Tohoku Math. J.
53 (2001), 81–93

MOMENT DECAY RATES OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

KAI LIU AND ANYUE CHEN

(Received April 27, 1999)

Abstract. The objective of this paper is to investigate the p-th moment asymptotic stability decay rates for certain finite-dimensional Itô stochastic differential equations. Motivated by some practical examples, the point of our analysis is a special consideration of general decay speeds, which contain as a special case the usual exponential or polynomial type one, to meet various situations. Sufficient conditions for stochastic differential equations (with variable delays or not) are obtained to ensure their asymptotic properties. Several examples are studied to illustrate our theory.

Introduction. So much effort has been devoted to the study of optimal control and filtering of stochastic differential equations. In practice, even from probability theory viewpoint, stability of stochastic differential equations is also important. There is an increasing requirement to study stability for a number of problems from, for instance, physics, biology and stochastic control, etc. in the sense of either p-th moment or almost sure. As a matter of fact, there exists an extensive literature in exponential stability of stochastic differential equations. We mention here Arnold [1], [2], Arnold, Oeljeklaus and Pardoux [3], Chappell [4] and Has'minskii [5] among others. On the other hand, as is well-known, not all stochastic systems are exponentially stable. However, it is worth pointing out that some of them are indeed stable but subject to a certain lower decay rate which is different from exponential decay, for instance, polynomial or logarithmic one. In particular, for stochastic differential equation theory itself it appears to be useful to extend the usual exponential stability concepts to a more general stable decay function. Let us start with the following examples for our motivation of the work.

EXAMPLE 0.1. Consider a one-dimensional Itô stochastic differential equation

$$dX_t = \frac{p}{1+t}X_t dt + (1+t)^{-p} dW_t\,, \quad t \geq 0\,, \tag{0.1}$$

with initial real data $X_0 = x_0 \in \boldsymbol{R}^1$, an $\mathcal{F}_0$-measurable random variable with finite second-order moment. Here $p > 1/2$ is a constant and W_t is a one-dimensional Brownian motion.

It is easy to obtain the explicit solution

$$X_t = (x_0 + W_t)(1+t)^{-p}\,, \quad t \geq 0\,.$$

2000 *Mathematics Subject Classification.* Primary 93E03; Secondary 60H10.

By a direct computation and using properties of one-dimensional Brownian motion, we get immediately that

$$\lim_{t\to\infty} \frac{\log EX_t(x_0)^2}{t} = 0 .$$

That is, the solution is not mean square stable with exponential decay. However, we can deduce that whenever $p > 1/2$, the solution is mean square stable with polynomial decay. Moreover, we have

$$\limsup_{t\to\infty} \frac{\log EX_t(x_0)^2}{\log t} \leq -(2p-1) .$$

EXAMPLE 0.2. Consider a scalar linear Itô equation

$$dX_t = -\frac{X_t}{(1+t)\log(1+t)}dt + e^{-t}X_t dW_t , \quad t \geq 0 , \tag{0.2}$$

with initial data $X_0 = x_0 \in \boldsymbol{R}^1$, an $\mathcal{F}_0$-measurable random variable with finite second-order moment, where W_t is a one-dimensional Brownian motion.

We can easily obtain the explicit solution

$$X_t = x_0 \exp\left\{ \int_0^t \left[-\frac{1}{(1+s)\log(1+s)} - \frac{1}{2}e^{-2s} \right] ds + \int_0^t e^{-s} dW_s \right\} .$$

Using the exponential martingale properties, it can be deduced that

$$\log EX_t(x_0)^2 = \log Ex_0^2 - 2\int_0^t \frac{1}{(1+s)\log(1+s)}ds + \frac{3}{2}\int_0^t e^{-2s} ds ,$$

which immediately implies that

$$\lim_{t\to\infty} \frac{\log EX_t(x_0)^2}{\log t} = 0 .$$

In other words, the solution on this occasion is not mean square stable with polynomial decay. However, we have the following logarithmic decay stability

$$\limsup_{t\to\infty} \frac{\log EX_t(x_0)^2}{\log\log t} \leq -2 .$$

Motivated by the examples above, in this paper we will carry out a Lyapunov function programme to study stability of stochastic differential systems with a general decay rate. There have been several expositions to treat these sort of decay rates different from exponential one. For instance, in [8] and [16] the polynomial type decay rate was studied to establish the stability of the traveling wave solutions of a class of determined hyperbolic systems with relaxation. For stochastic differential systems, the most original work on this aspect goes at least back to Has'minskii [5] for some possible consideration of certain decays different from exponential one. More recently, Mao [13], [15] studied the polynomial decay in a systematic way in the sense of pathwise with probability one for a class of stochastic differential equations with respect to Brownian motion. For a class of perturbed stochastic differential equations with respect to semimartingales, under some circumstances Mao [14] considered the same kind of decays once more in the sense of almost sure. For the general consideration of decay rate, in [12] a careful investigation has been carried out for non-autonomous stochastic differential equations with respect to Brownian motion in the sense of almost sure. In [9] and [10] some results above have also been generalized to cover infinite dimensional stochastic evolution equation cases. In this paper, we shall devote ourselves to the investigation of a class of non-autonomous stochastic differential equations for a general stability decay rate similarly to [12] but in a p-th moment sense.

1. Stability of Itô equations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ be a complete probability space with the filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual conditions, i.e., $\{\mathcal{F}\}_{t\geq 0}$ is right continuous and $\mathcal{F}_0$ contains all P-null sets. Let $W_t = (W_t^1, W_t^2, \dots, W_t^m)$ be an m-dimensional standard Brownian motion with $W_0 = 0$. Consider the following n-dimensional stochastic differential equation:

$$\begin{cases} dX_t = f(X_t, t)dt + g(X_t, t)dW_t\,, & t \geq 0\,, \\ X_0 = x_0\,, \end{cases} \tag{1.1}$$

where $f(x,t) = (f^1, \dots, f^n)^T : \boldsymbol{R}^n \times \boldsymbol{R}^+ \to \boldsymbol{R}^n$, $g(x,t) = (g^{ij})_{n\times m} : \boldsymbol{R}^n \times \boldsymbol{R}^+ \to \boldsymbol{R}^{n\times m}$ are two Borel measurable functions and x_0 is an $\mathcal{F}_0$-measurable $\boldsymbol{R}^n$-valued random vector. In particular, since we shall restrict ourselves to stability analysis, one always assumes throughout this paper that Equation (1.1) has a unique global solution which is denoted by $X_t(x_0) \in \boldsymbol{R}^n$. We note that under the conditions (1) and (2) of the following Theorems 1.1 and 1.2, the stochastic differential equations (1.1) has a unique global solution which is denoted by $X_t(x_0)$. In fact, for the family $\{V(t,x), \psi_1(t), \psi_2(t)$ (resp. $\psi_3(t))\}$ of functions as in Theorem 1.1 (resp. Theorem 1.2), set $U(t,x) = 1+V(t,x)$ and $\Psi(t) = \psi_1(t)+\psi_2(t)$ (resp. $\psi_3(t)$). Then $LU(t,x) \leq \Psi(t)U(t,x)$, where L is the differential generator associated with (1.1), and also $\lim_{|x|\to\infty} U(t,x) = \infty$. Namely, $U(t,x)$ is a radially unbounded Lyapunov function satisfying a sufficient condition for nonoccurrence of an explosion, which guarantees the pathwise uniqueness of a global solution for (1.1). For explosion criteria, see Has'minski [5, pp. 84–86 and p. 186] and Narita [6], [7].

Before proceeding to our stability arguments, let us firstly give the precise definition of the p-th moment stability with general decay rate $\lambda(t)$.

DEFINITION 1.1. Assume that $\lambda(t) \uparrow +\infty$, as $t \to +\infty$, and satisfies $\lambda(t+s) \leq \lambda(s)\lambda(t)$ for $s, t \in \boldsymbol{R}^+$ largely enough. Equation (1.1) is then said to be the p-th momently stable, $p > 0$, with decay $\lambda(t)$ of order $\gamma > 0$ if there exist a pair of positive constants $\gamma > 0$ and $C(x_0) > 0$ such that

$$E|X_t(x_0)|^p \leq C(x_0) \cdot \lambda(t)^{-\gamma}\,, \quad t \geq 0$$

holds for any $X_0 = x_0 \in \boldsymbol{R}^n$, an $\mathcal{F}_0$-measurable random vector, or equivalently,

$$\limsup_{t\to\infty} \frac{\log E|X_t(x_0)|^p}{\log \lambda(t)} \leq -\gamma\,.$$

REMARK. Clearly, replacing the decay function $\lambda(t)$ by e^t, $1+t$ and $\log(1+t)$ leads to the usual stability behavior with exponential, polynomial and logarithmic decays, respectively.

Let $C^{2,1}(\boldsymbol{R}^n \times \boldsymbol{R}^+; \boldsymbol{R}^+)$ denote the family of all functions $V(x,t) : \boldsymbol{R}^n \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ with continuous second-order partial derivatives in x and first-order partial derivatives in t. If $V(x,t) \in C^{2,1}(\boldsymbol{R}^n \times \boldsymbol{R}^+; \boldsymbol{R}^+)$, we define an operator L by

$$LV(x,t) := \frac{\partial}{\partial t}V(x,t) + \sum_{i=1}^{n} f^i(x,t)\frac{\partial}{\partial x_i}V(x,t) + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} g^{ik}(x,t)g^{jk}(x,t)\frac{\partial^2}{\partial x_i \partial x_j}V(x,t).$$

THEOREM 1.1. *Let $V(x,t) \in C^{2,1}(\boldsymbol{R}^n \times \boldsymbol{R}^+; \boldsymbol{R}^+)$ and $\psi_1(t)$, $\psi_2(t)$ be two continuous non-negative functions. Assume that for all $x \in \boldsymbol{R}^n$, $t \in \boldsymbol{R}^+$, there exist positive constants $p > 0$, $m > 0$ and real numbers ν, θ such that*

(1) $|x|^p \lambda(t)^m \leq V(x,t), \quad (x,t) \in \boldsymbol{R}^n \times \boldsymbol{R}^+;$

(2) $LV(x,t) \leq \psi_1(t) + \psi_2(t)V(x,t), \quad (x,t) \in \boldsymbol{R}^n \times \boldsymbol{R}^+;$

(3) $\displaystyle \limsup_{t\to\infty} \frac{\log\left(\int_0^t \psi_1(s)ds\right)}{\log \lambda(t)} \leq \nu, \quad \limsup_{t\to\infty} \frac{\int_0^t \psi_2(s)ds}{\log \lambda(t)} \leq \theta.$

Then, whenever $\gamma := m - \theta - \nu > 0$, the solution of Equation (1.1) *is the p-th momently stable with decay $\lambda(t)$. Moreover, we have*

$$\limsup_{t\to\infty} \frac{\log E|X_t(x_0)|^p}{\log \lambda(t)} \leq -\gamma. \tag{1.2}$$

PROOF. By Itô's formula and the definition of L, we can derive that

$$V(X_t,t) = V(x_0,0) + \int_0^t LV(X_s,s)ds + \int_0^t \sum_{i=1}^{n}\sum_{k=1}^{m} g^{ik}(X_s,s)\frac{\partial}{\partial x_i}V(X_s,s)dW_s^k. \tag{1.3}$$

Since the diffusion term

$$\int_0^t \sum_{i=1}^{n}\sum_{k=1}^{m} g^{ik}(X_s,s)\frac{\partial}{\partial x_i}V(X_s,s)dW_s^k$$

is a continuous martingale, it is easy to deduce, in addition to the condition (2), that

$$\begin{aligned} EV(X_t,t) &\leq EV(x_0,0) + \int_0^t E(LV(X_s,s))ds \\ &\leq EV(x_0,0) + \int_0^t (\psi_1(s) + \psi_2(s)EV(X_s,s))ds. \end{aligned}$$

So, by virtue of Gronwall's lemma, we derive that

$$EV(X_t,t) \leq \left[EV(x_0,0) + \int_0^t \psi_1(s)ds\right]\exp\left(\int_0^t \psi_2(s)ds\right),$$

which implies immediately that

$$\log(EV(X_t,t)) \leq \log\left[EV(x_0,0) + \int_0^t \psi_1(s)ds\right] + \int_0^t \psi_2(s)ds.$$

Therefore, by virtue of the conditions (2) and (3), for arbitrary $\varepsilon > 0$, whenever $t > 0$ largely enough, it deduces that

$$\log(EV(X_t, t)) \le \log[EV(x_0, 0) + \lambda(t)^{\nu+\varepsilon}] + \log \lambda(t)^{\theta+\varepsilon},$$

that is,

$$\limsup_{t\to\infty} \frac{\log(EV(X_t, t))}{\log \lambda(t)} \le (\nu + \varepsilon) + \theta + \varepsilon .$$

Letting $\varepsilon \to 0$ then gives

$$\limsup_{t\to\infty} \frac{\log(EV(X_t, t))}{\log \lambda(t)} \le \nu + \theta .$$

Finally, in view of the condition (1), we have

$$\begin{aligned} \limsup_{t\to\infty} \frac{\log E|X_t(x_0)|^p}{\log \lambda(t)} &\le \limsup_{t\to\infty} \frac{\log[\lambda(t)^{-m} EV(X_t, t)]}{\log \lambda(t)} \\ &\le -[m - (\nu + \theta)] \end{aligned}$$

as required. □

In order to obtain the second main result, we need the following extended Gronwall type lemma.

LEMMA 1.1. *Assume $h(t), u(t) \in B([0, T], \boldsymbol{R}^+)$, that is, $h(t)$ and $u(t)$ are two bounded Borel measurable non-negative functions. Let $w(t)$ be a continuous, non-negative and non-decreasing function defined on $[0, T]$, and $0 \le \alpha < 1$. Suppose*

$$h(t) \le w(t) + \int_0^t u(s) h(s-)^\alpha ds, \quad 0 \le t \le T .$$

Then

$$h(t) \le \left\{ w(t)^{1-\alpha} + (1 - \alpha) \int_0^t u(s) ds \right\}^{\frac{1}{1-\alpha}}, \quad 0 \le t \le T .$$

PROOF. See [15]. □

THEOREM 1.2. *Let $V(x, t) \in C^{2,1}(\boldsymbol{R}^n \times \boldsymbol{R}^+; \boldsymbol{R}^+)$ and $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ be three continuous non-negative functions. Assume that for all $x \in \boldsymbol{R}^n$ and $t \ge 0$, there exist positive constants $p > 0$, $m > 0$ and real numbers ν, θ, η, $0 \le \alpha < 1$ such that*

(1) $|x|^p \lambda(t)^m \le V(x, t), \quad (x, t) \in \boldsymbol{R}^n \times \boldsymbol{R}^+;$

(2) $LV(x, t) \le \psi_1(t) + \psi_2(t) V(x, t) + \psi_3(t) V(x, t)^\alpha, \quad (x, t) \in \boldsymbol{R}^n \times \boldsymbol{R}^+;$

(3) $$\limsup_{t\to\infty} \frac{\log\left(\int_0^t \psi_1(s) ds\right)}{\log \lambda(t)} \le \nu, \quad \limsup_{t\to\infty} \frac{\int_0^t \psi_2(s) ds}{\log \lambda(t)} \le \theta(1 - \alpha),$$

$$\limsup_{t\to\infty} \frac{\log\left(\int_0^t \psi_3(s) ds\right)}{\log \lambda(t)} \le \eta(1 - \alpha).$$

Then, whenever $\gamma := m - \theta - \nu \vee \eta > 0$, the solution of Equation (1.1) *is the p-th momently stable with decay rate $\lambda(t)$. Moreover, we have*

$$\limsup_{t\to\infty} \frac{\log E|X_t(x_0)|^p}{\log \lambda(t)} \le -\gamma . \tag{1.4}$$

PROOF. Using the same notation and a similar argument as in the proof of Theorem 1.1, we can get

$$EV(X_t, t) \le EV(x_0, 0) + \int_0^t E(LV(X_s, s))ds ,$$

which, together with the condition (2) and the Hölder inequality, immediately implies that

$$\begin{aligned} EV(X_t, t) &\le EV(x_0, 0) + \int_0^t (\psi_1(s) + \psi_2(s)EV(X_s, s) + \psi_3(s)EV^\alpha(X_s, s))ds \\ &\le EV(x_0, 0) + \int_0^t [\psi_1(s) + \psi_2(s)EV(X_s, s) + \psi_3(s)(EV(X_s, s))^\alpha]ds . \end{aligned} \tag{1.5}$$

So, by virtue of Gronwall's lemma, we easily derive that

$$EV(X_t, t) \le \left[EV(x_0, 0) + \int_0^t \psi_1(s)ds + \int_0^t \psi_3(s)(EV(X_s, s))^\alpha ds\right] \exp\left(\int_0^t \psi_2(s)ds\right) .$$

Once again invoking Lemma 2.1, we derive that

$$\begin{aligned} EV(X_t, t) \le \Bigg\{&\left[EV(x_0, 0) + \int_0^t \psi_1(s)ds\right]^{1-\alpha} \exp\left(\int_0^t \psi_2(s)ds\right) \\ &+ (1-\alpha)\exp\left(\int_0^t \psi_2(s)ds\right)\int_0^t \psi_3(s)ds\Bigg\}^{\frac{1}{1-\alpha}} . \end{aligned}$$

Therefore, noticing the conditions (2) and (3), for arbitrary $\varepsilon > 0$, whenever $t > 0$ largely enough, it is easy to deduce that

$$\begin{aligned} \log EV(X(t), t) \le &\frac{1}{1-\alpha} \log\{[EV(x_0, 0) + \lambda(t)^{\nu+\varepsilon}]^{1-\alpha} + \lambda(t)^{(1-\alpha)(\eta+\varepsilon)}\} \\ &+ (\theta + \varepsilon)\log \lambda(t) + \varepsilon , \end{aligned}$$

which, together with the condition (3), immediately implies that

$$\limsup_{t\to\infty} \frac{\log(EV(X_t, t))}{\log \lambda(t)} \le (\nu \vee \eta + \varepsilon) + \theta + \varepsilon .$$

Letting $\varepsilon \to 0$ then gives

$$\limsup_{t\to\infty} \frac{\log(EV(X_t, t))}{\log \lambda(t)} \le \nu \vee \eta + \theta .$$

Finally, by virtue of the condition (1), we have

$$\begin{aligned} \limsup_{t\to\infty} \frac{\log E|X_t(x_0)|^p}{\log \lambda(t)} &\le \limsup_{t\to\infty} \frac{\log[\lambda(t)^{-m}EV(X_t, t)]}{\log \lambda(t)} \\ &\le -[m - (\nu \vee \eta + \theta)] \end{aligned}$$

as required. □

Lastly, let us study several examples to close this section.

EXAMPLE 1.1. Let us first return to Example 0.1, i.e., consider a one-dimensional Itô stochastic differential equation

$$dX_t = -\frac{p}{1+t}X_t dt + (1+t)^{-p} dW_t\,, \quad t \geq 0$$

with initial data $X_0 = x_0 \in \boldsymbol{R}^1$, where $p > 1/2$ is a constant and W_t is a one-dimensional Brownian motion.

We construct the Lyapunov function as follows:

$$V(x,t) = (1+t)^{2p}x^2\,, \quad t \in \boldsymbol{R}^+\,,\ x \in \boldsymbol{R}^1\,.$$

It is easy to deduce that

$$LV(x,t) = 1\,.$$

Using Theorem 1.1, we can obtain that whenever $p > 1/2$, the solution is the second momently stable with polynomial decay. Moreover, we have

$$\limsup_{t\to\infty} \frac{1}{\log t} \log EX_t^2 \leq -(2p-1)\,.$$

EXAMPLE 1.2. Let us once again return to Example 0.2, i.e., consider a scalar linear Itô equation

$$dX_t = -\frac{X_t}{(1+t)\log(1+t)}dt + e^{-t}X_t dW_t\,, \quad t \geq 0$$

with initial data $X_0 = x_0 \in \boldsymbol{R}^1$, where W_t is a one-dimensional Brownian motion.

We construct the Lyapunov function as follows:

$$V(x,t) = (\log(1+t))^2 x^2\,, \quad t \in \boldsymbol{R}^+\,,\ x \in \boldsymbol{R}^1\,.$$

A direct computation easily deduces that

$$LV(x,t) = e^{-2t}V(x,t)\,.$$

Using Theorem 1.1, we can obtain that the solution is the second momently stable with logarithmic type decay. Moreover,

$$\limsup_{t\to\infty} \frac{1}{\log\log t} \log EX_t^2 \leq -2\,.$$

2. Stability of delay stochastic systems. Let $l > 0$ and denote by $C([-l,0],\boldsymbol{R}^n)$ the space of all continuous functions defined on $[-l,0]$ with values in $\boldsymbol{R}^n$. We introduce a norm over this space by

$$\|u\| = \max\{|u(s)| : -l \leq s \leq 0\}\,, \quad u \in C([-l,0],\boldsymbol{R}^n)\,.$$

At the moment, let $L^2(\Omega, \mathcal{F}_0, P; C([-l,0],\boldsymbol{R}^n))$ denote the family of all $\mathcal{F}_0$-measurable $C([-l,0],\boldsymbol{R}^n)$-valued random variable $\eta(t)$ with $E\|\eta\|^2 < \infty$.

In this section, we shall carry out a Lyapunov function approach to study the p-th moment stability for a class of stochastic differential equations with time delays:

$$\begin{cases} dX_t = f(X_t, X_{t-\delta(t)}, t)dt + g(X_t, X_{t-\delta(t)}, t)dW_t\,, & t \geq 0 \\ X_t = \eta(t)\,, \quad t \in [-l, 0] \end{cases} \tag{2.1}$$

with initial data $X_t = \eta(t) \in L^2(\Omega, \mathcal{F}_0, P; C([-l, 0], \boldsymbol{R}^n))$, $-l \leq t \leq 0$. Here $f : \boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^+ \to \boldsymbol{R}^n$, $g : \boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^+ \to \boldsymbol{R}^n$ are two measurable mappings and $\delta(\cdot) : [0, \infty) \to [0, l]$, $l \geq 0$, is a continuous function which shall play the role of variable delays. We shall also assume that the equation (2.1) has a unique global solution which is denoted by $X_t(x_0) \in \boldsymbol{R}^n$. In particular, we could also define the stability with a general decay rate of the solutions of the equation (2.1) in a totally similar way to Definition 1.1.

In order to obtain our main consequences, we need the following lemmas.

LEMMA 2.1. *Assume that $T > l > 0$ and $y(t)$ is a continuous, nonnegative function defined on $[-l, T]$. Let $w(t)$ be a continuous, nonnegative, nondecreasing function defined on $[0, T]$ and $u(t)$, $v(t)$ be two continuous nonnegative functions. Assume that*

$$y(t) \leq w(t) + \int_0^t u(s)y(s)ds + \int_0^t v(s)y(s - \delta(s))ds\,, \quad 0 \leq t \leq T\,. \tag{2.2}$$

Then for any $0 \leq t \leq T$

$$y(t) \leq \left(w(t) + \int_0^l v(s)ds \left[\sup_{-l \leq r \leq 0} y(r)\right]\right) \exp\left\{\int_0^t u(s)ds + \int_0^t v(s)ds\right\}. \tag{2.3}$$

PROOF. See [15]. □

LEMMA 2.2. *Assume that $T > l > 0$ and $y(t)$ is a continuous, nonnegative function defined on $[-l, T]$. Let $w(t)$ be a continuous, nonnegative, nondecreasing function defined on $[0, T]$ and $u(t)$, $v(t)$ be two continuous nonnegative functions. Let $0 \leq \alpha < 1$ and $\delta(t)$ be defined as above. Assume that*

$$y(t) \leq w(t) + \int_0^t u(s)y(s)ds + \int_0^t v(s)y(s - \delta(s))^\alpha ds\,.$$

Then

$$y(t) \leq \exp\left\{\left(\frac{1}{1-\alpha}\right)\int_0^t u(s)ds\right\}\left(N(t)^{1-\alpha} + (1-\alpha)2^\alpha \int_0^t v(s)ds\right)^{\frac{1}{1-\alpha}}, \tag{2.4}$$

where $N(t) = w(t) + [2 \sup_{-l \leq r \leq 0} y(r)]^\alpha \int_0^l v(s)ds$.

PROOF. See [15]. □

Suppose $V(x,t) \in C^{2,1}(\boldsymbol{R}^n \times \boldsymbol{R}^+; \boldsymbol{R}^+)$, and define the function $LV(x,y,t)$ as follows. For arbitrary $x, y \in \boldsymbol{R}^n$, $t \in \boldsymbol{R}^+$, we set

$$\begin{aligned} LV(x,y,t) := {} & \frac{\partial}{\partial t} V(x,t) + \sum_{i=1}^{n} f^i(x,y,t) \frac{\partial}{\partial x_i} V(x,t) \\ & + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{m} g^{ik}(x,y,t) g^{jk}(x,y,t) \frac{\partial^2}{\partial x_i \partial x_j} V(x,t) . \end{aligned}$$

THEOREM 2.1. *Let $V(x,t) \in C^{2,1}(\boldsymbol{R}^n \times \boldsymbol{R}^+; \boldsymbol{R}^+)$ and $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ be three continuous non-negative functions. Assume that for all $x, y \in \boldsymbol{R}^n$ and $t \geq 0$, there exist positive constants $c_1 > 0$, $c_2 > 0$, $p > 0$, $m > 0$ and real numbers ν, θ, γ such that*

(1) $c_1|x|^p \lambda(t)^m \leq V(x,t) \leq c_2 |x|^p \lambda(t)^m, \quad (x,t) \in \boldsymbol{R}^n \times \boldsymbol{R}^+;$

(2) $LV(x,y,t) \leq \psi_1(t) + \psi_2(t)V(x,t) + \psi_3(t)V(y,t), \quad x, y \in \boldsymbol{R}^n, \ t \in [0, +\infty);$

(3) $$\limsup_{t\to\infty} \frac{\log\left(\int_0^t \psi_1(s)ds\right)}{\log \lambda(t)} \leq \nu, \quad \limsup_{t\to\infty} \frac{\int_0^t \psi_2(s)ds}{\log \lambda(t)} \leq \theta,$$
$$\limsup_{t\to\infty} \frac{\int_0^t \psi_3(s)ds}{\log \lambda(t)} \leq \gamma.$$

Then the solution of Equation (2.1) *satisfies*

$$\limsup_{t\to\infty} \frac{\log E|X_t(\eta)|^p}{\log \lambda(t)} \leq -[m - (\nu + (c_2/c_1)\theta + (c_2/c_1)\lambda(l)^m \gamma)] .$$

PROOF. By Itô's formula and the definition of L, we can derive that

$$\begin{aligned} V(X_t,t) = {} & V(x_0,0) + \int_0^t LV(X_s, X_{s-\delta(s)}, s)ds \\ & + \int_0^t \sum_{i=1}^{n} \sum_{k=1}^{m} g^{ik}(X_s, X_{s-\delta(s)}, s) \frac{\partial}{\partial x_i} V(X_s, s) dW_s^k . \end{aligned} \tag{2.5}$$

Since the diffusion term

$$\int_0^t \sum_{i=1}^{n} \sum_{k=1}^{m} g^{ik}(X_s, X_{s-\delta(s)}, s) \frac{\partial}{\partial x_i} V(X_s, s) dW_s^k$$

is a continuous martingale, it is easy to deduce from the condition (2) that

$$\begin{aligned} EV(X_t,t) & \leq EV(x_0,0) + \int_0^t ELV(X_s, X_{s-\delta(s)}, s)ds \\ & \leq EV(x_0,0) + \int_0^t (\psi_1(s) + \psi_2(s)EV(X_s,s) \\ & \quad + \psi_3(s)EV(X_{s-\delta(s)}, s))ds \end{aligned} \tag{2.6}$$

for all $t \geq 0$. A direct application of the condition (1) and Definition 1.1 to (2.6) yields that

$$c_1 E|X_t|^p \lambda(t)^m \leq EV(x_0, 0) + \int_0^t (\psi_1(s) + c_2\psi_2(s)E|X_s|^p \lambda(s)^m + c_2\psi_3(s)\lambda(l)^m E|X_{s-\delta(s)}|^p \lambda(s-\delta(s))^m)ds\,,$$

which, in addition to Lemma 2.1, immediately yields that

$$\begin{aligned} c_1 E|X_t|^p \lambda(t)^m \leq &\Big[EV(x_0, 0) + \int_0^t \psi_1(s)ds \\ &+ (c_2/c_1)\lambda(l)^m \int_0^l \psi_3(s)ds\Big[\sup_{-l\leq r\leq 0} |\eta(r)|^p\Big]\Big] \\ &\cdot \exp\left(\int_0^t (c_2/c_1)\psi_2(s)ds + \int_0^t (c_2/c_1)\lambda(l)^m\psi_3(s)ds\right) \end{aligned}$$

for all $t \geq 0$. Therefore, for $t \in \boldsymbol{R}^+$ large enough, we derive for arbitrary $\varepsilon > 0$

$$\begin{aligned} &\log(c_1 E|X_t|^p \lambda(t)^m) \\ &\leq \log\left\{EV(x_0, 0) + (c_2/c_1)\lambda(l)^m \int_0^l \psi_3(s)ds\left[\sup_{-l\leq r\leq 0} |\eta(r)|^p\right] + \lambda(t)^{\nu+\varepsilon}\right\} \\ &\quad + (c_2/c_1)\int_0^t \psi_2(s)ds + (c_2/c_1)\lambda(l)^m \int_0^t \psi_3(s)ds \\ &\leq \log\left\{EV(x_0, 0) + (c_2/c_1)\lambda(l)^m \int_0^l \psi_3(s)ds\left[\sup_{-l\leq r\leq 0} |\eta(r)|^p\right] + \lambda(t)^{\nu+\varepsilon}\right\} \\ &\quad + (c_2/c_1)(\theta+\varepsilon)\log\lambda(t) + (c_2/c_1)\lambda(l)^m(\gamma+\varepsilon)\log\lambda(t)\,, \end{aligned}$$

which implies immediately that

$$\limsup_{t\to\infty} \frac{\log(c_1 E|X_t|^p \lambda(t)^m)}{\log\lambda(t)} \leq \nu + \varepsilon + (c_2/c_1)(\theta+\varepsilon) + (c_2/c_1)\lambda(l)^m(\gamma+\varepsilon)\,.$$

Letting $\varepsilon \to 0$ then gives

$$\limsup_{t\to\infty} \frac{\log(c_1 E|X_t|^p \lambda(t)^m)}{\log\lambda(t)} \leq \nu + (c_2/c_1)\theta + (c_2/c_1)\lambda(l)^m\gamma\,.$$

Finally, we have

$$\begin{aligned} \limsup_{t\to\infty} \frac{\log E|X_t(\eta)|^p}{\log\lambda(t)} &\leq \limsup_{t\to\infty} \frac{\log[\lambda(t)^{-m}(c_1 E|X_t|^p \lambda(t)^m)]}{\log\lambda(t)} \\ &\leq -[m - (\nu + (c_2/c_1)\theta + (c_2/c_1)\lambda(l)^m\gamma)] \end{aligned}$$

as required. □

THEOREM 2.2. *Let $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ be three continuous non-negative functions. Assume that for all $x \in \boldsymbol{R}^n$ and $t \geq 0$, there exist positive constants $c_1 > 0$, $c_2 > 0$, $p > 0$, $m > 0$ and real numbers θ, ν, ρ, $0 \leq \alpha < 1$ such that*

(1) $c_1|x|^p\lambda(t)^m \leq V(x,t) \leq c_2|x|^p\lambda(t)^m, \quad (x,t) \in \boldsymbol{R}^n \times \boldsymbol{R}^+$;

(2) $LV(x, y, t) \le \psi_1(t) + \psi_2(t)V(x, t) + \psi_3(t)V(y, t)^\alpha, \quad x, y \in \boldsymbol{R}^n, \ t \in [0, +\infty);$

(3) $\limsup_{t\to\infty} \dfrac{\log\left(\int_0^t \psi_1(s)ds\right)}{\log\lambda(t)} \le \theta, \quad \limsup_{t\to\infty} \dfrac{\int_0^t \psi_2(s)ds}{\log\lambda(t)} \le \nu(1-\alpha),$

$$\limsup_{t\to\infty} \frac{\log\left(\int_0^t \psi_3(s)ds\right)}{\log\lambda(t)} \le \rho(1-\alpha).$$

Then the solution of Equation (2.1) *satisfies*

$$\limsup_{t\to\infty} \frac{\log E|X_t(\eta)|^p}{\log\lambda(t)} \le -[m - ((c_2/c_1)\nu + \theta \vee \rho)].$$

PROOF. Using the same notation as in the proof of Theorem 2.1, we can derive from (2.6) and Lemma 2.2 that for arbitrary $\varepsilon > 0$

$$\begin{aligned} &c_1 E|X_t|^p\lambda(t)^m \\ &\le \exp\left\{\left(\frac{c_2}{c_1(1-\alpha)}\right)\int_0^t \psi_2(s)ds\right\}\left\{\left(EV(x_0, 0) + \left[2\sup_{-l\le r\le 0}|\eta(r)|^p\right]^\alpha\right.\right. \\ &\quad \left.\left.\cdot\,(c_2^\alpha/c_1)\lambda(l)^{\alpha m}\int_0^l \psi_3(s)ds + \int_0^t \psi_1(s)ds\right)^{1-\alpha} + (c_2^\alpha/c_1)2^\alpha\lambda(l)^{\alpha m}\int_0^t \psi_3(s)ds\right\}^{\frac{1}{1-\alpha}} \end{aligned}$$

for all $t > 0$ large enough.

Therefore, by virtue of the condition (3), we have

$$\begin{aligned} \log(c_1 E|X_t|^p\lambda(t)^m) &\le (c_2/c_1)(\nu+\varepsilon)\log\lambda(t) + \log\left[\left(EV(x_0, 0) + \lambda(t)^{\theta+\varepsilon}\right.\right. \\ &\quad \left. + (c_2^\alpha/c_1)\lambda(l)^{\alpha m}\left[2\sup_{-l\le r\le 0}|\eta(r)|^p\right]^\alpha \int_0^l \psi_3(s)ds\right)^{1-\alpha} \\ &\quad \left. + (c_2^\alpha/c_1)2^\alpha\lambda(l)^{\alpha m}\lambda(t)^{(1-\alpha)(\rho+\varepsilon)}\right]^{\frac{1}{1-\alpha}}, \end{aligned}$$

which implies immediately that

$$\limsup_{t\to\infty} \frac{\log(c_1 E|X_t|^p\lambda(t)^m)}{\log\lambda(t)} \le (c_2/c_1)(\nu+\varepsilon) + (\theta+\varepsilon) \vee (\rho+\varepsilon).$$

Letting $\varepsilon \to 0$ then gives

$$\limsup_{t\to\infty} \frac{\log(c_1 E|X_t|^p\lambda(t)^m)}{\log\lambda(t)} \le (c_2/c_1)\nu + \theta \vee \rho.$$

Finally, we have

$$\begin{aligned} \limsup_{t\to\infty} \frac{\log E|X_t(\eta)|^p}{\log\lambda(t)} &\le \limsup_{t\to\infty} \frac{\log[\lambda(t)^{-m}(c_1|X_t|^p\lambda(t)^m)]}{\log\lambda(t)} \\ &\le -[m - ((c_2/c_1)\nu + \theta \vee \rho)]. \end{aligned}$$

□

EXAMPLE 2.1. Consider a one-dimensional stochastic constant time delay equation

$$dX_t = -\frac{p}{1+t}X_t dt + \frac{1}{1+t}X_{t-l}dt + (1+t)^{-p}g(X_t)dW_t$$

with initial data $X_t = \eta(t)$, $t \in [-l, 0]$, where $g(\cdot) : \boldsymbol{R}^1 \to \boldsymbol{R}^1$ is a bounded, Lipschitz continuous function, W_t is a one-dimensional Brownian motion and p, l are two positive numbers.

We construct the Lyapunov function as follows:

$$V(x, t) = (1+t)^{2p}x^2, \quad t \in \boldsymbol{R}^+, \ x \in \boldsymbol{R}^1.$$

We can deduce that there exists a positive constant $M > 0$ such that

$$LV(x, y, t) \leq \frac{1}{1+t}V(x, t) + \frac{1}{1+t}V(y, t) + M.$$

Using Theorem 2.1, we can obtain that whenever $p > 1 + l^{2p}/2$, the solution is the second momently stable with polynomial type decay. Furthermore, we have

$$\limsup_{t\to\infty} \frac{\log E|X_t(\eta)|^2}{\log t} \leq -(2p - 2 - l^{2p}).$$

EXAMPLE 2.2. Let $l > 0$, $p > 0$ and $0 \leq \alpha < 1$. Assume that $\eta(t) : [-l, 0] \times \Omega \to \boldsymbol{R}^1$ is an $\mathcal{F}_0$-measurable process and $\delta(\cdot) : [0, \infty) \to [0, l]$ is the delay function. For $t \geq 0$, consider a stochastic Itô equation with variable time delays.

$$(2.7) \qquad \begin{cases} dX_t = -pX_t dt + e^{-\nu(1-\alpha)t}|X_{t-\delta(t)}|^\alpha dW_t, \\ X_t = \eta(t), \quad t \in [-l, 0], \end{cases}$$

where W_t is a one-dimensional Wiener process and $\nu \in \boldsymbol{R}^1$ is a certain real number.

We construct the Lyapunov function as follows:

$$V(x, t) = e^{2pt}x^2, \quad t \in \boldsymbol{R}^+, \ x \in \boldsymbol{R}^1.$$

A direct computation deduces that

$$\begin{aligned} LV(x, y, t) &= 2pV(x, t) - 2pV(x, t) + e^{(2p-2\nu)(1-\alpha)}V(y, t)^\alpha \\ &= e^{(2p-2\nu)(1-\alpha)}V(y, t)^\alpha. \end{aligned}$$

Using Theorem 2.2, we can therefore obtain that whenever $\nu > 0$, the solution is the second momently stable with exponential type decay. Furthermore,

$$\limsup_{t\to\infty} \frac{\log E|X_t(\eta)|^2}{t} \leq -2\nu.$$

Acknowledgments. The authors would very much like to thank the referee for his helpful suggestions. One of the authors, K. Liu also wants to acknowledge the financial support of EPSRC Grant GR/K 70397.

REFERENCES

[1] L. ARNOLD, Stochastic Differential Equations: Theory and Applications, John Wiley, New York-London-Sydney, 1974.

[2] L. ARNOLD, A formula connecting sample and moment stability of linear stochastic systems, SIAM J. Appl. Math. 44 (1984), 793–802.

[3] L. ARNOLD, E. OELJEKLAUS AND E. PARDOUX, Almost sure and moment stability for linear Itô equations, Lyapunov Exponents (Bremen, 1984), 129–159, Lecture Notes in Math. 1186, Springer-Verlag, Berlin-New York, 1986.

[4] M. CHAPPELL, Bounds for average Lyapunov exponents of gradient stochastic systems, Lyapunov Exponents (Bremen, 1984), 308–321, Lecture Notes in Math. 1186, Springer-Verlag, Berlin-New York, 1986.

[5] R. Z. HAS'MINSKII, Stochastic Stability of Differential Equations, Sijthoff and Noordfoff, Alphen aan den Rijn, The Netherlands, 1980.

[6] K. NARITA, No explosion criteria for stochastic differential equations, J. Math. Soc. Japan 34 (1982), 191–203.

[7] K. NARITA, Remarks on nonexplosion theorem for stochastic differential equations, Kodai Math. J. 5 (1982), 395–401.

[8] P. D. LAX, Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, SIAM, Philadelphia, 1973.

[9] K. LIU, On stability for a class of semilinear stochastic evolution equations, Stochastic Processes Appl. 70 (1997), 219–241.

[10] K. LIU, Lyapunov functionals and asymptotic stability of stochastic delay evolution equations, Stochastic and Stochastics Rep. 63 (1998), 1–26.

[11] K. LIU, P-th moment growth bounds of infinite-dimensional stochastic evolution equations, Proc. Roy. Soc. Edinburgh Sect. A 128 (1998), 107–121.

[12] K. LIU AND X. R. MAO, Stability of stochastic differential equations with a certain decay, Submitted.

[13] X. R. MAO, Almost sure polynomial stability for a class of stochastic differential equations, Quart. J. Math. Oxford Ser. (2) 43 (1992), 339–348.

[14] X. R. MAO, Polynomial stability for perturbed stochastic differential equations with respect to semimartingales, Stochastic Processes Appl. 41 (1992), 101–116.

[15] X. R. MAO, Stability of Stochastic Differential Equations with respect to Semimartingales, Pitman Res. Notes Math. Ser. 251, Longman Scientific and Technical, England, 1991.

[16] PAUL R. ZINGANO, Nonlinear stability with decay rate for traveling waves solutions of a hyperbolic system with relaxation, J. Differential Equations 130 (1996), 36–58.

DEPARTMENT OF PROBABILITY AND STATISTICS
THE UNIVERSITY OF SHEFFIELD
THE HICKS BUILDING, HOUNSFIELD ROAD
SHEFFIELD, S3 TRH
UNITED KINGDOM

E-mail address: k.Liu@sheffield.ac.uk

SCHOOL OF COMPUTING AND MATHEMATICAL SCIENCE
UNIVERSITY OF GREENWICH
30 PARK ROW, GREENWICH
LONDON SE10 9LS
UNITED KINGDOM

E-mail address: A.Chen@greenwich.ac.uk