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TIMELIKE SURFACES WITH CONSTANT MEAN CURVATURE IN LORENTZ THREE-SPACE

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Abstract. A cyclic surface in the Lorentz-Minkowski three-space is one that is foliated by circles. We classify all maximal cyclic timelike surfaces in this space, obtaining different families of non-rotational maximal surfaces. When the mean curvature is a non-zero constant, we prove that if the surface is foliated by circles in parallel planes, then it must be rotational. In particular, we obtain all timelike surfaces of revolution with constant mean curvature.

1. Introduction and statement of results. In Euclidean 3-space, a *cyclic* surface M is one that is foliated by pieces of circles, that is, M is defined by a one-parameter family of pieces of circles (cf. [2, 3]). It has been known that cyclic surfaces of constant mean curvature are completely determined. In fact, if the mean curvature vanishes on M, that is, in the minimal case, Enneper proved that the planes of the foliation must be parallel. Moreover, it is known that M is either rotational and hence is the catenoid ([13]) or one of the surfaces discovered by Enneper and Riemann in the last century ([2, 3, 16]). If the mean curvature is non-zero, Nitsche showed that M is a surface of revolution ([15]). It should be remarked that surfaces of revolution with non-zero constant mean curvature were classified by Delaunay in 1841 ([1]). A historical note on cyclic surfaces in Euclidean 3-space with constant mean curvature can be found in Nitsche's book ([14]).

In the Lorentz-Minkowski three-dimensional space $\boldsymbol{L}^3$ we may define the concept of a cyclic surface in the same manner as in the case of Euclidean ambient. Then cyclic spacelike maximal surfaces have been classified in [6] and [8]. In fact, besides the rotational surfaces that were determined in [5], there exist examples of non-rotational cyclic surfaces. One family of these plays the same role as that by Riemann's surfaces in Euclidean space. When the mean curvature is a non-zero constant, the situation is very different from the maximal case and is studied by the present author in [7]. It is proved that if the planes of the foliation are spacelike, then they must be parallel. Moreover, in this case, M is rotational. It should also be remarked that spacelike surfaces of revolution with constant mean curvature were classified in [4], while the (spacelike or timelike) cyclic surfaces in $\boldsymbol{L}^3$ with constant Gauss curvature were studied in [10]. A report on cyclic hypersurfaces in different spaces, including $\boldsymbol{L}^3$, can be found in [9].

In the present paper we shall study the cyclic timelike surfaces of constant mean curvature in $\boldsymbol{L}^3$. Our study goes as follows. In Section 2, we first introduce basic notation and recall local theory of surfaces in $\boldsymbol{L}^3$. Furthermore, we define the concept of a circle in $\boldsymbol{L}^3$. In

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Section 3, we classify constant mean curvature timelike surfaces of revolution, while maximal timelike surfaces of revolution were determined in [17]. In Sections 4 and 5, we prove the main results that can be summarized as follows:

Let M be a cyclic maximal timelike surface in $\boldsymbol{L}^3$. Then M is either rotational or one of the surfaces described in Theorem 1 *in Section* 4.

Let M be a cyclic timelike surface in $\boldsymbol{L}^3$ with non-zero constant mean curvature. If the planes of the foliation are parallel, then M is a surface of revolution.

2. Notation and preliminaries. Let $\boldsymbol{L}^3$ be the three-dimensional Lorentz-Minkowski space, that is, the three-dimensional real vector space $\boldsymbol{R}^3$ with the metric

$$\langle\, , \rangle = (dx_1)^2 + (dx_2)^2 - (dx_3)^2 ,$$

where (x_1, x_2, x_3) denotes the canonical coordinates in $\boldsymbol{R}^3$. Let M be a surface. An immersion $\boldsymbol{X} : M \to \boldsymbol{L}^3$ of M into $\boldsymbol{L}^3$ is timelike if the induced metric on M is a Lorentzian metric on each tangent plane. This is equivalent to that the unit normal vector ν is spacelike at each point of M. The Gauss map ν of $\boldsymbol{X}$ then assigns to each point of M a point of the timelike Lorentz sphere $x_1^2 + x_2^2 - x_3^2 = 1$ defined as follows. If $\boldsymbol{X} = \boldsymbol{X}(u, v)$ is a parametrization of M, then the unit normal vector field ν on M is given by

$$\nu = \frac{\boldsymbol{X}_u \wedge \boldsymbol{X}_v}{|\boldsymbol{X}_u \wedge \boldsymbol{X}_v|} ,$$

where $\boldsymbol{X}_u = \partial \boldsymbol{X}/\partial u$, $\boldsymbol{X}_v = \partial \boldsymbol{X}/\partial v$ and $\wedge$ stands for the Lorentzian cross product of $\boldsymbol{L}^3$.

The metric $\langle\, , \rangle$ on each tangent plane of M is determined by the first fundamental form

$$I = \langle d\boldsymbol{X}, d\boldsymbol{X}\rangle = Edu^2 + 2Fdudv + Gdv^2 ,$$

with differentiable coefficients

$$E = \langle \boldsymbol{X}_u, \boldsymbol{X}_u\rangle , \quad F = \langle \boldsymbol{X}_u, \boldsymbol{X}_v\rangle , \quad G = \langle \boldsymbol{X}_v, \boldsymbol{X}_v\rangle .$$

Since $\boldsymbol{X}$ is timelike, we have

$$\det I = EG - F^2 < 0 .$$

The shape operator of the immersion is represented by the second fundamental form

$$II = -\langle d\nu, d\boldsymbol{X}\rangle = edu^2 + 2fdudv + gdv^2 ,$$

with differentiable coefficients

$$e = \langle \nu, \boldsymbol{X}_{uu}\rangle , \quad F = \langle \nu, \boldsymbol{X}_{uv}\rangle , \quad G = \langle \nu, \boldsymbol{X}_{vv}\rangle .$$

With this notation, the mean curvature H is expressed in the local coordinate $\boldsymbol{X}$ by

$$H = \frac{1}{2}\,\frac{eG - 2fF + gE}{EG - F^2} . \tag{1}$$

Denote by $[\, , ,]$ the determinant in $\boldsymbol{L}^3$, that is,

$$[v_1, v_2, w] = \langle v_1 \wedge v_2, w\rangle \quad \text{for } v_1, v_2, w \in \boldsymbol{L}^3 ,$$

and put

$$W = (-\det I)^{1/2} = \sqrt{F^2 - EG}\,.$$

Note that M being timelike means that W is a positive real number. With this notation, (1) reads as

$$(2) \qquad -2HW^3 = P := E[\boldsymbol{X}_u, \boldsymbol{X}_v, \boldsymbol{X}_{vv}] - 2F[\boldsymbol{X}_u, \boldsymbol{X}_v, \boldsymbol{X}_{uv}] + G[\boldsymbol{X}_u, \boldsymbol{X}_v, \boldsymbol{X}_{uu}]\,.$$

To end this section, we define the concept of a circle in the Lorentz-Minkowski space. Motivated by the Euclidean case, a *circle* in $\boldsymbol{L}^3$ is defined to be the orbit of a point p away from a straight-line l under the action of the group of rotations in $\boldsymbol{L}^3$ that leaves l pointwise fixed. The group of rotations in $\boldsymbol{L}^3$ is well-known. In fact, consider an orthonormal basis (e_1, e_2, e_3) of $\boldsymbol{L}^3$ such that $\langle e_3, e_3\rangle = -1$. Depending on the causal character of the axis l of revolution, we have:

1. The axis l is timelike. If e_3 spans l, then the group of rotations around l that fixes l pointwise is given by $\{R_\theta; \theta \in \boldsymbol{R}\}$, where

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The circles of $\boldsymbol{L}^3$ corresponding to this case are written as

$$(3) \qquad \alpha(s) = c + r((\cos s)e_1 + (\sin s)e_2)\,,$$

where $r \neq 0$ and $c \in l$.

2. The axis l is spacelike. Suppose that l is generated by e_1. In this case, the group of rotations determined by l is given by

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix}.$$

The circles obtained are devided into two types:

$$(4) \qquad \alpha(s) = c + r((\cosh s)e_2 + (\sinh s)e_3) \quad \text{type I}\,,$$

and

$$(5) \qquad \alpha(s) = c + r((\sinh s)e_2 + (\cosh s)e_3) \quad \text{type II}\,,$$

where $r \neq 0$ and $c \in l$.

3. The axis l is null. If l is defined by the vector $e_2 + e_3$, then the rotations defined by l are given by

$$R_\theta = \begin{pmatrix} 1 & \theta & -\theta \\ -\theta & 1-\theta^2/2 & \theta^2/2 \\ -\theta & -\theta^2/2 & 1+\theta^2/2 \end{pmatrix}.$$

The circles are described as

$$\alpha(s) = c + se_1 + \frac{rs^2}{2}(e_2 + e_3)\,, \tag{6}$$

with $r \neq 0$.

In the particular case that (e_1, e_2, e_3) is the canonical frame in $\boldsymbol{L}^3$, the circles defined in these three cases are Euclidean circles, hyperbolas or parabolas, respectively. Finally, a surface $M \subset \boldsymbol{L}^3$ is said to be *rotational* (also a *surface of revolution*) if M is invariant by a group of rotations of $\boldsymbol{L}^3$.

3. Timelike surfaces of revolution with constant mean curvature. In this section, we shall determine the timelike surfaces of revolution with constant mean curvature. When the axis of revolution is non-degenerate, these surfaces of revolution will be described in terms of elliptic functions, while if the axis is null, the corresponding parametrizations of the surfaces will be obtained explicitly. After a Lorentz motion, we assume that the axis of revolution is the x_3-axis, the x_1-axis or the $\{x_2 = x_3, x_1 = 0\}$-line depending on if the axis is a timelike, spacelike or null line, respectively. We discuss separately each of these three cases:

1. *Surfaces of revolution with timelike axis.* We take the u-parameter of the foliation as the parameter along the x_3-axis. So we can parametrize M in the form

$$X(u, v) = (r(u)\cos v, r(u)\sin v, u)\,, \tag{7}$$

where $r > 0$ is a smooth function. A computation in (2) yields $W = r\sqrt{1 - r'^2}$ and $P = r^2(-1 + r'^2 - rr'')$. Thus the timelike character of M is equivalent to $r'^2 < 1$. From (2), it follows that r satisfies the equation

$$2rH = \frac{1}{(1 - r'^2)^{1/2}} + \frac{rr''}{(1 - r'^2)^{3/2}}\,. \tag{8}$$

Integrating this equation, we get

$$\frac{1}{\sqrt{1 - r'^2}} = Hr + \frac{a}{r}\,, \quad a \in \boldsymbol{R}\,. \tag{9}$$

An numerical solution of this equation is given in Figure 1. Therefore, the function $r = r(u)$ in (7) is defined by the following elliptic integral

$$\int^r \sqrt{\frac{-r^2 + (a + Hr^2)^2}{(a + Hr^2)^2}}\, dr = u + b\,, \quad a, b \in \boldsymbol{R}\,.$$

In the case that $r' = 0$, Equation (8) gives $r = 1/(2H)$. In this situation, the surface obtained is the right circular cylinder $x_1^2 + x_2^2 = 1/(4H^2)$.

In the maximal case, $H = 0$, Equation (9) is transformed into

$$r'^2 = \frac{a^2 - r^2}{a^2}\,, \quad a \neq 0\,,$$

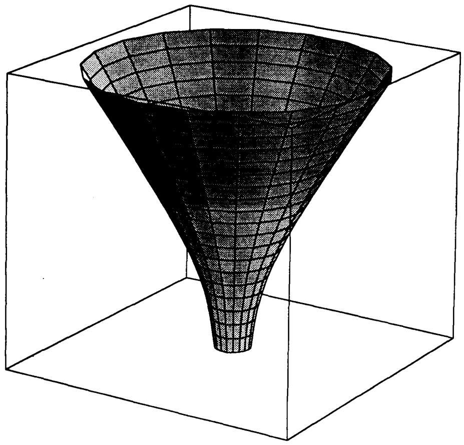

FIGURE 1. A timelike rotational surface with the x_3-line as axis of revolution. The mean curvature is $H = 1/2$, $r(0) = 1$, $r'(0) = 0.2$ and $0 \leq u \leq 10$.

whose solutions are given by

$$r(u) = a \sin\left(\frac{u+b}{a}\right), \quad a, b \in \boldsymbol{R}.$$

In this case, the parametrization of M is given by (see [17])

$$X(u, v) = \left(a \sin\left(\frac{u+b}{a}\right) \cos v, a \sin\left(\frac{u+b}{a}\right) \sin v, u\right).$$

To end the case that the axis is timelike, we characterize the right circular cylinder obtained when $r' = 0$.

PROPOSITION 1. *The cylinder $x_1^2 + x_2^2 = C^2$ is the only timelike surface of revolution with timelike axis and constant mean curvature in $\boldsymbol{L}^3$, which is also a constant mean curvature rotational surface with respect to the Euclidean metric.*

PROOF. After a homothety and a Lorentz motion of $\boldsymbol{L}^3$, we assume that the axis is the x_3-line and its parametrization is given by (7). Note that the parametrization (7) implies that M is a rotational surface with respect to the x_3-axis from the Euclidean viewpoint. A computation gives that the mean curvature h of M induced by the Euclidean metric of $\boldsymbol{R}^3$ is given by

$$2hr(1 + r'^2)^{3/2} = 1 + r'^2 - rr''. \tag{10}$$

To get a contradiction, assume that $r' \neq 0$ at some point. Around this point, we integrate Equation (10) to obtain

$$\frac{1}{\sqrt{1 + r'^2}} = hr + \frac{b}{r}, \quad b \in \boldsymbol{R}.$$

By combining this equation with (9), we have

$$\frac{r^2}{(a+Hr^2)^2}+\frac{r^2}{(b+hr^2)^2}=2\,.$$

This implies that r is constant, in contradiction with $r' \neq 0$. Thus, $r' = 0$ and the result follows. □

2. *Surfaces of revolution with spacelike axis.* A Lorentz motion allows us to assume that the axis of revolution is the x_1-axis. A surface of revolution with the x_1-line as axis is written in the following two forms (see (4) and (5)):

a. Surfaces of type I. In this case, the surface is parametrized by

$$X(u,v)=(u,r(u)\cosh v, r(u)\sinh v)\,, \tag{11}$$

where $r > 0$ is a differentiable function. Then, we have $W = r\sqrt{1+r'^2}$ and $P = r^2(-1-r'^2+rr'')$. By (2), we get the next equation for r:

$$2rH=\frac{1}{(1+r'^2)^{1/2}}-\frac{rr''}{(1+r'^2)^{3/2}}\,.$$

As a first step, we integrate this equation to get

$$\frac{1}{\sqrt{1+r'^2}}=Hr+\frac{a}{r}\,,\quad a\in \boldsymbol{R}\,. \tag{12}$$

which allows us to express the function $r = r(u)$ in (11) by the following elliptic integral:

$$\int^r\sqrt{\frac{r^2-(a+Hr^2)^2}{(a+Hr^2)^2}}\,dr=u+b\,,\quad a,b\in\boldsymbol{R}\,.$$

Figure 2 shows a numerical solution of this equation for $H = 1/2$. For instance, when $r' = 0$ in some interval, then $r = 1/2H$ and hence the surface obtained is the hyperbolic cylinder $x_2^2 - x_3^2 = 1/(4H^2)$ whose parametrization is given by (see Figure 3)

$$X(u,v)=\left(u,\frac{1}{2H}\cosh v,\frac{1}{2H}\sinh v\right)\,.$$

If we consider the maximal case, then (12) leads to

$$r'^2=\frac{r^2-a^2}{a^2}\,,\quad a\neq 0\,,$$

which has as solutions the functions

$$r(u)=a\cosh\left(\frac{u+b}{a}\right)\,,\quad a,b\in\boldsymbol{R}\,.$$

In this case, the parametrization obtained is (see [17]):

$$\boldsymbol{X}(u,v)=\left(u,\cosh\left(\frac{u+b}{a}\right)\cosh v,\cosh\left(\frac{u+b}{a}\right)\sinh v\right)\,.$$

b. Surfaces of type II. In this case, the parametrization of M is given by

$$\boldsymbol{X}(u,v)=(u,r(u)\sinh v, r(u)\cosh v)\,, \tag{13}$$

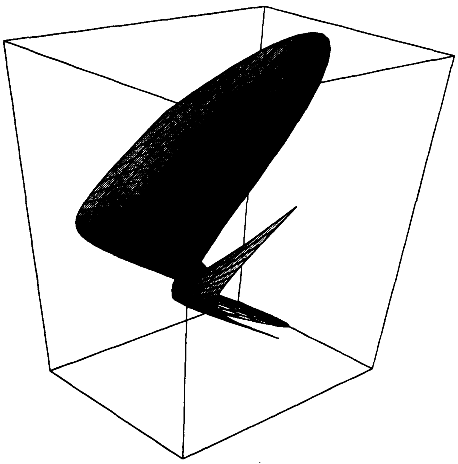

FIGURE 2. A timelike rotational surface of type I with the x_1-line as axis of revolution. The mean curvature is $H = 1/2, r(0) = 1, r'(0) = 1$ and $0 \le u \le 5$.

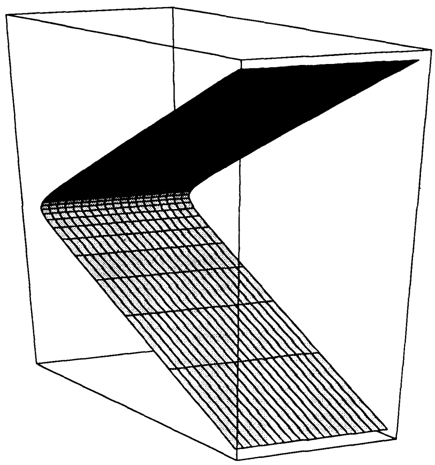

FIGURE 3. The surface $x_2^2 - x_3^2 = 1$: a timelike rotational surface with respect to the x_1-line and with $H = 1/2$.

where r is a differentiable function. A computation of the first and the second fundamental forms gives $W = r\sqrt{-1 + r'^2}$ and $P = r^2(1 - r'^2 + rr'')$. Since M is timelike, this implies that $r'^2 > 1$. Equation (2) gives

$$2rH = \frac{1}{(r'^2 - 1)^{1/2}} - \frac{rr''}{(r'^2 - 1)^{3/2}}.$$

Again, integrating this equation yields

$$\frac{1}{\sqrt{-1 + r'^2}} = Hr + \frac{a}{r}, \quad a \in \boldsymbol{R}.$$

Thus, the function $r = r(u)$ in (13) is defined by

$$\int^r \sqrt{\frac{r^2 + (a + Hr^2)^2}{(a + Hr^2)^2}}\, dr = u + b, \quad a, b \in \boldsymbol{R}.$$

See Figure 4 for a numerical solution of this equation. The maximal case reduces the equation to solve

$$r'^2 = \frac{r^2 + r^2}{a^2},$$

whose solutions are given by

$$r(u) = a \sinh\left(\frac{u + b}{a}\right), \quad a, b \in \boldsymbol{R}.$$

The parametrization of the corresponding timelike maximal surface is (see [17]):

$$X(u, v) = \left(u, \sinh\left(\frac{u + b}{a}\right) \sinh v, \sinh\left(\frac{u + b}{a}\right) \cosh v\right).$$

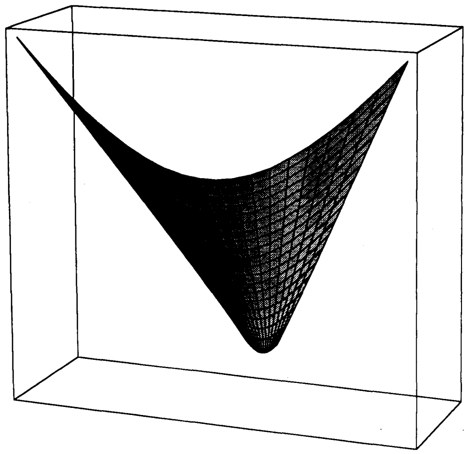

FIGURE 4. A timelike rotational surface of type II with the x_1-line as axis of revolution. The mean curvature is $H = 1/2$, $r(0) = 1$, $r'(0) = 2$ and $0 \leq u \leq 10$.

3. *Surfaces of revolution with null axis.* We consider the null line l of $\boldsymbol{L}^3$ defined by $x_1 = 0$, $x_2 - x_3 = 0$. A surface M which is rotational with respect to l is given by (see (6))

$$\boldsymbol{X}(u, v) = \left(v,\, g(u) + u + r(u)\frac{v^2}{2},\, g(u) - u + r(u)\frac{v^2}{2} \right),$$

where g and $r > 0$ are differentiable functions. The computation of W^2 and P gives $W^2 = 2(2r^2 - r')v^2 - 4g'$ and $P = (4rr' - r'')v^2 - 2g'' - 8rg'$. Now Equation (2) is a polynomial identity in v. The leading term is v^6 and from this equation we get $r' = 2r^2$. Solutions of this equation are given by

$$r(u) = \frac{1}{-2u + a}, \quad a \in \boldsymbol{R}.$$

Thus $W^2 = -4g'$. Since M is timelike, $g' < 0$ and (2) has the following expression:

$$g'' + \frac{4}{(-2u + a)} g' = H(-4g')^{3/2}, \quad g' < 0.$$

whose solutions are given by

$$g(u) = \frac{-2u + a}{4H(-b + 4aHu - 4Hu^2)} - \frac{\arctan\left(\sqrt{\dfrac{H}{b - a^2 H}}(2u - a)\right)}{4H^{3/2}\sqrt{b - a^2 H}} + c,$$

where $a, b, c \in \boldsymbol{R}$. Figure 5 shows an example for $H = 1/2$.

In the maximal case, we have

$$g'' + \frac{4}{(-2u + a)} g' = 0, \quad g' < 0,$$

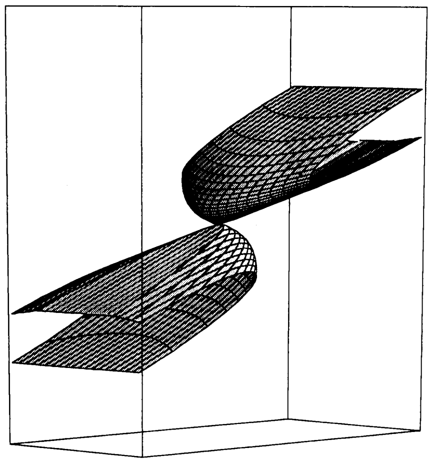

FIGURE 5. A timelike rotational surface with null line as axis of revolution. The mean curvature is $H = 1/2$. In this case, $r(u) = -1/(2u)$, $g(u) = 2u/(1+4u^2) - \arctan(2u)$ $(a = c = 0, b = 1/2)$.

and hence solutions are

$$g(u) = b + ca^2u - 2acu^2 + \frac{4c}{3}u^3 .$$

Notice that in this case, $W^2 = -4c(2u-a)^2 > 0$ and thus, this forces that $c < 0$.

4. Maximal timelike surfaces. In this section, we classify all maximal cyclic timelike surfaces of $\boldsymbol{L}^3$. The first assertion on this kind of surfaces is that the planes containing the circles of the foliation must be parallel. This fact is not a characteristic proper to the timelike surfaces. The same result holds also for spacelike surfaces. In fact, the proof uses only the property that the metric on the surface is *non-degenerate* (see [6]: essentially, it follows Enneper's ideas in [2], [3]).

PROPOSITION 2. *Let M be a cyclic maximal timelike surface in $\boldsymbol{L}^3$. Then the planes of the foliation are parallel.*

Now we are in a position to find all maximal cyclic timelike surfaces in $\boldsymbol{L}^3$. The following theorem is based on the calculation by Riemann (see [16] and [14]).

THEOREM 1. *Let M be a maximal timelike surface in $\boldsymbol{L}^3$ foliated by pieces of circles in parallel planes. Then either one of the following occurs*

1. *M is a surface of revolution.*
2. *M is determined by*

(a) *the parametrization* (20), *if the foliation planes are spacelike, or*

(b) *the parametrizations* (31) *and* (32), *if the foliation planes are timelike, or*

(c) *the parametrization* (33), *if the foliation planes are null, where f, g and r are described in the case* 3 *in the proof.*

PROOF. Since M is maximal, from (2) we have $P = 0$. We distinguish the cases by the causal character of the foliation planes as follows.

1. *The planes of the foliation are spacelike.* The parametrization of M is given by

$$X(u,v) = (f(u) + r(u)\cos v,\ g(u) + r(u)\sin v,\ u)\,, \tag{14}$$

where $r > 0$, and f and g are differentiable functions on u. A computation of P in (2) leads to

$$\begin{aligned} 0 &= r^2(2r'f' - rf'')\cos v + r^2(2r'g' - rg'')\sin v \\ &\quad + r^2(-1 + f'^2 + g'^2 + r'^2 - rr'')\,, \end{aligned} \tag{15}$$

which gives

$$2r'f' - rf'' = 0\,, \tag{16}$$

$$2r'g' - rg'' = 0\,, \tag{17}$$

$$-1 + f'^2 + g'^2 + r'^2 - rr'' = 0\,. \tag{18}$$

These differential equations can be solved in an analogous manner to that in the Euclidean case (see [14, p. 87]). A simple integration of (16) and (17) gives $f' = \lambda r^2$ and $g' = \mu r^2$ for some positive constants $\lambda, \mu \in \boldsymbol{R}$. Substituting these into (18), then yields

$$r^2(r^2)'' - [(r^2)']^2 - 2(\lambda^2 + \mu^2)r^6 + 2r^2 = 0\,. \tag{19}$$

Set now $x = (r^2)'$ and $y = y(x) = r^2$ as new independent and dependent variables. Then from (19) we get

$$xyy' - y^2 - 2(\lambda^2 + \mu^2)x^3 + 2x = 0\,,$$

whose solution is given by $y(x) = \pm\sqrt{x(4 + 4(\lambda^2 + \mu^2)x^2 + 8\delta x)}$, where $\delta \in \boldsymbol{R}$. Thus

$$r' = \pm\sqrt{(\lambda^2 + \mu^2)r^4 + 2\delta r^2 + 1}\,.$$

Take r as variable in the parametrization (14) of M. Then we get

$$f(t) = \lambda \int^t \frac{t^2}{\Delta} dt\,, \quad g(t) = \mu \int^t \frac{t^2}{\Delta} dt\,,$$

where

$$\Delta = \sqrt{(\lambda^2 + \mu^2)t^4 + 2\delta t^2 + 1}\,.$$

Therefore M is given by $X(t,v) = (x(t,v), y(t,v), z(t,v))$, where

$$\begin{aligned} x(t,v) &= a_0 + \lambda \int^t \frac{t^2}{\Delta} dt + t\cos v\,, \\ y(t,v) &= b_0 + \mu \int^t \frac{t^2}{\Delta} dt + t\sin v\,, \\ z(t,v) &= c_0 + \int^t \frac{dt}{\Delta}\,, \end{aligned} \tag{20}$$

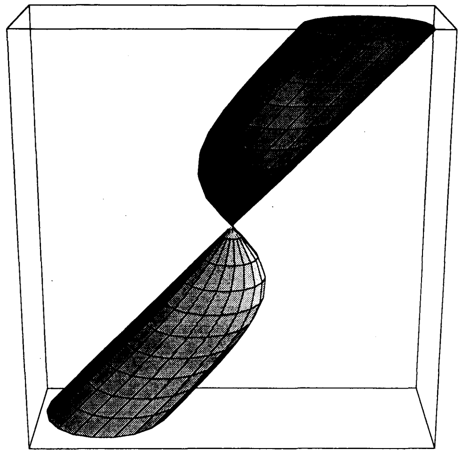

FIGURE 6. A maximal timelike non-rotational surface foliated by circles in spacelike planes. This surface corresponds with Example 1.

where a_0, b_0, c_0 are some constants. The surfaces of revolution studied in Section 3 are obtained now by putting $\lambda = \mu = 0$.

EXAMPLE 1. Put $a_0 = b_0 = c_0 = \lambda = 0$, $\mu = -\delta = 1$. Then $\Delta = \pm(t^2 - 1)$. In this case, $r(u) = \tanh(u + a)$, $f(u) = 0$ and $g(u) = b + u - \tanh(u + a)$. See Figure 6.

2. *The planes of the foliation are timelike.* Following (4) and (5), the surface M can be parametrized in two ways:

(21) $$X(u, v) = (u, f(u) + r(u)\cosh v, g(u) + r(u)\sinh v) \quad \text{type I},$$

(22) $$X(u, v) = (u, f(u) + r(u)\sinh v, g(u) + r(u)\cosh v) \quad \text{type II},$$

where $r > 0$, f and g are smooth functions.

For surfaces of type I, computing P in (2) yields

(23) $$\begin{aligned} 0 = {} & r^2(-2r'f' + rf'')\cosh v + r^2(2r'g' - rg'')\sinh v \\ & + r^2(-1 - f'^2 + g'^2 - r'^2 + rr''), \end{aligned}$$

which gives

(24) $$2r'f' - rf'' = 0,$$

(25) $$2r'g' - rg'' = 0,$$

(26) $$-1 - f'^2 + g'^2 - r'^2 + rr'' = 0.$$

Similarly, for surfaces of type II, we have

$$\begin{aligned} 0 &= r^2(-2g'r' + rg'')\cosh v + r^2(2r'f' - rf'')\sinh v \\ &\quad + r^2(1 + f'^2 - g'^2 - r'^2 + rr''), \end{aligned} \tag{27}$$

and

$$2r'g' - rg'' = 0, \tag{28}$$

$$2r'f' - rf'' = 0, \tag{29}$$

$$1 + f'^2 - g'^2 - r'^2 + rr'' = 0. \tag{30}$$

The procedure to solve these systems is analogous to that in solving (16) through (18). Notice that the first two equations (24) and (25) (resp. (28) and (29)) in these systems are identical with (16) and (17). The third equation (26) (resp. (30)) gives respectively

$$r^2(r^2)'' - [(r^2)']^2 + 2(-\lambda^2 + \mu^2)r^6 - 2r^2 = 0 \quad \text{type I},$$

$$r^2(r^2)'' - [(r^2)']^2 + 2(\lambda^2 - \mu^2)r^6 + 2r^2 = 0 \quad \text{type II}.$$

We can integrate both equations with respect to the variables x and $y = y(x)$ as in the previous case, and obtain

$$r' = \pm\sqrt{(\lambda^2 - \mu^2)r^4 + 2\delta r^2 - 1}, \quad \delta \in \boldsymbol{R} \quad \text{type I},$$

$$r' = \pm\sqrt{(-\lambda^2 + \mu^2)r^4 - 2\delta r^2 + 1}, \quad \delta \in \boldsymbol{R} \quad \text{type II}.$$

Put

$$\Delta = \sqrt{(\lambda^2 - \mu^2)t^4 + 2\delta t^2 - 1}.$$

Then the surface can be parametrized as

$$\begin{aligned} x(t, v) &= a_0 + \int^t \frac{dt}{\Delta}, \\ y(t, v) &= b_0 + \lambda \int^t \frac{t^2}{\Delta} dt + t\cosh v, \\ z(t, v) &= c_0 + \mu \int^t \frac{t^2}{\Delta} dt + t\sinh v, \end{aligned} \tag{31}$$

or

$$\begin{aligned} x(t, v) &= a_0 + \int^t \frac{dt}{\Delta}, \\ y(t, v) &= b_0 + \lambda \int^t \frac{t^2}{\Delta} dt + t\sinh v, \\ z(t, v) &= c_0 + \mu \int^t \frac{t^2}{\Delta} dt + t\cosh v, \end{aligned} \tag{32}$$

depending on if the surface is type I or type II, respectively. When $\lambda = \mu = 0$, the maximal surface is rotational (see Section 3).

EXAMPLE 2. Type I. Put $a_0 = b_0 = c_0 = 0$, $\lambda = \mu = 2\delta = 1$. Then we get

$$r(u) = \cosh(u+a)\,, \quad f(u) = g(u) = \frac{2u + \sinh 2u}{4} + b\,.$$

In this case, we have

$$\begin{aligned} W^2 =\, & \cosh^6 u(-\cosh v + \sinh v)^2 + \cosh^2 u\{1 + (\cosh^2 u + \cosh v \sinh u)^2 \\ & - (\cosh^2 u + \sinh u \sinh v)^2\}\,. \end{aligned}$$

EXAMPLE 3. Type II. Put $a_0 = b_0 = c_0 = \lambda = 0$, $\mu = -\delta = 1$. Then we get

$$r(u) = \tan(u+a)\,, \quad f(u) = \mathrm{cte}\,, \quad g(u) = -(u+a) + \tan(u+a) + b\,.$$

This surface is timelike for any value of the parameters. See Figure 8.

EXAMPLE 4. Type II. Put $a_0 = b_0 = c_0 = 0$, $\lambda = \mu = -2\delta = 1$. Then we get

$$r(u) = \sinh(u+a)\,, \quad f(u) = g(u) = \frac{-2u + \sinh 2u}{4} + \gamma\,.$$

In this case, we have

$$\begin{aligned} W^2 =\, & \sinh^6 u(\cosh v - \sinh v)^2 - \sinh^2 u\{1 - (\sinh^2 u + \cosh v \cosh u)^2 \\ & + (\sinh^2 u + \cosh u \sinh v)^2\}\,. \end{aligned}$$

3. *The planes of the foliation are null.* A parametrization of M is given by

$$X(u,v) = \left(f(u) + v,\ g(u) + r(u)\frac{v^2}{2},\ g(u) - u + r(u)\frac{v^2}{2} \right), \tag{33}$$

where f, g and $r > 0$ are differentiable functions. The calculation of P gives

$$0 = (4rr' - r'')v^2 + (4r'f' + 2rf'')v - 2rf'^2 - 8rg' - 2g''\,. \tag{34}$$

Therefore

$$r'' = 4rr'\,, \tag{35}$$

$$2r'f' + rf'' = 0\,, \tag{36}$$

$$rf'^2 + 4rg' + g'' = 0\,. \tag{37}$$

Equation (35) implies $r' = 2r^2 + a$ with $a \in \boldsymbol{R}$. Equation (36) is equivalent to

$$r^2 f' = b\,, \quad b \in \boldsymbol{R}\,.$$

When $b = 0$, the surface is rotational and it has been studied in Section 3. Thus, let us assume $b \neq 0$. Then (37) reads as

$$\frac{b^2}{r^3} + 4rg' + g'' = 0\,. \tag{38}$$

According to the sign of the constant a, we have the following:

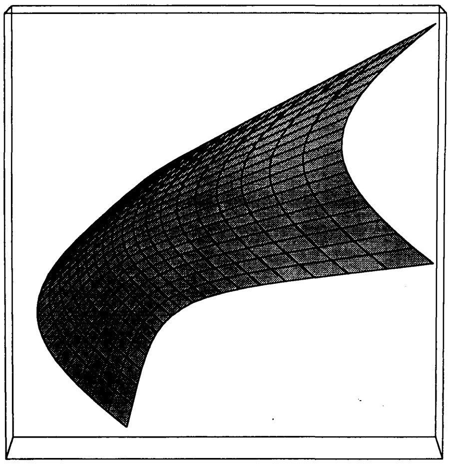

FIGURE 7. A maximal timelike non-rotational surface foliated by circles in timelike planes. This surface corresponds with Example 2.

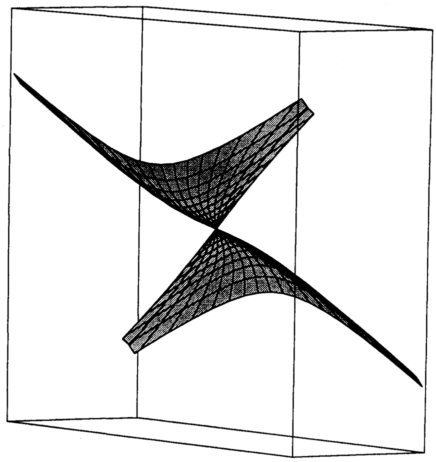

FIGURE 8. A maximal timelike non-rotational surface foliated by circles in timelike planes. This surface corresponds with Example 3.

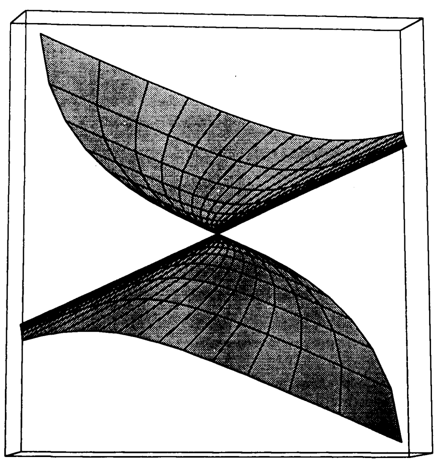

FIGURE 9. A maximal timelike non-rotational surface foliated by circles in timelike planes. This surface corresponds with Example 4.

1. $a = 0$. Then $r(u) = 1/(-2u + c)$, $c \in \boldsymbol{R}$. Equation (36) has a solution

$$f(u) = d + bc^2 u - 2bcu^2 + \frac{4}{3} bu^3, \quad d \in \boldsymbol{R}.$$

Integrating twice Equation (38), we have

$$g(u) = q + c^2 pu - \frac{c}{2}(b^2c^2 + 4p)u^2 + \frac{5b^2c^2 + 4p}{3} u^3 - 2b^2cu^4 + \frac{4}{5} b^2 u^5,$$

where $p, q \in \boldsymbol{R}$.

2. $a > 0$. Then we have

$$\begin{aligned} r(u) =& \sqrt{\frac{a}{2}}\tan(\sqrt{2a}(u+c))\,, \\ f(u) =& d - a^{-3/2}\operatorname{cosec}(\sqrt{2a}(u+c))\{\sqrt{2}b\cos(\sqrt{2a}(u+c)) \\ &+ 2\sqrt{a}b(u+c)\sin(\sqrt{2a}(u+c))\}\,, \\ g(u) =& q + \frac{1}{16a^{5/2}}\operatorname{cosec}(\sqrt{2a}(u+c))\{-9\sqrt{2}b^2\cos(\sqrt{2a}(u+c)) \\ &+ \sqrt{2}a^2 p\cos(\sqrt{2a}(u+c)) + \sqrt{2}(b^2 - pa^2)\cos(3\sqrt{2a}(u+c)) \\ &+ (8a^{5/2}p - 24\sqrt{a}b^2)(u+c)\sin(\sqrt{2a}(u+c))\}\,, \end{aligned}$$

where $c, d, p, q \in \boldsymbol{R}$.

3. $a < 0$. Then we have

$$\begin{aligned} r(u) =& \sqrt{\frac{-a}{2}}\tanh(\sqrt{-2a}(-u+c))\,, \\ f(u) =& d + a^{-3/2}\operatorname{cosech}(\sqrt{-2a}(u+c))(-\sqrt{2}b\cosh(\sqrt{-2a}(u+c)) \\ &+ 2\sqrt{-a}b(u+c)\sinh(\sqrt{-2a}(u+c)))\,, \\ g(u) =& q + \frac{1}{8a^{5/2}}\operatorname{cosech}(\sqrt{-2a}(u+c))\operatorname{sech}(\sqrt{-2a}(u+c))\{-\sqrt{2}b^2 - 2\sqrt{2}a^2 p \\ &- 3\sqrt{2}b^2\cosh(2\sqrt{-2a}(u+c)) + 2\sqrt{2}a^2 p\cosh(2\sqrt{-2a}(u+c)) \\ &+ 12\sqrt{-a}b^2(u+c)\sinh(2\sqrt{-2a}(u+c))\} \\ &- \frac{1}{a^{5/2}}2\sqrt{2}b^2\log(\sinh(\sqrt{-2a}(u+c))\tanh(\sqrt{-2a}(u+c))\,, \end{aligned}$$

where $c, d, p, q \in \boldsymbol{R}$.

EXAMPLE 5. Take $a = c = d = p = q = 0$ and $b = 1$. Then

$$r(u) = -\frac{1}{2u}\,, \quad f(u) = \frac{4}{3}u^3\,, \quad g(u) = \frac{4}{5}u^5\,.$$

In this case, we have $W^2 = -8u(2u^3 + v)$. Figure 10 shows this surface.

□

REMARK 1. We compare Theorem 1 with the case of the Euclidean ambient. Besides the rotational maximal surfaces (the catenoid), there exists a one-parameter family of non-rotational minimal surfaces discovered by Enneper and Riemann ([2, 3, 16]). In our case, we have obtained the surfaces of revolution and families of non-rotational maximal timelike surfaces. Moreover, the integration of Equations (16)–(17), (24)–(25) and (28)–(29) implies that the curve formed by the centers of the circles is contained in a plane. The same is true for surfaces of Riemann type in $\boldsymbol{R}^3$.

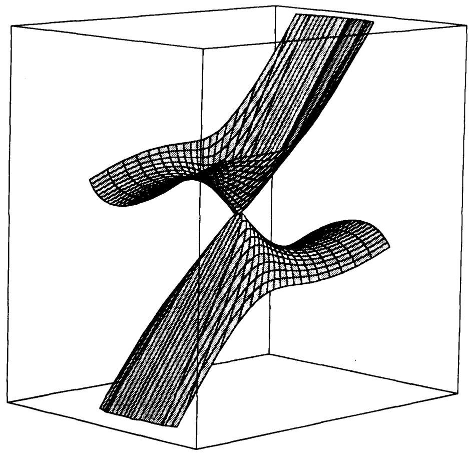

FIGURE 10. A maximal timelike non-rotational surface foliated by circles in null planes.

REMARK 2. The study of maximal spacelike surfaces in L^3 done in [6] employs the Weierstrass representation of the spacelike maximal surfaces ([5] and [12]). It should be remarked that for maximal timelike surfaces in L^3 there exists an analogous Weierstrass representation ([12] and [11]). However, our study on cyclic maximal timelike surfaces holds also for the spacelike case: the equation $P = 0$ in (2) is identical for both cases and the difference lies in the sign of $\det I = EG - F^2$. For instance, in Example 5, the surface M obtained is timelike (resp. spacelike) if the (u, v)-parameters lie in the region $\{u < 0, -2u^3 < v\} \cup \{u > 0, v < -2u^3\}$ (resp. $\{u < 0, v < -2u^3\} \cup \{u > 0, -2u^3 < v\}$).

5. Surfaces with non-zero constant mean curvature. In the case that the mean curvature is a non-zero constant, the content of Proposition 2 is different. More precisely, there exist timelike surfaces in L^3 with constant mean curvature which are foliated by circles in non-parallel planes. However, in this case, the surface is included in a surface of revolution. This phenomena also occurs in Euclidean 3-space, since the intersection between any smooth 1-parameter family of (not necessarily parallel) planes with an Euclidean sphere produces circles. The following fact was proved in [9].

PROPOSITION 3. *Let M be a timelike surface in L^3 with non-zero constant mean curvature which is foliated by pieces of circles. Then either the circles must be parallel or M is a subset of a Lorentz sphere.*

On the other hand, in contrast with the maximal case (see Theorem 1), we have the following result:

THEOREM 2. *Let M be a timelike surface in L^3 with non-zero constant mean curvature which is foliated by pieces of circles in parallel planes. Then M is a surface of revolution.*

PROOF. Without loss of generality, after a homothety of L^3, we may assume that the mean curvature of M is $H = 1/2$. Taking the square of (2), we have

$$4H^2W^6 = W^6 = P^2 . \tag{39}$$

Again, we discuss separately the following cases.

1. *The planes of the foliation are spacelike.* After a motion of L^3, a parametrization of M is given by (14). Equation (39) is a polynomial in $\sin nv$ and $\cos nv$. The value of P is given in (15) and a computation of W^2 yields

$$\begin{aligned} W^2 =& \frac{r^2}{2}(-f'^2 + g'^2)\cos 2v - (r^2 f'g')\sin 2v \\ & - (2r^2r'f')\cos v - (2r^2r'g')\sin v + r^2\left(1 - r'^2 - \frac{f'^2+g'^2}{2}\right). \end{aligned}$$

From the coefficients of $\cos 6v$ and $\sin 6v$ in (39) we get respectively

$$(f'^2 - g'^2)((f'^2 - g'^2)^2 - 12f'^2g'^2) = 0 ,$$
$$f'g'(-3(f'^2 - g'^2)^2 + 4f'^2g'^2) = 0 .$$

By using these equations, we obtain $f' = g' = 0$. Therefore the functions f and g are constant, proving that M is rotational.

2. *The planes of the foliation are timelike.* We consider the two possible parametrizations of M given in (21) and (22). First, we consider surfaces of type I. Then we get

$$\begin{aligned} W^2 =& \frac{r^2}{2}(f'^2 + g'^2)\cosh 2v - (r^2 f'g')\sinh 2v \\ & + (2r^2r'f')\cosh v - (2r^2r'g')\sinh v + r^2\left(1 + r'^2 + \frac{f'^2-g'^2}{2}\right). \end{aligned}$$

For surfaces of type II, we have

$$\begin{aligned} W^2 =& \frac{r^2}{2}(f'^2 + g'^2)\cosh 2v - (r^2 f'g')\sinh 2v \\ & + (2r^2r'g')\cosh v - (2r^2r'f')\sinh v + r^2\left(-1 + r'^2 + \frac{-f'^2+g'^2}{2}\right). \end{aligned}$$

Again, we use (39), where the values of P is calculated in (23) and (27). Equation (39) is now a hyperbolic trigonometric expression in $\sinh nv$ and $\cosh nv$. By a long computation of the coefficient of $\cosh 6v$ in both cases, we get

$$(f'^2 + g'^2)^3 + 12(f'^4g'^2 + f'^2g'^4) = 0 .$$

In conclusion, $f' = g' = 0$, i.e., M is a surface of revolution.

3. *The planes of the foliation are null.* After a motion in L^3, a parametrization is given by (33). The surface M is rotational provided the function f is constant. In our case, the calculation of W^2 yields

$$W^2 = 2(2r^2 - r')v^2 + 4rf'v - 4g' .$$

Then (39) is a polynomical equation in v. By using (34), from the leading coefficient, we get $r' = 2r^2$. With this data, the coefficients v^5 and v^4 become trivial. In conclusion, the coefficient of v^3 in (39) yields $64r^3 f'^3 = 0$. Therefore, $f' = 0$ and this implies that M is a surface of revolution. □

REFERENCES

[1] C. DELAUNAY, Sur la surface de révolution dont la courbure moyenne est constante, J. Math. Pures Appl. 6 (1841), 309–320.

[2] A. ENNEPER, Ueber die cyclischen Flächen, Nach. Königl. Ges. d. Wissensch. Göttingen, Math. Phys. Kl. (1866), 243–249.

[3] A. ENNEPER, Die cyklischen Flächen, Z. Math. Phys. 14 (1869), 393–421.

[4] J. I. HANO AND K. NOMIZU, Surfaces of revolution with constant mean curvature in Lorentz-Minkowski space, Tohoku Math. J. 36 (1984), 427–435.

[5] O. KOBAYASHI, Maximal surfaces in the 3-dimensional Minkowski space L^3, Tokyo J. Math. 6 (1983), 297–303.

[6] F. J. LÓPEZ, R. LÓPEZ AND R. SOUAM, Maximal surfaces of Riemann type in Lorentz-Minkowski space L^3, to appear in Michigan Math. J.

[7] R. LÓPEZ, Constant mean curvature surfaces foliated by circles in Lorentz-Minkowski space, Geom. Dedicata 76 (1999), 81–95.

[8] R. LÓPEZ, Constant mean curvature hypersurfaces foliated by spheres, Diff. Geom. Appl. 11 (1999), 245–256.

[9] R. LÓPEZ, Cyclic hypersurfaces of constant curvature, to appear in Advanced Studies in Pure Mathematics.

[10] R. LÓPEZ, Surfaces of constant Gauss curvature in Lorentz-Minkowski 3-space, Preprint, 1999.

[11] M. A. MAGID, The Bernstein problem for timelike surface, Yokohama Math. J. 37 (1989), 125–137.

[12] L. MCNERTNEY, One parameter families of surfaces with constant curvature in Lorentz 3-space, Ph. D. Thesis, Brown University, 1980.

[13] J. B. MEUSNIER, Mémoire sur la courbure des surfaces, Mém. des Savants étrangers 10 (1785), 504.

[14] J. C. C. NITSCHE, Lectures on Minimal Surfaces, Cambridge Univ. Press, Cambridge, 1989.

[15] J. C. C. NITSCHE, Cyclic surfaces of constant mean curvature, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. 1 (1989), 1–5.

[16] B. RIEMANN, Über die Flächen vom Kleinsten Inhalt be gegebener Begrenzung, Abh. Königl. Ges. d. Wissensch. Göttingen, Mathema. Kl. 13 (1868), 329–333.

[17] I. VAN DE WOESTIJNE, Minimal surfaces of the 3-dimensional Minkowski space, in Geometry and Topology of Submanifolds, II, World Scientific, Singapore, 1990.

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