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RELATING MULTIPLIERS AND TRANSPLANTATION FOR FOURIER-BESSEL EXPANSIONS AND HANKEL TRANSFORM

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Abstract. Proved are transference results that show connections between: a) multipliers for the Fourier-Bessel series and multipliers for the Hankel transform; b) maximal operators defined by Fourier-Bessel multipliers and maximal operators given by Hankel transform multipliers; c) Fourier-Bessel transplantation and Hankel transform transplantation. In some way the connections described in a) and b) can be seen as multi-dimensional extensions of the classical results of Igari, and Kenig and Tomas for the one dimensional Fourier transform. We prove our results for the non-modified Hankel transform in the power weight setting, and this allows to translate them also to the context of the modified Hankel transform. Together with Gilbert's transplantation theorem, our transference shows that harmonic analysis results for the Hankel transform of arbitrary order are consequences of corresponding results for the cosine expansions.

1. Introduction. There are a number of theorems relating L^p multipliers on $\boldsymbol{R}$ and its discrete subgroup $\boldsymbol{Z}$. One of them, which is a converse to a well-known theorem of de Leeuw, is the following result proved by Igari.

THEOREM ([11]). *Let $1 < p < \infty$ and assume that m is a bounded function on $\boldsymbol{R}$, continuous except on a set of Lebesgue measure zero. If $\{m(\varepsilon n)\} \in M_p(\boldsymbol{Z})$ for all sufficiently small $\varepsilon > 0$ and $\liminf_{\varepsilon\to 0^+} \|m(\varepsilon n)\|_{M_p(\boldsymbol{Z})} < \infty$, then $m \in M_p(\boldsymbol{R})$ and*

$$\|m\|_{M_p(\boldsymbol{R})} \le \liminf_{\varepsilon\to 0^+} \|m(\varepsilon n)\|_{M_p(\boldsymbol{Z})} .$$

Here $M_p(\boldsymbol{Z})$ and $M_p(\boldsymbol{R})$ denote the spaces of L^p-multipliers on $\boldsymbol{Z}$ and $\boldsymbol{R}$ respectively, and $\|m_n\|_{M_p(\boldsymbol{Z})}$ or $\|m\|_{M_p(\boldsymbol{R})}$ denote the multiplier norms of m_n or m, that are the norms of Fourier multiplier operators associated to m_n or m acting on $L^p(0, 2\pi)$ or $L^p(\boldsymbol{R})$.

Igari [12] then found another interesting relation between multipliers for the Jacobi polynomial expansions and the (modified) Hankel transform multipliers. This relation was successfully exploited by Gasper and Trebels [5] to furnish sufficient conditions for the Hankel transform multipliers by means of known sufficient conditions for the Jacobi multipliers.

A theorem relating maximal operators defined by Fourier multipliers on $\boldsymbol{R}$ and $\boldsymbol{Z}$ was proved by Kenig and Tomas [14], and Kanjin [13] then showed a similar relation between maximal operators defined by Jacobi multipliers and maximal operators defined by Hankel

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multipliers. The last relation was then used by Kanjin for proving a.e. convergence of spherical means for radial functions on $\boldsymbol{R}^n$ by using known L^p-estimates for maximal partial sum operators for Jacobi polynomial expansions.

The aim of this paper is, among others, to prove that similar relations of Igari and Kanjin hold true between multipliers for the Fourier-Bessel expansions and the Hankel transform multipliers. It is quite natural to expect such a relation, since in both, Fourier-Bessel expansions and Hankel transform of the same order, the same kind of Bessel function is involved. We mention at this point that the, somehow misterious, relation between Jacobi expansions and the Hankel transform is based on Hilb's asymptotic formula that asymptotically links Jacobi polynomials and Bessel functions.

Besides relations of Igari and Kanjin type we also show a relation between the transplantation for Fourier-Bessel series and that for the Hankel transform. Such type of relation has been recently proved in [22] for Jacobi series and Hankel transform. It should also be noted that in a series of papers, [19], [20] and [22], relations of Igari and Kanjin type, as well as the relation just mentioned, have also been proved for Laguerre series replacing Jacobi series.

Finally, we note that we use the non-modified Hankel transform in the polynomial weights setting. This easily allows to translate obtained results to the (unweighted) modified Hankel transform context. The results of Igari and Kanjin were proved in the (unweighted) modified Hankel transform setting.

Throughout this paper, by $[\alpha]$ we understand the integer part of α. As usual in such occasions the letter C will denote a positive constant that may vary from line to line.

2. Preliminaries and statement of results. Given $\nu > -1$ and f, a suitable function on $(0, \infty)$, its (non-modified) Hankel transform is defined by

$$\mathcal{H}_\nu f(x) = \int_0^\infty (xy)^{1/2} J_\nu(xy) f(y) dy\,, \quad x > 0\,,$$

where $J_\nu(x)$ denotes the Bessel function of the first kind and order ν, [16]. Then $(\mathcal{H}_\nu \circ \mathcal{H}_\nu) f = f$ and $\|\mathcal{H}_\nu f\|_{L^2(0,\infty)} = \|f\|_{L^2(0,\infty)}$ for any $f \in C_c^\infty(0, \infty)$, the space of C^∞ functions with compact support in $(0, \infty)$. These two facts are known in the literature for $\nu \geq -1/2$; in Lemma 2.7 we furnish a proof that works for any $\nu > -1$.

Given $1 \leq p < \infty$ and a real number a, by $L^{p,a}(0, \infty)$ we denote the weighted Lebesgue space of all (equivalence classes of) measurable functions g on $(0, \infty)$ for which the quantity

$$\|g\|_{p,a} = \left(\int_0^\infty |g(x)|^p x^a dx \right)^{1/p}$$

is finite. For $a = 0$ we simplify the notation writing $\|g\|_p$ and $L^p(0, \infty)$.

A bounded measurable function m on $(0, \infty)$ (genuine function, not equivalence class) is called a weighted $L^{p,a}$ Hankel multiplier provided that

$$\|\mathcal{H}_\nu(m \cdot \mathcal{H}_\nu f)\|_{p,a} \leq C \|f\|_{p,a}$$

with a constant C independent of f in $C_c^\infty(0, \infty)$. The least constant for which the above inequality holds is called the multiplier norm of m and is denoted by $\|m\|_{(p,a)}$.

Given $\nu > -1$, let $\lambda_n = \lambda_{n,\nu}$, $n = 1, 2, \dots$, denote the sequence of successive positive zeros of $J_\nu(x)$. Then

$$\int_0^1 J_\nu(\lambda_n x) J_\nu(\lambda_m x) x dx = \frac{1}{2}(J_{\nu+1}(\lambda_n))^2 \delta_{nm}, \quad n, m = 1, 2, \dots,$$

and the functions

$$\psi_n^\nu(x) = d_{n,\nu}(\lambda_n x)^{1/2} J_\nu(\lambda_n x), \quad d_{n,\nu} = \sqrt{2}\, |\lambda_n^{1/2} J_{\nu+1}(\lambda_n)|^{-1},$$

$n = 1, 2, \dots$, form a complete orthonormal system in $L^2((0,1), dx)$ (for completeness, see [10]). In particular,

$$\psi_n^{-1/2}(x) = \sqrt{2}\cos(\pi x(n - 1/2)), \quad \psi_n^{1/2}(x)\sqrt{2}\sin(\pi x n),$$

for $n = 1, 2, \dots$. The functions $\psi_n^\nu(x)$ are eigenfunctions of the differential operator (symmetric on $L^2(0,1)$)

$$L_\nu = \frac{d^2}{dx^2} + \frac{1/4 - \nu^2}{x^2}.$$

More precisely, we have

$$L_\nu \psi_n^\nu(x) = -\lambda_n^2 \psi_n^\nu(x). \tag{2.1}$$

To every appropriate function f on $(0, 1)$, for instance $f \in C_c^\infty(0, 1)$, we associate its Fourier-Bessel series

$$f(x) \sim \sum_1^\infty c_n^\nu \psi_n^\nu(x), \quad c_n^\nu = c_n^\nu(f) = \int_0^1 f(x)\psi_n^\nu(x) dx.$$

The last integral will be frequently denoted by $\langle f, \psi_n^\nu \rangle$. In general, we will write $\langle \cdot, \cdot \rangle_X$ for the inner product in a Hilbert space X.

A comprehensive study of Fourier-Bessel expansions is contained in Chapter XVII of Watson's monograph [24]. Slightly abusing the notation we will use the symbols $L^{p,a}$ and $\|\cdot\|_{p,a}$ in the same sense as before, but now restricted to functions defined on $(0, 1)$. A bounded sequence $\{m_n\}_{n=1}^\infty$ is called a weighted $L^{p,a}$ Fourier-Bessel multiplier provided

$$\left\| \sum_1^\infty m_n c_n^\nu \psi_n^\nu(x) \right\|_{p,a} \le C \|f\|_{p,a}$$

(with c_n^ν defined above) with a constant C independent of f in $C_c^\infty(0, 1)$ (for such an f the series $\sum_1^\infty m_n c_n^\nu \psi_n^\nu(x)$ converges pointwise, cf. Lemma 2.5). The least constant C satisfying the above inequality is called the multiplier norm of $\{m_n\}_{n=1}^\infty$ and it is denoted by $\|m_n\|_{(p,a)}$. We will use the asymptotics of the sequence $\{\lambda_n\}_{n=1}^\infty$:

$$\lambda_n = \pi(n + D_\nu + O(n^{-1})), \quad D_\nu = \frac{\nu}{2} - \frac{1}{4}, \tag{2.2}$$

and $\{d_{n,\nu}\}_{n=1}^\infty$:

$$d_{n,\nu} = \pi^{1/2}(1 + O(n^{-1})). \tag{2.3}$$

Also, we will use the well-known bounds for the Bessel function $J_\nu(t)$:

$$J_\nu(t) = O(t^\nu), \quad t \to 0^+, \tag{2.4}$$

and

$$J_\nu(t) = O(t^{-1/2}), \quad t \to \infty. \tag{2.5}$$

From now on we assume $\nu > -1$ to be fixed, and hence the Fourier-Bessel expansions and the Hankel transform we consider are both related to the index ν.

THEOREM 2.1. *Let $1 < p < \infty$, $a \in \boldsymbol{R}$ and $m(x)$ be a bounded function on $(0, \infty)$ continuous except on a set of Lebesgue measure zero. If $\{m(\varepsilon\lambda_n)\}$ is an $L^{p,a}(0, 1)$ Fourier-Bessel multiplier for all sufficiently small $\varepsilon > 0$ and $\liminf_{\varepsilon\to 0^+} \|m(\varepsilon\lambda_n)\|_{(p,a)}$ is finite, then $m(x)$ is an $L^{p,a}(0, \infty)$ Hankel transform multiplier and*

$$\|m(x)\|_{(p,a)} \le \liminf_{\varepsilon\to 0^+} \|m(\varepsilon\lambda_n)\|_{(p,a)}.$$

For $p = 1$ and $a = 0$ we have a weak type $(1, 1)$ substitute of Theorem 2.1. Such a substitute of Igari's result from [12] for Jacobi expansions was proved by Connett and Schwartz [4].

A bounded sequence $\{m_n\}_{n=1}^\infty$ is called a weak type $(1, 1)$ Fourier-Bessel multiplier provided

$$\left|\left\{x \in (0, 1);\ \left|\sum_1^\infty m_n \langle f, \psi_n^\nu\rangle \psi_n^\nu(x)\right| > s\right\}\right| \le \frac{C}{s}\|f\|_1$$

with a constant C independent of f in $C_c^\infty(0, 1)$ and $s > 0$ ($|A|$ denotes the Lebesgue measure of a measurable set $A \subset (0, \infty)$). The least constant C satisfying the above inequality is called the weak multiplier norm of $\{m_n\}_{n=1}^\infty$ and is denoted by $\|m_n\|_{weak}$. Similarly, a bounded measurable function m on $(0, \infty)$ is called a weak type $(1, 1)$ Hankel transform multiplier provided

$$|\{x \in (0, \infty);\ |\mathcal{H}_\nu(m \cdot \mathcal{H}_\nu f)(x)| > s\}| \le \frac{C}{s}\|f\|_1$$

with a constant C independent of f in $C_c^\infty(0, \infty)$ and $s > 0$. The least constant for which the above inequality holds is called the weak multiplier norm of m and is denoted by $\|m\|_{weak}$.

THEOREM 2.2. *Let $m(x)$ be a function on $(0, \infty)$ as in Theorem 2.1. If $\{m(\varepsilon\lambda_n)\}$ is a weak $L^1(0, 1)$ Fourier-Bessel multiplier for all sufficiently small $\varepsilon > 0$ and $\liminf_{\varepsilon\to 0^+} \|m(\varepsilon\lambda_n)\|_{weak}$ is finite, then $m(x)$ is a weak $L^1(0, \infty)$ Hankel transform multiplier and*

$$\|m(x)\|_{weak} \le \liminf_{\varepsilon\to 0^+} \|m(\varepsilon\lambda_n)\|_{weak}.$$

Given a bounded measurable function $m(x)$ on $(0, \infty)$, define the maximal multiplier operators

$$\mathcal{M}_m^* f(x) = \sup_{\varepsilon>0} |\mathcal{H}_\nu(m(\varepsilon\cdot)\mathcal{H}_\nu f)(x)|, \quad f \in C_c^\infty(0, \infty),$$

and

$$\tilde{\mathcal{M}}_m^* f(x) = \sup_{\varepsilon>0} \left| \sum_1^\infty m(\varepsilon\lambda_n) c_n^\nu(f) \psi_n^\nu(x) \right|, \quad f \in C_c^\infty(0, 1).$$

THEOREM 2.3. *Let* $1 < p < \infty$, $a \in \boldsymbol{R}$ *and* $m(x)$ *be a function on* $(0, \infty)$ *as in Theorem* 2.1. *If*

$$\|\tilde{\mathcal{M}}_m^* f\|_{p,a} \le C \|f\|_{p,a}$$

with a constant $C > 0$ *independent of* f *in* $C_c^\infty(0, 1)$, *then*

$$\|\mathcal{M}_m^* f\|_{p,a} \le C \|f\|_{p,a}$$

independently of f *in* $C_c^\infty(0, \infty)$ (*with the same constant* C).

Guy [8] showed that the size of the Hankel transform of any suitable function, when measured in the (weighted) L^p-norm, remains the same whatever the order of the Hankel transform is. More precisely, given $\nu, \mu \ge -1/2$, $1 < p < \infty$ and $-1 < a < p - 1$, there is a constant $C = C(\nu, \mu, p, a)$ such that for every appropriate function f

$$C^{-1} \|\mathcal{H}_\mu f\|_{p,a} \le \|\mathcal{H}_\nu f\|_{p,a} \le C \|\mathcal{H}_\mu f\|_{p,a}.$$

In another way, this can be expressed as

$$\|(\mathcal{H}_\nu \circ \mathcal{H}_\mu) f\|_{p,a} \le C \|f\|_{p,a}, \quad f \in C_c^\infty(0, \infty).$$

(The range of eligible a's can be enlarged to $-p(\nu + 1/2) - 1 < a < p(\mu + 3/2) - 1$, see [22]). The transplantation theorem for the Fourier-Bessel expansions says (cf. [6] for unweighted version; adding weights requires only minor modifications) that for μ, ν, p and a as above

$$\left\| \sum_1^\infty c_n^\mu(f) \psi_n^\nu(x) \right\|_{p,a} \le C \|f\|_{p,a}$$

with $C > 0$ independent of f in $C_c^\infty(0, 1)$.

We prove the following result.

THEOREM 2.4. *Let* $1 < p < \infty$, $a \in \boldsymbol{R}$ *and* $\nu, \mu > -1$. *If the Fourier-Bessel transplantation inequality*

$$\left\| \sum_1^\infty c_n^\mu(f) \psi_n^\nu(x) \right\|_{p,a} \le C \|f\|_{p,a}, \quad f \in C_c^\infty(0, 1),$$

holds true, then the Hankel transplantation inequality

$$\|(\mathcal{H}_\nu \circ \mathcal{H}_\mu) f\|_{p,a} \le C \|f\|_{p,a}, \quad f \in C_c^\infty(0, \infty),$$

is also satisfied (*with the same constant* C).

For the sake of completeness we now state and prove three auxiliary results. In the first lemma we show that the Fourier-Bessel series with a $C_c^\infty(0, 1)$ function f involved, is pointwise convergent and hence makes sense.

LEMMA 2.5. *Let $\nu, \mu > -1$ and $f \in C_c^\infty(0,1)$. Then the series*

$$\sum_1^\infty c_n^\mu(f)\psi_n^\nu(x)$$

converges absolutely for every x, $0 < x < 1$. Also, if $\{m_n\}_{n=1}^\infty$ is a bounded sequence, the series $\sum_1^\infty m_n c_n^\mu(f)\psi_n^\nu(x)$ converges for every x, $0 < x < 1$, and represents a continuous function on $(0,1)$.

PROOF. For any fixed positive constant $0 < c < 1$ we have

$$|\psi_n^\nu(x)| \le C \begin{cases} (nx)^{\nu+1/2}, & 0 < x < cn^{-1}, \\ 1, & cn^{-1} < x < 1. \end{cases} \tag{2.6}$$

This follows from (2.2)–(2.5). Next, using (2.1) we get

$$\langle f, \psi_n^\mu \rangle = -\lambda_{n,\mu}^{-2} \langle L_\mu f, \psi_n^\mu \rangle,$$

which gives $\langle f, \psi_n^\mu \rangle = O(n^{-2})$ and, together with (2.6), proves the lemma (on every $(\varepsilon, 1)$, $0 < \varepsilon < 1$, the series are uniformly convergent). □

A bit of comment is now appropriate on the question why $\mathcal{H}_\nu(m \cdot \mathcal{H}_\nu f)$ is well-defined for f in $C_c^\infty(0,\infty)$ and m is bounded when $-1 < \nu < -1/2$. Clearly $\mathcal{H}_\nu f(y)$ is a continuous function on $0 < y < \infty$ and, by using (2.4),

$$\mathcal{H}_\nu f(y) = O(y^{\nu+1/2}), \quad y \to 0^+. \tag{2.7}$$

Also, by using the asymptotic

$$\sqrt{t} J_\nu(t) = \sqrt{2/\pi} \left(\cos(t + a_\nu) + b_\nu \frac{\sin(t + c_\nu)}{t} + O(t^{-2}) \right), \quad t \to \infty,$$

(for certain a_ν, b_ν and c_ν), we get

$$\mathcal{H}_\nu f(y) = O(y^{-2}), \quad y \to \infty. \tag{2.8}$$

Hence, for any bounded function $m(y)$, the function $y \to (xy)^{1/2} J_\nu(xy) m(y) \mathcal{H}_\nu f(y)$ is Lebesgue integrable on $(0,\infty)$, and therefore $\mathcal{H}_\nu(m \cdot \mathcal{H}_\nu f)(x)$ exists for any x, $0 < x < \infty$. A similar comment applies to the definition of $(\mathcal{H}_\nu \circ \mathcal{H}_\mu) f(y)$ when at least one of the indices ν, μ lies in $(-1, -1/2)$.

LEMMA 2.6. *Let $\nu, \mu > -1$, $g \in C_c^\infty(0,\infty)$, $g_\alpha(x) = g(\alpha x)$, $\alpha > 0$, and take α so large that the support of g_α is contained in $(0,1)$. Given $N = 1, 2, \ldots$ and $K > 0$, there is a constant $C = C_{N,K}$ such that for $0 < x < K$ and large α*

$$\left| \sum_1^{N[\alpha]} \langle g_\alpha, \psi_n^\mu \rangle \psi_n^\nu \left(\frac{x}{\alpha} \right) \right| \le C x^{\nu+1/2}.$$

PROOF. We have

$$\langle g_\alpha, \psi_n^\mu \rangle = \frac{1}{\alpha} \int_0^\infty g(u) \psi_n^\mu(u/\alpha) du.$$

If g is supported in (m, M), $0 < m < M < \infty$, then for $0 < u \le M$, $0 < x \le K$ and $n \le N[\alpha]$ we have $u/\alpha \le cn^{-1}$ and $x/\alpha \le cn^{-1}$ with $c = N\max\{M, K\}$. Hence, by (2.6)

$$|\langle g_\alpha, \psi_n^\mu\rangle| \le C\alpha^{-\mu-3/2}n^{\mu+1/2}$$

and

$$\left|\psi_n^\nu\left(\frac{x}{\alpha}\right)\right| \le Cx^{\nu+1/2}\alpha^{-\nu-1/2}n^{\nu+1/2}.$$

Therefore,

$$\left|\sum_1^{N[\alpha]}\langle g_\alpha, \psi_n^\mu\rangle\psi_n^\nu\left(\frac{x}{\alpha}\right)\right| \le Cx^{\nu+1/2}\alpha^{-(\nu+\mu+2)}\sum_1^{N[\alpha]}n^{\nu+\mu+1} \le C_{N,K}x^{\nu+1/2}.$$

□

In the third lemma we prove the inversion formula and Plancherel's identity for any $C_c^\infty(0, \infty)$ function f and the Hankel transform $\mathcal{H}_\nu$ of any order $\nu > -1$. Known proofs of these two facts usually use the assumption $\nu \ge -1/2$. The argument we apply in the proof is standard but seems not to appear in the existing proofs of the inversion formula and Plancherel's identity for the Hankel transform.

LEMMA 2.7. *Let $\nu > -1$ and $f \in C_c^\infty(0, \infty)$. Then $\|\mathcal{H}_\nu f\|_2 = \|f\|_2$ and*

$$f(x) = \int_0^\infty (xy)^{1/2}J_\nu(xy)\mathcal{H}_\nu f(y)dy$$

for every $0 < x < \infty$.

PROOF. For any $N > 0$ the system

$$\frac{1}{\sqrt{N}}\psi_n^\nu\left(\frac{x}{N}\right), \quad n = 1, 2, \dots,$$

is a complete orthonormal system in $L^2((0, N), dx)$. Given $f \in C_c^\infty(0, \infty)$, take N so large that the support of f is contained in $(0, N)$. We claim that

$$f(x) = \sum_1^\infty \langle N^{-1/2}\psi_n^\nu(t/N), f(t)\rangle_{L^2((0,N),dt)} \cdot N^{-1/2}\psi_n^\nu(x/N), \tag{2.9}$$

for any $0 < x < N$. This follows from the fact that the series in (2.9) converges to f in $L^2((0, N), dx)$. Hence, we can choose a subsequence of partial sums

$$S_{N(k)}f(x) = \sum_1^{N(k)}\langle N^{-1/2}\psi_n^\nu(t/N), f(t)\rangle_{L^2((0,N),dt)} \cdot N^{-1/2}\psi_n^\nu(x/N),$$

$N(1) < N(2) < \cdots$, converging to f almost everywhere on $(0, N)$. Now, by scaling the result of Lemma 2.5, the series in (2.9), and hence the sequence $S_{N(k)}f(x)$, converges for every x in $(0, N)$ and represents a continuous function on $(0, N)$ and the claim follows.

Rewriting (2.9) gives

$$
\begin{aligned}
f(x) &= \sum_{1}^{\infty} \sqrt{x\frac{\lambda_n}{N}}\, J_\nu\left(x\frac{\lambda_n}{N}\right) \mathcal{H}_\nu f\left(\frac{\lambda_n}{N}\right) \cdot \frac{\lambda_{n+1}-\lambda_n}{N} \\
&\quad + \sum_{1}^{\infty} \frac{d_{n,\nu}^2 + \lambda_n - \lambda_{n+1}}{N} \sqrt{x\frac{\lambda_n}{N}}\, J_\nu\left(x\frac{\lambda_n}{N}\right) \mathcal{H}_\nu f\left(\frac{\lambda_n}{N}\right).
\end{aligned}
\tag{2.10}
$$

We now claim that the sum represented by the first series converges, when $N \to \infty$, to

$$
\int_0^\infty (xy)^{1/2} J_\nu(xy) \mathcal{H}_\nu f(y)dy\,,
$$

while the sum given by the second series approaches zero. To simplify the notation we write $h_x(y) = (xy)^{1/2} J_\nu(xy)\mathcal{H}_\nu f(y)$. By (2.4), (2.5), (2.7) and (2.8), $h_x(y) = O(y^{2\nu+1})$, $y \to 0^+$, and $h_x(y) = O(y^{-2})$, $y \to \infty$. Therefore, given large $M > 0$ and small $\delta > 0$, we have

$$
\sum_{MN+1}^{\infty} \left| h_x\left(\frac{\lambda_n}{N}\right) \cdot \frac{\lambda_{n+1}-\lambda_n}{N} \right| \le C\frac{1}{N} \sum_{MN+1}^{\infty} \left(\frac{n}{N}\right)^{-2} \le C\frac{1}{M}
$$

and, in case $-1 < \nu < -1/2$,

$$
\sum_{1}^{[\delta N]} \left| h_x\left(\frac{\lambda_n}{N}\right) \cdot \frac{\lambda_{n+1}-\lambda_n}{N} \right| \le C\frac{1}{N} \sum_{1}^{[\delta N]} \left(\frac{n}{N}\right)^{2\nu+1} \le C\delta^{2(\nu+1)}
$$

with C independent of $N \to \infty$, M and δ. If $\nu \ge -1/2$, $h_x(y)$ is bounded on $(0, \infty)$ and

$$
\begin{aligned}
&\left| \sum_{1}^{\infty} h_x\left(\frac{\lambda_n}{N}\right) \cdot \frac{\lambda_{n+1}-\lambda_n}{N} - \int_0^\infty h_x(y)dy \right| \\
&\quad \le \left| \sum_{1}^{MN} h_x\left(\frac{\lambda_n}{N}\right) \cdot \frac{\lambda_{n+1}-\lambda_n}{N} - \int_0^{\pi M} h_x(y)dy \right| \\
&\quad\quad + \sum_{MN+1}^{\infty} \left| h_x\left(\frac{\lambda_n}{N}\right) \cdot \frac{\lambda_{n+1}-\lambda_n}{N} \right| + \int_{\pi M}^{\infty} |h_x(y)|dy\,.
\end{aligned}
$$

The last three terms are sufficiently small for large N provided we first choose appropriate M. If $-1 < \nu < -1/2$, then, in addition to cutting off from infinity, we have to separate off from zero, since now $h_x(y)$ is unbounded there. We write

$$\begin{aligned}\left|\sum_1^\infty h_x\left(\frac{\lambda_n}{N}\right)\cdot\frac{\lambda_{n+1}-\lambda_n}{N}-\int_0^\infty h_x(y)dy\right| \\ \le\left|\sum_{[\delta N]<n<MN} h_x\left(\frac{\lambda_n}{N}\right)\cdot\frac{\lambda_{n+1}-\lambda_n}{N}-\int_{\pi\delta}^{\pi M} h_x(y)dy\right| \\ +\sum_1^{[\delta N]}\left|h_x\left(\frac{\lambda_n}{N}\right)\cdot\frac{\lambda_{n+1}-\lambda_n}{N}\right|+\int_0^{\pi\delta}|h_x(y)|dy \\ +\sum_{MN+1}^\infty\left|h_x\left(\frac{\lambda_n}{N}\right)\cdot\frac{\lambda_{n+1}-\lambda_n}{N}\right|+\int_{\pi M}^\infty|h_x(y)|dy\,.\end{aligned}$$

Again all three terms are sufficiently small for large N, provided we first choose appropriate M and δ.

To finish the proof of the inversion formula, it suffices to check that the remainder term in (2.10) approaches zero. Since $d_{n,\nu}^2+\lambda_n-\lambda_{n+1}=O(n^{-1})$ (this follows from (2.2) and (2.3)), by using (2.4), (2.5), (2.7) and (2.8), we bound the absolute value of the second series in (2.10) by a constant multiplied by

$$\frac{1}{N}\sum_1^N\frac{1}{n}\left(\frac{n}{N}\right)^{2\nu+1}+\frac{1}{N}\sum_{N+1}^\infty\frac{1}{N}\left(\frac{n}{N}\right)^{-2},$$

which is $o(1)$ as $N\to\infty$, considering separately the cases $-1<\nu<-1/2$, $\nu=-1/2$ and $-1/2<\nu$.

The proof of Plancherel's identity follows similar lines. For large N

$$\begin{aligned}\int_0^\infty|f(x)|^2dx&=\sum_1^\infty|\langle N^{-1/2}\psi_n^\nu(t/N),f(t)\rangle_{L^2((0,N),dt)}|^2\\&=\frac{1}{N}\sum_1^\infty d_{n,\nu}^2|\mathcal{H}_\nu f(\lambda_n/N)|^2\\&=\sum_1^\infty|\mathcal{H}_\nu f(\lambda_n/N)|^2(\lambda_{n+1}-\lambda_n)/N\\&\quad+\sum_1^\infty|\mathcal{H}_\nu f(\lambda_n/N)|^2(d_{n,\nu}^2-\lambda_{n+1}+\lambda_n)/N\,.\end{aligned}$$

Since, by (2.7) and (2.8), the function $\mathcal{H}_\nu f(y)$ is square integrable on $(0,\infty)$, the first sum above converges to

$$\int_0^\infty|\mathcal{H}_\nu f(y)|^2dy$$

when $N\to\infty$, while the sum of second series is $o(1)$ if $N\to\infty$. Clearly, we apply appropriate cutting off at infinity and, if necessary, at zero. □

REMARK. Careful analysis of the proofs of Theorems 2.1 through 2.3 furnished in the next section reveals that in these theorems the multiplier sequence $\{m(\varepsilon\lambda_n)\}$ can be replaced by $\{m(\varepsilon n)\}$. This is important for a possible applications since, for instance, measuring smoothness of the sequence $\{m(\varepsilon n)\}$ for given m is much easier than doing the same for $\{m(\varepsilon\lambda_n)\}$. Similar remark applies to Theorems 4.1 through 4.3 stated in Section 4.

3. Proof of theorems. In this section Theorems 2.1 through 2.4 are proved. Throughout, given $g \in C_c^\infty(0,\infty)$ and $\alpha > 0$, we use g_α to denote the function defined by $g_\alpha(x) = g(\alpha x)$, $x > 0$. Also, we assume α to be sufficiently large; in particular, we assume g_α to be supported in $(0, 1)$. We would like to point out here that in several places of the proofs the case $-1 < \nu < -1/2$ requires additional efforts (this has been already seen in the proof of Lemma 2.7).

PROOF OF THEOREM 2.1. Fix $g \in C_c^\infty(0,\infty)$. By assumption,

$$\left\| \sum_1^\infty m(\lambda_n/\alpha)\langle g_\alpha, \psi_n^\nu\rangle \psi_n^\nu(x) \right\|_{p,a} \le \|m(\lambda_n/\alpha)\|_{(p,a)} \|g_\alpha\|_{p,a}$$

and a change of variables then gives

$$\|F_\alpha\|_{p,a} \le \|m(\lambda_n/\alpha)\|_{(p,a)} \|g\|_{p,a},$$

where we denote

$$F_\alpha(x) = \chi_{(0,\alpha)}(x) \sum_1^\infty m(\lambda_n/\alpha)\langle g_\alpha, \psi_n^\nu\rangle \psi_n^\nu(x/\alpha), \quad x \in (0,\infty).$$

Let $L = \liminf_{\alpha\to\infty} \|m(\lambda_n/\alpha)\|_{(p,a)}$. There exists a sequence $0 < \alpha_1 < \alpha_2 < \cdots, \alpha_j \to \infty$, such that $L = \lim_{j\to\infty} \|m(\lambda_n/\alpha_j)\|_{(p,a)}$, and

$$\|F_{\alpha_j}\|_{p,a} \le (L + 1/j)\|g\|_{p,a}, \quad j \in N. \tag{3.1}$$

On the other hand, since m is bounded, $|m(x)| \le B$, $x \in (0,\infty)$, we have

$$\|F_{\alpha_j}\|_2 \le B\|g\|_2, \quad j \in N. \tag{3.2}$$

From (3.1) and (3.2) it follows that there exists a subsequence of $\{\alpha_j\}_{j\in N}$ (call it again $\{\alpha_j\}_{j\in N}$), such that $\{F_{\alpha_j}\}_{j\in N}$ converges weakly to a function F both, in $L^2(0,\infty)$ and $L^{p,a}(0,\infty)$. Moreover, (3.1) also gives

$$\|F\|_{p,a} \le L\|g\|_{p,a}.$$

To finish the proof we show that

$$F(x) = \mathcal{H}_\nu(m \cdot \mathcal{H}_\nu g)(x) \tag{3.3}$$

for almost every x in $(0,\infty)$.

For any given $N = 1, 2, \ldots$, we will use the decomposition

$$F_\alpha(x) = \chi_{(0,\alpha)}(x)\left(\sum_{1}^{N[\alpha]} + \sum_{N[\alpha]+1}^{\infty}\right) m(\lambda_n/\alpha)\langle g_\alpha, \psi_n^\nu\rangle \psi_n^\nu(x/\alpha)$$
$$= F_\alpha^N(x) + H_\alpha^N(x), \quad x \in (0, \infty).$$

Using (2.1) and the symmetry and homogeneity of the differential operator L_ν leads to

$$\langle g_\alpha, \psi_n^\nu\rangle = -(\alpha/\lambda_n)^2 \langle (L_\nu g)_\alpha, \psi_n^\nu\rangle, \quad n \in \boldsymbol{N}.$$

Fix $N \in \boldsymbol{N}$. Applying the above identity and Parseval's identity then gives

$$\begin{aligned}\int_0^\infty |H_\alpha^N(x)|^2 dx &= \alpha \sum_{N[\alpha]+1}^{\infty} |m(\lambda_n/\alpha)|^2 |\langle g_\alpha, \psi_n^\nu\rangle|^2 \\ &\le CB\alpha \sum_{N[\alpha]+1}^{\infty} |\alpha/n|^4 |\langle (L_\nu g)_\alpha, \psi_n^\nu\rangle|^2 \\ &\le C\frac{\alpha}{N^4}\sum_{1}^{\infty} |\langle (L_\nu g)_\alpha, \psi_n^\nu\rangle|^2 \\ &= \frac{C}{N^4}\int_0^\infty |L_\nu g(x)|^2 dx.\end{aligned}$$

Hence, we conclude that

$$\int_0^\infty |H_\alpha^N(x)|^2 dx = O(N^{-4})$$

uniformly in $\alpha \to \infty$.

Now, by invoking the diagonal argument, we can find a subsequence of $\{\alpha_j\}_{j\in \boldsymbol{N}}$ (call it $\{\alpha_j\}_{j\in \boldsymbol{N}}$) such that for every $N \in \boldsymbol{N}$, $\{H_{\alpha_j}^N\}_{j\in \boldsymbol{N}}$ is weakly convergent to a function H^N in $L^2(0, \infty)$. Clearly, $\|H^N\|_2 = O(N^{-2})$, and hence, for an increasing sequence $\{N_k\}_{k\in \boldsymbol{N}}$ of positive integers, $\{H^{N_k}\}_{k\in \boldsymbol{N}}$ converges to zero almost everywhere on $(0, \infty)$. By defining $F^{N_k} = F - H^{N_k}$, $k \in \boldsymbol{N}$, it is clear that $F_{\alpha_j}^{N_k} \to F^{N_k}$ weakly in $L^2(0, \infty)$, for every $k \in \boldsymbol{N}$. Moreover, $\{F^{N_k}\}_{k\in \boldsymbol{N}}$ converges to F almost everywhere on $(0, \infty)$.

We now prove that

$$\lim_{j\to\infty} F_{\alpha_j}^{N_k}(x) = \int_0^{\pi N_k} m(y)\mathcal{H}_\nu g(y)(xy)^{1/2} J_\nu(xy)dy \tag{3.4}$$

for every $x \in (0, \infty)$. Then, by weak convergence in $L^2(0, \infty)$, $\langle F_{\alpha_j}^{N_k}, \chi_{(r,s)}\rangle \to \langle F^{N_k}, \chi_{(r,s)}\rangle$ for arbitrary $0 < r < s < \infty$. Hence, by using the Lebesgue dominated convergence theorem (this is possible by means of Lemma 2.6), we will get

$$F^{N_k}(x) = \int_0^{\pi N_k} m(y)\mathcal{H}_\nu g(y)(xy)^{1/2} J_\nu(xy)dy$$

a.e. in $(0, \infty)$. Therefore, by letting $k \to \infty$, (3.3) will follow. Thus the proof of Theorem 2.1 will be finished.

To prove (3.4), fix again $N \in \boldsymbol{N}$ and $x \in (0, \infty)$ and write for α large

$$
\begin{aligned}
F_\alpha^N(x) &= \sum_1^{N[\alpha]} m(\lambda_n/\alpha)\langle g_\alpha, \psi_n^\nu\rangle \psi_n^\nu(x/\alpha) \\
&= \sum_1^{N[\alpha]} m(\lambda_n/\alpha)\mathcal{H}_\nu g(\lambda_n/\alpha)\Big(\frac{x}{\alpha}\lambda_n\Big)^{1/2} J_\nu\left(\frac{x}{\alpha}\lambda_n\right)\frac{d_{n,\nu}^2}{\alpha} \\
&= \sum_1^{N[\alpha]} m(\lambda_n/\alpha)\mathcal{H}_\nu g(\lambda_n/\alpha)\Big(\frac{x}{\alpha}\lambda_n\Big)^{1/2} J_\nu\left(\frac{x}{\alpha}\lambda_n\right)\cdot\frac{\lambda_{n+1}-\lambda_n}{\alpha} \\
&\quad + \sum_1^{N[\alpha]} m(\lambda_n/\alpha)\mathcal{H}_\nu g(\lambda_n/\alpha)\Big(\frac{x}{\alpha}\lambda_n\Big)^{1/2} J_\nu\left(\frac{x}{\alpha}\lambda_n\right)\cdot\frac{d_{n,\nu}^2-\lambda_{n+1}+\lambda_n}{\alpha}.
\end{aligned} \tag{3.5}
$$

Since $d_{n,\nu}^2 - \lambda_{n+1} + \lambda_n = O(n^{-1})$, by using (2.4) and (2.7) we bound the absolute value of the second sum in (3.5) by a constant multiplied by

$$\frac{1}{\alpha^{2\nu+2}} \sum_1^{N[\alpha]} n^{2\nu}.$$

Considering separately the cases $-1 < \nu < -1/2$, $\nu = -1/2$ and $\nu > -1/2$ easily shows that this bound is $o(1)$ as $\alpha \to \infty$. Now, it remains to note that the first sum in (3.5) approaches

$$\int_0^{\pi N} m(y)\mathcal{H}_\nu g(y)(xy)^{1/2} J_\nu(xy)dy.$$

This is clear when $\nu \geq -1/2$, since then the integrand is a bounded Riemann integrable function on $(0, \pi N)$ ($m(y)$ is such by assumption). If $-1 < \nu < -1/2$, then by (2.4) and (2.7), the integrand is $O(y^{2\nu+1})$ as $y \to 0^+$ and hence is Lebesgue integrable on $(0, \pi N)$. Proceeding as in the proof of Lemma 2.7, we take a $\delta > 0$ sufficiently small, cut off from zero and prove the claim. This finishes the proof of (3.4), and hence (3.3) and Theorem 2.1. □

PROOF OF THEOREM 2.2. Fix $g \in C_c^\infty(0, \infty)$ and $s > 0$. By assumption,

$$\left|\left\{x \in (0,1); \left|\sum_1^\infty m(\lambda_n/\alpha)\langle g_\alpha, \psi_n^\nu\rangle\psi_n^\nu(x)\right| > s\right\}\right| \leq \frac{1}{s}\|m(\lambda_n/\alpha)\|_{weak}\|g_\alpha\|_1,$$

which implies

$$|\{x \in (0,\infty); |F_\alpha(x)| > s\}| \leq \frac{1}{s}\|m(\lambda_n/\alpha)\|_{weak}\|g\|_1,$$

where F_α has the same meaning as in the proof of Theorem 2.1. Let $L = \liminf_{\alpha\to\infty} \|m(\lambda_n/\alpha)\|_{weak}$. We now choose a sequence $0 < \alpha_1 < \alpha_2 < \cdots$, $\alpha_j \to \infty$, and a function F in $L^2(0, \infty)$ such that $L = \lim_{j\to\infty} \|m(\lambda_n/\alpha_j)\|_{weak}$,

$$|\{x \in (0,\infty); |F_{\alpha_j}(x)| > s\}| \leq \frac{(L+1/j)}{s}\|g\|_1, \quad j \in \boldsymbol{N}, \tag{3.6}$$

and $\{F_{\alpha_j}\}_{j\in N}$ converges weakly to F in $L^2(0,\infty)$. Proceeding as in the proof of Theorem 2.1 (and keeping the notation), we prove that

$$F(x) = \mathcal{H}_\nu(m \cdot \mathcal{H}_\nu g)(x)$$

a.e. in $(0,\infty)$. So it remains only to check that

$$|\{x \in (0,\infty); |F(x)| > s\}| \le \frac{L}{s}\|g\|_1, \quad s > 0. \tag{3.7}$$

Recall that we have to our disposal a subsequence of $\{\alpha_j\}_{j\in N}$ (called again $\{\alpha_j\}_{j\in N}$), an increasing sequence $\{N_k\}_{k\in N}$ of positive integers, the decomposition

$$F_{\alpha_j} = F_{\alpha_j}^{N_k} + H_{\alpha_j}^{N_k},$$

and the $L^2(0,\infty)$ functions F^{N_k} satisfying the following properties:

(i) F^{N_k} converges to F a.e., $k \to \infty$;

(ii) for every $k = 1, 2, \dots$, $F_{\alpha_j}^{N_k}$ converges to F^{N_k} a.e., $j \to \infty$;

(iii) $\|H_{\alpha_j}^{N_k}\|_2^2 = O(N_k^{-4})$ uniformly in $j = 1, 2, \dots$.

We use the above properties to show (3.7). Fix $\delta > 0$. Fatou's lemma then gives

$$|\{|F(x)| > s\}| \le \liminf_{k\to\infty} |\{|F^{N_k}(x)| > s\}|.$$

Hence, for a subsequence of $\{N_k\}$ (call it again $\{N_k\}$)

$$|\{|F(x)| > s\}| \le |\{|F^{N_k}(x)| > s\}| + \delta. \tag{3.8}$$

Fix $k = 1, 2, \dots$. Fatou's lemma again gives

$$|\{|F^{N_k}(x)| > s\}| \le \liminf_{j\to\infty} |\{|F_{\alpha_j}^{N_k}(x)| > s\}|.$$

Hence, for a subsequence of $\{\alpha_j\}$ (call it again $\{\alpha_j\}$)

$$|\{|F^{N_k}(x)| > s\}| \le |\{|F_{\alpha_j}^{N_k}(x)| > s\}| + \delta. \tag{3.9}$$

By invoking the diagonal argument, we can assume that (3.9) holds for every $k, j \in \{1, 2, \dots\}$. Combining (3.8) and (3.9) then gives

$$|\{|F(x)| > s\}| \le |\{|F_{\alpha_j}^{N_k}(x)| > s\}| + 2\delta \tag{3.10}$$

for every $k, j \in \{1, 2, \dots\}$. We now have

$$\begin{aligned} |\{|F_{\alpha_j}^{N_k}(x)| > s\}| &= |\{|F_{\alpha_j}(x) - H_{\alpha_j}^{N_k}(x)| > s\}| \\ &\le |\{|F_{\alpha_j}(x)| > s(1-\delta)\}| + |\{|H_{\alpha_j}^{N_k}(x)| > s\delta\}|. \end{aligned} \tag{3.11}$$

By Chebyshev's inequality

$$|\{|H_{\alpha_j}^{N_k}(x)| > s\delta\}| \le \|H_{\alpha_j}^{N_k}\|_2^2/(s\delta)^2.$$

Hence $|\{|H_{\alpha_j}^{N_k}(x)| > s\delta\}|$ can be made arbitrarily small for sufficiently large k, uniformly in $j = 1, 2, \dots$. Let

$$|\{|H_{\alpha_j}^{N_k}(x)| > s\delta\}| \le \delta \tag{3.12}$$

for $k \geq k_o$ and $j = 1, 2, \ldots$. Combining (3.10), (3.11) and (3.12) now gives

$$|\{|F(x)| > s\}| \leq |\{|F_{\alpha_j}(x)| > s(1-\delta)\}| + 3\delta \,.$$

Finally, by using arbitrariness of δ and (3.6), letting $j \to \infty$ shows (3.7) and finishes the proof of Theorem 2.2. □

PROOF OF THEOREM 2.3. To prove Theorem 2.3 we first define

$$\mathcal{M}_{m,\varepsilon} f(x) = \mathcal{H}_\nu(m(\varepsilon\cdot)\mathcal{H}_\nu f(\cdot))(x)\,, \quad f \in C_c^\infty(0,\infty)\,,$$

and

$$\tilde{\mathcal{M}}_{m,\varepsilon} f(x) = \sum_1^\infty m(\varepsilon\lambda_n)\langle f, \psi_n^\nu\rangle \psi_n^\nu(x)\,, \quad f \in C_c^\infty(0,1)\,.$$

The following linearization result is a reformulation of [14, Lemma 1] to our situation.

LEMMA 3.1. *Let $1 < p < \infty$, $1/p + 1/q = 1$ and $a \in \boldsymbol{R}$. Assume that m is a function as in Theorem* 2.1. *Then*

(i) $\|\mathcal{M}_m^* f\|_{p,a} \leq C\|f\|_{p,a}$ *for all $f \in C_c^\infty(0,\infty)$ if and only if, with the same constant $C > 0$,*

$$\left\|\sum_1^K \mathcal{M}_{m,R_k}(g_k)\right\|_{q,a} \leq C\left\|\sum_1^K |g_k|\right\|_{q,a}, \tag{3.13}$$

for every finite sequence $\{g_k\}_{k=1}^K$ of $C_c^\infty(0,\infty)$ functions and every finite sequence $\{R_k\}_{k=1}^K$ of positive numbers.

(ii) $\|\tilde{\mathcal{M}}_m^* f\|_{p,a} \leq C\|f\|_{p,a}$ *for all $f \in C_c^\infty(0,1)$ if and only if, with the same constant $C > 0$,*

$$\left\|\sum_1^K \tilde{\mathcal{M}}_{m,R_k}(h_k)\right\|_{q,a} \leq C\left\|\sum_1^K |h_k|\right\|_{q,a}, \tag{3.14}$$

for every finite sequence $\{h_k\}_{k=1}^K$ of $C_c^\infty(0,1)$ functions and every finite sequence $\{R_k\}_{k=1}^K$ of positive numbers.

Thus, we are now reduced to showing that (3.14) implies (3.13). Let $K \in \boldsymbol{N}$. Choose a sequence of functions $g_k \in C_c^\infty(0,\infty)$ and a sequence of numbers $R_k > 0$, $k = 1, 2, \ldots, K$. To simplify the notation, write $g_{k,\alpha}$ in place of $(g_k)_\alpha$, and assume α is so large that all $g_{k,\alpha}$, $k = 1, 2, \ldots, K$, are supported in $(0,1)$. By assumption, for every subset S of $\{1, 2, \ldots, K\}$

$$\left\|\sum_{k\in S} \tilde{\mathcal{M}}_{m,R_k}(g_{k,\alpha})\right\|_{q,a} \leq C\left\|\sum_{k\in S} |g_{k,\alpha}|\right\|_{q,a}$$

and a change of variables then gives

$$\left\|\sum_{k\in S} F_{k,\alpha R_k}\right\|_{q,a} \leq C\left\|\sum_{k\in S} |g_k|\right\|_{q,a}, \tag{3.15}$$

where we let to denote

$$F_{k,\alpha}(x) = \chi_{(0,\alpha)}(x) \sum_1^\infty m(\lambda_n/\alpha)\langle g_{k,\alpha}, \psi_n^\nu\rangle \psi_n^\nu(x/\alpha) .$$

In particular, in (3.15) we can consider an S consisting of a single element. Also, (3.15) holds true, by Parseval's identity, for $q = 2$ and $a = 0$. Now we proceed as in the proof of Theorem 2.1: first we choose an α-sequence good for any $k = 1, 2, \dots, K$ and corresponding weak limits, and then we obtain required vector valued inequalities. □

PROOF OF THEOREM 2.4. Since the present proof mimics the proof of Theorem 2.1, we will skip a large part of details. Fix $g \in C_c^\infty(0, \infty)$ and, for sufficiently large α, consider the function

$$G_\alpha(x) = \chi_{(0,\alpha)}(x) \sum_1^\infty \langle g_\alpha, \psi_n^\mu\rangle \psi_n^\nu(x/\alpha) , \quad x \in (0, \infty) .$$

By assumption,

$$\|G_\alpha\|_{p,a} \le C\|g\|_{p,a} .$$

On the other hand, Plancherel's identity leads to $\|G_\alpha\|_2 = \|g\|_2$.

There exist a sequence $0 < \alpha_1 < \alpha_2 < \cdots, \alpha_j \to \infty$, and a function $G \in L^{p,a}(0, \infty) \cap L^2(0, \infty)$ such that $\{G_{\alpha_j}\}_{j \in N}$ converges weakly to G in $L^{p,a}(0, \infty)$ and $L^2(0, \infty)$. Moreover, $\|G\|_{p,a} \le C\|g\|_{p,a}$. It is now sufficient to show that

$$G(x) = (\mathcal{H}_\nu \circ \mathcal{H}_\mu)g(x) \tag{3.16}$$

a.e. on $(0, \infty)$. This will be achieved by showing that

$$\lim_{\alpha \to \infty} \sum_1^{N[\alpha]} \langle g_\alpha, \psi_n^\mu\rangle \psi_n^\nu(x/\alpha) = \int_0^{\pi N} (xy)^{1/2} J_\nu(xy)\mathcal{H}_\mu g(y)dy , \tag{3.17}$$

for every given $x \in (0, \infty)$ and $N \in N$, and

$$\int_0^\alpha \left| \sum_{N[\alpha]+1}^\infty \langle g_\alpha, \psi_n^\mu\rangle \psi_n^\nu(x/\alpha)\right|^2 dx = O(N^{-4}) \tag{3.18}$$

uniformly in $\alpha \to \infty$.

We start with proving (3.18). By using Parseval's identity, we have

$$\begin{aligned}\int_0^\alpha \left| \sum_{N[\alpha]+1}^\infty \langle g_\alpha, \psi_n^\mu\rangle \psi_n^\nu(x/\alpha)\right|^2 dx &= \alpha \int_0^1 \left| \sum_{N[\alpha]+1}^\infty \langle g_\alpha, \psi_n^\mu\rangle \psi_n^\nu(x)\right|^2 dx \\ &= \alpha \int_0^1 \left| \sum_{N[\alpha]+1}^\infty \left(\frac{\alpha}{\lambda_{n,\mu}}\right)^2 \langle (L_\mu g)_\alpha, \psi_n^\mu\rangle \psi_n^\nu(x)\right|^2 dx\end{aligned}$$

$$= \alpha \sum_{N[\alpha]+1}^{\infty} \left| \left(\frac{\alpha}{\lambda_{n,\mu}} \right)^2 \langle (L_\mu g)_\alpha, \psi_n^\mu \rangle \right|^2$$
$$\le C \frac{\alpha}{N^4} \sum_{N[\alpha]+1}^{\infty} |\langle (L_\mu g)_\alpha, \psi_n^\mu \rangle|^2$$
$$\le \frac{C}{N^4} \int_0^\infty |L_\mu g(y)|^2 dy .$$

Returning to (3.17), note that

$$\langle g_\alpha, \psi_n^\mu \rangle = \frac{d_{n,\mu}}{\alpha} \mathcal{H}_\mu g(\lambda_{n,\mu}/\alpha) .$$

Hence, by using (2.3),

$$\sum_1^{N[\alpha]} \langle g_\alpha, \psi_n^\mu \rangle \psi_n^\nu \left(\frac{x}{\alpha} \right)$$
$$= \sum_1^{N[\alpha]} (\pi^{1/2} + O(n^{-1})) \left(\frac{x}{\alpha} \lambda_{n,\nu} \right)^{1/2} J_\nu \left(\frac{x}{\alpha} \lambda_{n,\nu} \right) (\pi^{1/2} + O(n^{-1})) \mathcal{H}_\mu g \left(\frac{\lambda_{n,\mu}}{\alpha} \right) \cdot \frac{1}{\alpha}$$
$$= \frac{\pi}{\alpha} \sum_1^{N[\alpha]} \left(\frac{x}{\alpha} \lambda_{n,\nu} \right)^{1/2} J_\nu \left(\frac{x}{\alpha} \lambda_{n,\nu} \right) \mathcal{H}_\mu g \left(\frac{\lambda_{n,\mu}}{\alpha} \right) + R_{N,\alpha}(x) .$$

The absolute value of the remainder $R_{N,\alpha}(x)$ is bounded by a constant multiplied by

$$\alpha^{-(\nu+\mu+2)} \sum_1^{N[\alpha]} n^{\nu+\mu} . \tag{3.19}$$

Considering separately the cases $-2 < \nu + \mu < -1$, $\nu + \mu = -1$ and $\nu + \mu > -1$ easily shows that this is $o(1)$ as $\alpha \to \infty$. Hence it remains to check that

$$\begin{aligned} &\lim_{\alpha \to \infty} \frac{\pi}{\alpha} \sum_1^{N[\alpha]} \left(\frac{x}{\alpha} \lambda_{n,\nu} \right)^{1/2} J_\nu \left(\frac{x}{\alpha} \lambda_{n,\nu} \right) \mathcal{H}_\mu g \left(\frac{\lambda_{n,\mu}}{\alpha} \right) \\ &= \int_0^{\pi N} (xy)^{1/2} J_\nu(xy) \mathcal{H}_\mu g(y) dy . \end{aligned} \tag{3.20}$$

The mean value theorem allows to write

$$\left(\frac{x}{\alpha} \lambda_{n,\nu} \right)^{1/2} J_\nu \left(\frac{x}{\alpha} \lambda_{n,\nu} \right) - \left(\frac{x}{\alpha} \lambda_{n,\mu} \right)^{1/2} J_\nu \left(\frac{x}{\alpha} \lambda_{n,\mu} \right)$$
$$= \frac{x}{\alpha} (\lambda_{n,\nu} - \lambda_{n,\mu}) \frac{d}{dy} (y^{1/2} J_\nu(y)) \bigg|_{y=y_o} ,$$

where y_o is a value between $x\lambda_{n,\nu}/\alpha$ and $x\lambda_{n,\mu}/\alpha$. Differential properties of Bessel functions give

$$\frac{d}{dy} (y^{1/2} J_\nu(y)) = (\nu + 1/2) y^{\nu - 1/2} y^{-\nu} J_\nu(y) - y^{\nu+3/2} y^{-(\nu+1)} J_{\nu+1}(y) .$$

Therefore

$$\left|\frac{d}{dy}(y^{1/2}J_\nu(y))\right| = O(y^{\nu-1/2}), \quad y\to 0^+$$

and

$$\frac{\pi}{\alpha}\sum_1^{N[\alpha]}\left(\frac{x}{\alpha}\lambda_{n,\nu}\right)^{1/2} J_\nu\left(\frac{x}{\alpha}\lambda_{n,\nu}\right)\mathcal{H}_\mu g\left(\frac{\lambda_{n,\mu}}{\alpha}\right)$$
$$= \frac{\pi}{\alpha}\sum_1^{N[\alpha]}\left(\frac{x}{\alpha}\lambda_{n,\mu}\right)^{1/2} J_\nu\left(\frac{x}{\alpha}\lambda_{n,\mu}\right)\mathcal{H}_\mu g\left(\frac{\lambda_{n,\mu}}{\alpha}\right) + P_{N,\alpha}(x),$$

where, as it is easily seen, the absolute value of the remainder $P_{N,\alpha}(x)$ is bounded by a constant multiplied by (3.19). Hence, again $P_{N,\alpha}(x) = o(1)$ as $\alpha\to\infty$, and (3.20) follows (cutting off from zero is necessary in case $-1<\nu<-1/2$; the argument has been already discussed in the proof of Lemma 2.7 or Theorem 2.1). The proof of Theorem 2.4 is completed. □

4. Applications. The modified Hankel transform H_ν, $\nu>-1$, is defined by

$$H_\nu f(x) = \int_0^\infty \frac{J_\nu(xy)}{(xy)^\nu} f(y)dm_\nu(y), \quad x>0,$$

where $dm_\nu(y) = y^{2\nu+1}dy$ and f is a suitable function on $(0,\infty)$. If $\nu = (n-2)/2$, $n = 2,3,\dots$, the modified Hankel transform H_ν replaces the Fourier transform of radial functions in $\boldsymbol{R}^n$. Clearly, both Hankel transforms are related to each other by

$$\mathcal{H}_\nu f(x) = x^{\nu+1/2}H_\nu((\cdot)^{-(\nu+1/2)}f(\cdot))(x). \tag{4.1}$$

Hence, by Lemma 2.7, the inversion formula

$$f(x) = \int_0^\infty \frac{J_\nu(xy)}{(xy)^\nu} H_\nu f(y)dm_\nu(y), \quad x>0,$$

and Plancherel's identity

$$\|H_\nu f\|_{L^2(dm_\nu)} = \|f\|_{L^2(dm_\nu)}$$

hold for any $f\in C_c^\infty(0,\infty)$ and $\nu>-1$. Related to the continuous transformation H_ν are discrete expansions with respect to the complete and orthonormal in $L^2((0,1),dm_\nu)$ system of functions

$$\phi_n^\nu(x) = c_{n,\nu}J_\nu(\lambda_n x)/(\lambda_n x)^\nu, \quad n=1,2,\dots,$$

where $c_{n,\nu} = d_{n,\nu}\lambda_n^{\nu+1/2}$. Multipliers, weak multipliers, multiplier norms, maximal multiplier operators for the modified Hankel transform H_ν and for $\{\phi_n^\nu\}$-expansions are defined analogously to the previous situation (we abuse slightly the notation by using the symbols $\|\cdot\|_{(p,a)}$ and $\|\cdot\|_{weak}$; now they refer to H_ν or $\{\phi_n^\nu\}$-expansions). Then we have the following analogues of Theorems 2.1 through 2.3.

THEOREM 4.1. *Let $1<p<\infty$, $a\in\boldsymbol{R}$ and $m(x)$ be a function on $(0,\infty)$ as in Theorem 2.1. If $\{m(\varepsilon\lambda_n)\}$ is a $\{\phi_n^\nu\}$-multiplier in $L^p((0,1),x^a dm_\nu)$ for all sufficiently small $\varepsilon>0$*

and $\liminf_{\varepsilon\to 0^+} \|m(\varepsilon\lambda_n)\|_{(p,a)}$ *is finite, then* $m(x)$ *is an* H_ν*-multiplier in* $L^p((0,\infty), x^a dm_\nu)$ *and*

$$\|m(x)\|_{(p,a)} \le \liminf_{\varepsilon\to 0^+} \|m(\varepsilon\lambda_n)\|_{(p,a)} .$$

THEOREM 4.2. *Let* $m(x)$ *be a function on* $(0,\infty)$ *as in Theorem* 2.1. *If* $\{m(\varepsilon\lambda_n)\}$ *is a weak* $L^1((0,1), dm_\nu)$ *Fourier-Bessel multiplier for all sufficiently small* $\varepsilon > 0$ *and* $\liminf_{\varepsilon\to 0^+} \|m(\varepsilon\lambda_n)\|_{weak}$ *is finite, then* $m(x)$ *is a weak* $L^1((0,\infty), dm_\nu)$ H_ν*-Hankel transform multiplier and*

$$\|m(x)\|_{weak} \le \liminf_{\varepsilon\to 0^+} \|m(\varepsilon\lambda_n)\|_{weak} .$$

THEOREM 4.3. *Let* $1 < p < \infty$, $a \in \boldsymbol{R}$ *and* $m(x)$ *be a function on* $(0,\infty)$ *as in Theorem* 2.1. *If*

$$\|\tilde{M}_m^* f\|_{L^p((0,1),x^a dm_\nu)} \le C\|f\|_{L^p((0,1),x^a dm_\nu)}$$

with a constant $C > 0$ *independent of* f *in* $C_c^\infty(0,1)$, *then*

$$\|M_m^* f\|_{L^p((0,\infty),x^a dm_\nu)} \le C\|f\|_{L^p((0,\infty),x^a dm_\nu)}$$

independently of f *in* $C_c^\infty(0,\infty)$ *(with the same constant C).*

Wing [25] proved that the partial sum operators

$$S_N^\psi f(x) = \sum_1^N \langle f, \psi_n^\nu\rangle_{L^2((0,1),dx)} \psi_n^\nu(x)$$

for the $\{\psi_n^\nu\}$-expansions, $\nu \ge -1/2$, are uniformly bounded with $N \to \infty$ in any $L^p((0,1), dx)$, $1 < p < \infty$. Benedek and Panzone [2] then extended this result to $-1 < \nu < -1/2$ and the p-range $2/(2\nu+3) < p < -2/(2\nu+1)$. Theorem 2.1 thus gives (with $m = \chi_{(0,1)}$ and $a = 0$).

COROLLARY 4.4. *Let either* $\nu \ge -1/2$ *and* $1 < p < \infty$ *or* $-1 < \nu < -1/2$ *and* $2/(2\nu+3) < p < -2/(2\nu+1)$. *Then the Hankel transform partial sum operators*

$$S_R^{\mathcal{H}} f(x) = \mathcal{H}_\nu(\chi_{(0,R)} \cdot \mathcal{H}_\nu f)(x), \quad R > 0,$$

are uniformly bounded in $L^p((0,\infty), dx)$.

Observe that uniform boundedness of $S_R^{\mathcal{H}} f$ in the case $\nu \ge -1/2$ is known as Wing's theorem [26], but the result in the case $-1 < \nu < -1/2$ seems to be new.

A slight modification of Wing's argument from [25] shows that the partial sum operators

$$S_N^\phi f(x) = \sum_1^N \langle f, \phi_n^\nu\rangle_{L^2((0,1),dm_\nu)} \phi_n^\nu(x)$$

for the $\{\phi_n^\nu\}$-expansions, $\nu \ge -1/2$, are uniformly bounded with $N \to \infty$ in any $L^p((0,1), dm_\nu)$, $4(\nu+1)/(2\nu+3) < p < 4(\nu+1)/(2\nu+1)$. Another modification, of Benedek and Panzone's result [2], then extends this result to $-1 < \nu < -1/2$ and the p-range $1 < p < \infty$. Theorem 4.1 hence gives (with $m = \chi_{(0,1)}$ and $a = 0$).

COROLLARY 4.5. *Let either* $\nu \geq -1/2$ *and* $4(\nu+1)/(2\nu+3) < p < 4(\nu+1)/(2\nu+1)$ *or* $-1 < \nu < -1/2$ *and* $1 < p < \infty$. *Then the modified Hankel transform partial sum operators*

$$S_R^H f(x) = H_\nu(\chi_{(0,R)} \cdot H_\nu f)(x), \quad R > 0,$$

are uniformly bounded in $L^p((0,\infty), dm_\nu)$.

Observe that the uniform boundedness of $S_R^H f$ in the case $\nu \geq -1/2$ is known as Herz' theorem [9], but, as before, the result for the case $-1 < \nu < -1/2$ seems to be new. Let us also mention that an alternative approach to the conclusion of Corollary 4.4 in the case $\nu \geq -1/2$ is the following: modifying Wing's argument allows proving uniform boundedness of S_N^ψ in $L^p((0,1), x^a dx)$ with a in the A_p-range $-1 < a < p-1$. Hence, by Theorem 2.1, the conclusion of Corollary 4.4 holds in the weighted setting, and the relation (4.1) then shows uniform boundedness of S_R^H, $R > 0$, within the corresponding p-interval. Therefore, we could say that both, Wing and Herz results for the Hankel transforms, are consequences of weighted results for $\{\psi_n^\nu\}$-expansions.

Boundedness of maximal operator associated to the partial sums for the modified Hankel transform was proved by Prestini [17]. An alternative proof of this result was given by Kanjin [13], by transfering a corresponding result for the partial sums of Jacobi expansions to the Hankel transform setting. As we already mentioned, our transference result, Theorem 2.3 (or, rather, its modified version, Theorem 4.3) substitutes Jacobi by Fourier-Bessel expansions in Kanjin's result. In fact, estimates of maximal operators for the partial sums of Fourier-Bessel expansions were known since the paper of Gilbert [6]. In Theorem 1 of [6] a general maximal transplantation theorem was proved that allowed, for instance, to transplant L^p Carleson-Hunt maximal inequalities for the trigonometric system to fairly general systems (with Fourier-Bessel expansions included). This result was put into weighted setting in [7] (see also [23] where a direct approach is presented following Prestini's ideas). Therefore, the aforementioned transplanted L^p estimates can be used, via the transference results from Theorem 2.3 and Theorem 4.3, to obtain the corresponding results for the Hankel transforms.

Gilbert's result [6, Theorem 1], and its weighted extension [7, Theorem 1] also give a weighted transplantation theorem for Fourier-Bessel expansions (the unweighted case is stated as Theorem A and Theorem B in [6]). To be precise, for $\nu, \mu \geq -1/2$, $1 < p < \infty$ and $-1 < a < p-1$, the transplantation inequality

$$\left\| \sum_1^\infty c_n^\mu(f) \psi_n^\nu(x) \right\|_{p,a} \leq C \|f\|_{p,a}$$

holds true with $C > 0$ independent of $f \in C_c^\infty(0,1)$. Consequently, Theorem 2.4 gives a weighted transplantation inequality for the Hankel transform:

$$\|(\mathcal{H}_\nu \circ \mathcal{H}_\mu) f\|_{p,a} \leq C \|f\|_{p,a}, \quad f \in C_c^\infty(0,\infty).$$

This inequality is known as Guy's transplantation theorem (cf. [8] and also Schindler's paper [18] for an interesting alternative proof based on an explicit kernel formula).

The above remarks led to the following general conclusions. Harmonic analysis results for the Hankel transform of arbitrary order $\nu \geq -1/2$ are consequences of corresponding results for the cosine expansions. This is possible first, by transplanting cosine expansions results (that correspond to the case $\nu = -1/2$) to the Fourier-Bessel $\{\psi_n^\nu\}$-expansions with arbitrary $\nu \geq -1/2$, and then by transferring this result from $\{\psi_n^\nu\}$-expansions to the Hankel transform $\mathcal{H}_\nu$ setting. The first step is done by using Gilbert's transplantation result, and the second step is done by using our transference results. In case we use weighted setting for cosine expansions, usually with the A_p-range of power weights, then weighted version of Gilbert's theorem and our transference results give the corresponding weighted results for $\mathcal{H}_\nu$. Consequently, these can be translated to the modified Hankel transform H_ν setting by using (4.1).

REFERENCES

[1] A. BENEDEK AND R. PANZONE, Mean convergence of series of Bessel functions, Rev. Un. Mat. Argentina 26 (1972), 42–61.

[2] A. BENEDEK AND R. PANZONE, On mean convergence of Fourier-Bessel series of negative order, Stud. Appl. Math. 50 (1971), 281–292.

[3] A. BENEDEK AND R. PANZONE, Pointwise convergence of series of Bessel functions, Rev. Un. Mat. Argentina 26 (1972), 167–186.

[4] W. C. CONNETT AND A. L. SCHWARTZ, Weak type multipliers for Hankel transforms, Pacific J. Math. 63 (1976), 125–129.

[5] G. GASPER AND W. TREBELS, Jacobi and Hankel multipliers of type (p, q), $1 < p < q < \infty$, Math. Ann. 237 (1978), 243–251.

[6] J. E. GILBERT, Maximal theorems for some orthogonal series, I, Trans. Amer. Math. Soc. 145 (1969), 495–515.

[7] J. J. GUADALUPE, M. PEREZ, F. J. RUIZ AND J. L. VARONA, Two notes on convergence and divergence a.e. of Fourier series with respect to some orthogonal systems, Proc. Amer. Math. Soc. 116 (1992), 457–464.

[8] D. L. GUY, Hankel multiplier transformations and weighted p-norms, Trans. Amer. Math. Soc. 95 (1960), 137–189.

[9] C. S. HERZ, On the mean inversion of Fourier and Hankel transforms, Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 996–999.

[10] H. HOCHSTADT, The mean convergence of Fourier-Bessel series, SIAM Rev. 9 (1967), 211–218.

[11] S. IGARI, Functions of L^p-multipliers, Tôhoku Math. J. 21 (1969), 304–320.

[12] S. IGARI, On the multipliers of Hankel transform, Tôhoku Math. J. 24 (1972), 201–206.

[13] Y. KANJIN, Convergence and divergence almost everywhere of spherical means for radial functions, Proc. Amer. Math. Soc. 103 (1988), 1063–1069.

[14] C. E. KENIG AND P. A. TOMAS, Maximal operators defined by Fourier multipliers, Studia Math. 68 (1980), 79–83.

[15] K. DE LEEUW, On L_p-multipliers, Ann. Math. 81 (1965), 364–379.

[16] N. N. LEBEDEV, Special functions and its applications, Dover, New York, 1972.

[17] E. PRESTINI, Almost everywhere convergence of the spherical partial sums for radial functions, Monatsh. Math. 105 (1988), 207–216.

[18] S. SCHINDLER, Explicit integral transform proofs of some transplantation theorems for the Hankel transform, SIAM J. Math. Anal. 4 (1973), 367–384.

[19] K. STEMPAK, On connections between Hankel, Leguerre and Heisenberg multipliers, J. London Math. Soc. (2) 51 (1995), 286–298.

[20] K. STEMPAK, Transplanting maximal inequalities between Laguerre and Hankel multipliers, Monatsh. Math. 122 (1996), 187–197.

[21] K. STEMPAK, A transplantation theorem for Fourier-Bessel coefficients, Anal. Math. 24 (1998), 311–318.

[22] K. STEMPAK, On connections between Hankel, Laguerre and Jacobi transplantation, preprint, 1998.
[23] K. STEMPAK, On convergence and divergence of Fourier-Bessel series, preprint, 1999.
[24] G. N. WATSON, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 1958.
[25] G. M. WING, The mean convergence of orthogonal series, Amer. J. Math. 72 (1950), 792–808.
[26] G. M. WING, On the L^p theory of Hankel transforms, Pacific J. Math. 1 (1951), 313–319.
[27] A. H. ZEMANIAN, Generalized integral transformations, Interscience Publishers, New York, 1968.

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