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WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS WITH TEST FUNCTIONS IN THE WEAK-L^n SPACE

Dedicated to Professor John G. Heywood on his sixtieth birthday

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Abstract. We show the existence of weak solutions of the Navier-Stokes equations with test functions in the weak-L^n space. As an application, we give a new criterion on uniqueness and regularity of weak solutions which covers the previous results.

Introduction. Let Ω be any domain in the Euclidean n-space $\boldsymbol{R}^n$ ($n \geq 2$) with boundary $\partial\Omega$. Consider the Navier-Stokes equations in $\Omega \times (0, T)$:

$$\text{(N-S)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } x \in \Omega\,,\ 0 < t < T\,, \\ \operatorname{div} u = 0 \quad \text{in } x \in \Omega\,,\ 0 < t < T\,, \\ u = 0 \quad \text{on } \partial\Omega\,, \\ u|_{t=0} = a\,, \end{cases}$$

where $u = u(x,t) = (u^1(x,t), \dots, u^n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the pressure of the fluid at a point $(x,t) \in \Omega \times (0,T)$, respectively, while $a = a(x) = (a^1(x), \dots, a^n(x))$ is the given initial velocity vector field. For simplicity, we assume that the external force has a scalar potential and is included into the pressure gradient.

The purpose of this paper is to enlarge the space of test functions in the definition of weak solutions. In [19] and [13], Leray and Hopf proved the existence of weak solutions with test functions in $C_{0,\sigma}^\infty(\Omega)$, where the subscript σ means solenoidal vector fields. By the density argument, one can extend the space of test functions to $H_{0,\sigma}^1(\Omega)$. To define the integral $\int_\Omega u \cdot \nabla u \cdot \phi dx$ for arbitrary weak solutions u with test functions ϕ, we need $\phi \in L^n(\Omega)$. Hence, if $n \leq 4$, by the Sobolev embedding we have $H_{0,\sigma}^1(\Omega) \subset L^r(\Omega)$ for $2 \leq r \leq 2n/(n-2)$, so the space $H_{0,\sigma}^1(\Omega)$ suffices to be taken as test functions. In the case $n \geq 5$, however, we need to choose ϕ in $H_{0,\sigma}^1(\Omega) \cap L^n(\Omega)$. On the other hand, from the viewpoint of scaling invariance introduced by Caffarelli-Kohn-Nirenberg [4], it is important to find a solution of (N-S) in $L^n(\Omega)$. Giga-Miyakawa [8] and Kato [14] constructed a solution u in $C([0,T]; L^n(\Omega))$, which is necessarily unique and regular. It seems to be reasonable to take the space of test functions as large as possible so that smooth solutions can be obtained under minimum additional assumption on weak solutions. Masuda [21] proved the existence of weak solutions with test functions in $C([0,T]; H_{0,\sigma}^1(\Omega) \cap L^n(\Omega))$ and applied it to generalize

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the uniqueness criterion due to Foias [5] and Serrin [24]. Indeed, it is shown in [21] that, for each $n \geq 2$, if u is a weak solution of (N-S) in $L^s(0, T; L^q(\Omega))$ for $2/s + n/q \leq 1$ with $q > n$, then u is unique.

Recently, Kozono-Yamazaki [17] constructed a smooth solution in $C((0, T); L^n_w(\Omega))$, where $L^n_w(\Omega)$ denotes the weak-L^n space. In the present paper, we prove the existence of weak solutions with test functions in $C([0, T]; H^1_{0,\sigma}(\Omega) \cap L^n_w(\Omega))$. As an application, we give a new criterion on uniqueness and regularity of weak solutions which covers Foias-Serrin-Masuda's result. Up to the present, the class $C([0, T]; L^n(\Omega))$ is the largest space that enables us to obtain *both* uniqueness and regularity of weak solutions. More precisely, uniqueness is ensured in the class $L^\infty(0, T; L^n(\Omega))$ ([26], [15]), while its regularity is still an open question ([1], [2], [16]). Our class is larger than $C([0, T]; L^n(\Omega))$. The crucial difference between the usual and the weak L^n-spaces stems from the fact that $C_0^\infty(\Omega)$ is *not dense* in the latter space. To get around this difficulty, we impose a certain assumption on the distribution functions determined by test functions, which plays a substitutive role for the density property.

In Section 1, we shall state our main results. Section 2 is devoted to some lemmas for the proof of the main theorems. In Section 3, we shall show the existence of our weak solutions. Finally in Sections 4 and 5, we shall prove the new criterion on uniqueness and regularity, respectively.

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1. Results. Before stating our results, we first introduce some function spaces. Let $C^\infty_{0,\sigma}$ denote the set of all C^∞ vector functions $\phi = (\phi^1, \dots, \phi^n)$ with compact support in Ω, such that $\operatorname{div} \phi = 0$. L^r_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the L^r-norm $\|\cdot\|_r$. $(\cdot, \cdot)$ denotes the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued) L^r-space over Ω, where $1 < r < \infty$. $H^1_{0,\sigma}$ denotes the closure of $C^\infty_{0,\sigma}$ with respect to the norm

$$\|\phi\|_{H^1} = \|\phi\|_2 + \|\nabla\phi\|_2 ,$$

where $\nabla\phi = (\partial\phi^i/\partial x_j)$, $i, j = 1, \dots, n$.

L^r_w denotes the weak-L^r space over Ω with the quasi-norm $\|\cdot\|_{r,w}$ defined by

$$\|\phi\|_{r,w} \equiv \sup_{R>0} R\mu\{x \in \Omega; |\phi(x)| > R\}^{1/r} ,$$

where μ is the Lebesgue measure. For $1 < r < \infty$, there is another norm equivalent to this $\|\cdot\|_{r,w}$ (see Bergh-Löfström [3, p. 8]), so we may regard L^r_w as a Banach space. For an interval I in $\boldsymbol{R}^1$ and a Banach space X, $L^p(I; X)$ and $C^m(I; X)$ denote the usual Banach spaces of functions of L^p and C^m-class on I with values in X, respectively, where $1 \leq p \leq \infty$, $m = 0, 1, \dots$.

We next introduce an assumption on the initial data a and then state our definition of a weak solution of (N-S).

ASSUMPTION 1. The initial data $a = a(x)$ is in L^2_σ.

As our space of test functions, we define the space S by
$S \equiv \{\Phi \in H^1(0,T; H^1_{0,\sigma}) \cap C([0,T]; L^n_w); \Phi$ satisfies the following condition (1.1)$\}$;

$$\lim_{R\to\infty}\left(\sup_{0\le t\le T} R\mu\{x\in\Omega; |\Phi(x,t)| > R\}^{\frac{1}{n}}\right) = 0. \tag{1.1}$$

Concerning the relation between S and the usual space of test functions, we have

PROPOSITION 1. *Every function* $\Phi \in C([0,T]; L^n)$ *satisfies* (1.1).

For the proof, see the Appendix.

DEFINITION. Suppose that Assumption 1 holds. A measurable function u on $\Omega \times (0,T)$ is called a weak solution of (N-S) if

(i) $u \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1_{0,\sigma})$;

(ii) For every $\Phi \in S$ with $\Phi(T) = 0$,

$$\int_0^T \{-(u, \partial_t\Phi) + (\nabla u, \nabla\Phi) + (u\cdot\nabla u, \Phi)\}dt = (a, \Phi(0)).$$

REMARKS. (1) For u and Φ as above, the integral $\int_0^T (u\cdot\nabla u, \Phi)dt$ is well-defined. Indeed, by Lemma 2.1 below there holds that

$$\int_0^T |(u\cdot\nabla u, \Phi)|dt \le C \sup_{0<t<T} \|\Phi(t)\|_{n,w} \int_0^T \|\nabla u\|_2^2 dt.$$

(2) Masuda [21] defined test functions Φ in $H^1(0,T; H^1_{0,\sigma} \cap L^n)$. By Proposition 1, we see that our space S is larger than that of Masuda.

(3) After redefining its value of $u(t)$ on a set of measure zero in the interval $(0,T)$, we see that $u(t)$ is continuous for t in the weak topology of L^2_σ. By a weak solution we mean a weak solution redefined in this manner.

Our theorem on the existence of weak solutions now reads:

THEOREM 1. *Suppose that Assumption* 1 *holds. Then there exists a weak solution* u *of* (N-S) *such that*

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau \le \|a\|_2^2, \quad 0 \le t \le T; \tag{1.2}$$

$$\|u(t) - a\|_2 \to 0 \quad \text{as } t \downarrow +0. \tag{1.3}$$

We next proceed to the uniqueness criterion. To this end, we impose the following assumption.

ASSUMPTION 2. For each $m = 1, 2, \ldots$, there is a bounded operator J_m on $H^1_{0,\sigma} \cap L^n_w$ such that the following properties (i), (ii) and (iii) hold:

(i) J_m, $m = 1, 2, \ldots$, are uniformly bounded as

$$\sup_{m=1,2,\ldots} \|J_m\|_{B(H^1_{0,\sigma}\cap L^n_w)} \equiv M < \infty,$$

where $\|\cdot\|_{B(X)}$ denotes the norm of bounded operators on X;

(ii) For every $\phi \in H^1_{0,\sigma} \cap L^n_w$, we have

$$J_m\phi \in L^n \quad \text{for all } m = 1, 2, \ldots$$

with

$$\|J_m\phi\|_n \leq C_m \|\phi\|_2 ,$$

where C_m is a constant depending on $m = 1, 2, \ldots$;

(iii) For every $\phi \in H^1_{0,\sigma} \cap L^n_w$, $\{J_m\phi\}_{m=1}^{\infty}$ satisfies

$$J_m\phi \to \phi \quad \text{in } H^1_{0,\sigma} \quad \text{and} \quad J_m\phi \to \phi \quad \text{weakly* in } L^n_w$$

as $m \to \infty$.

The assumption above is satisfied at least in the following cases.

PROPOSITION 2. *Assumption* 2 *is satisfied if one of the following conditions is satisfied.*

(0) $2 \leq n \leq 4$;

(1) *Ω is the whole space $\boldsymbol{R}^n$ ($n \geq 2$);*

(2) *Ω is the half space $\boldsymbol{R}^n_+$ ($n \geq 2$);*

(3) *Ω is a bounded domain in $\boldsymbol{R}^n$ ($n \geq 2$) with $C^{2+\mu}$ ($\mu > 0$)-boundary $\partial\Omega$;*

(4) *Ω is an exterior domain in $\boldsymbol{R}^n$ ($n \geq 2$), i.e., a domain having a compact complement $\boldsymbol{R}^n \setminus \Omega$ with $C^{2+\mu}$ ($\mu > 0$)-boundary $\partial\Omega$.*

For the proof, see the Appendix.

Our theorems on the uniqueness and regularity of weak solutions now read as follows.

THEOREM 2. *Suppose that Assumptions* 1 *and* 2 *hold. Then there is an absolute constant $\varepsilon_0 > 0$ with the following property. Let u and v be weak solutions of* (N-S). *Suppose that there is a non-negative L^2-function $M = M(t)$ on* $(0, T)$ *such that*

$$\sup_{R \geq M(t)} R\mu\{x \in \Omega; |u(x,t)| > R\}^{\frac{1}{n}} \leq \varepsilon_0 \quad \text{for almost every } t \in (0,T) . \tag{1.4}$$

Assume that v satisfies the energy inequality

$$\|v(t)\|_2^2 + 2\int_0^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|a\|_2^2, \quad 0 \leq t \leq T . \tag{1.5}$$

Then we have $u \equiv v$ on $[0, T]$.

REMARKS. (1) The constant ε_0 in (1.4) depends only on n, but not on T and $a \in L^2_\sigma$.

(2) In Theorem 2, v need not fulfill the property (1.4) assumed for u, but satisfies the energy inequality (1.5). On the other hand, it should be remarked that (1.4) assures a stronger property than (1.5). Indeed, u satisfies necessarily the energy *identity*

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2 d\tau = \|u(s)\|_2^2, \quad 0 \leq s \leq t \leq T .$$

See (4.18) below.

As for regularity, we impose the following assumption on the domain Ω:

ASSUMPTION 3. The domain Ω satisfies one of the conditions (1), (2), (3) and (4) in Proposition 2.

THEOREM 3. *Suppose that Assumptions* 1 *and* 3 *hold. Then every weak solution* u *with the property* (1.4) *belongs to* $C^2(\Omega \times (0, T))$.

REMARKS. (1) By Serrin [23], [24] and Masuda [21], it is known that if the weak solution u is in $L^s(0, T; L^q)$ for $2/s + n/q = 1$ with $q > n$, then $u \equiv v$ in Theorem 2 and u belongs to $C^2(\Omega \times (0, T))$. For such u, we can define $M(t)$ in (1.4) as

$$M(t) \equiv \left(\frac{\|u(t)\|_q^q}{\varepsilon_0^n} \right)^{\frac{1}{q-n}} .$$

Obviously, there holds

$$\int_0^T M(t)^2 dt = \varepsilon_0^{\frac{2n}{n-q}} \int_0^T \|u(t)\|_q^s dt < \infty .$$

Moreover, by the Chebischev inequality, we have

$$R\mu\{x \in \Omega; |u(x, t)| > R\}^{\frac{1}{n}} \le R^{1-\frac{q}{n}} \|u(t)\|_q^{\frac{q}{n}} \quad \text{for all } R > 0 ,$$

which yields

$$\sup_{R \ge M(t)} R\mu\{x \in \Omega; |u(x, t)| > R\}^{\frac{1}{n}} \le M(t)^{1-\frac{q}{n}} \|u(t)\|_q^{\frac{q}{n}} = \varepsilon_0$$

and (1.4) is fulfilled. Hence our theorems cover the previous criterion on the uniqueness and regularity in the class $L^s(0, T; L^q)$ so far as Assumptions 2 and 3 are satisfied.

(2) von Wahl [29] and Giga [7] showed that, under Assumption 3, if the weak solution u is in $C([0, T]; L^n)$, then u belongs to $C^2(\Omega \times (0, T))$. For $u \in C([0, T]; L^n)$, we obtain from Proposition 1 a constant R_0 depending ε_0 such that

$$\sup_{R \ge R_0} R\mu\{x \in \Omega; |u(x, t)| > R\}^{\frac{1}{n}} < \varepsilon_0 \quad \text{for all } t \in (0, T) .$$

Hence, by taking $M(t) \equiv R_0$, we see that Theorem 3 covers also the result on regularity proved by von Wahl and Giga.

(3) It is an open question whether every weak solution $u \in L^\infty(0, T; L^n)$ is regular or not (see [16]). Struwe [27] showed that if $\sup_{0<t<T} \|u(t)\|_n < \varepsilon_0$, then u is a unique smooth solution. From Theorem 3 we obtain a larger class of smooth solutions, namely, if the weak solution u satisfies

$$\limsup_{R \to \infty} \left(\sup_{0<t<T} R\mu\{x \in \Omega; |u(x, t)| > R\}^{\frac{1}{n}} \right) < \varepsilon_0 ,$$

then u belongs to $C^2(\Omega \times (0, T))$. This implies that if $u(t)$ lies uniformly in $(0, T)$ near the closure of $L^n_w \cap L^\infty$ in the norm of L^n_w, then u is the smooth solution. In other words, every weak solution in $L^\infty(0, T; L^n_w)$ whose *local* singularities in L^n_w are uniformly small in $(0, T)$ becomes regular.

2. Preliminaries. In this section we shall prove some lemmas for later use. In what follows we shall denote various constants by C. In particular, $C = C(*, \dots, *)$ denotes constants depending only on the quantities appearing in the parentheses.

Let us first estimate the nonlinear term of (N-S).

LEMMA 2.1. (i) *Let u, v be in $H^1_{0,\sigma}$ and ϕ be in L^n_w. Then the coupling $(u \cdot \nabla v, \phi)$ of the integral in the definition of a weak solution of* (N-S) *is well-defined with*

$$|(u \cdot \nabla v, \phi)| \le C \|\nabla u\|_2 \|\nabla v\|_2 \|\phi\|_{n,w}, \tag{2.1}$$

where $C = C(n)$.

(ii) *Let u, v, w be in $H^1_{0,\sigma}$. Suppose that u is decomposed as $u = u_0 + u_1$ with $u_0 \in L^n_w$ and $u_1 \in L^\infty$. Then there holds*

$$(u \cdot \nabla v, w) = -(u \cdot \nabla w, v), \tag{2.2}$$

$$(v \cdot \nabla u, u) = 0. \tag{2.3}$$

(iii) *Let u, v, w be in $L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; H^1_{0,\sigma})$. Suppose that u is decomposed as $u = u_0 + u_1$ with $u_0 \in L^\infty(0, T; L^n_w)$ and $u_1 \in L^2(0, T; L^\infty)$. Then there holds*

$$\int_0^T (u \cdot \nabla v, w) d\tau = -\int_0^T (u \cdot \nabla w, v) d\tau, \tag{2.4}$$

$$\int_0^T (v \cdot \nabla u, u) d\tau = 0. \tag{2.5}$$

PROOF. (i) Let us denote by $L^{p,q}$ ($1 < p < \infty$, $1 \le q \le \infty$) the Lorentz space over Ω with the norm $\|\cdot\|_{L^{p,q}}$. By the Hölder inequality in $L^{p,q}$ ([18, Proposition 2.1]), there holds

$$|(u \cdot \nabla v, \phi)| \le C \|u\|_{L^{2n/(n-2),2}} \|\nabla v\|_{L^{2,2}} \|\phi\|_{L^{n,\infty}},$$

where $C = C(n)$. Since H^1_0 is continuously embedded into $L^{2n/(n-2),2}$ ([18, Proposition 2.2]) and since $L^{2,2} = L^2$, $L^{n,\infty} = L^n_w$, the estimate above yields

$$|(u \cdot \nabla v, \phi)| \le C \|\nabla u\|_2 \|\nabla v\|_2 \|\phi\|_{n,w} \tag{2.6}$$

with $C = C(n)$. This implies (2.1).

(ii) Since $C^\infty_{0,\sigma}$ is dense in $H^1_{0,\sigma}$, there is a sequence v_k, $k = 1, 2, \dots$, in $C^\infty_{0,\sigma}$ such that $v_k \to v$, $\nabla v_k \to \nabla v$ in L^2. By integration by parts, we have

$$(u \cdot \nabla v_k, w) = -(u \cdot \nabla w, v_k) \quad \text{for all } k = 1, 2, \dots. \tag{2.7}$$

By (2.1) there holds

$$
\begin{aligned}
&|(u \cdot \nabla v_k, w) - (u \cdot \nabla v, w)| \\
&\quad \le |(u_0 \cdot \nabla(v_k - v), w)| + |(u_1 \cdot \nabla(v_k - v), w)| \\
&\quad \le C\|u_0\|_{n,w}\|\nabla v_k - \nabla v\|_2\|\nabla w\|_2 + C\|u_1\|_\infty\|\nabla v_k - \nabla v\|_2\|w\|_2 \\
&\quad \to 0 \quad \text{as } k \to \infty .
\end{aligned}
\tag{2.8}
$$

Similarly, we have

$$
\begin{aligned}
&|(u \cdot \nabla w, v_k) - (u \cdot \nabla w, v)| \\
&\quad \le C\|u_0\|_{n,w}\|\nabla w\|_2\|\nabla v_k - \nabla v\|_2 + C\|u_1\|_\infty\|\nabla w\|_2\|v_k - v\|_2 \\
&\quad \to 0 \quad \text{as } k \to \infty .
\end{aligned}
\tag{2.9}
$$

Letting $k \to \infty$ in (2.7), from (2.8) and (2.9), we obtain (2.2).

In the same manner, we have by integration by parts

$$
(v_k \cdot \nabla u, u) = -(v_k \cdot \nabla u, u) = 0 \quad \text{for all } k = 1, 2, \dots . \tag{2.10}
$$

By (2.1) there holds

$$
\begin{aligned}
&|(v_k \cdot \nabla u, u) - (v \cdot \nabla u, u)| \\
&\quad \le |((v_k - v) \cdot \nabla u, u_0)| + |(v_k - v) \cdot \nabla u, u_1)| \\
&\quad \le C\|\nabla v_k - \nabla v\|_2\|\nabla u\|_2\|u_0\|_{n,w} + C\|v_k - v\|_2\|\nabla u\|_2\|u_1\|_\infty \\
&\quad \to 0 \quad \text{as } k \to \infty .
\end{aligned}
\tag{2.11}
$$

Letting $k \to \infty$ in (2.10), we obtain (2.3) from (2.11).

(iii) By (2.1) and the Hölder inequality, we have

$$
\begin{aligned}
\int_0^T |(u \cdot \nabla v, w)| d\tau \le{}& C\|u_0\|_{L^\infty(0,T;L^n_w)} \left(\int_0^T \|\nabla v\|_2^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^T \|\nabla w\|_2^2 d\tau\right)^{\frac{1}{2}} \\
&+ C\|w\|_{L^\infty(0,T;L^2)} \left(\int_0^T \|u_1\|_\infty^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^T \|\nabla v\|_2^2 d\tau\right)^{\frac{1}{2}} .
\end{aligned}
$$

Hence each term of integrals in (2.4) and (2.5) is well-defined. Then the proofs of (2.4) and (2.5) follow from (ii). □

The following lemma may be regarded as a generalization of the one by Masuda [21, Lemma 2.5].

LEMMA 2.2 (Masuda). *For any $\varepsilon > 0$ and any $\Phi \in S$, there exist a constant $C = C(\varepsilon, \Phi)$, an integer N and functions ψ_i, $i = 1, 2, \dots, N$, in L^2 such that the inequality*

$$
\begin{aligned}
\int_0^T |(u \cdot \nabla v, \Phi)| d\tau \le{}& \varepsilon \int_0^T (\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u\|_2^2) d\tau \\
&+ C \sum_{i=1}^N \int_0^T |(u(\tau), \psi_i)|^2 d\tau
\end{aligned}
$$

holds for all $u, v \in L^2(0, T; H^1_{0,\sigma})$.

PROOF. We follow the argument in Masuda [21, Lemma 2.5]. Let $\eta = \eta(t)$ be a smooth monotone decreasing function on $[0, \infty)$ so that $0 \le \eta \le 1$, $\eta(t) \equiv 1$ for $0 \le |t| \le 1$ and $\eta(t) \equiv 0$ for $|t| \ge 2$. We also take $\zeta \in C_0^\infty(\boldsymbol{R}^n)$ in such a way that $\zeta(x) = \eta(|x|)$ for $x \in \boldsymbol{R}^n$. For $R > 0$ we define η_R and ζ_R by $\eta_R(t) \equiv \eta(t/R)$ and $\zeta_R(x) \equiv \zeta(x/R)$, respectively.

Let us take $\Phi \in S$. By (1.1), for every $\varepsilon' > 0$ there is $R_{\varepsilon'} > 0$ such that

$$R\mu\{x \in \Omega; |\Phi(x,t)| > R\}^{\frac{1}{n}} < \varepsilon' \quad \text{for all } R \ge R_{\varepsilon'} \text{ and all } t \in (0,T)\,. \tag{2.12}$$

Let us fix such $R_{\varepsilon'}$. We decompose Φ as

$$\begin{aligned} \Phi &= (1 - \eta_{R_{\varepsilon'}}(|\Phi(x,t)|))\Phi(x,t) + \eta_{R_{\varepsilon'}}(|\Phi(x,t)|)\Phi(x,t) \\ &\equiv \Phi_0(x,t) + \Phi_1(x,t)\,. \end{aligned} \tag{2.13}$$

Then there holds $\Phi_0 \in L^\infty(0,T; L^2 \cap L^n_w)$ with

$$\sup_{0<t<T} \|\Phi_0(t)\|_{n,w} \le \varepsilon'\,. \tag{2.14}$$

In fact, since $|\Phi(x,t)| \ge |\Phi_0(x,t)|$ for all $(x,t) \in \Omega \times (0,T)$, we have by (2.12) that

$$\begin{aligned} \|\Phi_0(t)\|_{n,w} &= \sup_{0<R<\infty} R\mu\{x \in \Omega; |\Phi_0(x,t)| > R\}^{\frac{1}{n}} \\ &\le \max\left\{ \sup_{0<R<R_{\varepsilon'}} R\mu\{x \in \Omega; |\Phi_0(x,t)| > R\}^{\frac{1}{n}},\ \sup_{R_{\varepsilon'} \le R < \infty} R\mu\{x \in \Omega; |\Phi_0(x,t)| > R\}^{\frac{1}{n}} \right\} \\ &\le \max\left\{ R_{\varepsilon'}\mu\{x \in \Omega; |\Phi(x,t)| > R_{\varepsilon'}\}^{\frac{1}{n}},\ \sup_{R_{\varepsilon'} \le R < \infty} R\mu\{x \in \Omega; |\Phi(x,t)| > R\}^{\frac{1}{n}} \right\} \\ &= \sup_{R_{\varepsilon'} \le R < \infty} R\mu\{x \in \Omega; |\Phi(x,t)| > R\}^{\frac{1}{n}} < \varepsilon' \end{aligned}$$

for all $t \in (0,T)$, which implies (2.14).

As for Φ_1, we have

$$\Phi_1 \in L^\infty(0,T; L^r) \quad \text{for all } 2 \le r \le \infty\,. \tag{2.15}$$

Taking $m = 1, 2, \dots$, we next decompose Φ_1 as

$$\begin{aligned} \Phi_1(x,t) &= (1 - \zeta_m(x))\Phi_1(x,t) + \zeta_m(x)\Phi_1(x,t) \\ &\equiv \Phi^m_{1,0}(x,t) + \Phi^m_{1,1}(x,t)\,. \end{aligned}$$

Since it holds for every $(x,t), (x,t') \in \Omega \times (0,T)$ and every $m = 1, 2, \dots$ that

$$|\Phi^m_{1,0}(x,t) - \Phi^m_{1,0}(x,t')| \le \sup_{\tau \in \boldsymbol{R}^1} (\eta(\tau) + |\tau\eta'(\tau)|)|\Phi(x,t) - \Phi(x,t')|\,,$$

we have $\Phi^m_{1,0} \in C([0,T]; L^n_w)$. Using the inequality $1 - \zeta_m(x) \ge 1 - \zeta_{m+1}(x)$ for all $x \in \boldsymbol{R}^n$, we see that for each fixed $t \in [0,T]$, there holds

$$\|\Phi^m_{1,0}(t)\|_{n,w} \ge \|\Phi^{m+1}_{1,0}\|_{n,w}\,, \quad m = 1, 2, \dots\,.$$

Obviously, (2.15) yields

$$\|\Phi^m_{1,0}(t)\|_{n,w} \le \|\Phi^m_{1,0}(t)\|_n \to 0 \quad \text{as } m \to \infty \text{ for each fixed } t \in [0,T].$$

Then it follows from the Dini theorem that there is an integer $m_{\varepsilon'}$ such that

$$\sup_{0 \le t \le T} \|\Phi^{m_{\varepsilon'}}_{1,0}(t)\|_{n,w} < \varepsilon'. \tag{2.16}$$

Let us fix such $m_{\varepsilon'}$. By (2.1), (2.14) and (2.16) we have

$$\int_0^T |(u\cdot\nabla v, \Phi_0)|d\tau \le C\varepsilon' \int_0^T \|\nabla u\|_2\|\nabla v\|_2 d\tau, \tag{2.17}$$

$$\int_0^T |(u\cdot\nabla v, \Phi^m_{1,0})|d\tau \le C\varepsilon' \int_0^T \|\nabla u\|_2\|\nabla v\|_2 d\tau, \tag{2.18}$$

where $C = C(n)$ is independent of u, v, ε'.

To handle $\Phi^m_{1,1}$, notice that

$$\operatorname{supp} \Phi^m_{1,1}(\cdot, t) \subset \Omega_{2m} \equiv \Omega \cap \{|x| < 2m\}$$

and that $\Phi^m_{1,1} \in L^\infty(\Omega \times (0,T))$ with $\|\phi^m_{1,1}\|_{L^\infty(\Omega\times(0,T))} \le 2R_{\varepsilon'}$. Hence by the Schwarz inequality there holds

$$\begin{aligned}\int_0^T |(u\cdot\nabla v, \Phi^m_{1,1})|d\tau &\le 2R_{\varepsilon'}\int_0^T \|u\|_{L^2(\Omega_{2m})}\|\nabla v\|_{L^2(\Omega_{2m})}d\tau \\ &\le \varepsilon'\int_0^T \|\nabla v\|_2^2 d\tau + C_{\varepsilon'}\int_0^T \|u\|^2_{L^2(\Omega_{2m})}d\tau.\end{aligned} \tag{2.19}$$

Since $H^1(\Omega_{2m})$ is compactly embedded into $L^2(\Omega_{2m})$, it follows from the Friedrichs inequality that there is an integer $N_{\varepsilon'}$ such that

$$\begin{aligned}\|u\|^2_{L^2(\Omega_{2m})} &\le \frac{\varepsilon'}{C_{\varepsilon'}}(\|\nabla u\|^2_{L^2(\Omega_{2m})} + \|u\|^2_{L^2(\Omega_{2m})}) + \sum_{i=1}^{N_{\varepsilon'}}\left(\int_{\Omega_{2m}} u\cdot\phi_i dx\right)^2 \\ &\le \frac{\varepsilon'}{C_{\varepsilon'}}(\|\nabla u\|_2^2 + \|u\|_2^2) + \sum_{i=1}^{N_{\varepsilon'}}|(u, \chi_{\Omega_{2m}}\phi_i)|^2,\end{aligned} \tag{2.20}$$

where $\{\phi_i\}_{i=1}^\infty$ is the complete orthonormal system in $L^2(\Omega_{2m})$ and $\chi_{\Omega_{2m}}$ is the characteristic function on Ω_{2m}. Defining $\psi_i \equiv \chi_{\Omega_{2m}}\phi_i$, $i = 1, 2, \dots, N_{\varepsilon'}$, we have $\psi_i \in L^2$ and (2.19)–(2.20) yield

$$\begin{aligned}\int_0^T |(u\cdot\nabla v, \Phi^m_{1,1})|d\tau &\le \varepsilon'\int_0^T (\|\nabla v\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2)d\tau \\ &\quad + C_{\varepsilon'}\sum_{i=1}^{N_{\varepsilon'}}\int_0^T |(u, \psi_i)|^2 d\tau.\end{aligned} \tag{2.21}$$

Since ε' is arbitrarily taken, the desired estimate follows from (2.17), (2.18) and (2.21). This proves Lemma 2.2.

Let us assume that the domain Ω is as in Assumption 3. For such Ω, the L^r-theory of the Stokes operator A_r is established. Recall first the Helmholtz decomposition:

$$L^r = L^r_\sigma \oplus G^r \text{ (direct sum)}, \quad 1 < r < \infty,$$

where $G^r = \{\nabla p \in L^r; p \in L^r_{loc}(\bar{\Omega})\}$. For the proof, see Fujiwara-Morimoto [6] and Simader-Sohr [25]. Let P_r denote the projection operator from L^r onto L^r_σ along G^r. The Stokes operator A_r on L^r_σ is then defined by $A_r = -P_r\Delta$ with domain $D(A_r) = \{u \in H^{2,r}(\Omega); u|_{\partial\Omega} = 0\} \cap L^r_\sigma$. We regard $D(A_r)$ as a Banach space with the graph norm $\|u\|_{D(A_r)} \equiv \|u\|_r + \|A_r u\|_r$. It is known that

(2.22) $\quad (L^r_\sigma)^*$ (the dual space of L^r_σ) $= L^{r'}_\sigma$, $\quad A^*_r$ (the adjoint operator of A_r) $= A_{r'}$,

where $1/r + 1/r' = 1$. Moreover, we have

LEMMA 2.3 (Giga-Sohr). *Suppose that Assumption* 3 *holds. Then we have the following.*

(i) $-A_r$ *generates a uniformly bounded holomorphic semigroup* $\{e^{-tA_r}\}_{t\ge 0}$ *of class* C^0 *in* L^r_σ.

(ii) *By* (i) *above, we can define the fractional power* A^α_r *for* $\alpha > 0$, *and there is a continuous embedding* $D(A^\alpha_r) \subset H^{2\alpha,r}$ *with*

(2.23) $$\|u\|_{H^{2\alpha,r}} \le C(\|u\|_r + \|A^\alpha_r u\|_r) \quad \textit{for all } u \in D(A^\alpha_r)$$

with $C = C(n, r, \alpha)$, *where* $H^{\beta,r}$ *denotes the space of Bessel potentials over* Ω.

(iii) *Let* $1 < s < \infty$ *and* $1 < r < \infty$. *Suppose that* $w \in L^s(0, T; D(A_r))$ *with* $\partial_t w \in L^s(0, T; L^r_\sigma)$ *and* $w(0) = b \in D(A_r)$. *Then we have*

$$w \in L^{s_0}(0, T; L^{r_0}_\sigma)$$

for $1 < s \le s_0 < \infty$, $1 < r < r_0 < \infty$ *with* $2/s_0 + n/r_0 = 2/s + n/r - 2$,

$$\nabla w \in L^{s_1}(0, T; L^{r_1})$$

for $1 < s \le s_1 < \infty$, $1 < r < r_1 < \infty$ *or* $1 < s < s_1 < \infty$, $1 < r \le r_1 < \infty$ *with* $2/s_1 + n/r_1 = 2/s + n/r - 1$. *Moreover, there hold the estimates*

(2.24) $$\|w\|_{L^{s_0}(0,T;L^{r_0})} \le C_0(\|\partial_t w\|_{L^s(0,T;L^r)} + \|(A_r + 1)w\|_{L^s(0,T;L^r)} + \|b\|_{D(A_r)}),$$

(2.25) $$\|\nabla w\|_{L^{s_1}(0,T;L^{r_1})} \le C_1(\|\partial_t w\|_{L^s(0,T;L^r)} + \|(A_r + 1)w\|_{L^s(0,T;L^r)} + \|b\|_{D(A_r)}),$$

where $C_j = C_j(s, r, s_j, r_j)$, $j = 0, 1$, *is independent of* T.

For the proof, see Giga-Sohr [9, Theorem 3.1] and [10, Lemma 5.2]. Although they proved the result for $1 < r < n/2$, with A_r replaced by $A_r + 1$, the above estimates can be proved for all $1 < r < \infty$ in the same way as in [9] and [10]. In fact, restriction on $1 < r < n/2$ is necessary only for getting (2.24) and (2.25) including the case $T = \infty$ in exterior domains. This is related to the sharp estimate $\|D^2 u\|_r \le C\|A_r u\|_r$ for all $u \in D(A_r)$ which holds in exterior domains for $1 < r < n/2$. However, the estimate $\|D^2 u\|_r \le C\|(A_r + 1)u\|_r$ is true for all $1 < r < \infty$. It should be noted that the above estimates hold even in the case $s_0 = s$ and $s_1 = s$ provided $r < r_0$, $r < r_1$. (2.25) holds even for $r = r_1$ provided $s < s_1$.

Let us introduce the space $X^{s,r}(0, T)$ by

$$X^{s,r}(0,T) \equiv \{w \in L^s(0,T; D(A_r));\ \partial_t w \in L^s(0,T; L^r_\sigma)\},$$

which is a Banach space with the norm

$$\|w\|_{X^{s,r}(0,T)} \equiv \|\partial_t w\|_{L^s(0,T;L^r)} + \|(A_r+1)w\|_{L^s(0,T;L^r)}.$$

LEMMA 2.4. *Suppose that Assumption* 3 *holds.*

(i) *Let* $u_0 \in L^\infty(0,T; L^n_w)$ *and* $v \in X^{s,r}(0,T)$ *for* $1 < s < \infty$, $1 < r < n$. *Then we have* $u_0 \cdot \nabla v \in L^s(0,T; L^r)$ *with*

$$\|u_0 \cdot \nabla v\|_{L^s(0,T;L^r)} \le C\|u_0\|_{L^\infty(0,T;L^n_w)}\|v\|_{X^{s,r}(0,T)}, \tag{2.26}$$

where $C = C(n,s,r)$ *is independent of* T.

(ii) *Let* $u_1 \in L^2(0,T; L^\infty)$ *and* $v \in X^{s,r}(0,T)$ *for* $1 < s < 2$, $1 < r < \infty$ *with* $v(0) \in D(A_r)$. *Then we have* $u_1 \cdot \nabla v \in L^s(0,T; L^r)$ *with*

$$\|u_1 \cdot \nabla v\|_{L^s(0,T;L^r)} \le C\|u_1\|_{L^2(0,T;L^\infty)}(\|v\|_{X^{s,r}(0,T)} + \|v(0)\|_{D(A_r)}), \tag{2.27}$$

where $C = C(n,s,r)$ *is independent of* T.

PROOF. (i) Taking r_0 and r_1 so that $1 < r_0 < r < r_1 < n$, we have $1/r = (1-\theta)/r_0 + \theta/r_1$ for some $0 < \theta < 1$. Let us define q_j, $j = 0, 1$, by $1/q_j = 1/r_j - 1/n$. By (2.23) and the Sobolev embedding $H^{2,r_j} \subset H^{1,q_j}$, we have

$$\begin{aligned}\|u \cdot \nabla v\|_{r_j,w} &\le \|u\|_{n,w}\|\nabla v\|_{q_j,w}\\ &\le C\|u\|_{n,w}\|v\|_{H^{2,r_j}}\\ &\le C\|u\|_{n,w}\|(A_{r_j}+1)v\|_{r_j}\end{aligned}$$

for all $u \in L^n_w$ and all $v \in D(A_{r_j})$ with $C = C(n, r_0, r_1)$. This implies that for each fixed $u \in L^n_w$, the map

$$v \in D(A_{r_j}) \mapsto u \cdot \nabla v \in L^{r_j}_w, \quad j = 0, 1$$

is a bounded operator with bound $\le C\|u\|_{n,w}$. Applying the Marcinkiewicz interpolation inequality, we see that

$$v \in D(A_r) \mapsto u \cdot \nabla v \in L^r$$

defines a bounded operator with bound $\le C\|u\|_{n,w}$. Hence we have

$$\|u \cdot \nabla v\|_r \le C\|u\|_{n,w}\|(A_r+1)v\|_r, \quad v \in D(A_r)$$

with $C = C(n, r)$. Then (2.26) is an immediate consequence of the above estimate.

(ii) Taking $s_1 \in (1, \infty)$ so that $1/s_1 = 1/s - 1/2$, we have by the Hölder inequality

$$\|u_1 \cdot \nabla v\|_{L^s(0,T;L^r)} \le \|u_1\|_{L^2(0,T;L^\infty)}\|\nabla v\|_{L^{s_1}(0,T;L^r)}.$$

Applying (2.25) with $r = r_1$ to this estimate, we obtain (2.27). □

3. Existence of weak solutions; Proof of Theorem 1. We shall construct a weak solution via the method of retarded mollifier according to Caffarelli-Kohn-Nirenberg [4].

Let $h(\geq 0) \in C_0^\infty(\boldsymbol{R}^1)$ and $\rho(\geq 0) \in C_0^\infty(\boldsymbol{R}^n)$ such that

$$\operatorname{supp} h \subset [1,2], \quad \int_1^2 h(\tau)d\tau = 1, \quad \operatorname{supp}\rho \subset \{x \in \boldsymbol{R}^n; |x| < 1\}, \quad \int_{|x|<1} \rho(x)dx = 1,$$

respectively. For $\delta > 0$ we set

$$h_\delta(\tau) \equiv \frac{1}{\delta} h\left(\frac{\tau}{\delta}\right), \quad \rho_\delta(x) \equiv \frac{1}{\delta^n}\rho\left(\frac{x}{\delta}\right).$$

For $u \in L^2(0,T; H^1_{0,\sigma})$, we define the retarded mollifier $\Psi_\delta[u]$ by

$$\Psi_\delta[u](x,t) \equiv \int_{-\infty}^{\infty} h_\delta(t-\tau)\left(\int_{\boldsymbol{R}^n} \rho_\delta(x-y)\tilde{u}(y,\tau)dy\right)d\tau,$$

where $\tilde{u}(y,\tau) = u(y,\tau)$ for $(y,\tau) \in \Omega \times (0,T)$, and $= 0$ otherwise. It is easy to see that for every $u \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1_{0,\sigma})$ there holds $\Psi_\delta[u] \in BC(\boldsymbol{R}^n \times \boldsymbol{R}^1)$, BC denoting the class of bounded and continuous functions, with $\operatorname{div}\Psi_\delta[u] = 0$ and

$$\|\Psi_\delta[u]\|_{L^\infty(0,T;L^2)} \leq \|u\|_{L^\infty(0,T;L^2)}, \quad \|\nabla\Psi_\delta[u]\|_{L^2(0,T;L^2)} \leq \|\nabla u\|_{L^2(0,T;L^2)}. \tag{3.1}$$

For each $m = 1, 2, \ldots$, consider the following approximation

$$\begin{cases} u_m' + A_2 u_m + P_2(\Psi_\delta[u_m]\cdot\nabla u_m) = 0, & 0 < t < T, \\ u_m(0) = a, \end{cases} \tag{3.2}$$

where $\delta = T/m$. It follows from Caffarelli-Kohn-Nirenberg [4, Appendix] that there exists a solution u_m of (3.2) such that

$$u_m \in C([0,T); L^2_\sigma) \cap L^2(0,T; H^1_{0,\sigma}); \tag{3.3}$$

$$u_m' \in L^2(0,T; (H^1_{0,\sigma})^*); \tag{3.4}$$

$$\|u_m(t)\|_2^2 + 2\int_0^t \|\nabla u_m\|_2^2 d\tau = \|a\|_2^2, \quad 0 \leq t \leq T. \tag{3.5}$$

By (3.5) we see that the sequence $\{u_m\}_{m=1}^\infty$ is bounded in $L^\infty(0,T; L^2) \cap L^2(0,T; H^1_{0,\sigma})$. Hence the weak compactness theorem yields a subsequence of $\{u_m\}_{m=1}^\infty$, which we denote also by $\{u_m\}_{m=1}^\infty$ for simplicity, and a limit u such that

$$u_m \to u \quad \text{weakly-star in } L^\infty(0,T; L^2) \text{ and weakly in } L^2(0,T; H^1_{0,\sigma}). \tag{3.6}$$

Moreover, for every $\phi \in C_{0,\sigma}^\infty$, we have by (3.1) and (3.5)

$$\left|\int_s^t (Au_m, \phi)d\tau\right| \leq \left(\int_s^t \|\nabla u_m\|_2^2 d\tau\right)^{\frac{1}{2}}\left(\int_s^t \|\nabla\phi\|_2^2 d\tau\right)^{\frac{1}{2}} \leq \|a\|_2\|\nabla\phi\|_2|t-s|^{\frac{1}{2}},$$

$$\begin{aligned} \left|\int_s^t (P(\Psi_\delta[u_m]\cdot\nabla u_m), \phi)d\tau\right| &= \left|\int_s^t (\Psi_\delta[u_m]\cdot\nabla\phi, u_m)d\tau\right| \\ &\leq \|\nabla\phi\|_\infty\|\Psi_\delta[u_m]\|_{L^\infty(0,T;L^2)}\|u_m\|_{L^\infty(0,T;L^2)}|t-s| \\ &\leq \|\nabla\phi\|_\infty\|a\|_2^2|t-s| \end{aligned}$$

for all $m = 1, 2, \ldots$ and all $0 \leq s \leq t \leq T$.

Integrating (3.2) in time on (s, t), we obtain from these estimates

$$|(u_m(t), \phi) - (u_m(s), \phi)| \le \|a\|_2 \|\nabla\phi\|_2 |t-s|^{\frac{1}{2}} + \|\nabla\phi\|_\infty \|a\|_2^2 |t-s|$$

for all $0 \le s \le t \le T$ and all $\phi \in C_{0,\sigma}^\infty$. Since $C_{0,\sigma}^\infty$ is dense in L_σ^2 and since we have $\sup_{0<\tau<T} \|u_m(\tau)\|_2 \le \|a\|_2$ for all $m = 1, 2, \dots$, the above estimate implies that for each $\phi \in L_\sigma^2$ the sequence $\{(u_m(t), \phi)\}_{m=1}^\infty$ forms a *uniformly bounded* and *equi-continuous* family of continuous functions on $[0, T]$. By the Ascoli-Arzelá theorem, we may assume that for every $\phi \in L_\sigma^2$

$$(u_m(t), \phi) \to (u(t), \phi) \quad \text{uniformly in } t \in [0, T] \text{ as } m \to \infty .$$

By (3.1) and the definition of $\Psi_\delta[u]$, we also have for each $\phi \in L_\sigma^2$

$$(3.7) \qquad (\Psi_\delta[u_m](t), \phi) \to (u(t), \phi) \quad \text{uniformly in } t \in [0, T] \text{ as } m \to \infty .$$

To prove that the limit u is the desired weak solution, it suffices to show that

$$(3.8) \qquad \int_0^T (\Psi_\delta[u_m] \cdot \nabla u_m, \Phi) d\tau \to \int_0^T (u \cdot \nabla u, \Phi) d\tau \quad \text{for all } \Phi \in S .$$

Indeed, we have

$$(3.9) \qquad \begin{aligned} &\left| \int_0^T (\Psi_\delta[u_m] \cdot \nabla u_m, \Phi) d\tau - \int_0^T (u \cdot \nabla u, \Phi) d\tau \right| \\ &\le \int_0^T |((\Psi_\delta[u_m] - u) \cdot \nabla u_m, \Phi)| d\tau + \int_0^T |(u \cdot \nabla(u_m - u), \Phi)| d\tau \\ &\equiv I_1 + I_2 . \end{aligned}$$

By Lemma 2.2, for every $\varepsilon > 0$, there are a constant $C = C(\varepsilon)$, an integer N and functions $\psi_i, i = 1, 2, \dots, N$, in L^2 such that

$$\begin{aligned} I_1 \le \varepsilon &\int_0^T (\|\nabla\Phi_\delta[u_m] - \nabla u\|_2^2 + \|\nabla u\|_2^2 + \|\Phi_\delta[u_m] - u\|_2^2) d\tau \\ &+ C \sum_{i=1}^N \int_0^T |(\Psi_\delta[u_m] - u, \psi_i)|^2 d\tau . \end{aligned}$$

By (3.1), (3.5) and (3.7), we have

$$\limsup_{m\to\infty} I_1 \le 3(1+T)\|a\|_2^2 \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, this implies that

$$(3.10) \qquad \lim_{m\to\infty} I_1 = 0 .$$

By [18, Proposition 2.2], we have a continuous inclusion $H_0^1 \subset L^{2n/(n-2),2}$ and hence it follows from the Hölder inequality in the Lorentz space ([18, Proposition 2.1]) that

$$w^j \equiv u^j \Phi \in L^2(0, T, L^2) (= L^2(\Omega \times (0, T))$$

with $\|w^j\|_{L^2(\Omega\times(0,T))} \le C \|\nabla u\|_{L^2(0,T;L^2)} \|\Phi\|_{L^\infty(0,T;L_w^n)}$, where u^j denotes the j-th component of u. Since $C_0^\infty(\Omega \times (0, T))$ is dense in $L^2(\Omega \times (0, T))$, there is a sequence w_k^j,

$k = 1, 2, \ldots,$ in $C_0^\infty(\Omega \times (0,T))$ such that $w_k^j \to w^j$ in $L^2(\Omega \times (0,T))$ as $k \to \infty$. Hence, by integration by parts we have

$$
\begin{aligned}
I_2 &\le \left| \int_0^T \int_\Omega \sum_{j=1}^n (w^j - w_k^j) \cdot \frac{\partial}{\partial x_j}(u_m - u)\,dx\,d\tau \right| + \left| \int_0^T \int_\Omega \sum_{j=1}^n \frac{\partial w_k^j}{\partial x_j}(u_m - u)\,dx\,d\tau \right| \\
&\le \left(\int_0^T \int_\Omega \sum_{j=1}^n |w^j - w_k^j|^2 dx\,d\tau \right)^{\frac{1}{2}} \left(\int_0^T \int_\Omega |\nabla u_m - \nabla u|^2 dx\,d\tau \right)^{\frac{1}{2}} \\
&\quad + \left| \int_0^T \int_\Omega \sum_{j=1}^n \frac{\partial w_k^j}{\partial x_j}(u_m - u)\,dx\,d\tau \right| \\
&\le 2\|a\|_2 \left(\int_0^T \int_\Omega \sum_{j=1}^n |w^j - w_k^j|^2 dx\,d\tau \right)^{\frac{1}{2}} + \left| \int_0^T \int_\Omega \sum_{j=1}^n \frac{\partial w_k^j}{\partial x_j}(u_m - u)\,dx\,d\tau \right|
\end{aligned}
$$

for all $m, k = 1, 2, \ldots$. Since $u_m \to u$ weakly* in $L^\infty(0,T;L^2)$ as $m \to \infty$ and since $w_k \to w$ in $L^2(\Omega \times (0,T))$, by letting $m \to \infty$ and then $k \to \infty$ in the above estimate, we obtain

$$\lim_{m\to\infty} I_2 = 0\,. \tag{3.11}$$

Then (3.8) follows from (3.9), (3.10) and (3.11). This completes the proof of Theorem 1.

4. Uniqueness of weak solutions; Proof of Theorem 2. Let u be a weak solution of (N-S) with (1.4). Then u can be decomposed as

$$u = u_0 + u_1\,, \tag{4.1}$$

where $u_0 \in L^\infty(0,T;L^2 \cap L^n_w)$ and $u_1 \in L^2(0,T;L^2 \cap L^\infty)$ with

$$\sup_{0<t<T} \|u_0(t)\|_{n,w} \le \varepsilon_0\,. \tag{4.2}$$

Indeed, we define u_0 and u_1 as

$$
u_0(x,t) = \begin{cases} u(x,t) & \text{on } \{(x,t) \in \Omega \times (0,T);\ |u(x,t)| \ge M(t)\}\,, \\ 0 & \text{on } \{(x,t) \in \Omega \times (0,T);\ |u(x,t)| < M(t)\}\,, \end{cases}
$$

$$
u_1(x,t) = \begin{cases} 0 & \text{on } \{(x,t) \in \Omega \times (0,T);\ |u(x,t)| \ge M(t)\}\,, \\ u(x,t) & \text{on } \{(x,t) \in \Omega \times (0,T);\ |u(x,t)| < M(t)\}\,. \end{cases}
$$

Since $M \in L^2(0,T)$, we see easily $u_1 \in L^2(0,T;L^\infty) \cap L^\infty(0,T;L^2)$ with

$$\|u_1\|_{L^2(0,T;L^\infty)} \le \left(\int_0^T M(t)^2 dt \right)^{\frac{1}{2}}.$$

Similarly to (2.14), we have by (1.4)

$$
\begin{aligned}
\|u_0(t)\|_{n,w} &= \sup_{0<R<\infty} R\mu\{x \in \Omega; |u_0(x,t)| > R\}^{\frac{1}{n}} \\
&= \max\left\{ \sup_{0<R<M(t)} R\mu\{x \in \Omega; |u_0(x,t)| > R\}^{\frac{1}{n}}, \sup_{M(t)\le R<\infty} R\mu\{x \in \Omega; |u_0(x,t)| > R\}^{\frac{1}{n}} \right\} \\
&\le \max\left\{ M(t)\mu\{x \in \Omega; |u(x,t)| \ge M(t)\}^{\frac{1}{n}}, \sup_{M(t)\le R<\infty} R\mu\{x \in \Omega; |u(x,t)| > R\}^{\frac{1}{n}} \right\} \\
&= \sup_{M(t)\le R<\infty} R\mu\{x \in \Omega; |u(x,t)| > R\}^{\frac{1}{n}} \le \varepsilon_0
\end{aligned}
$$

for all $0 < t < T$, which implies (4.2).

Now taking a mollifier $\rho_h(\tau) = (1/h)\rho(\tau/h)$, $(h > 0, \tau \in \mathbf{R}^1)$, we define $u_{h,m}$ by

$$
u_{h,m}(\tau) \equiv \int_0^t \rho_h(\tau - \sigma) J_m u(\sigma) d\sigma , \quad m = 1, 2, \dots , \quad 0 < t < T ,
$$

where $\{J_m\}_{m=1}^{\infty}$ is the family of bounded operators on $H^1_{0,\sigma} \cap L^n_w$ in Assumption 2. Since $u = u_0 + u_1 \in L^2(0, T; L^n_w)$, we have by Assumption 2 (i)–(ii) and Proposition 1 that $u_{h,m} \in S$ for all $h > 0$ and all $m = 1, 2, \dots$. Choosing $u_{h,m}$ as a test function associated with the weak solution v, we obtain

$$
\begin{aligned}
\int_0^t \{-(v, \partial_\tau u_{h,m}) + (\nabla v, \nabla u_{h,m}) + (v \cdot \nabla v, u_{h,m})\} d\tau \qquad (4.3)\\
= -(v(t), u_{h,m}(t)) + (a, u_{h,m}(0)) .
\end{aligned}
$$

Let us define u_h be as

$$
u_h(\tau) \equiv \int_0^t \rho_h(\tau - \sigma) u(\sigma) d\sigma .
$$

Since $u_h \in C([0, t]; H^1_{0,\sigma} \cap L^n_w)$, we have by Assumption 2 (ii) that for each $h > 0$ and $\tau \in [0, t]$

$$
\|\nabla u_{h,m}(\tau) - \nabla u_h(\tau)\|_2 \le (M + 1)(\|u_h(\tau)\|_{H^1} + \|u_h(\tau)\|_{n,w})
$$

with

$$
\|\nabla u_{h,m}(\tau) - \nabla u_h(\tau)\|_2 \le \|J_m u_h(\tau) - u_h(\tau)\|_{H^1} \to 0 \quad \text{as } m \to \infty .
$$

By the Lebesgue convergence theorem, there holds

$$
\int_0^t (\nabla v, \nabla u_{h,m}) d\tau \to \int_0^t (\nabla v, \nabla u_h) d\tau \quad \text{as } m \to \infty . \qquad (4.4)
$$

We obtain from (2.1) and Assumption 2 (i) that for every $\tau \in [0, t]$ and every $m = 1, 2, \dots$

$$
\begin{aligned}
&|(v \cdot \nabla v(\tau), u_{h,m}(\tau) - u_h(\tau))| \\
&\le C\|\nabla v(\tau)\|_2^2 \|u_{h,m}(\tau) - u_h(\tau)\|_{n,w} \\
&\le C(M + 1)\left(\sup_{0\le\sigma\le t} \|u_h(\sigma)\|_{H^1} + \sup_{0\le\sigma\le t} \|u_h(\sigma)\|_{n,w} \right) \|\nabla v(\tau)\|_2^2
\end{aligned}
$$

with $C = C(n)$. Furthermore, by Assumption 2 (iii), there holds

$$u_{h,m}(\tau) \to u_h(\tau) \quad \text{weakly* in } L^n_w \text{ as } m \to \infty$$

for each fixed $\tau \in [0, t]$. In the same way as in (2.6), we have $v \cdot \nabla v(\tau) \in L^{n',1}$ and hence from the above convergence we obtain

$$(v \cdot \nabla v(\tau), u_{h,m}(\tau) - u_h(\tau)) \to 0 \quad \text{as } m \to \infty$$

for almost every $\tau \in [0, t]$. Then the Lebesgue convergence theorem yields

$$\int_0^t (v \cdot \nabla v, u_{h,m}) d\tau \to \int_0^t (v \cdot \nabla v, u_h) d\tau \quad \text{as } m \to \infty. \tag{4.5}$$

Letting $m \to \infty$ in (4.3), we obtain from (4.4) and (4.5) that

$$\int_0^t \{-(v, \partial_\tau u_h) + (\nabla v, \nabla u_h) + (v \cdot \nabla v, u_h)\} d\tau = -(v(t), u_h(t)) + (a, u_h(0)). \tag{4.6}$$

For the weak solution v, there is a sequence v_k, $k = 1, 2, \ldots$, in $H^1(0, T; H^1_{0,\sigma} \cap L^n)$ such that $v_k \to v$ in $L^2(0, T; H^1_{0,\sigma})$ (see Masuda [21]). Let us define $v_{k,h}$ and v_h by

$$v_{k,h}(\tau) = \int_0^t \rho_h(\tau - \sigma) v_k(\sigma) d\sigma, \quad v_h(\tau) = \int_0^t \rho_h(\tau - \sigma) v(\sigma) d\sigma.$$

Similarly to (4.3), we have

$$\begin{aligned} &\int_0^t \{-(u, \partial_\tau v_{k,h}) + (\nabla u, \nabla v_{k,h}) + (u \cdot \nabla u, v_{k,h})\} d\tau \\ &\qquad = -(u(t), v_{k,h}(t)) + (a, v_{k,h}(0)). \end{aligned} \tag{4.7}$$

By (2.1) and (4.1), there holds

$$\begin{aligned} &\int_0^t |(u \cdot \nabla u, v_{k,h} - v_h)| d\tau \\ &\quad \le \int_0^t |(u_0 \cdot \nabla u, v_{k,h} - v_h)| d\tau + \int_0^t |(u_1 \cdot \nabla u, v_{k,h} - v_h)| d\tau \\ &\quad \le \sup_{0 \le \sigma \le T} \|u_0(\sigma)\|_{n,w} \left(\int_0^t \|\nabla u\|_2^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\nabla v_{k,h} - \nabla v_h\|_2^2 d\tau \right)^{\frac{1}{2}} \\ &\qquad + \sup_{0 \le \sigma \le t} \|v_{k,h}(\sigma) - v_h(\sigma)\|_2 \left(\int_0^t \|u_1\|_\infty^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\nabla u\|_2^2 d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

Since $v_{k,h} \to v_h$ in $H^1(0, t; H^1_{0,\sigma})$ as $k \to \infty$, the above estimate yields

$$\int_0^t (u \cdot \nabla u, v_{k,h}) d\tau \to \int_0^t (u \cdot \nabla u, v_h) d\tau \quad \text{as } k \to \infty.$$

Hence, letting $k \to \infty$ in (4.7), we see easily

$$\int_0^t \{-(u, \partial_\tau v_h) + (\nabla u, \nabla v_h) + (u \cdot \nabla u, v_h)\} d\tau = -(u(t), v_h(t)) + (a, v_h(0)). \tag{4.8}$$

By the symmetry of ρ_h, we have $\int_0^t (v, \partial_\tau u_h) d\tau = -\int_0^t (u, \partial_\tau v_h) d\tau$, and hence adding (4.6) and (4.8), we obtain

$$\int_0^t \{(\nabla v, \nabla u_h) + (\nabla u, \nabla v_h) + (v \cdot \nabla v, u_h) + (u \cdot \nabla u, v_h)\} d\tau \\ = -(v(t), u_h(t)) - (u(t), v_h(t)) + (a, u_h(0) + v_h(0)) . \tag{4.9}$$

Let $u_{0,h}$ and $u_{1,h}$ be as

$$u_{i,h}(\tau) = \int_0^t \rho_h(\tau - \sigma) u_i(\sigma) d\sigma , \quad i = 0, 1 ,$$

where u_0 and u_1 are as in (4.1). Since $u_0 \in L^\infty(0, T; L^n_w)$, we have $u_{0,h} \to u_0$ in $L^r(0, t; L^n_w)$ for all $1 \le r < \infty$. Hence there is a subsequence of $\{u_{0,h}\}_{h>0}$, which we denote also by $\{u_{0,h}\}_{h>0}$ for simplicity, such that $u_{0,h}(\tau) \to u_0(\tau)$ in L^n_w for almost every $\tau \in [0, t]$ as $h \to 0$, which yields by (2.1)

$$(v \cdot \nabla v(\tau), u_{0,h}(\tau)) \to (v \cdot \nabla v(\tau), u_0(\tau)) \quad \text{for almost every } \tau \in [0, t] \text{ as } h \to 0 .$$

Moreover, by (4.2)

$$|(v \cdot \nabla v(\tau), u_{0,h}(\tau))| \le C \|\nabla v(\tau)\|_2^2 \|u_{0,h}(\tau)\|_{n,w} \le \varepsilon_0 \|\nabla v(\tau)\|_2^2$$

for all $h > 0$ and all $\tau \in [0, t]$. Hence the Lebesgue convergence theorem states

$$\int_0^t (v \cdot \nabla v, u_{0,h}) d\tau \to \int_0^t (v \cdot \nabla v, u_0) d\tau \quad \text{as } h \to 0 . \tag{4.10}$$

Since $u_{1,h} \to u_1$ in $L^2(0, T; L^\infty)$ as $h \to 0$, we have by the Schwarz inequality

$$\left| \int_0^t (v \cdot \nabla v, u_{1,h}) d\tau - \int_0^t (v \cdot \nabla v, u_1) d\tau \right| \\ \le \sup_{0 \le \sigma \le t} \|v(\sigma)\|_2 \left(\int_0^t \|\nabla v\|_2^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|u_{1,h} - u_1\|_\infty^2 d\tau \right)^{\frac{1}{2}} \\ \to 0 \quad \text{as } h \to 0 . \tag{4.11}$$

Then (4.10) and (4.11) yield

$$\int_0^t (v \cdot \nabla v, u_h) d\tau \to \int_0^t (v \cdot \nabla v, u) d\tau \quad \text{as } h \to 0 . \tag{4.12}$$

Since $v_h \to v$ in $L^2(0, t; H^1_{0,\sigma})$, we have by (2.1) and (4.2) that

$$\left| \int_0^t (u_0 \cdot \nabla u, v_h) d\tau - \int_0^t (u_0 \cdot \nabla u, v) d\tau \right| \\ \le \varepsilon_0 \left(\int_0^t \|\nabla u\|_2^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\nabla v_h - \nabla v\|_2^2 d\tau \right)^{\frac{1}{2}} \\ \to 0 \quad \text{as } h \to 0 . \tag{4.13}$$

Furthermore, there is a subsequence of $\{v_h\}_{h>0}$, which we denote also by $\{v_h\}_{h>0}$ for simplicity, such that $v_h(\tau) \to v(\tau)$ in $H^1_{0,\sigma}$ for almost every $\tau \in [0, t]$ as $h \to 0$, which yields

$$|(u_1 \cdot \nabla u(\tau), v_h(\tau) - v(\tau)| \le \|u_1(\tau)\|_\infty \|\nabla u(\tau)\|_2 \|v_h(\tau) - v(\tau)\|_2 \to 0$$

as $h \to 0$ for almost every $\tau \in [0, t]$ together with the uniform estimate with respect to h:

$$|(u_1 \cdot \nabla u(\tau), v_h(\tau) - v(\tau)| \le 2\|v\|_{L^\infty(0,T;L^2)} \|u_1(\tau)\|_\infty \|\nabla u(\tau)\|_2 , \quad 0 \le \tau \le t .$$

Since the right hand side of the above is summable on $(0, t)$, the Lebesgue convergence theorem yields

$$\int_0^t (u_1 \cdot \nabla u(\tau), v_h(\tau))d\tau \to \int_0^t (u_1 \cdot \nabla u(\tau), v(\tau))d\tau \quad \text{as } h \to 0 . \tag{4.14}$$

By (4.13) and (4.14), we have

$$\int_0^t (u \cdot \nabla u(\tau), v_h(\tau))d\tau \to \int_0^t (u \cdot \nabla u(\tau), v(\tau))d\tau \quad \text{as } h \to 0 . \tag{4.15}$$

Now letting $h \to 0$ in (4.9), we obtain from (4.12), (4.15) and the standard argument that

$$\int_0^t \{2(\nabla u, \nabla v) + (v \cdot \nabla v, u) + (u \cdot \nabla u, v)\}d\tau = -(v(t), u(t)) + \|a\|_2^2 . \tag{4.16}$$

For every $\phi \in H^1_{0,\sigma}$, we have by (2.1) and (2.2)

$$\begin{aligned} |(u \cdot \nabla u, \phi)| &= |(u \cdot \nabla \phi, u)| = |(u \cdot \nabla \phi, u_0 + u_1)| \\ &\le C(\|\nabla u\|_2 \|u_0\|_{n,w} + \|u\|_2 \|u_1\|_\infty)\|\nabla \phi\|_2 , \end{aligned} \tag{4.17}$$

which yields $u \cdot \nabla u \in L^2(0, T; (H^1_{0,\sigma})^*)$, X^* denoting the dual space of X. Then it follows from Temam [28, Chapter 3, Lemma 1.2] that u satisfies the energy identity:

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau = \|a\|_2^2 , \quad 0 \le t \le T . \tag{4.18}$$

Adding (4.16) (multiplied by -2), (4.18) and (1.5), we have by (2.4) and (2.5) that

$$\begin{aligned} &\|v(t) - u(t)\|_2^2 + 2\int_0^t \|\nabla v - \nabla u\|_2^2 d\tau \\ &\le 2\int_0^t \{(v \cdot \nabla v, u) + (u \cdot \nabla u, v)\}d\tau \\ &= 2\int_0^t ((v - u) \cdot \nabla v, u)d\tau \\ &= 2\int_0^t ((v - u) \cdot \nabla (v - u), u)d\tau , \quad 0 \le t \le T , \end{aligned}$$

which yields

$$\|w(t)\|_2^2 + 2\int_0^t \|\nabla w\|_2^2 d\tau \le 2\left|\int_0^t (w \cdot \nabla w, u)d\tau\right| , \quad 0 \le t \le T , \tag{4.19}$$

here $w \equiv v - u$. Applying (4.1) and (4.2) to the right hand side of (4.19), we obtain from 2.1) and the Schwarz inequality

$$\begin{aligned}
&\|w(t)\|_2^2 + 2\int_0^t \|\nabla w\|_2^2 d\tau \\
&\le 2\int_0^t |(w \cdot \nabla w, u_0)| d\tau + 2\int_0^t |(w \cdot \nabla w, u_1)| d\tau \\
&\le C_* \sup_{0\le\sigma\le T} \|u_0(\sigma)\|_{n,w} \int_0^t \|\nabla w\|_2^2 d\tau + \int_0^t \|\nabla w\|_2^2 d\tau + \int_0^t \|w\|_2^2 \|u_1\|_\infty^2 d\tau \\
&\le (C_*\varepsilon_0 + 1)\int_0^t \|\nabla w\|_2^2 d\tau + \int_0^t \|w\|_2^2 \|u_1\|_\infty^2 d\tau ,
\end{aligned}$$

where $C_* = C_*(n)$. Let us now define ε_0 in (1.4) as

$$\varepsilon \equiv 1/C_* .$$

Then from the above estimate, we get

$$\|w(t)\|_2^2 \le \int_0^t \|w(\tau)\|_2^2 \|u_1(\tau)\|_\infty^2 d\tau , \quad 0 \le t \le T ,$$

and the Gronwall inequality yields that

$$\|w(t)\|_2^2 \le \|w(0)\|_2^2 \exp\left(\int_0^t \|u_1(\tau)\|_\infty^2 d\tau\right) \quad \text{for all } 0 \le t \le T .$$

This implies that

$$u \equiv v \quad \text{on } [0, T] ,$$

which proves Theorem 2.

5. Regularity of weak solutions; Proof of Theorem 3. In this section, we assume that the domain Ω satisfies Assumption 3. Let us recall the Stokes operator A_r and the Banach space $X^{s,r}(0, T)$ introduced in Section 2. For u as in (4.1) with (4.2), we first consider the following Stokes equations with the perturbed convection term $u \cdot \nabla v$:

$$\text{(P-S)} \qquad \begin{cases} \dfrac{dv}{dt} + A_r v + P_r(u \cdot \nabla v) = 0 , & 0 < t < T , \\ v(0) = b . \end{cases}$$

Then we have

LEMMA 5.1. *Suppose that Ω satisfies Assumption 3. For $1 < s < 2$ and $n/2 < r < n$ with $2 < 2/s + n/r \le 3$, there is a constant $\varepsilon_0 = \varepsilon_0(n, s, r)$ with the following property. If $u \in L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; H^1_{0,\sigma})$ and if u satisfies (1.4), then for every $b \in D(A_r) \cap L^2_\sigma$ there is a unique solution v of (P-S) in $X^{s,r}(0, T)$. Moreover, such v has the additional properties:*

$$\text{(5.1)} \qquad v \in L^{s_0}(0, T; L^{r_0}_\sigma)$$

for $1 < s \le s_0 < \infty$, $1 < r < r_0 < \infty$ *with* $2/s_0 + n/r_0 = 2/s + n/r - 2$;

$$v \in C([0, T); L^2_\sigma) \cap L^2(0, T; H^1_{0,\sigma}) ; \tag{5.2}$$

$$\|v(t)\|_2^2 + 2\int_0^t \|\nabla v(\tau)\|_2^2 d\tau = \|b\|_2^2 \quad \textit{for all } 0 < t < T . \tag{5.3}$$

PROOF. Representing v as $v(t) = e^t w(t)$ in (P-S), we may solve the following equation for w:

$$\begin{cases} \dfrac{dw}{dt} + \tilde{A}_r w + P_r(u \cdot \nabla w) = 0 , & 0 < t < T , \\ w(0) = b , \end{cases} \tag{P-S$'$}$$

where $\tilde{A} \equiv A + 1$.

We shall make use of the following successive approximation $\{w_j\}_{j=0}^{\infty}$:

$$w_0(t) = e^{-t\tilde{A}_r} b , \tag{5.4}$$

$$\begin{cases} \dfrac{dw_{j+1}}{dt} + \tilde{A}_r w_{j+1} = -P_r(u \cdot \nabla w_j) , & 0 < t < T , \\ w_{j+1}(0) = b . \end{cases} \tag{5.5}$$

For each $j = 0, 1, \dots$, we can find a unique solution w_{j+1} of (5.5) in $X^{s,r}(0, T)$. In fact, for $j = 0$, since $b \in D(A_r)$, there holds $w_0 \in X^{s,r}(0, T)$ with $\|w_0\|_{X^{s,r}(0,T)} \le 2\|b\|_{D(A_r)}$. Suppose that $w_j \in X^{s,r}(0, T)$. Since u can be decomposed as in (4.1) with (4.2), we have by Lemma 2.4 that

$$\begin{aligned} &\|u \cdot \nabla w_j\|_{L^s(0,T;L^r)} \\ &\quad \le C(\|u_0 \cdot \nabla w_j\|_{L^s(0,T;L^r)} + \|u_1 \cdot \nabla w_j\|_{L^s(0,T;L^r)} \\ &\quad \le C\{\|u_0\|_{L^\infty(0,T;L^n_w)} \|w_j\|_{X^{s,r}(0,T)} \\ &\qquad + \|u_1\|_{L^2(0,T;L^\infty)} (\|w_j\|_{X^{s,r}(0,T)} + \|b\|_{D(A_r)})\} \\ &\quad \le C(\varepsilon_0 + \|u_1\|_{L^2(0,T;L^\infty)})(\|w_j\|_{X^{s,r}(0,T)} + \|b\|_{D(A_r)}) , \end{aligned} \tag{5.6}$$

where $C = C(n, s, r)$ is independent of j and T. Then it follows from Giga-Sohr [10, Theorem 2.9] that there exists a unique solution w_{j+1} of (5.5) in $X^{s,r}(0, T)$ with

$$\begin{aligned} &\|w_{j+1}\|_{X^{s,r}(0,T)} \\ &\quad \le C_*\{(\varepsilon_0 + \|u_1\|_{L^2(0,T;L^\infty)})(\|w_j\|_{X^{s,r}(0,T)} + \|b\|_{D(A_r)}) + \|b\|_{D(A_r)}\} , \end{aligned} \tag{5.7}$$

where $C_* = C_*(n, s, r)$ is independent of j and T.

Now we take ε_0 in (4.2) and $\delta > 0$ in such a way that

$$\varepsilon_0 \equiv 1/4C_* , \tag{5.8}$$

$$\left(\int_t^{t+\delta} \|u_1(\tau)\|_\infty^2 d\tau \right)^{\frac{1}{2}} \le 1/4C_* \quad \text{for all } 0 \le t \le T . \tag{5.9}$$

Set $\hat{w}_j \equiv w_j - w_{j-1} (w_{-1} = 0)$. Similarly to (5.7), we obtain from (5.8) and (5.9)

$$\begin{aligned}\|\hat{w}_{j+1}\|_{X^{s,r}(0,\delta)} &\le C_*(\varepsilon_0 + \|u_1\|_{L^2(0,\delta;L^\infty)})\|\hat{w}_j\|_{X^{s,r}(0,\delta)} \\ &\le \frac{1}{2}\|\hat{w}_j\|_{X^{s,r}(0,\delta)} ,\end{aligned}$$

which yields

$$\|\hat{w}_j\|_{X^{s,r}(0,\delta)} \le \left(\frac{1}{2}\right)^{j-1} \|b\|_{D(A_r)}, \quad j = 0, 1, \dots .$$

Notice that $\hat{w}_j(0) = 0$ for all $j = 1, 2, \dots$. Since $w_j = \sum_{k=0}^{j} \hat{w}_k$, there exists a limit w of w_j as $j \to \infty$ in $X^{s,r}(0, \delta)$. Letting $j \to \infty$ in (5.5), we see easily that w is a solution of (P-S$'$) on the interval $(0, \delta)$.

Next, we proceed to solve (P-S$'$) beyond δ. Since $w \in X^{s,r}(0, \delta)$, we can take $\sigma \in (\delta/2, \delta)$ so that $w(\sigma) \in D(A_r)$. Since (P-S$'$) is a linear equation for w, as in the similar manner to the above, we can construct a solution $\tilde{w}$ of (P-S$'$) in $X^{s,r}(\sigma, \sigma + \delta)$ with $\tilde{w}(\sigma) = w(\sigma)$. By the uniqueness, there holds $\tilde{w} \equiv w$ on $[\sigma, \delta]$, which yields a solution w of (P-S$'$) in $X^{s,r}(0, 3\delta/2)$. Proceeding this argument inductively, within finitely many steps, we get the solution w of (P-S$'$) in $X^{s,r}(0, T)$.

Finally we shall show that the solution w satisfies

$$w \in C([0, T); L^2_\sigma) \cap L^2(0, T; H^1_{0,\sigma}) \tag{5.10}$$

with

$$\|w(t)\|_2^2 + 2\int_0^t (\|\nabla w\|_2^2 + \|w\|_2^2)d\tau = \|b\|_2^2 \quad \text{for all } 0 < t < T . \tag{5.11}$$

To this end, we need to return to the approximation solution $\{w_j\}_{j=0}^\infty$ in (5.5). Since $b \in L^2_\sigma$, we see easily that w_0 satisfies (5.10). Suppose that w_j is in the class of (5.10). Then by (4.17), we have $P(u \cdot \nabla w_j) \in L^2(0, T; (H^1_{0,\sigma})^*)$ and again by Temam [28, Chapter 3, Lemma 1.2], w_{j+1} belongs to the class of (5.10). Moreover, by (2.2) and (2.3), there holds

$$\|w_{j+1}(t)\|_2^2 + 2\int_0^t (\|\nabla w_{j+1}\|_2^2 + \|w_{j+1}\|_2^2)d\tau = \|b\|_2^2, \quad 0 \le t \le T .$$

By the weak compactness, there is a limit $\tilde{w}$ of a subsequence of $\{w_j\}_{j=0}^\infty$ in $L^\infty(0, T; L^2_\sigma) \cap L^2(0, T, H^1_{0,\sigma})$. It is easy to see that $w \equiv \tilde{w}$. Since $P(u \cdot \nabla w) \in L^2(0, T; (H^1_{0,\sigma})^*)$, we have $w \in C([0, T]; L^2_\sigma)$, which yields (5.10) and (5.11). Defining $v(t) = e^t w(t)$, we conclude from the investigation above and (2.24) that v is the desired solution of (P-S). This proves Lemma 5.1.

LEMMA 5.2. *Suppose that Ω satisfies Assumption* 3. *For* $2 \le r < \infty$, *there is a constant* $\varepsilon_0 = \varepsilon_0(n, r)$ *such that if u is a weak solution of* (N-S) *with* (1.4), *then there holds* $u(t) \in L^2_\sigma \cap D(A_r)$ *for almost all* $t \in (0, T)$.

PROOF. Let us take q so that $\max\{n, r\} < q < \infty$. By (4.1) and (4.2), there is a subset $J \subset [0, T]$ with $\mu(J) = 0$ such that if $t \in (0, T) \setminus J$, then

$$u(t) = u_0(t) + u_1(t),$$

where $u_0(t) \in L^2_\sigma \cap L^n_{w,\sigma}$ with $\|u_0(t)\|_{n,w} \le \varepsilon_0$ and $u_1(t) \in L^2_\sigma \cap L^q_\sigma$. Here $L^n_{w,\sigma}$ is defined by the real interpolation as $L^n_{w,\sigma} = (L^{p_0}_\sigma, L^{p_1}_\sigma)_{\theta,\infty}$ for $1/n = (1-\theta)/p_0 + \theta/p_1$, $0 < \theta < 1$. Then it follows from Kozono-Yamazaki [17, Theorem 2] that under the suitable choice of ε_0, there exist $t_* = t_*(t)$ and a smooth solution v of (N-S) on the interval $(t, t+t_*)$ with $v(t) = u(t)$ such that

$$v \in C((t, t+t_*); D(A_p)), \quad 2 \le p \le q; \tag{5.12}$$

$$v \in C([t, t+t_*); L^p_\sigma), \quad 2 \le p < n \tag{5.13}$$

with the energy identity

$$\|v(t')\|_2^2 + 2\int_t^{t'} \|\nabla v\|_2^2 d\tau = \|u(t)\|_2^2, \quad t \le t' \le t + t_*.$$

By Theorem 2, we have $u \equiv v$ on $[t, t+t_*)$, from which together with (5.12)–(5.13) it follows that $u(s) \in D(A_r) \cap L^2_\sigma$ for all $s \in (t, t+t_*)$. Since t can be taken arbitrarily in $(0, T) \setminus J$, we get the desired result. □

Completion of the proof of Theorem 3. We shall reduce our problem to the classical regularity criterion of Serrin [23]. Let s and r be as in Lemma 5.1. By Lemma 5.2, for every $\sigma > 0$, there is $0 < \delta < \sigma$ such that $u(\delta) \in D(A_r) \cap L^2_\sigma$. Then it follows from Lemma 5.1 that there exists a unique solution v of (P-S) in $X^{s,r}(\delta, T)$ with $v(\delta) = u(\delta)$. By (5.1) we have

$$v \in L^{s_0}(\delta, T; L^{r_0}) \quad \text{for } 2/s_0 + n/r_0 \le 1 \text{ with } r_0 > n. \tag{5.14}$$

Since v satisfies (5.3) with 0 and b replaced by δ and $u(\delta)$, respectively, as in the proof of Theorem 2, it is easy to conclude that $u \equiv v$ on $[\delta, T)$. Now by (5.14), the regularity criterion of Serrin [23] assures that $u \in C^2(\Omega \times (\sigma, T))$. Since $\sigma > 0$ can be taken arbitrarily small, we obtain the desired regularity. This proves Theorem 3.

6. Appendix.

PROOF OF PROPOSITION 1. Let $\Phi \in C([0, T]; L^n)$. We take a function $\chi \in C^\infty([0, \infty))$, $0 \le \chi \le 1$ with $\chi(r) = 1$ for $r \ge 1$, $\chi(r) = 0$ for $0 \le r \le 1/2$, and $|\chi'(r)| \le 3$. For $R > 0$ we set $\chi_R(r) \equiv \chi(r/R)$. Obviously there holds

$$\begin{aligned} R\mu\{x \in \Omega; |\Phi(x,t)| > R\}^{\frac{1}{n}} &\le \left(\int_{\{x\in\Omega; |\phi(x,t)|>R\}} |\Phi(x,t)|^n dx\right)^{\frac{1}{n}} \\ &\le \left(\int_\Omega |\chi_R(|\Phi(x,t)|)\Phi(x,t)|^n dx\right)^{\frac{1}{n}} \\ &= \|\chi_R(|\Phi(\cdot,t)|)\Phi(\cdot,t)\|_n \\ &\equiv f_R(t) \end{aligned} \tag{6.1}$$

for all $R > 0$. For each fixed $R > 0$, f_R is a continuous function on $[0, T]$. Indeed, we have

$$\begin{aligned} |f_R(t) - f_R(t')| &= | \, \|\chi_R(|\Phi(\cdot,t)|)\Phi(\cdot,t)\|_n - \|\chi_R(|\Phi(\cdot,t')|)\Phi(\cdot,t')\|_n| \\ &\le \|\chi_R(|\Phi(\cdot,t)|)\Phi(\cdot,t) - \chi_R(|\Phi(\cdot,t')|)\Phi(\cdot,t')\|_n \,. \end{aligned} \tag{6.2}$$

Since

$$|\chi_R(|\xi|)\xi - \chi_R(|\xi'|)\xi'| \le \sup_{r>0}(\chi(r) + |r\chi'(r)|)|\xi - \xi'|$$

for all $\xi, \xi' \in \boldsymbol{R}^n$ and all $R > 0$, we obtain from (6.2)

$$|f_R(t) - f_R(t')| \le C\|\Phi(\cdot,t) - \Phi(\cdot,t')\|_n$$

for all $R > 0$ and all $t, t' \in [0, T]$. Since $\Phi \in C([0, T]; L^n)$, there holds $f_R \in C([0, T])$ for all $R > 0$. Clearly, for each fixed $t \in [0, T]$, $f_R(t)$ is monotone decreasing with respect to R and by the Lebesgue convergence theorem there holds

$$f_R(t) \to 0 \quad \text{as } R \to \infty \,.$$

Now it follows from the Dini theorem that

$$f_R \to 0 \quad \textit{uniformly} \text{ on } [0, T] \text{ as } R \to \infty \,,$$

from which and (6.1) we obtain

$$\lim_{R\to\infty} \sup_{0\le t\le T} (R\mu\{x \in \Omega; |\Phi(x,t)| > R\}^{\frac{1}{n}}) = 0 \,.$$

This proves Proposition 1.

PROOF OF PROPOSITION 2. By the Sobolev embedding $H^1 \subset L^2 \cap L^{2n/(n-2)}$, when $2 \le n \le 4$, we have $H^1 \subset L^n$, so we may define $J_m \equiv$ identity for $m = 1, 2, \ldots$.

Let Ω be as in (1)–(4). In such cases, the L^r-theory of the Stokes operators A_r is established and we may choose $\{J_m\}_{m=1}^{\infty}$ as

$$J_m \equiv \left(1 + \frac{1}{m}A_2\right)^{-\frac{n-2}{4}} \,, \quad m = 1, 2, \ldots \,.$$

Since we have $A_2\phi = A_r\phi$ for $\phi \in D(A_2) \cap D(A_r)$, we may regard A_2 as an operator also defined on L^r_σ, even on $L^r_{w,\sigma}$. Then by Lemma 2.3 (i) and the real interpolation, $\{J_m\}_{m=1}^{\infty}$ is a family of bounded operators on $L^n_{w,\sigma}$ with $\sup_{m=1,\ldots} \|J_m\|_{\boldsymbol{B}(L^n_{w,\sigma})} \equiv N < \infty$. For $\phi \in H^1_{0,\sigma} \cap L^n_w$, there holds $\phi = P\phi$ and we have

$$\begin{aligned} \|J_m\phi\|_{n,w} &\le N\|\phi\|_{n,w} \,, \\ \|J_m\phi\|_{H^1} &= \|J_m\phi\|_2 + \|A^{\frac{1}{2}}J_m\phi\|_2 \\ &\le \|J_m\|_{\boldsymbol{R}(L^2_\sigma)}(\|\phi\|_2 + \|A^{\frac{1}{2}}\phi\|_2) \\ &\le \|\phi\|_{H^1} \end{aligned}$$

for all $m = 1, 2, \ldots$. Hence (i) follows from the resonance theorem.

To show (ii), we make use of the continuous embedding $D(A_2^{\frac{n-2}{4}}) \subset L^n$, which is derived from (2.23). For every $\phi \in H^1_{0,\sigma} \cap L^n_w$, such an embedding yields $J_m\phi \in L^n$ with the estimate

$$\|J_m\phi\|_n \le C(\|J_m\phi\|_2 + \|A_2^{\frac{n-2}{4}} J_m\phi\|_2) \le C(1 + m^{\frac{n-2}{4}})\|\phi\|_2$$

with a constant C independent of ϕ and m.

Finally, we prove (iii). Obviously, there holds

$$\|J_m\phi - \phi\|_{H^1} \to 0 \quad \text{as } m \to \infty .$$

Since we may regard A as a densely-defined, m-accretive operator on $L^{n',1}_\sigma$, the closure of $C^\infty_{0,\sigma}$ in $L^{n',1}$, there holds

$$J_m\psi \to \psi \quad \text{in } L^{n',1} \text{ for every } \psi \in L^{n',1}_\sigma .$$

Hence for every $\phi \in H^1_{0,\sigma} \cap L^n_w$ and every $\psi \in L^{n',1}$, we have

$$(J_m\phi, \psi) = (J_m\phi, P\psi) = (\phi, J_mP\psi) \to (\phi, P\psi) = (\phi, \psi) ,$$

which implies

$$J_m\phi \to \phi \quad \text{weakly* in } L^n_w .$$

This proves Proposition 2.

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