quad \text{where } \psi(x) = u\varphi(x)\,;$$

$(5.8)_1, (5.8)_2$ $$\mathcal{N} = \mathcal{N}' = \omega P\,, \quad \text{where } \omega(x) = \frac{1}{2u} Y \log\frac{1}{1-\psi}\,;$$

$(5.9)_1, (5.9)_2$ $$\mathcal{R} = \chi P\,, \quad \text{where } \chi(x) = Y\omega + 3\omega - u\omega^2\,;$$

(5.10) $$\mathcal{H}^{-1} = I + \frac{\varphi}{1-\psi}P\,.$$

ψ, ω and χ are of class C^∞ in Ω.

REMARK 5.1. A question is how to obtain φ when χ is given. An answer is the following. If we have a solution ω to $(5.9)_2$ of class C^∞ in a neighborhood of $x = 0$, then

(5.11) $$u\varphi(x) = 1 - \exp\left\{-2u\int_0^1 \omega(sx)s\,ds\right\}\,.$$

$(5.9)_2$ is a Riccati type equation for ω. If we set

$$\lambda(t) = t\exp\left\{-u\int_0^t \omega(sx)s\,ds\right\}\,,$$

then $(5.9)_2$ is transformed to a Sturm-Liouville type equation for λ:

$(5.9)_3$ $$\frac{\partial^2\lambda}{\partial t^2} = -u\chi(tx)\lambda\,.$$

$(5.9)_2$ is integrated by quadrature if and only if $(5.9)_3$ is so. If $\chi = \alpha$ is a constant for example, then $\omega = -2\alpha\sigma'(\alpha u)/\sigma(\alpha u)$, $u\varphi = 1 - \sigma(\alpha u)^2$ and g is of constant curvature (see (5.4)).

REMARK 5.2. If φ in (5.6) depends only on u and if the Taylor series at $u = 0$ contains a non-vanishing term, there exists a neighborhood of $x = 0$ in which an equality

(5.12) $$\mathcal{H} = I - \hat{\varphi}(u)\mathcal{R}$$

holds, where $\hat{\varphi}$ is a function of class C^∞ in an open interval containing $u = 0$.

PROOF. Let the Taylor series of φ, ω, χ at $u = 0$ be

$$\varphi = \sum_{j=0}^N u^j\varphi_j + O(|u|^{N+1})\,, \quad \omega = \sum_{j=0}^N u^j\omega_j + O(|u|^{N+1})\,, \quad \chi = \sum_{j=0}^N u^j\chi_j + O(|u|^{N+1})\,.$$

From (5.8), (5.9), we see that $\mathcal{N}_{(m)} = \mathcal{R}_{(m)} = O$ if m is odd and

$$\mathcal{N}_{(2j)} = \omega_j u^{j-1} P\,, \quad \mathcal{R}_{(2j)} = \chi_j u^{j-1} P \quad (j \geq 1)\,. \tag{5.13}$$

Now, if r is the smallest number j such that $\varphi_j \neq 0$ and if r is finite, then $\omega - (r+1)\varphi_r u^r$ and $\chi - (2r^2 + 5r + 3)\varphi_r u^r$ are of $O(|u|^{r+1})$. If we set $\hat{\varphi} = \varphi/\chi$, then $\hat{\varphi}$ is of class C^∞ in a neighborhood of $u = 0$ and $\hat{\varphi}(0) = (2r^2 + 5r + 3)^{-1}$. We have (5.12) with this $\hat{\varphi}$. q.e.d.

REMARK 5.3. Contrary to Remark 5.2, suppose that all derivatives of φ vanish at $u = 0$. Then $g_{ab} - \gamma_{ab}$, ${\Gamma_s}^b{}_a$ and $R^b{}_{ras}$ are of $O(|x|^\infty)$. ω, χ are also of $O(|u|^\infty)$ and φ/χ may not be defined on the set $\{u = 0\}$. So, (5.12) does no longer hold. Also, if φ is not a function of single variable, (5.12) does no longer hold in general.

REMARK 5.4. Let $g^{(0)}$ be a conformally flat metric $e^{2f(v)}\gamma_{ab}dy^a dy^b$ in a neighborhood of $y = 0$ in cartesian coordinates y, where $v = \gamma_{ab}y^a y^b$ and $f(v)$ is a function of class C^∞ in a neighborhood of $v = 0$ such that $f(0) = 0$. Let x be the normal coordinate system for $g^{(0)}$ with origin at $y = 0$ such that $x^a = y^a + O(|y|^2)$. Then, $\gamma_{ab}x^a x^b$ depends only on $\gamma_{ab}y^a y^b$ and the metric tensor of $g^{(0)}$ with respect to x is of the form (5.6).

PROOF. Let $\lambda = \lambda(u)$ be a solution of class C^1 in a neighborhood of $u = 0$ to

$$2u\frac{d\lambda}{du} + \lambda = \exp\{-f(u\lambda^2)\} \quad \text{or} \quad \lambda(u) = \int_0^1 \exp\{-f(\theta u \lambda(\theta u)^2)\}\frac{d\theta}{\sqrt{4\theta}}\,. \tag{5.14}$$

Note that λ is unique, of class C^∞ and $\lambda(0) = 1$. Next, let $U = U(v)$ be the function satisfying

$$v = U\lambda(U)^2\,, \quad U(0) = 0\,. \tag{5.15}$$

We have $U'(0) = 1$. Given $(y^1, \dots, y^n)$ in a neighborhood of $y = 0$, define $(x^1, \dots, x^n)$ to be

$$x^a = y^a/\lambda(U(\gamma_{pq}y^p y^q))\,. \tag{5.16}$$

The mapping $y \to x$ is of class C^∞ in a neighborhood of $y = 0$, $x^a = y^a + O(|y|^3)$ and $\gamma_{ab}x^a x^b = U(\gamma_{pq}y^p y^q)$. Define $\varphi(u)$ to be

$$\varphi(u) = \{1 - \lambda(u)^2 e^{2f(u\lambda(u)^2)}\}/u \quad \text{for } u \neq 0 \quad \text{and} \quad \varphi(0) = -4f'(0)/3\,. \tag{5.17}$$

Then, φ is of class C^∞ in a neighborhood of $u = 0$ and $\exp\{2f(\gamma_{pq}y^p y^q)\}\gamma_{ab}dy^a dy^b = g_{ab}dx^a dx^b$, where g_{ab} is given in (5.6) with this φ. q.e.d.

6. Power series expansions.

6.1. In this section, we represent $\mathcal{H}_{(m)}$, $(\mathcal{H}^{-1})_{(m)}$ and $(\det \mathcal{H}^\lambda)_{(m)}$ by means of $\{\mathcal{R}_{(r)}\}$ (see Definition 3.1). First of all, we reproduce two lemmas and a proposition proved by Günther.

LEMMA 6.1 ([Gü1], [Gü2, Appendix 1]). *$\mathcal{R}_{(0)} = \mathcal{R}_{(1)} = O$ and*

$$(6.1)\qquad \begin{aligned}&(A, B)\text{-element of } \mathcal{R}_{(m)}\\ &= \frac{1}{(m-2)!} x^{j_1} \cdots x^{j_m} (\nabla_{j_1} \cdots \nabla_{j_{m-2}} R^B{}_{j_{m-1} A j_m})(0) \quad (m \geq 2)\,.\end{aligned}$$

6.2. Covariant metric tensor $g_{ab}(x)$.

LEMMA 6.2 ([Gü1], [Gü2, Appendix 1]). *$\mathcal{S}_{(0)} = I$, $\mathcal{S}_{(1)} = O$ and*

$$(6.2)\qquad \mathcal{S}_{(m)} = \sum_{h=1}^{[m/2]} \sum_{r_j \geq 2, r_1+\cdots+r_h=m} (-1)^h a_{r_1 r_2 \cdots r_h} \mathcal{R}_{(r_1)} \mathcal{R}_{(r_2)} \cdots \mathcal{R}_{(r_h)} \quad (m \geq 2)\,,$$

where $a_{r_1 r_2 \cdots r_h}$ are the coefficients defined in $(a.7)$ with $\kappa = 1$.

PROOF. (2.5) is the same as $(a.4')$ if we set $\mathcal{V} = \mathcal{S}$, $\mathcal{B} = -\mathcal{R}$ and $\kappa = 1$. So, $(a.7)$ implies (6.2). Note that $\Psi_1 = \Psi(0) = O$. q.e.d.

Define furthermore $\{b_{r_1 r_2 \cdots r_h}\}$ in the following way with $\kappa = 1$:

$$(6.3)\qquad \begin{aligned}&b_r = 2a_r\,,\\ &b_{r_1 r_2 \cdots r_h} = b_{r_h \cdots r_2 r_1} = a_{r_1 r_2 \cdots r_h} + a_{r_h \cdots r_2 r_1} + \textstyle\sum_{m=1}^{h-1} a_{r_1 r_2 \cdots r_m} a_{r_h \cdots r_{m+1}} \quad (h \geq 2)\,.\end{aligned}$$

PROPOSITION 6.1 ([Gü1], [Gü2, Appendix 1]). *$G_{(0)} = \gamma$, $G_{(1)} = O$, and*

$$(6.4)\qquad G_{(m)} = \sum_{h=1}^{[m/2]} \sum_{r_j \geq 2, r_1+\cdots+r_h=m} (-1)^h b_{r_1 r_2 \cdots r_h} \mathcal{R}_{(r_1)} \mathcal{R}_{(r_2)} \cdots \mathcal{R}_{(r_h)} \gamma \quad (m \geq 2)\,.$$

PROOF. This follows from (2.3), (6.2) and $\gamma\, {}^t\mathcal{R} = \mathcal{R}\gamma$. q.e.d.

(6.4) is very practical because $b_{r_1 r_2 \cdots r_h}$ are explicitly defined.

6.3. Powers of $\det(g_{ac}(x) g^{cb}(0))$. First, homogeneous parts of $\mathcal{N}$ are expanded by means of those of $\mathcal{R}$.

PROPOSITION 6.2. *$\mathcal{N}_{(0)} = \mathcal{N}_{(1)} = O$ and*

$$(6.5)\qquad \mathcal{N}_{(m)} = \sum_{h=1}^{[m/2]} \sum_{r_j \geq 2, r_1+\cdots+r_h=m} c_{r_1 r_2 \cdots r_h} \mathcal{R}_{(r_1)} \mathcal{R}_{(r_2)} \cdots \mathcal{R}_{(r_h)} \quad (m \geq 2)\,,$$

where $c_{r_1 r_2 \cdots r_h}$ are the coefficients defined in $(a.13)$ with $\kappa = 1$.

PROOF. (2.4) is the same as $(a.8')$ if we set $\mathcal{C} = \mathcal{R}$ and $\kappa = 1$. Since $\Gamma_j{}^a{}_k(0) = 0$, we have $\mathcal{N}_{(0)} = \mathcal{N}_{(1)} = O$. And $(a.12)$ implies (6.5). q.e.d.

PROPOSITION 6.3. *Denote $\theta(x) = \log\det(g_{ac}(x) g^{cb}(0))_{a,b=1}^n$ and by $\theta_{(m)}$ the homogeneous part of degree m in the Taylor series at $x = 0$. Then,*

$$(6.6)\qquad \theta_{(0)} = \theta_{(1)} = 0\,, \quad \theta_{(m)} = -\frac{2}{m} \operatorname{tr} \mathcal{N}_{(m)} \quad (m \geq 2)\,.$$

PROOF. Since $Y\mathcal{H} = -2\mathcal{N}'\mathcal{H}$ by (2.2), we have $Y\theta = -2\operatorname{tr}\mathcal{N}' = -2\operatorname{tr}\mathcal{N}$, so

$$\theta(x) = -2\int_0^1 \operatorname{tr}\mathcal{N}(tx)\frac{dt}{t}$$

and we have (6.6). q.e.d.

Now, we expand $(\det(GG(0)^{-1}))^{\lambda} = e^{\lambda\theta}$ for any complex number λ.

PROPOSITION 6.4. *Let $D^{(\lambda)}_{(m)}$ be the homogeneous part of degree m in the Taylor series of $\{\det(g_{ac}(x)g^{cb}(0))^{n}_{a,b=1}\}^{\lambda}$ at $x = 0$. Then, $D^{(\lambda)}_{(0)} = 1$, $D^{(\lambda)}_{(1)} = 0$ and*

$$D^{(\lambda)}_{(m)} = \sum_{\alpha_j \geq 0, 2\alpha_2+3\alpha_3+\cdots+m\alpha_m=m} \frac{1}{\alpha_2!\cdots\alpha_m!}(\lambda\theta_{(2)})^{\alpha_2}(\lambda\theta_{(3)})^{\alpha_3}\cdots(\lambda\theta_{(m)})^{\alpha_m} \quad (m \geq 2)\,. \tag{6.7}$$

This can be applied to the volume form ($\lambda = 1/2$) and also to the 0-th Hadamard coefficient of the Laplace-Beltrami operator ($\lambda = -1/4$).

REMARK 6.1. $Y\theta = -2\mathrm{tr}\mathcal{N}'$ due to a well-known equality $\partial_a\theta = 2\Gamma_a{}^s{}_s$. However, (6.7) is obtained not from (6.4) but from (2.2) and (2.4).

6.4. Contravariant metric tensor $g^{ab}(x)$.

LEMMA 6.3. $(\mathcal{S}^{-1})_{(0)} = I$, $(\mathcal{S}^{-1})_{(1)} = O$, *and*

$$(\mathcal{S}^{-1})(m) = \sum_{h=1}^{[m/2]} \sum_{r_j \geq 2, r_1+\cdots+r_h=m} d_{r_h\cdots r_2 r_1}\mathcal{N}_{(r_1)}\mathcal{N}_{(r_2)}\cdots\mathcal{N}_{(r_h)} \quad (m \geq 2)\,, \tag{6.8}$$

where $d_{r_h\cdots r_2 r_1}$ are the coefficients defined in (a.3).

PROOF. Equation $Y(\mathcal{S}^{-1}) = \mathcal{N}\mathcal{S}^{-1}$ is of type (a.1′) if we set $\mathcal{U} = \mathcal{S}^{-1}$, $\mathcal{A} = \mathcal{N}$ but $\mathcal{U}$, $\mathcal{A}$ are in reverse order. So, (a.3) yields (6.8), where $\Phi_1 = O$ because $\mathcal{N}_{(0)} = \mathcal{N}_{(1)} = O$. q.e.d.

Define furthermore $\{e_{r_1 r_2\cdots r_h}\}$ in the following way.

$$\begin{aligned} &e_r = 2d_r\,, \\ &e_{r_1 r_2\cdots r_h} = e_{r_h\cdots r_2 r_1} = d_{r_1 r_2\cdots r_h} + d_{r_h\cdots r_2 r_1} + \sum_{m=1}^{h-1} d_{r_1 r_2\cdots r_m} d_{r_h\cdots r_{m+1}} \quad (h \geq 2)\,. \end{aligned} \tag{6.9}$$

PROPOSITION 6.5. $(G^{-1})_{(0)} = \gamma^{-1}$, $(G^{-1})_{(1)} = O$, *and*

$$(G^{-1})_{(m)} = \sum_{h=1}^{[m/2]} \sum_{r_j \geq 2, r_1+\cdots+r_h=m} e_{r_1 r_2\ldots r_h}\gamma^{-1}\mathcal{N}_{(r_1)}\mathcal{N}_{(r_2)}\cdots\mathcal{N}_{(r_h)} \quad (m \geq 2)\,. \tag{6.10}$$

PROOF. This follows immediately from (6.8), $G^{-1} = {}^t\mathcal{S}^{-1}\gamma^{-1}\mathcal{S}^{-1}$ and $\mathcal{N}\gamma = \gamma\,{}^t\mathcal{N}$. q.e.d.

REMARK 6.2. We can represent $(G^{-1})_{(m)}$ by means of $\{\mathcal{R}_{(r)}\}$ if we combine (6.10) with (6.5). Since $c_{r_1 r_2\cdots r_h}$, $d_{r_1 r_2\cdots r_h}$, $e_{r_1 r_2\cdots r_h}$ are positive, $(G^{-1})_{(m)}$ is represented as a sum of products of $\mathcal{R}_{(r)}$ with positive coefficients. We can also represent $(G^{-1})_{(m)}$ by making use of (6.4) and $\gamma G^{-1} = I + (I - G\gamma^{-1}) + (I - G\gamma^{-1})^2 + \cdots$.

REMARK 6.3. To compute $c_{r_1 r_2 \cdots r_h}$ appearing in (6.5) for large h from $(a.13)$, we have a difficulty of combinatorial character because (2.4) is non-linear. However, once we know $\{\mathcal{N}_{(r)}\}$, all expansions are done in a unified way. We can modify (6.10) to have a formula for $G_{(m)}$. We have only to multiply $(-1)^h$, delete γ^{-1} and multiply γ from the right in the summand.

7. An algebraic structure of $V_{(2)}$.

7.1. By Lemma 3.1, $\mathcal{N}$, $\mathcal{R}$, $\mathcal{H} - I$ belong to V, and $\mathcal{S}_{(m)} \in V_{(m)}$ for $m = 2, 3, 4$ (see the end of Section 5.1).

LEMMA 7.1. *$V_{(0)} = V_{(1)} = \{O\}$. If $p \geq 2$, we have*

$$\dim V_{(p)} = \frac{np - n}{2} \binom{n + p - 1}{p + 1}. \tag{7.1}$$

PROOF. $V_{(0)} = \{O\}$ because a constant matrix $\mathcal{A}$ annihilates all x if and only if $\mathcal{A} = O$. Next, given an $\mathcal{A} \in V_{(1)}$, $\mathcal{A}(x)\gamma$ is symmetric. Linearizing $x\mathcal{A}(x)\gamma = 0$, we have $x\mathcal{A}(y)\gamma + y\mathcal{A}(x)\gamma = 0$ for all $x, y \in \boldsymbol{R}^n$. So, $y\mathcal{A}(x)\gamma\, {}^t y = 0$ for all x, y. Then, any eigenvalue of $\mathcal{A}(x)\gamma$ is zero. This implies that $\mathcal{A}(x)\gamma = O$ for all x, therefore $V_{(1)} = \{O\}$.

Suppose that $p \geq 2$. Any element of $V_{(p)}$ is written as $\mathcal{A} = \sum_{j=1}^{N} \varphi_{(j)} e_{(j)} \gamma^{-1}$. Here, $\{e_{(j)}\}_{j=1}^{N}$ is a basis of $n \times n$ symmetric real matrices ($N = (n^2 + n)/2$) and $\varphi_{(j)} \in H_{(p)}$, where $H_{(p)}$ is the vector space of homogeneous polynomials of degree p in n variables, so

$$\dim H_{(p)} = d_p = \binom{n + p - 1}{p}.$$

For each a, let $\psi_{(a)} = \sum_{j=1}^{N} l_{(a,j)} \varphi_{(j)}$ be the a-th component of $x\mathcal{A}$. $l_{(a,j)}$ is a linear form of x, non-vanishing for at least one j if a is fixed. If $\psi_{(a)} = 0$, coefficients of $\varphi_{(j)}$ satisfy some d_{p+1} linear conditions. When a runs from 1 to n, we have nd_{p+1} conditions. Therefore, $\dim V_{(p)} = Nd_p - nd_{p+1}$ and we have (7.1). q.e.d.

7.2. Let $W_{(2)}$ be the vector space consisting of sets of real numbers $\{\rho_{hjkl}\}$ satisfying

$$\rho_{hjkl} = \rho_{klhj} = -\rho_{jhkl}, \quad \rho_{hjkl} + \rho_{hklj} + \rho_{hljk} = 0 \tag{7.2}$$

for all h, j, k, l from 1 to n. The components below are linearly independent:

$$\begin{gathered} \rho_{pqpq} \quad \text{for } 1 \leq p < q \leq n; \quad \rho_{prqr} \quad \text{for } 1 \leq p, q, r \leq n,\ p < q,\ p \neq r \neq q; \\ \rho_{pqrs},\ \rho_{prqs} \quad \text{for } 1 \leq p < q < r < s \leq n. \end{gathered} \tag{7.3}$$

So, $\dim W_{(2)} = \dim V_{(2)} = (n^4 - n^2)/12$. Let us find an isomorphism from $W_{(2)}$ to $V_{(2)}$.

T_oM has the canonical basis $\{\varepsilon_a\}_{a=1}^{n}$ and the scalar product $g_o(u, v) = \gamma_{ab} u^a v^b$ ($\varepsilon_a = (\partial_a)_o$, $u = u^a \varepsilon_a$). Let $\Lambda^2 = \Lambda^2(T_oM)$ be the space of bivectors endowed with the scalar product $\langle\,,\rangle$ which is the linear extension of

$$\langle u \wedge v, z \wedge w \rangle = g_o(u, z) g_o(v, w) - g_o(u, w) g_o(v, z).$$

Let $\mathcal{Q}$ be the vector space of symmetric endomorphisms of Λ^2 ($\langle Qf, f'\rangle = \langle f, Qf'\rangle$ if $Q \in \mathcal{Q}$). Given $(Q, f) \in \mathcal{Q} \times \Lambda^2$, define an $n \times n$ matrix $Q^+(f)$ to be such that

$$g_o(wQ^+(f), z) = \langle Qf, z \wedge w\rangle \quad (z, w \in T_oM)\,.$$

Note that the mapping $Q \in \mathcal{Q} \mapsto Q^+ \in \mathrm{Hom}(\Lambda^2, \mathrm{End}(T_oM))$ is linear and injective.

Let $\mathcal{Q}^{(b)}$ be the vector space which consists of $Q \in \mathcal{Q}$ satisfying the first Bianchi identity

$$wQ^+(u \wedge v) + uQ^+(v \wedge w) + vQ^+(w \wedge u) = 0 \quad \text{for all } u, v, w \in T_oM\,. \tag{7.4}$$

Since $\dim \mathcal{Q} = (n^2 - n)/4 \times (n^2 - n + 2)/2$ and (7.4) implies $\binom{n}{4}$ conditions,

$$\dim \mathcal{Q}^{(b)} = \dim \mathcal{Q} - \binom{n}{4} = \frac{n^4 - n^2}{12}\,.$$

LEMMA 7.2. *$\mathcal{Q}^{(b)}$ is isomorphic to $W_{(2)}$ and also to $V_{(2)}$.*

PROOF. (see Willmore [Wi], §3.11). A bijection $W_{(2)} \leftrightarrow \mathcal{Q}^{(b)}$ is given by

$$\langle Q(\varepsilon_k \wedge \varepsilon_l), \varepsilon_h \wedge \varepsilon_j\rangle = \rho_{hjkl}\,. \tag{7.5}$$

Next, given a $Q \in \mathcal{Q}^{(b)}$, we define $\mathcal{A} \in V_{(2)}$ to be $\mathcal{A}(x) = xQ^+(\cdot \wedge x)$ or

$$u\mathcal{A}(x) = xQ^+(u \wedge x)\,. \tag{7.6}$$

(We identify the coordinate vector x with $x^r\varepsilon_r \in T_oM$). The inverse mapping $\mathcal{A} \to Q$ is given by

$$wQ^+(u \wedge v) = (2/3)\{u\mathcal{A}(v, w) - v\mathcal{A}(u, w)\}\,, \tag{7.7}$$

where $\mathcal{A}(u, v) = (1/2)\{\mathcal{A}(u+v) - \mathcal{A}(u) - \mathcal{A}(v)\} = \mathcal{A}(v, u)$. Q is well-defined and $Q \in \mathcal{Q}^{(b)}$. This is because $g_o(z\mathcal{A}(u, v), w) = g_o(u\mathcal{A}(z, w), v)$ owing to $x\mathcal{A}(x) = 0$. q.e.d.

An explicit basis of $V_{(2)}$ is given as follows. Set $E_{(p,q)} = (\delta_{ap}\delta_{bq} + \delta_{aq}\delta_{bp})_{a,b=1}^n$ and

$$\begin{aligned} \mathcal{A}_{(p,q,r,s)} = \ & (2x^px^rE_{(q,s)} + 2x^qx^sE_{(p,r)} - x^px^qE_{(r,s)} - x^rx^sE_{(p,q)} \\ & -x^qx^rE_{(p,s)} - x^px^sE_{(q,r)})\gamma^{-1}\,. \end{aligned} \tag{7.8}$$

$\mathcal{A}_{(p,q,r,s)}$ belongs to $V_{(2)}$ for all p, q, r, s from 1 to n. Since $\mathcal{R}_{(2)} = -3G_{(2)}\gamma^{-1}$, we have

PROPOSITION 7.1. (i) *If $\{\rho_{hjkl}\} \in W_{(2)}$ and if the homogeneous part of degree 2 of a metric tensor is such that*

$$\begin{aligned} -3G_{(2)}\gamma^{-1} = \ & \sum(\rho_{pqrs}\mathcal{A}_{(p,q,r,s)} + \rho_{prqs}\mathcal{A}_{(p,r,q,s)}) + \sum \frac{\rho_{prqr}}{2}\mathcal{A}_{(p,r,q,r)} \\ & + \sum \frac{\rho_{pqpq}}{2}\mathcal{A}_{(p,q,p,q)}\,, \end{aligned} \tag{7.9}$$

summations being extended over ranges of indices in the list (7.3), *then the curvature tensor of this metric satisfies $R_{hjkl}(o) = \rho_{hjkl}$.*

(ii) *$R_{hjkl}(o)$ are represented by means of $\mathcal{R}_{(2)}$ in the following way.*

$$R_{hjkl}(o) = (2/3)g_o(\varepsilon_k\mathcal{R}_{(2)}(\varepsilon_l, \varepsilon_j) - \varepsilon_l\mathcal{R}_{(2)}(\varepsilon_k, \varepsilon_j), \varepsilon_h)\,. \tag{7.10}$$

For the proof, we have only to remark that $\langle Qf, f\rangle = 6f^{p,q}f^{r,s} - 6f^{p,s}f^{q,r}$ if $\mathcal{R}_{(2)} = A_{(p,q,r,s)}$, where $f \in \Lambda^2$, $f^{p,q} = \langle f, \varepsilon^p \wedge \varepsilon^q\rangle$ and $\varepsilon^p = \gamma^{pa}\varepsilon_a$.

7.3. Let us show a structure of V from a viewpoint of real analysis for the case $n = 2, 3, 4$. An arbitrary element $\mathcal{A}$ of V is of the form

$$\mathcal{A}\gamma = \sum_{a=1}^{n} f_a E_{(a,a)} - \sum_{1 \le p < q \le n} h_{pq} E_{(p,q)}, \quad h_{qp} = h_{pq}, \tag{7.11}$$

where f_a, h_{pq} are real-valued functions of class C^∞. Condition $x\mathcal{A} = 0$ is equivalent to

$$x^a \text{ divides } \sum_{p(\neq a)} x^p h_{pa} \quad \text{and} \quad f_a = \sum_{p(\neq a)} x^p h_{pa}/x^a \quad (1 \le a \le n). \tag{7.12}$$

If $n = 2$, there exists a function φ which we can prescribe arbitrarily such that

$$h_{12} = x^1 x^2 \varphi, \quad f_1 = (x^2)^2 \varphi, \quad f_2 = (x^1)^2 \varphi. \tag{7.13}$$

Therefore, V is isomorphic to $C^\infty_{x^1,x^2}$.

If $n = 3$, let (a, b, c) be any one of $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$. Then,

$$\begin{aligned} h_{bc} &= x^b x^c \varphi_a(x^b, x^c) + x^a(x^b \psi_b + x^c \psi_c - x^a \psi_a), \\ f_a &= (x^b)^2 \varphi_c(x^a, x^b) + (x^c)^2 \varphi_b(x^c, x^a) + 2x^b x^c \psi_a, \end{aligned} \tag{7.14}$$

where φ_a and ψ_a are of two and three variables, respectively, which we can prescribe arbitrarily. So, V is isomorphic to the direct sum of $C^\infty_{x^2,x^3}$, $C^\infty_{x^3,x^1}$, $C^\infty_{x^1,x^2}$ and three spaces $C^\infty_{x^1,x^2,x^3}$.

If $n = 4$, then f_a and h_{pq} are of the following form:

$$\begin{aligned} f_a &= \sum_{p(\neq a)} (x^p)^2 \varphi_{ap} + \sum_{p(\neq a)} \sum_{q(\neq a,p)} x^p x^q \theta_{a,pq} + 2(x^1x^2x^3x^4/x^a)\omega_a, \\ h_{pq} &= x^p x^q \varphi_{pq} + \sum_{l(\neq p,q)} x^l (x^p \theta_{p,ql} + x^q \theta_{q,pl} - x^l \theta_{l,pq}) \\ &\quad + x^r x^s (2x^p \omega_p + 2x^q \omega_q - x^r \omega_r - x^s \omega_s + \kappa_{pq}). \end{aligned} \tag{7.15}$$

Here, $\varphi_{pq}(x^p, x^q)$ $(= \varphi_{qp})$ are functions of two variables, $\theta_{p,qr}(x^p, x^q, x^r)$ $(= \theta_{p,rq})$ are of three variables and ω_a, κ_{pq} are of four variables. κ_{pq} satisfy $\kappa_{pq} = \kappa_{qp} = \kappa_{rs}$ and $\kappa_{pq} + \kappa_{pr} + \kappa_{ps} = 0$ if p, q, r, s are distinct. (p, q, r, s are distinct in the last sum on the right hand side of the equation for h_{pq}). We can prescribe φ_{pq}, $\theta_{p,qr}$, ω_a and κ_{12}, κ_{13} arbitrarily. Therefore, V is isomorphic to the direct sum of six spaces of functions of two variables, twelve spaces of functions of three variables and six spaces of functions of four variables.

REMARK 7.1. Kowalski and Belger [KB] obtained a very precise result in the real analytic framework by taking into account of all commutators as $[\nabla_h, \nabla_j]$, $[[\nabla_h, \nabla_j], \nabla_k]$, $\cdots$. We might say that our V is a C^∞ version of tensors $\{R^{(k)}\}_{k=0}^\infty$ introduce in [KB]. Commutator relations are implicit in our framework.

A. Preliminaries on ordinary and partial differential equations. Let $\mathcal{M}_n$ be the ring of $n \times n$ real matrices as above and $C^k\mathcal{M}_n$ $(0 \le k \le \infty)$ be the set of functions of t with values in $\mathcal{M}_n$ of class C^k in an open interval containing $t = 0$. We do not specify the interval of t in what follows.

Let Ω be a star-shaped open neighborhood of $x=0$ in $\boldsymbol{R}^n$. Denote by $C^k(\Omega, \mathcal{M}_n)$ the set of functions of x of class C^k in Ω with values in $\mathcal{M}_n$.

A.1. Given a $\Phi \in C^0\mathcal{M}_n$, find an element U of $C^1\mathcal{M}_n$ which satisfies

$$U'(t) = U(t)\Phi(t)\,, \quad U(0) = I\,. \tag{$a.1$}$$

This is interpreted as an integral equation of Volterra type. The series

$$U = I + J[I] + J^2[I] + \cdots \quad \left(J[U](t) = \int_0^t U(s)\Phi(s)ds\right)$$

converges, $\|U(t)\| \le \exp\{|t|\varphi(|t|)\}$ ($\varphi(|t|) = \sup\{\|\Phi(s)\|; -|t| \le s \le |t|\}$), and it is a unique solution to $U = I + J[U]$ in the space $C^0\mathcal{M}_n$. So, it is a unique solution to (a.1) in the space $C^1\mathcal{M}_n$. We can verify that $U \in C^{k+1}\mathcal{M}_n$ if $\Phi \in C^k\mathcal{M}_n$ ($0 \le k \le \infty$).

The next question is to write down the Taylor series of U assuming that of Φ. We set

$$\Phi(t) = \sum_{m=0}^N t^m \Phi_{m+1} + O(|t|^{N+1})\,, \quad U(t) = \sum_{m=0}^N t^m U_m + O(|t|^{N+1})\,.$$

The recurrence relation for $\{U_m\}$ is as follows:

$$U_0 = I\,, \quad U_m = \frac{1}{m}\sum_{\mu=0}^{m-1} U_\mu \Phi_{m-\mu} \quad (m \ge 1)\,. \tag{$a.2$}$$

PROPOSITION A.1. *We have $U_0 = I$ and for $m \ge 1$*

$$U_m = \sum_{h=1}^m \sum_{r_j \ge 1, r_1 + \cdots + r_h = m} d_{r_1 r_2 \cdots r_h} \Phi_{r_1}\Phi_{r_2}\cdots\Phi_{r_h}\,, \tag{$a.3_1$}$$

where

$$d_{r_1, r_2 \cdots r_h} = 1 \Big/ \prod_{\mu=1}^h \sum_{j=1}^\mu r_j\,. \tag{$a.3_2$}$$

This is proved by induction with respect to m.

A.1′. Given an $\mathcal{A} \in C^k(\Omega, \mathcal{M}_n)$ ($k \ge 1$) such that $\mathcal{A}(0) = O$, find an element $\mathcal{U}$ of $C^k(\Omega, \mathcal{M}_n)$ which satisfies

$$Y\mathcal{U}(x) = \mathcal{U}(x)\mathcal{A}(x)\,, \quad \mathcal{U}(0) = I\,. \tag{$a.1'$}$$

Let an element $\mathcal{U}$ of $C^1(\Omega, \mathcal{M}_n)$ be a solution to (a.1′). For $x \in \Omega$ and $0 \le t \le 1$, we set $U(t,x) = \mathcal{U}(tx)$, $\Phi(t,x) = (1/t)\mathcal{A}(tx)$. Then $U(\cdot, x)$ is a unique solution to (a.1) in the space $C^1\mathcal{M}_n$ and it is of class C^k with respect to x in Ω provided that $\mathcal{A} \in C^k(\Omega, \mathcal{M}_n)$ and $\mathcal{A}(0) = O$. So, $\mathcal{U}$ is a unique solution to (a.1′) in the space $C^k(\Omega, \mathcal{M}_n)$ ($k \ge 1$).

A.2. Given a $\Psi \in C^1\mathcal{M}_n$ and a real number κ greater than -1, find an element V of $C^2\mathcal{M}_n$ which satisfies

$$tV''(t) + (1+\kappa)V'(t) = V(t)\Psi(t)\,, \quad V(0) = I\,. \tag{$a.4$}$$

This is rewritten as an integral equation $V = I + K[V]$, where

$$K[V](t) = t\int_0^1 k_\kappa(\theta)V(\theta t)\Psi(\theta t)d\theta \quad \text{with} \quad k_\kappa(\theta) = \frac{1-\theta^\kappa}{\kappa} \text{ if } \kappa \neq 0, \quad k_0(\theta) = \log\frac{1}{\theta}.$$

A scalar equation $tv''(t) + (1+\kappa)v'(t) = v(t)$, $v(0) = 1$, has the solution

$$(a.5) \qquad v(t) = \Gamma(1+\kappa)t^{-\frac{\kappa}{2}}I_\kappa(\sqrt{4t}) = \sum_{j=0}^{\infty} t^j\Gamma(1+\kappa)/(j!\Gamma(j+1+\kappa)),$$

where $I_\kappa(z)$ is the modified Bessel function (see [Wa]). The series

$$V = I + K[I] + K^2[I] + \cdots$$

converges, $\|V(t)\| \le v(|t|\psi(|t|))$ $(\psi(|t|) = \sup\{\|\Psi(s)\|; -|t| \le s \le |t|\})$, and it is a solution to $V = I + K[V]$, which is unique in the space $C^0\mathcal{M}_n$. So, it is a unique solution to $(a.4)$ in the space $C^2\mathcal{M}_n$. Furthermore, $V \in C^{k+1}\mathcal{M}_n$ if $\Psi \in C^k\mathcal{M}_n$ $(1 \le k \le \infty)$. We set

$$\Psi(t) = \sum_{m=0}^{N} t^m\Psi_{m+1} + O(|t|^{N+1}), \qquad V(t) = \sum_{m=0}^{N} t^m V_m + O(|t|^{N+1}).$$

The recurrence relation for $\{V_m\}$ is as follows.

$$(a.6) \qquad V_m = \frac{1}{m^2+\kappa m}\sum_{\mu=1}^{m} V_{m-\mu}\Psi_\mu \quad (m \ge 1).$$

PROPOSITION A.2 ([Gü1], [Gü2, Appendix 1]).

$$(a.7_1) \qquad V_0 = I, \quad V_m = \sum_{h=1}^{m}\ \sum_{r_j\ge 1, r_1+\cdots+r_h=m} a_{r_1r_2\cdots r_h}\Psi_{r_1}\Psi_{r_2}\cdots\Psi_{r_h} \quad (m \ge 1),$$

where

$$(a.7_2) \qquad a_{r_1r_2\cdots r_h} = \prod_{\mu=1}^{h}\lambda\left(\sum_{j=1}^{\mu} r_j\right), \quad \lambda(r) = \frac{1}{r^2+\kappa r}.$$

This is verified also by induction. We find neither $(a.3)$ nor $(a.7)$ in any textbook of ordinary differential equations. (6.2) is equivalent to $(a.7)$, which must be a useful discovery by Günther.

A.2$'$. Given a $\mathcal{B} \in C^k(\Omega, \mathcal{M}_n)$ $(k \ge 2)$ such that $\mathcal{B}(0) = O$ and a real number κ greater than -1, find an element $\mathcal{V}$ of $C^k(\Omega, \mathcal{M}_n)$ which satisfies

$$(a.4') \qquad YY\mathcal{V}(x) + \kappa Y\mathcal{V}(x) = \mathcal{V}(x)\mathcal{B}(x), \quad \mathcal{V}(0) = I.$$

Let $\mathcal{V}$ of $C^2(\Omega, \mathcal{M}_n)$ be a solution to $(a.4')$. For $x \in \Omega$ and $0 \le t \le 1$, we set $V(t, x) = \mathcal{V}(tx)$, $\Psi(t, x) = (1/t)\mathcal{B}(tx)$. Then $V(\cdot, x)$ is a unique solution to $(a.4)$ in the space $C^2\mathcal{M}_n$, which is of class C^k with respect to x in Ω provided that $\mathcal{B} \in C^k(\Omega, \mathcal{M}_n)$ and $\mathcal{B}(0) = O$. So, $\mathcal{V}$ is a unique solution to $(a.4')$ in the space $C^k(\Omega, \mathcal{M}_n)$ $(k \ge 2)$. See Section A.3$'$ for detail on higher order smoothness.

A.3. Given a $P \in C^1\mathcal{M}_n$ such that $P(0) = O$ and a real number κ greater than -1, find an element N of $C^1\mathcal{M}_n$ satisfying

$$(a.8)\qquad tN'(t) + \kappa N(t) = P(t) + N(t)^2, \quad N(0) = O.$$

In the linear theory of regular singular points, we assume any condition on neither $P(0)$ nor $N(0)$ (see Sibuya [Si]). However then, $N(0)$ is not determined because of the non-linearity. We are interested only in the case where $P(0) = N(0) = O$. N satisfies an integral equation

$$(a.9)\qquad N = L[P + N^2], \quad \text{where} \quad L[Q](t) = \int_0^1 \theta^{\kappa-1} Q(\theta t) d\theta.$$

Scalar equation $tn'(t) + \kappa n(t) = t + n(t)^2$, $n(0) = 0$, has the solution (see $(a.5)$)

$$(a.10)\qquad n(t) = \sqrt{t} J_{\kappa+1}(\sqrt{4t})/J_\kappa(\sqrt{4t}) = tv'(-t)/v(-t) = \sum_{p=1}^{\infty} \frac{4t}{c_p^2 - 4t},$$

where $\{c_p\}_{p=1}^{\infty}$ are positive zeros of $J_\kappa(z)$ (see [Wa, p. 498, equality (3)]). Taylor series of $n(t)$ is of positive coefficients and convergent if $|t| < c_1^2/4$. Set

$$N_0 = L[P], \quad N_{j+1} = L[P + N_j^2], \quad N(t) = \lim_{j\to\infty} N_j(t).$$

Then, $N(t)$ converges, $\|N(t)\| \le n(|t|p(|t|))$ ($p(|t|) = \sup\{\|P'(s)\|; -|t| \le s \le |t|\}$), and it is a unique solution to $(a.9)$ in the space $C^1\mathcal{M}_n$ as long as $|t|p(|t|) < c_1^2/4$. So, it is a unique solution to $(a.8)$ in the space $C^1\mathcal{M}_n$. Uniqueness follows from Lemma 2.2. Furthermore, $N \in C^k\mathcal{M}_n$ if $P \in C^k\mathcal{M}_n$ $(1 \le k \le \infty)$.

We set

$$P(t) = \sum_{m=1}^{N} t^m P_m + O(|t|^{N+1}), \quad N(t) = \sum_{m=1}^{N} t^m N_m + O(|t|^{N+1}).$$

The recurrence relation for $\{N_m\}$ is the following.

$$(a.11)\qquad N_1 = \frac{1}{1+\kappa} P_1, \quad N_m = \frac{1}{m+\kappa}\left(P_m + \sum_{\mu=1}^{m-1} N_\mu N_{m-\mu}\right) \quad (m \ge 2).$$

N_m can be written as

$$(a.12)\qquad N_m = \sum_{h=1}^{m} \sum_{r_j \ge 1, r_1+\cdots+r_h = m} c_{r_1 r_2 \cdots r_h} P_{r_1} P_{r_2} \cdots P_{r_h} \quad (m \ge 1).$$

A formula for $c_{r_1 r_2 \cdots r_h}$ will be given in Proposition A.3 below. The notation in $(a.13)$ is defined in the following way. For the moment, a set of consecutive r integers is said to be an *interval of length* r. Given a positive integer h, we set $H_1 = \{q\}_{q=1}^{h}$. By induction with respect to h, we are going to construct a sequence $\Delta = \{H_p\}_{p=1}^{2h-1}$ of non-empty subintervals $H_2, \ldots, H_{2h-1}$ of H_1, and denote the set of all Δ's by $\mathcal{I}_h$.

If $h = 1$, we set $\varDelta = \{1\}$, $\mathcal{I}_1 = \{\{1\}\}$. If $h \geq 2$, we take an arbitrary positive integer j smaller than h, set $H_2 = \{q\}_{q=1}^{j}$ and $H_{2j+1} = \{q\}_{q=j+1}^{h}$. They are of lengths j and $h-j$, respectively. Let $\{H_p\}_{p=2}^{2j}$ be an element of $\mathcal{I}_j$ consisting of non-empty subintervals of H_2 obtained by induction hypothesis, and $\{H_p\}_{p=2j+1}^{2h-1}$ be an element of $\mathcal{I}_{h-j}$ consisting of non-empty subintervals of H_{2j+1} obtained by induction hypothesis. In this way, we obtain an element $\varDelta = \{H_p\}_{p=1}^{2h-1}$ of $\mathcal{I}_h$.

If $\varDelta \in \mathcal{I}_h$, then $\varDelta$ contains $\{1\}, \{2\}, \dots, \{h\}$. If $H_k \in \varDelta$ is of length r and $r \geq 2$, then $\{H_p\}_{p=k}^{k+2r-2} \in \mathcal{I}_r$ and $H_{k+1}, \dots, H_{k+2r-2}$ are non-empty subintervals of H_k. If $H_p, H_q \in \varDelta$ and $p < q$, then either H_q is a proper subinterval of H_p or $H_p \cap H_q = \emptyset$.

The number of $\varDelta$'s contained in $\mathcal{I}_h$ is equal to

$$\frac{(2h-2)!}{(h-1)!h!}.$$

PROPOSITION A.3. *For every* $(r_1, \dots, r_h)$ *with* $r_j \geq 1$,

$$c_{r_1 r_2 \cdots r_h} = \sum_{\varDelta \in \mathcal{I}_h} \prod_{H \in \varDelta} \left(\kappa + \sum_{j \in H} r_j \right)^{-1} = c_{r_h \cdots r_2 r_1}. \tag{$a.13$}$$

PROOF. $N_1 = (1/(\kappa + 1))P_1$ from $(a.9)$. Assume $(a.12)$ for all m and we prove $(a.12)$ by induction with respect to h. The term with $h = 1$ comes only from $L[P]$ on the right hand side of $(a.9)$. So, $c_r = 1/(\kappa + r)$ for all $r \geq 1$, proving $(a.13)$ for $h = 1$. Suppose that $h \geq 2$ and that $(a.13)$ is true up to $h - 1$. Terms with h indices come only from $L[N^2]$ and

$$c_{r_1 r_2 \cdots r_h} = \frac{1}{\kappa + r_1 + \cdots + r_h} \sum_{j=1}^{h-1} c_{r_1 r_2 \cdots r_j} c_{r_{j+1} \cdots r_h}.$$

Induction hypothesis applied to $c_{r_1 r_2 \cdots r_j}$ and $c_{r_{j+1} \cdots r_h}$ shows $(a.13)$ for h. q.e.d.

A.3$'$. Given a $\mathcal{C} \in C^k(\Omega, \mathcal{M}_n)$ $(k \geq 1)$ such that $\mathcal{C}(0) = O$ and a real number κ greater than -1, find an element $\mathcal{N}$ of $C^k(\Omega, \mathcal{M}_n)$ which satisfies

$$Y\mathcal{N}(x) + \kappa\mathcal{N}(x) = \mathcal{C}(x) + \mathcal{N}(x)^2, \quad \mathcal{N}(0) = O. \tag{$a.8'$}$$

There exists a star-shaped neighborhood Ω' of $x = 0$ contained in Ω such that $\mathcal{N}$ is unique in the space $C^1(\Omega', \mathcal{M}_n)$ and obtained by applying $(a.8)$ to $N(t, x) = \mathcal{N}(tx)$, $P(t, x) = \mathcal{C}(tx)$.

Let us verify that $\mathcal{N}$ is of class C^k. This is true possibly except at $x = 0$ because Y is hyperbolic except at $x = 0$. For a non-zero real number λ and a constant vector $a \in \boldsymbol{R}^n$, set $\mathcal{N}_\lambda(x, a) = \{\mathcal{N}(x + \lambda a) - \mathcal{N}(x)\}/\lambda$ and $\mathcal{C}_\lambda(x, a)$ analogously. $(a.9)$ implies that

$$\mathcal{N}_\lambda(x, a) = \int_0^1 \theta^k \{\mathcal{C}_{\theta\lambda}(\theta x, a) + \mathcal{N}_{\theta\lambda}(\theta x, a)\mathcal{N}(\theta x) + \mathcal{N}(\theta x)\mathcal{N}_{\theta\lambda}(\theta x, a) + \theta\lambda\mathcal{N}_{\theta\lambda}(\theta x, a)^2\} d\theta.$$

For any ε satisfying $0 < \varepsilon < (\kappa + 1)/2$, there exists a positive number δ such that $\|\mathcal{N}(x)\| < \varepsilon$ if $|x| < \delta$ because $\mathcal{N}(0) = O$. For $a, \varepsilon, \delta, x, \theta$ fixed, $\mathcal{C}_{\theta\lambda}(\theta x, a) \to a^s(\partial_s \mathcal{C})(\theta x)$ as $\lambda \to 0$ because $\mathcal{C}$ is of class C^1 by assumption. Contraction mapping argument shows that $\mathcal{N}_\lambda(x, a)$

remains bounded as long as $|\lambda|$ is small and has a limit as $\lambda \to 0$ (for a smaller δ if necessary). The limit is equal to $a^s \partial_s \mathcal{N}(x)$. So, $\mathcal{N}$ is of class C^1 in the ball $|x| < \delta$ and

$$\partial_s \mathcal{N}(x) = \int_0^1 \theta^\kappa \{(\partial_s \mathcal{C})(\theta x) + (\partial_s \mathcal{N})(\theta x)\mathcal{N}(\theta x) + \mathcal{N}(\theta x)(\partial_s \mathcal{N})(\theta x)\} d\theta\,,$$

or

$$Y \partial_s \mathcal{N} + (\kappa + 1)\partial_s \mathcal{N} = \partial_s \mathcal{C} + (\partial_s \mathcal{N})\mathcal{N} + \mathcal{N}(\partial_s \mathcal{N})\,.$$

We can repeat this procedure k times because κ simply augments by 1 each time we differentiate $\mathcal{N}$. Finally, $\mathcal{N} \in C^k(\Omega', \mathcal{M}_n)$ provided that $\mathcal{C} \in C^k(\Omega', \mathcal{M}_n)$ and $\mathcal{C}(0) = O$. Therefore, $\mathcal{N}$ is a unique solution to $(a.8')$ in the space $C^k(\Omega', \mathcal{M}_n)$.

A.4. Real analyticity of solutions. Suppose that we have a matrix-valued power series $\Phi(t) = \sum_{j=0}^\infty t^j \Phi_j$ and a scalar power series $\psi(t) = \sum_{j=0}^\infty t^j \psi_j$. We say that ψI is a majorant series of Φ and denote $\Phi \ll \psi I$ if $\psi_j \geq 0$ and if $\|\Phi_j\| \leq \psi_j$ for every j. If ψ converges absolutely in a disk $\{t \in \boldsymbol{C}; |t| < \rho\}$, then Φ converges in norm and it is holomorphic in this disk. And hence Φ is real analytic in the interval $-\rho < t < \rho$.

PROPOSITION A.4. *If $\mathcal{A}$ (or $\mathcal{B}$) is real analytic in a star-shaped neighborhood Ω of $x = 0$, then the solution $\mathcal{U}$ to $(a.1')$ (resp. $\mathcal{V}$ to $(a.4')$) is also real analytic in Ω. If $\mathcal{C}$ is real analytic in Ω, then the solution $\mathcal{N}$ to $(a.8')$ is real analytic in a star-shaped neighborhood Ω' of $x = 0$ contained in Ω.*

PROOF. We can find the solutions to equations $(a.1)$, $(a.4)$ and $(a.8)$ in individual cases involving a constant μ:

$$(a.14_1) \qquad \begin{gathered} U(t) = u_0(t)I \quad \text{if } \Phi(t) = \varphi_0(t)I\,, \quad V(t) = v_0(t)I \quad \text{if } \Psi(t) = \psi_0(t)I\,, \\ N(t) = n_0(t)I \quad \text{if } P(t) = p_0(t)I\,, \end{gathered}$$

where

$(a.14_2)$

$$\begin{gathered} u_0(t) = \frac{1}{1-\mu t}\,, \quad \varphi_0(t) = \frac{\mu}{1-\mu t}\,, \quad v_0(t) = \frac{1}{(1-\mu t)^{\kappa+1}}\,, \quad \psi_0(t) = \frac{a+bt}{(1-\mu t)^2}\,, \\ n_0(t) = \frac{\mu t}{1-\mu t}\,, \quad p_0(t) = \frac{(1+\kappa)\mu t}{1-\mu t}\,, \quad a = (1+\kappa)^2\mu\,, \quad b = (1+\kappa)\mu^2\,. \end{gathered}$$

Also, we remark that the coefficients $d_{r_1 \cdots r_h}$ in $(a.3)$, $a_{r_1 \cdots r_h}$ in $(a.7)$ and $c_{r_1 \cdots r_h}$ in $(a.12)$ are positive. Therefore, if μ is a positive number, we can prove that

$$(a.15) \qquad \begin{gathered} U(t) \ll u_0(t)I \quad \text{if } \Phi(t) \ll \varphi_0(t)I\,, \quad V(t) \ll v_0(t)I \quad \text{if } \Psi(t) \ll \psi_0(t)I\,, \\ N(t) \ll n_0(t)I \quad \text{if } P(t) \ll p_0(t)I\,. \end{gathered}$$

Next, given matrix functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, which are real analytic by hypothesis, for equations $(a.1')$, $(a.4')$, $(a.8')$, respectively, we can choose a positive number μ in such a way that

$$(a.16) \qquad \begin{gathered} \mathcal{A}(x) \ll \varphi_0(x^1 + \cdots + x^n)I\,, \quad \mathcal{B}(x) \ll \psi_0(x^1 + \cdots + x^n)I\,, \\ \mathcal{C}(x) \ll p_0(x^1 + \cdots + x^n)I\,, \end{gathered}$$

respectively. Applying $(a.15)$ to this, we see that

$$(a.17)\qquad \begin{aligned} \mathcal{U}(x) \ll u_0(x^1 + \cdots + x^n)I\,, \quad \mathcal{V}(x) \ll v_0(x^1 + \cdots + x^n)I\,, \\ \mathcal{N}(x) \ll n_0(x^1 + \cdots + x^n)I\,, \end{aligned}$$

respectively. Consequently, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{N}$ are real analytic in the star-shaped neighborhood $|x^1| + \cdots + |x^n| < 1/\mu$ of $x = 0$ (see Petrovsky [P]). q.e.d.

REMARK A.1. The domain of definition of $N(t)$ is in general smaller than that of $P(t)$ because of the non-linearity of $(a.8)$, although they coincide in $(a.14)$.

For a fixed p $(p \neq 0)$, suppose that $\mathcal{R}(tp)$ be of class C^∞ or real analytic in an interval $0 \leq t < T_1$, the solution $\mathcal{N}(tp)$ to (2.4) be of class C^∞ or real analytic in $0 \leq t < T_2$, and that T_1, T_2 be optimal. Naturally, $0 < T_2 \leq T_1$. On the other hand, $\mathcal{S}(tp)$ is of class C^∞ or real analytic in $0 \leq t < T_1$ because (2.5) is a linear equation, and $\mathcal{S}(tp)$ is non-singular for small t because $\mathcal{S}(0) = I$. From (2.1), $\mathcal{N}(tp) = -t\mathcal{S}(tp)^{-1}(\partial/\partial t)[\mathcal{S}(tp)]$. Therefore, $\det \mathcal{S}(T_2 p) = 0$ if $T_2 < T_1$. For example, $T_1 = +\infty$ for $v(-t)$ and $T_2 = c_1^2/4$ for $n(t)$ (see $(a.5)$, $(a.10)$). T_2 is the smallest positive zero of $v(-t)$ (see also (5.4)).

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