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HOMOLOGICAL INFINITENESS OF DECORATED TORELLI GROUPS AND TORELLI SPACES

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Abstract. We prove that the rational homology of decorated Torelli groups and Torelli spaces are infinite dimensional when the genus of the reference surface is at least seven, thereby extended one of the main results of [2].

1. Introduction. The purpose of the present note is to extend results of [2] concerning the rational homology of Torelli groups and Torelli spaces. Let Σ_g be a closed orientable surface of genus g. Throughout this paper, we will assume $g \geq 2$. Let $\Gamma_{g,r}^n$ be the mapping class group of Σ_g relative to n marked points and r embedded disks. Namely it is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g which fix marked points and embedded disks pointwise. The action of $\Gamma_{g,r}^n$ on the first integral homology of Σ_g induces a surjective homomorphism $\Gamma_{g,r}^n \to Sp(2g, \boldsymbol{Z})$, where $Sp(2g, \boldsymbol{Z})$ is the Siegel modular group of degree g. The *Torelli group* $\mathcal{I}_{g,r}^n$ is defined to be its kernel so that we have an extension

$$1 \to \mathcal{I}_{g,r}^n \to \Gamma_{g,r}^n \to Sp(2g, \boldsymbol{Z}) \to 1 .$$

Henceforth we omit the decorations n and r when they are zero.

According to a result of Johnson [6] (see also [3]), one has

$$H_1(\mathcal{I}_{g,r}^n, \boldsymbol{Q}) \cong H_1(\Sigma_g, \boldsymbol{Q})^{\oplus(n+r)} \oplus \wedge^3 H_1(\Sigma_g, \boldsymbol{Q})/H_1(\Sigma_g, \boldsymbol{Q})$$

for $g \geq 3$. On the contrary, Mess [7] showed that $\mathcal{I}_2$ is a free group on infinitely many generators and that $H_3(\mathcal{I}_3, \boldsymbol{Z})$ contains a free abelian group of infinite rank (he attributes the latter result to Johnson and Millson). The author [2] showed that $H_*(\mathcal{I}_{g,r}^n, \boldsymbol{Q})$ is infinite dimensional if g is sufficiently large relative to $n+r$. In particular, $H_*(\mathcal{I}_g, \boldsymbol{Q})$, $H_*(\mathcal{I}_g^1, \boldsymbol{Q})$, and $H_*(\mathcal{I}_{g,1}, \boldsymbol{Q})$ are proved to be infinite dimensional for $g \geq 7$. For further results concerning the (co)homology of Torelli groups, see [5, 8] and references therein. In this note, we will prove:

THEOREM. *For $g \geq 7$ and $n, r \geq 0$, the rational homology $H_*(\mathcal{I}_{g,r}^n, \boldsymbol{Q})$ of the Torelli group $\mathcal{I}_{g,r}^n$ is infinite dimensional.*

Let $\mathcal{T}_{g,r}^n$ be the Teichmüller space of genus g with $n + r$ marked points and r tangent vectors. Namely it is the space of all conformal structures on Σ_g up to isotopies that fix $n + r$ marked points $p_1, \dots, p_{n+r}$ and act trivially on the tangent space $T_{p_i}\Sigma_g$ for $n+1 \leq i \leq n+r$. It is a complex manifold of dimension $3g - 3 + n + 2r$. The Torelli group $\mathcal{I}_{g,r}^n$ acts on $\mathcal{T}_{g,r}^n$

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freely so that the quotient space $T^n_{g,r} = \mathcal{T}^n_{g,r}/\mathcal{I}^n_{g,r}$ is a complex manifold which is called the *Torelli space*. Since $\mathcal{T}^n_{g,r}$ is contractible, $T^n_{g,r}$ is the Eilenberg-MacLane space $K(\mathcal{I}^n_{g,r}, 1)$ so that there is a canonical isomorphism $H_*(T^n_{g,r}, \boldsymbol{Z}) \cong H_*(\mathcal{I}^n_{g,r}, \boldsymbol{Z})$. As a consequence of the theorem, we obtain:

COROLLARY. *For $g \geq 7$ and $n, r \geq 0$, the rational homology $H_*(T^n_{g,r}, \boldsymbol{Q})$ of the Torelli space $T^n_{g,r}$ is infinite dimensional.*

The author thanks to Richard Hain for valuable comments. Theorem and Corollary have been announced in [1].

2. Proof.

PROPOSITION 1. *If $H_*(\mathcal{I}^n_g, \boldsymbol{Q})$ is infinite dimensional, then so is $H_*(\mathcal{I}^{n+1}_g, \boldsymbol{Q})$.*

Let $x_i \in \Sigma_g$ $(1 \leq i \leq n+1)$ be the marked points of the reference surface Σ_g and set $\Sigma^n_g = \Sigma_g \setminus \{x_1, x_2, \dots, x_n\}$. Then the Torelli groups $\mathcal{I}^{n+1}_g$ and $\mathcal{I}^n_g$ fit into an extension

$$1 \to \pi_1(\Sigma^n_g, x_{n+1}) \to \mathcal{I}^{n+1}_g \to \mathcal{I}^n_g \to 1 \tag{1}$$

(see [5]). We wish to apply the Hochschild-Serre spectral sequence to (1). To do so, the following lemma is required:

LEMMA. *Define a positive integer $l(g, n)$ by*

$$l(g, n) = \min\{i \in \boldsymbol{Z} \mid \dim_{\boldsymbol{Q}} H_i(\mathcal{I}^n_g, \boldsymbol{Q}) = \infty\}.$$

If $l(g, n)$ is defined, then

$$\dim_{\boldsymbol{Q}} H_i(\mathcal{I}^n_g, H_1(\Sigma^n_g, \boldsymbol{Q})) \begin{cases} < \infty & i < l(g, n), \\ = \infty & i = l(g, n). \end{cases}$$

PROOF. Let $\iota : \Sigma^n_g \hookrightarrow \Sigma_g$ be the inclusion and $\iota_* : H_1(\Sigma^n_g, \boldsymbol{Q}) \to H_1(\Sigma_g, \boldsymbol{Q})$ the induced homomorphism on the first homology. It follows from the definition of $\mathcal{I}^n_g$ that

$$H_i(\mathcal{I}^n_g, H_1(\Sigma_g, \boldsymbol{Q})) \cong H_i(\mathcal{I}^n_g, \boldsymbol{Q})^{\oplus 2g}. \tag{2}$$

If $n = 0$ or 1, then ι_* is an isomorphism of $\boldsymbol{Q}\mathcal{I}^n_g$-modules and there is nothing to prove. Henceforth we assume $n \geq 2$. Since ι_* is surjective, it yields a short exact sequence of $\boldsymbol{Q}\mathcal{I}^n_g$-modules:

$$0 \to \ker \iota_* \hookrightarrow H_1(\Sigma^n_g, \boldsymbol{Q}) \xrightarrow{\iota_*} H_1(\Sigma_g, \boldsymbol{Q}) \to 0. \tag{3}$$

For $1 \leq i \leq n$, choose a small circle $\gamma_i \subset \Sigma_g$ centered at x_i and let $[\gamma_i] \in H_1(\Sigma^n_g, \boldsymbol{Q})$ be the homology class represented by γ_i. Then $\ker \iota_*$ is generated by those $[\gamma_i]$. It is easy to see that the action of $\mathcal{I}^n_g$ on $H_1(\Sigma^n_g, \boldsymbol{Q})$ fixes each $[\gamma_i]$ and hence $\mathcal{I}^n_g$ acts on $\ker \iota_*$ trivially. Since $\dim_{\boldsymbol{Q}} \ker \iota_* = n - 1$, this implies

$$H_i(\mathcal{I}^n_g, \ker \iota_*) \cong H_i(\mathcal{I}^n_g, \boldsymbol{Q})^{\oplus(n-1)}. \tag{4}$$

In view of (2) and (4), the lemma follows from the long exact sequence

$$\cdots \to H_i(\mathcal{I}_g^n, \ker \iota_*) \to H_i(\mathcal{I}_g^n, H_1(\Sigma_g^n, \boldsymbol{Q})) \to$$
$$H_i(\mathcal{I}_g^n, H_1(\Sigma_g, \boldsymbol{Q})) \to H_{i-1}(\mathcal{I}_g^n, \ker \iota_*) \to \cdots$$

associated to the short exact sequence (3). □

PROOF OF PROPOSITION 1. Let us consider the Hochschild-Serre spectral sequence for the group extension (1):

$$E^2_{i,j} = H_i(\mathcal{I}_g^n, H_j(\Sigma_g^n, \boldsymbol{Q})) \Rightarrow H_{i+j}(\mathcal{I}_g^{n+1}, \boldsymbol{Q}) .$$

Write $l = l(g, n)$ for simplicity. Observe that $E^2_{l,0} = H_l(\mathcal{I}_g^n, \boldsymbol{Q})$ is infinite dimensional by the definition of l, while $E^2_{i,j}$ is finite dimensional for all $i < l$ and j by Lemma. It follows that $E^r_{l,0}$ is infinite dimensional for all $r \geq 2$. But the spectral sequence converges at the E^3-term when $n \geq 1$ and at the E^4-term when $n = 0$ by the dimensional reasons, hence verifying the proposition. □

The following proposition was proved in [2]:

PROPOSITION 2. *If $H_*(\mathcal{I}_g^n, \boldsymbol{Q})$ is infinite dimensional, then $H_*(\mathcal{I}_{g,r}^n, \boldsymbol{Q})$ is infinite dimensional for all $r \geq 0$.*

Propositions 1 and 2 imply the theorem, since $H_*(\mathcal{I}_g, \boldsymbol{Q})$ is infinite dimensional for $g \geq 7$ as was shown in [2].

REMARK. Since $H_*(\mathcal{I}_2, \boldsymbol{Q})$ and $H_*(\mathcal{I}_3, \boldsymbol{Q})$ are infinite dimensional, $H_*(\mathcal{I}_{2,r}^n, \boldsymbol{Q})$ and $H_*(\mathcal{I}_{3,r}^n, \boldsymbol{Q})$ are also infinite dimensional for all $n, r \geq 0$.

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