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TOTALLY REAL TOTALLY GEODESIC SUBMANIFOLDS OF COMPACT 3-SYMMETRIC SPACES

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Abstract. We prove that a half dimensional, totally real and totally geodesic submanifold of a compact Riemannian 3-symmetric space is expressed as an orbit of a Lie subgroup of the isometry group of the ambient manifold. Moreover, we associate such submanifolds with graded Lie algebras of the second kind.

1. Introduction. It has been known, for instance see Kobayashi and Nomizu [KNo], that a (complete) totally geodesic submanifold of a homogeneous Riemannian manifold is also homogeneous. In particular, a totally geodesic submanifold of a Riemannian symmetric space is expressed as an orbit of a Lie subgroup of the isometry group. However, unless the ambient manifold is a symmetric space, such property does not hold in general (cf. [To3]).

Let $(G/K, \langle\,,\rangle)$ be a Riemannian 3-symmetric space such that G is compact and $\langle\,,\rangle$ is a bi-invariant metric on G. The purpose of this paper is to describe a class of totally geodesic submanifolds which are expressed as orbits of Lie subgroups of the isometry group of $(G/K, \langle\,,\rangle)$. In particular, we shall prove that a half dimensional, totally real and totally geodesic submanifold of $(G/K, \langle\,,\rangle)$, with respect to the canonical almost complex structure, is expressed as an orbit of a Lie subgroup of G (see Proposition 3.2 and Proposition 3.3).

To be more precise, let $\mathfrak{g}^*$ be a noncompact simple Lie algebra and $\mathfrak{g}^* = \mathfrak{a} + \mathfrak{m}^*$ a Cartan decomposition, where $\mathfrak{a}$ is a Lie subalgebra. Take a gradation

$$\mathfrak{g}^* = \mathfrak{g}^*{}_{-1} + \mathfrak{g}^*{}_0 + \mathfrak{g}^*{}_1$$

of the first kind so that the characteristic element Z is contained in $\mathfrak{m}^*$. Then $\sigma = \mathrm{Ad}(\exp \pi\sqrt{-1}Z)$ is an involutive automorphism of the compact simple Lie algebra $\mathfrak{g} = \mathfrak{a} + \mathfrak{m}$ ($\mathfrak{m} = \sqrt{-1}\mathfrak{m}^*$). Let G and A be Lie groups whose Lie algebras are $\mathfrak{g}$ and $\mathfrak{a}$, respectively. Then Takeuchi [T] proved that the A-orbit through $\{K\} \in G/K$ (K is the fixed point set of σ) is a half dimensional, totally real and totally geodesic submanifold of the compact Hermitian symmetric space G/K and that all such submanifolds are obtained in this way.

By a similar method we shall construct, in this paper, a half dimensional, totally real and totally geodesic submanifold of a compact Riemannian 3-symmetric space $(G/K, \langle\,,\rangle)$ (see Section 5). Moreover we shall classify such submanifolds in the case where $\mathrm{rk}\, G = \mathrm{rk}\, K$ and the dimension of the center of K is not zero (see Theorem 5.5).

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2. Preliminaries. Let G be a Lie group and K a closed subgroup of G such that $\mathrm{Ad}(K)$ is compact. Then G/K admits a G-invariant Riemannian metric $\langle\, ,\rangle$. We call $(G/K, \langle\, ,\rangle)$ a *Riemannian* 3-*symmetric space* if it is not isometric to a Riemannian symmetric space and there exists an automorphism σ of order 3 on G such that the following properties are satisfied:

(i) ${G^{\sigma}}_0 \subset K \subset G^{\sigma}$, where G^{σ} is the set of fixed points of σ and ${G^{\sigma}}_0$ the identity component of G^{σ}, and

(ii) the transformation of G/K induced by σ is an isometry.

Let $(G/K, \langle\, ,\rangle)$ be a Riemannian 3-symmetric space with an automorphism σ, which we denote by $(G/K, \langle\, ,\rangle, \sigma)$. Define an isometry J of the tangent space $(T_o(G/K), \langle\, ,\rangle)$ at $o = \{K\}$ by

$$\sigma = -\frac{1}{2}\,\mathrm{Id} + \frac{\sqrt{3}}{2} J \quad (\mathrm{Id} = \text{the identity map of } T_o(G/K))\,.$$

Then it is known that J induces a G-invariant almost complex structure on G/K (denoted by the same symbol as J), which is called the *canonical almost complex structure* (see Gray [G]).

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Choose a subspace $\mathfrak{p}$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is an $\mathrm{Ad}(K)$- and σ-invariant decomposition. Under the canonical identification of $\mathfrak{p}$ with $T_o(G/K)$ we have the following (see [G]).

LEMMA 2.1. *For $X, Y \in \mathfrak{p}$, we have*

$$[JX, JY]_{\mathfrak{k}} = [X, Y]_{\mathfrak{k}}\,, \quad [JX, Y]_{\mathfrak{p}} = -J[X, Y]_{\mathfrak{p}}\,.$$

Next we shall describe an inner automorphism of order 3 on a compact simple Lie algebra. Let $\mathfrak{g}_{\boldsymbol{C}}$ be a semisimple Lie algebra over $\boldsymbol{C}$ and $\mathfrak{h}_{\boldsymbol{C}}$ a Cartan subalgebra of $\mathfrak{g}_{\boldsymbol{C}}$. Let Δ denote the set of non-zero roots of $\mathfrak{g}_{\boldsymbol{C}}$ with respect to $\mathfrak{h}_{\boldsymbol{C}}$ and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ a fundamental root system of Δ for some lexicographic order. We choose root vectors $\{E_\alpha\}$ $(\alpha \in \Delta)$ so that for $\alpha, \beta \in \Delta$

$$\begin{aligned} &B(E_\alpha, E_{-\alpha}) = 1\,, \\ &[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}\,, \quad N_{\alpha,\beta} = -N_{-\alpha,-\beta} \in \boldsymbol{R}\,, \end{aligned} \tag{2.1}$$

where B is the Killing form of $\mathfrak{g}_{\boldsymbol{C}}$. Then it is known (cf. [H]) that

$$N_{\alpha,\beta} = N_{\gamma,\alpha} \quad (\alpha + \beta + \gamma = 0)\,. \tag{2.2}$$

We set $H_\alpha = [E_\alpha, E_{-\alpha}]$ and denote by Δ^+ the set of positive roots of Δ with respect to the order. Let $\beta + n\alpha$ $(p \le n \le q)$ denote the α-series containing β. Then

$$(N_{\alpha,\beta})^2 = \frac{q(1-p)}{2}\alpha(H_\alpha)\,. \tag{2.3}$$

As is well-known, a Lie algebra

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta^+} (\boldsymbol{R}(E_\alpha - E_{-\alpha}) + \boldsymbol{R}\sqrt{-1}(E_\alpha + E_{-\alpha})) \tag{2.4}$$

is a compact real form of $\mathfrak{g}_C$. Here $\mathfrak{h} = \sum_{\alpha \in \Delta^+} \boldsymbol{R}\sqrt{-1}H_\alpha$. We define $H_j \in \mathfrak{h}_C$ $(j = 1, \dots, l)$ by

$$\alpha_i(H_j) = \delta_{ij}\,, \quad i, j = 1, \dots, l\,.$$

Assume that $\mathfrak{g}$ is simple. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the following is known (cf. Wolf and Gray [WG], Helgason [H]).

LEMMA 2.2. *Let σ be an inner automorphism of order* 3 *on G. Then σ is conjugate to* $\mathrm{Ad}(g)$, *where an element g of G has one of the following forms*:

(1) $g_0 = \exp(2\pi\sqrt{-1}/3)H_k\,, \quad (m_k = 3)$,

(2) $g_1 = \exp(2\pi\sqrt{-1}/3)H_i\,, \quad (m_i = 2)$,

(3) $g_2 = \exp(2\pi\sqrt{-1}/3)(H_i + H_j)\,, \quad (m_i = m_j = 1)$,

(4) $g_3 = \exp(2\pi\sqrt{-1}/3)H_n\,, \quad (m_n = 1)$.

Here $\delta = \sum_{i=1}^{l} m_i\alpha_i$ is the highest root of Δ.

REMARK 2.3. (1) In the case (4), we can see that the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is (Hermitian) symmetric.

(2) Let $\mathfrak{z}$ be the center of $\mathfrak{g}^\sigma$. If $\sigma = \mathrm{Ad}(g_i)$ $(i = 0, 1, 2)$, then the dimension of $\mathfrak{z}$ is equal to i.

3. Totally geodesic submanifolds. In this section we shall give several examples of totally geodesic submanifolds of a compact Riemannian 3-symmetric space $(G/K, \langle\, ,\rangle, \sigma)$, which are expressed as orbits of some Lie subgroups of G.

Throughout this section we use the same notation as in Section 2 and suppose that G is compact and $\langle\, ,\rangle$ is a G-invariant metric induced from a bi-invariant metric of G. Moreover we assume that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is an orthogonal decomposition with respect to $\langle\, ,\rangle$. Let ∇ denote the Levi-Civita connection of $(G/K, \langle\, ,\rangle)$.

LEMMA 3.1. *An affine connection $\bar{\nabla}$ of G/K defined by the following identity is canonical* (*see* [KNo] *for the definition of the canonical connection*):

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{2} J\{(\nabla_X J)(Y)\}\,,$$

where J is the canonical almost complex structure of $(G/K, \langle\, ,\rangle, \sigma)$.

PROOF. For $X \in \mathfrak{p}$ we define a vector field X_* around $o = \{K\}$ by

$$(X_*)_{(\exp x)\cdot o} = d\exp x(X)\,,$$

where $x \in \mathfrak{p}$ and $|x|$ is small enough. Since $\langle\, ,\rangle$ is bi-invariant, we have $(\nabla_{X_*}Y_*)_o = (1/2)[X, Y]_{\mathfrak{p}}$ (cf. Nomizu [N]). Hence it follows from Lemma 2.1 that

$$\begin{aligned}(\bar{\nabla}_{X_*}Y_*)_o &= \frac{1}{2}[X, Y]_{\mathfrak{p}} - \frac{1}{2}J\left(\frac{1}{2}[X, JY]_{\mathfrak{p}} - \frac{1}{2}J[X, Y]_{\mathfrak{p}}\right)\\ &= 0\,.\end{aligned}$$

This completes the proof. □

PROPOSITION 3.2. (1) *Let N be a maximal connected totally real* (*with respect to J*) *and totally geodesic submanifold of* $(G/K, \langle\, ,\rangle, \sigma)$ *such that* $2\dim N = \dim(G/K)$. *Then N is expressed as an orbit of a Lie subgroup of G.*

(2) *The same property holds in the case where N is a maximal almost complex and totally geodesic submanifold.*

PROOF. Let N be a connected totally real and totally geodesic submanifold with $2\dim N = \dim(G/K)$. We may assume that $o \in N$. For vector fields X, Y of G/K which are tangent to N at any point of N, the vector field $\nabla_X Y$ is tangent to N and $(\nabla_X J)Y$ is normal to N. Therefore it follows that $\bar{\nabla}_X Y$ is tangent to N.

Let T and R be the torsion and the curvature tensors of $\bar{\nabla}$ at o, respectively. Then the above argument implies that

$$T(V, V) \subset V\,, \quad R(V, V)V \subset V \quad (V = T_oN \subset \mathfrak{p})\,.$$

As is well-known, for $X, Y, Z \in \mathfrak{p}$, we have

$$T(X, Y) = -[X, Y]_{\mathfrak{p}}\,, \quad R(X, Y)Z = -[[X, Y]_{\mathfrak{k}}, Z]\,.$$

Thus we can see that $\mathfrak{a} = V + [V, V]_{\mathfrak{k}}$ is a Lie subalgebra of $\mathfrak{g}$. Let A be the connected Lie subgroup of G corresponding to $\mathfrak{a}$. Since $A \cdot o$ is totally geodesic (cf. Sagle [S], [To1]) and $N \subset A \cdot o$, we get the proposition.

In the case where N is almost complex, we can prove the proposition by a similar argument. □

Let N be a totally real and totally geodesic submanifold of $(G/K, \langle\, ,\rangle, \sigma)$ such that $o \in N$ and $2\dim N = \dim(G/K)$. As above we set $V = T_oN \subset \mathfrak{p}$ and $\mathfrak{a} = V + [V, V]_{\mathfrak{k}}$.

PROPOSITION 3.3. *Suppose that G is semisimple and* G/K *is almost effective. Then* $(\mathfrak{g}, \mathfrak{a})$ *is a local symmetric pair.*

PROOF. Let U denote the orthogonal complement of $[V, V]_{\mathfrak{k}}$ in $\mathfrak{k}$ with respect to $\langle\, ,\rangle$. Set $\mathfrak{m} = JV + U$. Then we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{a}$. Because $\langle\, ,\rangle$ is $\mathrm{Ad}(G)$-invariant, we get

$$[\mathfrak{a}, \mathfrak{m}] \subset \mathfrak{m}\,, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}\,. \tag{3.1}$$

Since $\mathfrak{g}$ is a compact semisimple Lie algebra, it is sufficient to show that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{a}$.

First it follows from Lemma 2.1 that

$$[JX, JY] = [JX, JY]_{\mathfrak{k}} + [JX, JY]_{\mathfrak{p}} = [X, Y]_{\mathfrak{k}} - [X, Y]_{\mathfrak{p}} \quad (X, Y \in V)\,.$$

So we have $[JV, JV] \subset \mathfrak{a}$.

It is clear that $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}}$ is an ideal of $\mathfrak{g}$. Since G/K is almost effective, there is no nontrivial ideal contained in $\mathfrak{k}$. Thus $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}} = \mathfrak{k}$. Also, by Lemma 2.1 we obtain

$$\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}} = [V, V]_{\mathfrak{k}} + [JV, V]_{\mathfrak{k}}.$$

Moreover, we have by (3.1)

$$\langle [JV, V]_{\mathfrak{k}}, [V, V]_{\mathfrak{k}} \rangle = 0.$$

Therefore we get

$$U = [JV, V]_{\mathfrak{k}}. \tag{3.2}$$

From Lemma 2.1 and (3.2) it follows that

$$\begin{aligned}\langle [[JV, V]_{\mathfrak{k}}, JV], JV \rangle &= \langle [JV, V]_{\mathfrak{k}}, [JV, JV]_{\mathfrak{k}} \rangle \\ &= \langle [JV, V]_{\mathfrak{k}}, [V, V]_{\mathfrak{k}} \rangle = 0,\end{aligned}$$

that is,

$$[U, JV] \subset V. \tag{3.3}$$

Finally it follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned}[U, U] = [[JV, V], U]_{\mathfrak{k}} &\subset [[U, V], JV]_{\mathfrak{k}} + [[U, JV], V]_{\mathfrak{k}} \\ &\subset [JV, JV]_{\mathfrak{k}} + [V, V]_{\mathfrak{k}} = [V, V]_{\mathfrak{k}}.\end{aligned}$$

Hence we obtain the proposition. □

EXAMPLE 3.4. Let $\mathfrak{g}$ be a compact simple Lie algebra as (2.4) and G a Lie group whose Lie algebra is $\mathfrak{g}$. Let σ be an inner automorphism of order 3 on G. Then by Lemma 2.2 we may assume that there is an element $\sqrt{-1}H$ in $\mathfrak{h}$ such that $\sigma = \mathrm{Ad}(\exp \sqrt{-1}H)$. We set $\mathfrak{k} = \mathfrak{g}^\sigma$ and $\Delta^+(H) = \{\alpha \in \Delta^+; \alpha(H) \in 2\pi \boldsymbol{Z}\}$. Then the orthogonal complement $\mathfrak{p}$ of $\mathfrak{g}^\sigma$ with respect to the Killing form B is written as

$$\mathfrak{p} = \sum_{\alpha \in \Delta^+ \setminus \Delta^+(H)} (\boldsymbol{R}(E_\alpha - E_{-\alpha}) + \boldsymbol{R}\sqrt{-1}(E_\alpha + E_{-\alpha})). \tag{3.4}$$

Now we consider a Riemannian 3-symmetric space $(G/G^\sigma, -B)$. Let A be the connected Lie subgroup of G corresponding to a Lie subalgebra

$$\mathfrak{a} = \sum_{\alpha \in \Delta^+} \boldsymbol{R}(E_\alpha - E_{-\alpha}).$$

It is easy to see that

$$J(E_\alpha - E_{-\alpha}) = \varepsilon\sqrt{-1}(E_\alpha + E_{-\alpha}) \quad (\varepsilon = 1 \text{ or } -1, \alpha \in \Delta^+ \setminus \Delta^+(H)).$$

Therefore we see that $A \cdot o$ is a totally real and totally geodesic submanifold of $(G/G^\sigma, -B)$ with $2\dim(A \cdot o) = \dim(G/G^\sigma)$.

4. Isometry groups. Let G be a compact simple Lie group and $\langle\, , \rangle$ a bi-invariant metric on G. We assume that $(G/K, \langle\, , \rangle, \sigma)$ is a Riemannian 3-symmetric space and σ is an inner automorphism of type (2) or (3) in Lemma 2.2. We devote this section to the proof of the following theorem.

THEOREM 4.1. *If G/K is effective, then G is isomorphic to the identity component of the isometry group of a Riemannian 3-symmetric space.*

According to Theorem 3.6 of [To2], if $(G/K, \langle\,,\rangle)$ is not isometric to a Riemannian symmetric space, then the theorem is true. So we shall prove that $(G/K, \langle\,,\rangle)$ is not locally symmetric.

As in Example 3.4, we have

$$\mathfrak{p} = \sum_{\alpha\in\Delta^+\setminus\Delta^+(H)} (\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha)\,, \tag{4.1}$$

where $A_\alpha = (E_\alpha - E_{-\alpha})$ and $B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$. Therefore we get

$$\mathfrak{k} = \mathfrak{h} + \sum_{\alpha\in\Delta^+(H)} (\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha)\,. \tag{4.2}$$

We denote the Levi-Civita connection of $(G/K, \langle\,,\rangle)$ by ∇ and its curvature tensor by R.

LEMMA 4.2. *If there exist roots α, β of $\Delta^+ \setminus \Delta^+(H)$ such that*

$$\alpha+\beta \in \Delta^+ \setminus \Delta^+(H)\,, \quad \beta \pm 2\alpha \notin \Delta\,, \quad \pm(\alpha-\beta) \notin \Delta^+ \setminus \Delta^+(H)\,,$$

then $\nabla R \neq 0$.

PROOF. Let X_* ($X \in \mathfrak{p}$) be a local vector field of G/K defined in Lemma 3.1. Then we have

$$(\nabla_{X_*}Y_*)_o = \frac{1}{2}[X, Y]_\mathfrak{p} \quad (X, Y \in \mathfrak{p})\,. \tag{4.3}$$

Now, we compute $(\nabla_{A_{\alpha*}}R)_o(A_{\alpha*}, B_{\alpha*}, A_{\beta*})$. As is well-known, the space $(G/K, \langle\,,\rangle)$ is naturally reductive. So R is expressed as follows (cf. [KNo]).

$$\begin{aligned}(R(X,Y)Z)_o = &-[[X,Y]_\mathfrak{k}, Z] - \frac{1}{2}[[X,Y]_\mathfrak{p}, Z]_\mathfrak{p} \\ &+\frac{1}{4}[X,[Y,Z]_\mathfrak{p}]_\mathfrak{p} - \frac{1}{4}[Y,[X,Z]_\mathfrak{p}]_\mathfrak{p} \quad (X, Y, Z \in \mathfrak{p})\,.\end{aligned} \tag{4.4}$$

From (4.1), (4.2), (4.3) and the condition of the lemma, we get

$$(\nabla_{A_{\alpha*}}A_{\alpha*})_o = (\nabla_{A_{\alpha*}}B_{\alpha*})_o = 0\,, \quad (\nabla_{A_{\alpha*}}A_{\beta*})_o = (1/2)N_{\alpha,\beta}A_{\alpha+\beta}\,.$$

Therefore it follows from (4.3) that

$$\begin{aligned}(\nabla_{A_{\alpha*}}R)_o(A_{\alpha*}, B_{\alpha*}, A_{\beta*}) &= \{\nabla_{A_{\alpha*}}(R(A_\alpha, B_\alpha)A_\beta)_*\}_o - R(A_\alpha, B_\alpha)(\nabla_{A_{\alpha*}}A_{\beta*})_o \\ &= \frac{1}{2}[A_\alpha, R(A_\alpha, B_\alpha)A_\beta]_\mathfrak{p} - \frac{1}{2}R(A_\alpha, B_\alpha)[A_\alpha, A_\beta]_\mathfrak{p}\,.\end{aligned}$$

Considering (2.1), (2.2) and (4.4) together with the condition of the lemma, we can see

$$\begin{aligned} R(A_\alpha, B_\alpha)A_\beta &= -[[A_\alpha, B_\alpha], A_\beta] + \frac{1}{4}[A_\alpha, [B_\alpha, A_\beta]_{\mathfrak{p}}]_{\mathfrak{p}} - \frac{1}{4}[B_\alpha, [A_\alpha, A_\beta]_{\mathfrak{p}}]_{\mathfrak{p}} \\ &= -[2\sqrt{-1}H_\alpha, A_\beta] + \frac{1}{4}N_{\alpha,\beta}N_{\alpha,-(\alpha+\beta)}B_\beta + \frac{1}{4}N_{\alpha,\beta}N_{\alpha,-(\alpha+\beta)}B_\beta \\ &= -\left\{2\alpha(H_\beta) + \frac{1}{2}(N_{\alpha,\beta})^2\right\} B_\beta \,. \end{aligned}$$

Similarly, we obtain

$$R(A_\alpha, B_\alpha)[A_\alpha, A_\beta]_{\mathfrak{p}} = \left\{-2\alpha(H_\alpha + H_\beta) + \frac{1}{2}(N_{\alpha,\beta})^2\right\} N_{\alpha,\beta}B_{\alpha+\beta} \,.$$

Consequently, we have

$$\begin{aligned} (\nabla_{A_{\alpha *}} R)_o(A_{\alpha *}, B_{\alpha *}, A_{\beta *}) &= -\left\{\alpha(H_\beta) + \frac{1}{4}(N_{\alpha,\beta})^2\right\}[A_\alpha, B_\beta]_{\mathfrak{p}} \\ &\quad - \left\{-\alpha(H_\alpha + H_\beta) + \frac{1}{4}(N_{\alpha,\beta})^2\right\} N_{\alpha,\beta}B_{\alpha+\beta} \\ &= N_{\alpha,\beta}\left\{\alpha(H_\alpha) - \frac{1}{2}(N_{\alpha,\beta})^2\right\} B_{\alpha+\beta} \,. \end{aligned}$$

It follows from (2.3) that $(N_{\alpha,\beta})^2 = ((1-p)/2)\alpha(H_\alpha)$ ($p = 0$ or -1). We have thus obtained the lemma. □

By the assumption on σ, we can set

(i) $\sigma = \mathrm{Ad}(\exp(2\pi\sqrt{-1}/3)(H_i + H_j))$ ($i < j, m_i = m_j = 1$) or

(ii) $\sigma = \mathrm{Ad}(\exp(2\pi\sqrt{-1}/3)H_k))$ ($m_k = 2$),

where $\sum_{a=1}^l m_a\alpha_a$ is the highest root. So we have

$$\Delta^+ \setminus \Delta^+(H) = \left\{\gamma = \sum_{a=1}^l n_a\alpha_a \in \Delta^+ \,\middle|\, n_i = 1 \text{ or } n_j = 1\right\}$$

in the case (i), and

$$\Delta^+ \setminus \Delta^+(H) = \left\{\gamma = \sum_{a=1}^l n_a\alpha_a \in \Delta^+ \,\middle|\, 1 \leq n_k \leq 2\right\}$$

in the case (ii).

It is easy to see (by a case-by-case check) that there exist α, β satisfying the condition of Lemma 4.2. For example, in the case (i), it is immediate to see that

$$\alpha = \alpha_1 + \cdots + \alpha_{j-1}\,, \quad \beta = \alpha_j + \cdots + \alpha_l$$

satisfy the condition of Lemma 4.2.

We have thus obtained Theorem 4.1.

REMARK 4.3. Suppose that σ is an inner automorphism as in (1) of Lemma 2.2. Then we can also check that there are roots α and β satisfying the condition of Lemma 4.2, excluding $(G_2/SU(3), \langle\ ,\ \rangle, \sigma)$.

5. Classification. Let $(G/K, \langle\ ,\ \rangle, \sigma)$ be a compact irreducible Riemannian 3-symmetric space, where σ is an inner automorphism of type (2) or (3) in Lemma 2.2. Then we may assume that G is a compact simple Lie group (cf. [G]) and $\langle\ ,\ \rangle$ is equal to $-B$. In this section we shall classify a half dimensional (maximal), totally real and totally geodesic submanifold N of $(G/K, \langle\ ,\ \rangle, \sigma)$. In the following, we call a pair $((G/K, -B, \sigma), N)$ a *TRG pair* for simplicity.

In order to construct a TRG pair we recall the notion of graded Lie algebras. Let $\mathfrak{g}^*$ be a noncompact simple Lie algebra whose complexification is simple. Let τ denote a Cartan involution of $\mathfrak{g}^*$ and

$$\mathfrak{g}^* = \mathfrak{a} + \mathfrak{m}^*, \quad \tau|_{\mathfrak{a}} = 1, \ \tau|_{\mathfrak{m}^*} = -1, \tag{5.1}$$

the corresponding Cartan decomposition. Now let us take a gradation of the second kind on $\mathfrak{g}^*$:

$$\begin{aligned} &\mathfrak{g}^* = \mathfrak{g}^*{}_{-2} + \mathfrak{g}^*{}_{-1} + \mathfrak{g}^*{}_0 + \mathfrak{g}^*{}_1 + \mathfrak{g}^*{}_2 \quad (\mathfrak{g}^*{}_1 \neq \{0\}), \\ &\tau(\mathfrak{g}^*{}_p) = \mathfrak{g}^*{}_{-p}, \quad p = 0, \pm 1, \pm 2. \end{aligned} \tag{5.2}$$

Then there is a unique element $Z \in \mathfrak{g}^*$ such that

$$\{X \in \mathfrak{g}^* \mid [Z, X] = pX\} = \mathfrak{g}^*{}_p, \quad p = 0, \pm 1, \pm 2. \tag{5.3}$$

The element Z is called a *characteristic element*. From (5.1), (5.2) and (5.3) we see that $Z \in \mathfrak{m}^* \cap \mathfrak{g}^*{}_0$. Moreover we have

$$\begin{aligned} &\mathfrak{g}^*{}_0 = \mathfrak{g}^*{}_0 \cap \mathfrak{a} + \mathfrak{g}^*{}_0 \cap \mathfrak{m}^*, \\ &\mathfrak{g}^*{}_p + \mathfrak{g}^*{}_{-p} = (\mathfrak{g}^*{}_p + \mathfrak{g}^*{}_{-p}) \cap \mathfrak{a} + (\mathfrak{g}^*{}_p + \mathfrak{g}^*{}_{-p}) \cap \mathfrak{m}^*, \quad p = 1, 2. \end{aligned}$$

We define a Lie subalgebra $\mathfrak{k}$ and a subspace $\mathfrak{p}$ of the compact simple Lie algebra $\mathfrak{g} = \mathfrak{a} + \sqrt{-1}\mathfrak{m}^*$ as follows.

$$\begin{aligned} &\mathfrak{k} = \mathfrak{g}^*{}_0 \cap \mathfrak{a} + \sqrt{-1}(\mathfrak{g}^*{}_0 \cap \mathfrak{m}^*), \\ &\mathfrak{p} = \bigoplus_{p=1}^{2} \{(\mathfrak{g}^*{}_p + \mathfrak{g}^*{}_{-p}) \cap \mathfrak{a} + \sqrt{-1}((\mathfrak{g}^*{}_p + \mathfrak{g}^*{}_{-p}) \cap \mathfrak{m}^*)\}. \end{aligned} \tag{5.4}$$

Noting (5.3) and (5.4), the following lemma is easy to see.

LEMMA 5.1. *$\sigma = \mathrm{Ad}(\exp(2\pi\sqrt{-1}/3)Z)$ is an automorphism of order 3 on $\mathfrak{g}$. Moreover, $\mathfrak{k}$ equals $\mathfrak{g}^\sigma$, the set of fixed points of σ, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a σ-invariant orthogonal decomposition with respect to B.*

By Lemma 5.1 it is immediate that

$$J = \frac{2}{\sqrt{3}}\left(\sigma + \frac{1}{2}\mathrm{Id}\right) : \mathfrak{p} \to \mathfrak{p}$$

is a complex structure on $\mathfrak{p}$. Now we write J explicitly. For

$$X_+ + X_- \in (\mathfrak{g}^*{}_1 + \mathfrak{g}^*{}_{-1}) \cap \mathfrak{a}\,, \quad X_\pm \in \mathfrak{g}^*{}_{\pm 1}\,,$$

it follows from (5.1) and (5.2) that $\tau(X_\pm) = X_\mp$. Therefore, by a straightforward computation, we have

$$J(X_+ + X_-) = \sqrt{-1}(X_+ - X_-) \in \sqrt{-1}((\mathfrak{g}^*{}_1 + \mathfrak{g}^*{}_{-1}) \cap \mathfrak{m}^*)\,.$$

Similarly, for $X_+ + X_- \in (\mathfrak{g}^*{}_2 + \mathfrak{g}^*{}_{-2}) \cap \mathfrak{a}$ we get

$$J(X_+ + X_-) = -\sqrt{-1}(X_+ - X_-) \in \sqrt{-1}((\mathfrak{g}^*{}_2 + \mathfrak{g}^*{}_{-2}) \cap \mathfrak{m}^*)\,.$$

Hence we have an orthogonal decomposition $\mathfrak{p} = V + JV$ $(V = \mathfrak{a} \cap \mathfrak{p})$ with respect to B. Note that $\mathfrak{a} = V + [V, V]_{\mathfrak{k}}$.

Let G be a Lie group whose Lie algebra equals $\mathfrak{g}$, and K the set of fixed points of an automorphism σ defined in Lemma 5.1. Moreover we denote by A the connected Lie subgroup of G corresponding to $\mathfrak{a}$. Then by the above argument we get a TRG pair $((G/K, -B, \sigma), A \cdot o)$. In what follows, we call $((G/K, -B, \sigma), A \cdot o)$ thus obtained a TRG pair *corresponding to* a graded Lie algebra $(\mathfrak{g}^*, \tau, Z)$ of the second kind with the characteristic element Z. Since K is a centralizer of a toral subgroup of G, the group K is connected and G/K is simply connected.

Conversely, let $((G/K, \langle\, ,\rangle, \sigma), N)$ be a TRG pair. We assume that $o \in N$ and set $V = T_oN$. Then by Proposition 3.3, $(\mathfrak{g}, \mathfrak{a})$ $(\mathfrak{a} = V + [V, V]_{\mathfrak{k}})$ is a local symmetric pair. let $\mathfrak{g} = \mathfrak{a} + \mathfrak{m}$ denote the canonical decomposition of $\mathfrak{g}$. As in the proof of Proposition 3.3, we have

$$\mathfrak{m} = \mathfrak{m} \cap \mathfrak{p} + \mathfrak{m} \cap \mathfrak{k} = JV + [V, JV]_{\mathfrak{k}}\,.$$

LEMMA 5.2. *Let* $\mathfrak{g}^* = \mathfrak{a} + \mathfrak{m}^*$ $(\mathfrak{m}^* = \sqrt{-1}\mathfrak{m})$ *be the noncompact dual of* $\mathfrak{g}$ *and* $\mathfrak{z}$ *the center of* $\mathfrak{k}$. *Then there exists an element* $\sqrt{-1}Z$ *in* $[V, JV]_{\mathfrak{k}} \cap \mathfrak{z} (= \mathfrak{m} \cap \mathfrak{z})$ *such that*

$$\sigma = \mathrm{Ad}\left(\exp \frac{2\pi}{3}\sqrt{-1}Z\right)$$

and the eigenvalues of $\mathrm{ad}(Z) : \mathfrak{g}^* \to \mathfrak{g}^*$ *are* $0, \pm 1$ *and* ± 2.

PROOF. Note that $\dim \mathfrak{z} = 1$ or 2. First, we suppose $\dim \mathfrak{z} = 1$. Since σ is inner, there is a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which is contained in $\mathfrak{k}$. Let Δ be the set of nonzero roots with respect to $(\mathfrak{g}_C, \mathfrak{h}_C)$ and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ a fundamental root system. Let $\delta = \sum_{i=1}^l m_i\alpha_i$ be the highest root. From Lemma 2.2 together with Remark 2.3 it follows that

$$\sigma = \mathrm{Ad}\left(\exp \frac{2\pi}{3}\sqrt{-1}H_i\right) \quad \text{for some } i \text{ with the property } m_i = 2\,.$$

Clearly the eigenvalues of $\mathrm{ad}(H_i)$ are 0, ± 1 and ± 2. So it is sufficient to prove that $\mathfrak{z} = \boldsymbol{R}\sqrt{-1}H_i$ is contained in $[V, JV]_{\mathfrak{k}}$. To see this, we decompose $\sqrt{-1}H_i$ as

$$\sqrt{-1}H_i = A_1 + A_2\,, \quad A_1 \in \mathfrak{k} \cap \mathfrak{m}\,, \; A_2 \in \mathfrak{k} \cap \mathfrak{a}\,.$$

Since

$$\sqrt{-1}H_i \in \mathfrak{z}\,, \quad [\mathfrak{m}, \mathfrak{m}] = \mathfrak{a}\,, \; [\mathfrak{a}, \mathfrak{m}] = \mathfrak{m}\,,$$

we see that $A_1, A_2 \in \mathfrak{z}$. Because $\dim \mathfrak{z} = 1$, either A_1 or A_2 is zero. Now we suppose that $A_1 = 0$ (namely $\mathfrak{z} \subset \mathfrak{a}$). Since $[\mathfrak{a} \cap \mathfrak{k}, V] \subset V$, it is easy to see that for $X \in V$

$$JX = \frac{2}{\sqrt{3}}\left(\sigma(X) + \frac{1}{2}X\right) \in V\,,$$

which is a contradiction. Hence we conclude that $\mathfrak{z} \subset [V, JV]_{\mathfrak{k}}$.

Next, we assume that $\dim \mathfrak{z} = 2$. In this case we can set

$$\sigma = \mathrm{Ad}\left(\exp \frac{2\pi}{3}\sqrt{-1}(H_i + H_j)\right) \quad (m_i = m_j = 1)\,, \quad \mathfrak{z} = \boldsymbol{R}\sqrt{-1}H_i + \boldsymbol{R}\sqrt{-1}H_j\,.$$

We define a subset Δ_k $(k = 1, 2, 3)$ of Δ^+ as follows:

$$\begin{aligned} \Delta_1 &= \{\alpha \in \Delta^+;\ \alpha(H_i) = 1,\ \alpha(H_j) = 0\}\,, \\ \Delta_2 &= \{\alpha \in \Delta^+;\ \alpha(H_j) = 1,\ \alpha(H_i) = 0\}\,, \\ \Delta_3 &= \{\alpha \in \Delta^+;\ \alpha(H_i) = 1,\ \alpha(H_j) = 1\}\,. \end{aligned} \tag{5.5}$$

Then we have an orthogonal decomposition

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3\,, \tag{5.6}$$

where

$$\mathfrak{p}_k = \sum_{\alpha \in \Delta_k} (\boldsymbol{R}(E_\alpha - E_{-\alpha}) + \boldsymbol{R}\sqrt{-1}(E_\alpha + E_{-\alpha}))\,.$$

By an argument similar to the above, we see that $\mathfrak{z} = \mathfrak{z} \cap \mathfrak{m} + \mathfrak{z} \cap \mathfrak{a}$ and $\mathfrak{z} \not\subset \mathfrak{a}$. So we may suppose that there are nonzero elements Z_1 and Z_2 subject to the following condition:

$$\mathfrak{z} = \boldsymbol{R}Z_1 + \boldsymbol{R}Z_2\,, \quad Z_1 \in \mathfrak{z} \cap \mathfrak{m}\,, \ Z_2 \in \mathfrak{z} \cap \mathfrak{a}\,.$$

Set

$$Z_1 = \sqrt{-1}(aH_i + bH_j)\,, \quad Z_2 = \sqrt{-1}(cH_i + dH_j)\,.$$

If $c = d$, since $\sqrt{-1}(H_i + H_j) \in \mathfrak{a}$, this case reduces to the previous one. Hence we assume that $c \neq d$. For any $X \in \mathfrak{p}$ we denote by X_k $(k = 1, 2, 3)$ the $\mathfrak{p}_k$-component of X. From (5.5) and (5.6) we can see that for $X \in V$

$$\begin{aligned} \mathrm{Ad}(\exp tZ_2)(X) = {} & \cos ct \cdot X_1 + \sin ct \cdot Y_1 + \cos dt \cdot X_2 \\ & + \sin dt \cdot Y_2 + \cos(c+d)t \cdot X_3 + \sin(c+d)t \cdot Y_3 \in V\,, \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} Y_1 &= [\sqrt{-1}H_i, X_1]\,, \quad Y_2 = [\sqrt{-1}H_j, X_2]\,, \\ Y_3 &= [\sqrt{-1}H_i, X_3](= [\sqrt{-1}H_j, X_3])\,. \end{aligned} \tag{5.8}$$

We note that

$$J(X_k) = Y_k \quad (k = 1, 2)\,, \quad J(X_3) = -Y_3\,. \tag{5.9}$$

If $cd \neq 0$ and $c + d \neq 0$, then c, d and $c + d$ are mutually distinct. So we have from (5.7) that $X_k, Y_k \in V$ $(k = 1, 2, 3)$. However, by (5.9) the vectors Y_k are in JV. Hence $cd = 0$ or $c + d = 0$.

We assume that $c = 0$ and $d \neq 0$. Then it follows from (5.7) and (5.9) that

$$X_1\,,\quad (X_2 + X_3)\,,\quad (Y_2 + Y_3) \in V\,,\quad \text{and}\quad Y_1\,,\quad (Y_2 - Y_3)\,,\quad (X_2 - X_3) \in JV\,.$$

So, since $[Z_1, X_2 + X_3] = bY_2 + (a+b)Y_3 \in [\mathfrak{m}, V]_{\mathfrak{p}} = JV$, we have $b = -(a+b)$, namely $\sqrt{-1}(2H_i - H_j) \in \mathfrak{m}$. Set $Z = (H_j - 2H_i)$. Then it follows that

$$\mathrm{Ad}\left(\exp\frac{2\pi}{3}\sqrt{-1}Z\right) = \mathrm{Ad}\left(\exp\frac{2\pi}{3}\sqrt{-1}(H_i + H_j)\right)(=\sigma)$$

and the eigenvalues of $\mathrm{ad}(Z)$ are 0, ± 1 and ± 2. By making use of (5.5) and (5.8), for example, we can check that

$$\begin{aligned}
&[Z, X_1 \pm \sqrt{-1}Y_1] = 2(\pm X_1 + \sqrt{-1}Y_1)\,,\\
&[Z, (X_2 + X_3) \pm \sqrt{-1}(Y_2 - Y_3)] = \mp(X_2 + X_3) - \sqrt{-1}(Y_2 - Y_3)\,,\\
&[Z, (Y_2 + Y_3) \pm \sqrt{-1}(X_2 - X_3)] = \pm(Y_2 + Y_3) + \sqrt{-1}(X_2 - X_3)\,.
\end{aligned}$$

In the case that $c \neq 0$ and $d = 0$, the vector $Z = (H_i - 2H_j)$ satisfies the condition of the lemma.

Finally we consider the case $c + d = 0$. It follows from (5.7) and (5.9) that

$$X_3\,,\quad (X_1 + X_2)\,,\quad (Y_1 - Y_2) \in V\,,\quad \text{and}\quad Y_3\,,\quad (Y_1 + Y_2)\,,\quad (X_1 - X_2) \in JV\,.$$

Similarly, as above we can see that $\sqrt{-1}Z = \sqrt{-1}(H_i + H_j)$ is in $\mathfrak{m}$. We have thus obtained the lemma. □

From Lemma 5.2 we have a graded simple Lie algebra $(\mathfrak{g}^*, \tau, Z)$ of the second kind

$$\mathfrak{g}^* = \mathfrak{g}^*{}_{-2} + \mathfrak{g}^*{}_{-1} + \mathfrak{g}^*{}_0 + \mathfrak{g}^*{}_1 + \mathfrak{g}^*{}_2$$

with the characteristic element Z. It is easy to see that $((G/K, \langle\,,\rangle, \sigma), N)$ is a TRG pair corresponding to $(\mathfrak{g}^*, \tau, Z)$. Therefore we have

PROPOSITION 5.3. *Any TRG pair is obtained as a TRG pair corresponding to a graded simple Lie algebra of the second kind.*

Now, we say that two TRG pairs $((G/K, \langle\,,\rangle_G, \sigma), N)$ and $((\bar{G}/\bar{K}, \langle\,,\rangle_{\bar{G}}, \bar{\sigma}), \bar{N})$ are *equivalent* if there exists an isometry $f : (G/K, \langle\,,\rangle_G) \to (\bar{G}/\bar{K}, \langle\,,\rangle_{\bar{G}})$ such that $f(N) = \bar{N}$.

REMARK 5.4. Let ϕ be an isomorphism between graded simple Lie algebras $(\mathfrak{g}^*, \tau, Z)$ and $(\bar{\mathfrak{g}}^*, \bar{\tau}, \bar{Z})$, i.e., $\phi : \mathfrak{g}^* \to \bar{\mathfrak{g}}^*$ is an isomorphism such that

$$\phi(Z) = \bar{Z}\,,\quad \bar{\tau}\circ\phi = \phi\circ\tau\,.$$

Let $\mathfrak{g}^* = \mathfrak{a} + \mathfrak{m}^*$ and $\bar{\mathfrak{g}}^* = \bar{\mathfrak{a}} + \bar{\mathfrak{m}}^*$ be the Cartan decompositions corresponding to τ and $\bar{\tau}$, respectively. We define an isomorphism ϕ' from $\mathfrak{g} = \mathfrak{a} + \mathfrak{m}$ to $\bar{\mathfrak{g}} = \bar{\mathfrak{a}} + \bar{\mathfrak{m}}$ ($\mathfrak{m} = \sqrt{1}\mathfrak{m}^*$, $\bar{\mathfrak{m}} = \sqrt{-1}\bar{\mathfrak{m}}^*$) as follows:

$$\phi'(X + \sqrt{-1}Y) = \phi(X) + \sqrt{-1}\phi(Y)\quad (X \in \mathfrak{a},\ Y \in \mathfrak{m}^*)\,.$$

Then it is easy to see that $\phi'(\mathfrak{a}) = \bar{\mathfrak{a}}$ and $\phi'(\mathfrak{k}) = \bar{\mathfrak{k}}$ (see (5.4) for the definitions of $\mathfrak{k}$ and $\bar{\mathfrak{k}}$). Therefore an isomorphism between graded Lie algebras induces an equivalence between TRG pairs.

Suppose that $((G/K, \langle\, ,\rangle_G, \sigma), N)$ and $((\bar{G}/\bar{K}, \langle\, ,\rangle_{\bar{G}}, \bar{\sigma}), \bar{N})$ are TRG pairs corresponding to $(\mathfrak{g}^*, \tau, Z)$ and $(\bar{\mathfrak{g}}^*, \bar{\tau}, \bar{Z})$, respectively. Moreover we assume that G/K and $\bar{G}/\bar{K}$ are effective, and $o \in N$, $\bar{o} \in \bar{N}$ ($\bar{o} = \{\bar{K}\}$). As before we set

$$\mathfrak{a} = T_oN + [T_oN, T_oN]\,, \quad \bar{\mathfrak{a}} = T_{\bar{o}}\bar{N} + [T_{\bar{o}}\bar{N}, T_{\bar{o}}\bar{N}]\,.$$

If these TRG pairs are equivalent, then there exists an isometry $\varphi : (G/K, \langle\, ,\rangle_G) \to (\bar{G}/\bar{K}, \langle\, ,\rangle_{\bar{G}})$ such that

$$\varphi(o) = \bar{o}\,, \quad \varphi(N) = \bar{N}\,. \tag{5.10}$$

Since G acts effectively on G/K, we can regard G as a Lie group of isometries. Then it follows from Theorem 4.1 that $\iota_\varphi(G) = \bar{G}$ ($\iota_\varphi(g) = \varphi \circ g \circ \varphi^{-1}$). So, by (5.10), we have

$$\iota_\varphi(K) = \bar{K}\,, \quad \iota_{\varphi_*}(\mathfrak{a}) = \bar{\mathfrak{a}}\,.$$

Set $\sqrt{-1}Z' = \iota_{\varphi_*}(\sqrt{-1}Z)$. Then it is easy to see that

$$\phi : \mathfrak{g}^* \to \bar{\mathfrak{g}}^*;\ (X + Y) \mapsto \iota_{\varphi_*}(X) - \sqrt{-1}\iota_{\varphi_*}(\sqrt{-1}Y) \quad (X \in \mathfrak{a},\ \sqrt{-1}Y \in \mathfrak{m}^*)$$

is an isomorphism from $(\mathfrak{g}^*, \tau, Z)$ to $(\bar{\mathfrak{g}}^*, \bar{\tau}, Z')$.

If the dimension of the center of $\mathfrak{k}$ (and $\bar{\mathfrak{k}}$) is one, then $\sqrt{-1}Z' = \sqrt{-1}\bar{Z}$ or $-\sqrt{-1}\bar{Z}$. Note that $\bar{\tau}$ provides an isomorphism between $(\bar{\mathfrak{g}}^*, \bar{\tau}, \bar{Z})$ and $(\bar{\mathfrak{g}}^*, \bar{\tau}, -\bar{Z})$. Summing up the above argument together with Remark 5.4, we obtain

THEOREM 5.5. *The above construction of TRG pairs from graded Lie algebras of the second kind induces a surjection*:

$$\begin{gathered}\{\textit{A graded simple Lie algebra } (\mathfrak{g}^*, \tau, Z);\ \mathfrak{g}^*{}_C \textit{ is simple}\}/\sim \\ \downarrow \\ \{\textit{TRG pairs}\}/\sim\ .\end{gathered}$$

In particular, we have a one-to-one correspondence:

$$\begin{gathered}\{(\mathfrak{g}^*, \tau, Z);\ \mathfrak{g}^*{}_C \textit{ is simple and } \dim(\textit{the center of } \mathfrak{g}^*{}_0) = 1\}/\sim \\ \updownarrow \\ \{\textit{TRG pair } ((G/K, \langle\, ,\rangle, \sigma), N);\ \dim(\textit{the center of } K) = 1\}/\sim\ .\end{gathered}$$

REMARK 5.6. Kaneyuki [K] classifies the graded simple Lie algebras of the second kind.

REMARK 5.7. The surjection in Theorem 5.5 is not injective. To see this, set $(\mathfrak{g}^*, \mathfrak{a}) = (\mathfrak{sl}(4, \boldsymbol{R}), \mathfrak{so}(4))$. Then it is known that $\mathfrak{g}^*$ is a normal real form of $\mathfrak{sl}(4, \boldsymbol{C})$ and $\mathfrak{h} = \sum_{i=1}^{3} \boldsymbol{R}(E_{i,i} - E_{i+1,i+1})$ is a maximal abelian subalgebra of $\mathfrak{g}^*$ contained in $\mathfrak{m}^*$. Here, we denote by $E_{i,i}$ a diagonal matrix whose i-th diagonal element equals 1 and the others are zero. Let $e_i(H)$ $(1 \leq i \leq 4)$ be the i-th diagonal element of $H \in \mathfrak{h}$. Then $\{\alpha_1, \alpha_2, \alpha_3\}$ $(\alpha_i = e_i - e_{i+1})$ is a basis of the root system of $\mathfrak{g}^*$ with respect to $\mathfrak{h}$.

Put

$$Z_1 = H_1 + H_3 , \quad Z_2 = H_1 - 2H_3 \quad \text{(see Section 2 for the definition of } H_i) .$$

Let $({\mathfrak{g}^{*(i)}}_k)_{-2 \leq k \leq 2}$ $(i = 1, 2)$ be the gradation of $\mathfrak{g}^*$ whose characteristic element is Z_i. Then we have $\dim({\mathfrak{g}^{*(1)}}_1) = 4$, $\dim({\mathfrak{g}^{*(2)}}_1) = 3$ and

$$\mathrm{Ad}\left(\exp \frac{2\pi\sqrt{-1}}{3} Z_1\right) = \mathrm{Ad}\left(\exp \frac{2\pi\sqrt{-1}}{3} Z_2\right) .$$

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