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CONFLUENCE PROCESSES IN DEFINING MANIFOLDS FOR PAINLEVÉ SYSTEMS

Dedicated to Professor Norio Shimakura on his sixtieth birthday

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Abstract. For each Painlevé system, we have a manifold, called the defining manifold, on which the system defines a uniform foliation. In this paper, we describe confluence processes in these manifolds as deformations of manifolds compatible to those in Painlevé systems.

1. Introduction. The purpose of this paper is to describe confluence processes in the defining manifolds for Painlevé systems, namely, to show a hierarchy of these manifolds.

Each Painlevé equation P_J $(J = VI, V, \dots, I)$ is equivalent to a Hamiltonian system $(H_J) : dx/dt = \partial H_J/\partial y, dy/dt = -\partial H_J/\partial x$, where H_J is a polynomial of x and y whose coefficients are rational functions of t. Thus the Hamiltonian system (H_J) is called the *J-th Painlevé system.*

For each Painlevé system, there is a manifold E_J, called the *defining manifold for the J-th Painlevé system*, on which the system defines a *uniform foliation*. The manifold E_J is a fiber space over the t-space $B_J = \boldsymbol{P} - \{\text{the fixed singular points}\}$ ($\boldsymbol{P}$ denotes the complex projective line), containing as a fiber subspace the trivial fiber space $\boldsymbol{C}^2 \times B_J (\ni (x, y, t))$ on which the system (H_J) defines a nonsingular foliation. It should be noted that the foliation on $\boldsymbol{C}^2 \times B_J (\ni (x, y, t))$ is not uniform because the solution of (H_J) may have movable singularities, but that on E_J is uniform, namely, (i) every leaf is transversal to fibers, (ii) for every $P_0 \in E_J$, any curve in B_J with the starting point $\pi_J(P_0)$ (π_J denotes the projection from E_J onto B_J) can be lifted on the leaf passing through the point P_0 ([4]). Each fiber $E_J(t)$ over $t \in B_J$, called the *space of initial conditions*, consists of mutually disjoint $\boldsymbol{C}^2$ and several divisors each of which is isomorphic to $\boldsymbol{C}$ ([4]). The manifold E_J $(J \neq I)$ is described as a patching of several copies of $\boldsymbol{C}^2 \times B_J$ by certain birational and symplectic identifications, and the Hamiltonian on each chart $V_J(*) = \boldsymbol{C}^2 \times B_J \ni (x(*), y(*), t)$ is a polynomial of $x(*)$ and $y(*)$ whose coefficients are rational functions of t holomorphic in B_J ([6], [7]). The Hamiltonian system defined on the whole manifold E_J is called the *extended J-th Painlevé system.* Note that the space of initial conditions is not compact, but there is no other Hamiltonian system holomorphic on E_J and meromorphic on some compactification of E_J than the extended J-th Painlevé system ([6], [8]).

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On the other hand, we know certain confluence processes in Painlevé systems ([1]). Thus it is natural to ask if there exist confluence processes in the defining manifolds E_J's compatible to those in Painlevé systems. In this paper, we establish them as deformations of manifolds.

This paper is organized as follows. Section 2 is devoted to some preliminaries which are relevant to state and prove our results. The main results of this paper are given in Section 3, and proved in the following sections.

2. Preliminaries. In this section, we recall some known facts and recall a lemma which is used in the proofs of our results.

2.1. We first review the forms of the Hamiltonian functions for Painlevé systems ([1]):

$$\begin{aligned}
H_{VI}(x,y,t) =& \frac{1}{t(t-1)}[x(x-1)(x-t)y^2 - \{\kappa_0(x-1)(x-t) \\
&+ \kappa_1 x(x-t) + (\kappa_t - 1)x(x-1)\}y + \kappa(x-t)] \\
&\left(\kappa := \frac{1}{4}[(\kappa_0 + \kappa_1 + \kappa_t - 1)^2 - \kappa_\infty^2]\right), \\
H_V(x,y,t) =& \frac{1}{t}[x(x-1)^2y^2 - \{\kappa_0(x-1)^2 + \kappa_t x(x-1) - \eta t x\}y + \kappa(x-1)] \\
&\left(\kappa := \frac{1}{4}\{(\kappa_0 + \kappa_t)^2 - \kappa_\infty^2\}\right), \\
H_{IV}(x,y,t) =& 2xy^2 - \{x^2 + 2tx + 2\kappa_0\}y + \kappa_\infty x, \\
H_{III'}(x,y,t) =& \frac{1}{t}\left[x^2y^2 - \{\eta_\infty x^2 + \kappa_0 x - \eta_0 t\}y + \frac{1}{2}\eta_\infty(\kappa_0 + \kappa_\infty)x\right], \\
H_{III}(x,y,t) =& \frac{1}{t}[2x^2y^2 - \{2\eta_\infty t x^2 + (2\kappa_0 + 1)x - 2\eta_0 t\}y + \eta_\infty(\kappa_0 + \kappa_\infty)tx], \\
H_{II}(x,y,t) =& \frac{1}{2}y^2 - \left(x^2 + \frac{t}{2}\right)y - \left(\alpha + \frac{1}{2}\right)x, \\
H_I(x,y,t) =& \frac{1}{2}y^2 - 2x^3 - tx.
\end{aligned}$$

Here x, y, and t are variables and the other Greek letters stand for complex constants.

2.2. We next give descriptions of the defining manifolds E_J's ($J \neq I$) ([6], [7]). In the following, we distinguish several coordinate systems by labels such as $00, 0\infty, 1\infty$ and so on. The coordinate system $(x(00), y(00), t) = (x, y, t)$ is the original one of $V(00) \times B_J = \boldsymbol{C}^2 \times B_J$ on which the Hamiltonian system (H_J) is defined. We note that $(x(0\infty), y(0\infty))$, for example, is a coordinate system which is appropriate to observe a 1-parameter family of leaves passing through the point $(x, y) = (0, \infty)$.

2.2.1. The manifold E_{VI} for the VI-th Painlevé system is obtained by glueing six copies $V(*) \times B_{VI} \ni (x(*), y(*), t)$ of $\boldsymbol{C}^2 \times B_{VI}$ via the following symplectic identifications:

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty)), \quad y(00) = 1/y(0\infty),$$

$$x(00) = 1 + y(1\infty)(\kappa_1 - x(1\infty)y(1\infty)), \quad y(00) = 1/y(1\infty),$$
$$x(00) = t + y(t\infty)(\kappa_t - x(t\infty)y(t\infty)), \quad y(00) = 1/y(t\infty),$$
$$x(00) = 1/x(\infty 0+), \quad y(00) = x(\infty 0+)(\kappa(+) - x(\infty 0+)y(\infty 0+)),$$
$$x(\infty 0+) = y(\infty 0-)(\kappa_\infty - x(\infty 0-)y(\infty 0-)), \quad y(\infty 0+) = 1/y(\infty 0-),$$

where

$$B_{VI} = \boldsymbol{C} - \{0, 1\}, \quad \kappa(+) = (\kappa_0 + \kappa_1 + \kappa_t - 1 + \kappa_\infty)/2,$$

and $V(00) \times B_{VI}$ is the original space on which the Hamiltonian function $H_{VI}(x, y, t)$ is defined.

2.2.2. The manifold E_V for the V-th Painlevé system in the case $\eta \neq 0$ is obtained by glueing five copies $V(*) \times B_V \ni (x(*), y(*), t)$ of $\boldsymbol{C}^2 \times B_V$ via the following symplectic identifications:

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty)), \quad y(00) = 1/y(0\infty),$$
$$x(00) = 1 + x(1\infty), \quad y(00) = -\frac{\eta t}{x(1\infty)^2} + \frac{\kappa_t + 1}{x(1\infty)} + y(1\infty),$$
$$x(00) = 1/x(\infty 0+), \quad y(00) = x(\infty 0+)(\kappa(+) - x(\infty 0+)y(\infty 0+)),$$
$$x(\infty 0+) = y(\infty 0-)(\kappa_\infty - x(\infty 0-)y(\infty 0-)), \quad y(\infty 0+) = 1/y(\infty 0-),$$

where

$$B_V = \boldsymbol{C} - \{0\}, \quad \kappa(+) = (\kappa_0 + \kappa_t + \kappa_\infty)/2,$$

and $V(00) \times B_V$ is the original space on which the Hamiltonian function $H_V(x, y, t)$ is defined.

2.2.3. The manifold E_{IV} for the IV-th Painlevé system is obtained by glueing four copies $V(*) \times B_{IV} \ni (x(*), y(*), t)$ of $\boldsymbol{C}^2 \times B_{IV}$ via the following symplectic identifications:

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty)), \quad y(00) = 1/y(0\infty),$$
$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)(\kappa_\infty - x(\infty 0)y(\infty 0)),$$
$$x(\infty 0) = x(\infty\infty), \quad y(\infty 0) = -\frac{1/2}{x(\infty\infty)^3} - \frac{t}{x(\infty\infty)^2} + \frac{2\kappa_\infty - \kappa_0 + 1}{x(\infty\infty)} + y(\infty\infty),$$

where $B_{IV} = \boldsymbol{C}$ and $V(00) \times B_{IV}$ is the original space on which the Hamiltonian function $H_{IV}(x, y, t)$ is defined.

2.2.4. The manifold $E_{III'}$ for the modified III-rd Painlevé system in the case $\eta_0\eta_\infty \neq 0$ is obtained by glueing four copies $V(*) \times B_{III'} \ni (x(*), y(*), t)$ of $\boldsymbol{C}^2 \times B_{III'}$ via the following symplectic identifications:

$$x(00) = x(0\infty), \quad y(00) = -\frac{\eta_0 t}{x(0\infty)^2} + \frac{\kappa_0 + 1}{x(0\infty)} + y(0\infty),$$
$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)((\kappa_0 + \kappa_\infty)/2 - x(\infty 0)y(\infty 0)),$$
$$x(\infty 0) = x(\infty\eta_\infty), \quad y(\infty 0) = -\frac{\eta_\infty}{x(\infty\eta_\infty)^2} + \frac{\kappa_\infty}{x(\infty\eta_\infty)} + y(\infty\eta_\infty),$$

where $B_{III'} = \boldsymbol{C} - \{0\}$ and $V(00) \times B_{III'}$ is the original space on which the Hamiltonian function $H_{III'}(x, y, t)$ is defined.

2.2.5. The manifold E_{III} for the III-rd Painlevé system in the case $\eta_0\eta_\infty \neq 0$ is obtained by glueing four copies $V(*) \times B_{III} \ni (x(*), y(*), t)$ of $\boldsymbol{C}^2 \times B_{III}$ via the following symplectic identifications:

$$x(00) = x(0\infty), \quad y(00) = -\frac{\eta_0 t}{x(0\infty)^2} + \frac{\kappa_0 + 1}{x(0\infty)} + y(0\infty),$$

$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)((\kappa_0 + \kappa_\infty)/2 - x(\infty 0)y(\infty 0)),$$

$$x(\infty 0) = x(\infty\eta_\infty t), \quad y(\infty 0) = -\frac{\eta_\infty t}{x(\infty\eta_\infty t)^2} + \frac{\kappa_\infty}{x(\infty\eta_\infty t)} + y(\infty\eta_\infty t),$$

where $B_{III} = \boldsymbol{C} - \{0\}$ and $V(00) \times B_{III}$ is the original space on which the Hamiltonian function $H_{III}(x, y, t)$ is defined.

2.2.6. The manifold E_{II} for the II-nd Painlevé system is obtained by glueing three copies $V(*) \times B_{II} \ni (x(*), y(*), t)$ of $\boldsymbol{C}^2 \times B_{II}$ via the following symplectic identifications:

$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)(-\alpha - 1/2 - x(\infty 0)y(\infty 0)),$$

$$x(\infty 0) = x(\infty\infty), \quad y(\infty 0) = -\frac{2}{x(\infty\infty)^4} - \frac{t}{x(\infty\infty)^2} - \frac{2\alpha}{x(\infty\infty)} + y(\infty\infty),$$

where $B_{II} = \boldsymbol{C}$ and $V(00) \times B_{II}$ is the original space on which the Hamiltonian function $H_{II}(x, y, t)$ is defined.

2.2.7. We remark that each Hamiltonian function $H_J(*)$ on the chart $V(*) \times B_J$ is a polynomial of $x(*)$ and $y(*)$.

2.3. We now give the confluence processes in Painlevé systems ([1], pp. 142–144).

2.3.1. The confluence process from the VI-th Painlevé system (H_{VI}) to the V-th one (H_V) is given by the following diagram:

$$\kappa_1 \to \eta\varepsilon^{-1} + \kappa_t + 1, \quad \kappa_t \to -\eta\varepsilon^{-1},$$

$$(x, y, H_{VI}, t) \to (x, y, \varepsilon^{-1}H_{VI\to V}(\varepsilon), 1 + \varepsilon t).$$

This means the following: By the change of parameters, variables, and functions

$$\kappa_1 = \eta\varepsilon^{-1} + k_t + 1, \quad \kappa_t = -\eta\varepsilon^{-1},$$

$$x = X, \quad y = Y, \quad H_{VI} = \varepsilon^{-1}H_{VI\to V}, \quad t = 1 + \varepsilon T,$$

the VI-th Painlevé system $dx/dt = \partial H_{VI}/\partial y,\ dy/dt = -\partial H_{VI}/\partial x$ is changed to a Hamiltonian system $dX/dT = \partial H_{VI\to V}/\partial Y,\ dY/dT = -\partial H_{VI\to V}/\partial X$, because

$$dy \wedge dx - dH_{VI} \wedge dt = dY \wedge dX - dH_{VI\to V} \wedge dT.$$

The new Hamiltonian $H_{VI\to V}$ is a function of X, Y, T depending also on parameters k_t, κ_0, κ_∞, η, ε. The above diagram implies that the new Hamiltonian in which X, Y, T, k_t is rewritten by x, y, t, κ_t, respectively, is equal to the function εH_{VI}, where $x, y, t, \kappa_1, \kappa_t$ are replaced by $x, y, 1 + \varepsilon t, \eta\varepsilon^{-1} + \kappa_t + 1, -\eta\varepsilon^{-1}$, respectively.

We see

$$H_{VI\to V}(\varepsilon)=\frac{1}{t(1+\varepsilon t)}[x(x-1)(x-1-\varepsilon t)y^2-\{\kappa_0(x-1)(x-1-\varepsilon t)\\ +\kappa_t x(x-1)-(\eta+\varepsilon\kappa_t+\varepsilon)tx\}y+\kappa(\varepsilon)(x-1-\varepsilon t)],$$

where $\kappa(\varepsilon):=[(\kappa_0+\kappa_t)^2-{\kappa_\infty}^2]/4$. Notice that $H_{VI\to V}(\varepsilon)$ is a polynomial of x and y, and it tends to H_V as $\varepsilon\to 0$. This property holds in each of the following processes.

2.3.2. The confluence process from (H_V) to (H_{IV}) is given by:

$$\eta\to-\varepsilon^{-2},\quad \kappa_t\to\varepsilon^{-2}+2\kappa_\infty-\kappa_0,\quad \kappa_\infty\to-\varepsilon^{-2},$$

$$(x,y,H_V,t)\to\left(\frac{\varepsilon}{\sqrt{2}}x,\frac{\sqrt{2}}{\varepsilon}y,\frac{1}{\sqrt{2}\varepsilon}H_{V\to IV}(\varepsilon),1+\sqrt{2}\varepsilon t\right).$$

2.3.3. The confluence process from (H_V) to $(H_{III'})$ is given by:

$$\kappa_0\to\eta_\infty\varepsilon^{-1},\quad \kappa_t\to\kappa_0,\quad \kappa_\infty\to-\eta_\infty\varepsilon^{-1}+\kappa_\infty,\quad \eta\to\eta_0\varepsilon,$$

$$(x,y,H_V,t)\to(1+\varepsilon x,\varepsilon^{-1}y,H_{V\to III'}(\varepsilon),t).$$

2.3.4. The confluence process from (H_{IV}) to (H_{II}) is given by:

$$\kappa_0\to\varepsilon^{-6}/2,\quad \kappa_\infty\to-\alpha-1/2,$$

$$(x,y,H_{IV},t)\to(\varepsilon^{-3}(1+2^{2/3}\varepsilon^2x),\ 2^{-2/3}\varepsilon y,\ 2^{2/3}\varepsilon^{-1}H_{IV\to II}(\varepsilon),\ -\varepsilon^{-3}+2^{-2/3}\varepsilon t).$$

2.3.5. The confluence process from (H_{III}) to (H_{II}) is given by:

$$\eta_0\to-\varepsilon^{-3}/4,\quad \eta_\infty\to\varepsilon^{-3}/4,$$

$$\kappa_0\to-\varepsilon^{-3}/2-2\alpha-1,\quad \kappa_\infty\to\varepsilon^{-3}/2,$$

$$(x,y,H_{III},t)\to(1+2\varepsilon x,(\varepsilon^{-1}/2)y,\varepsilon^{-2}H_{III\to II}(\varepsilon),1+\varepsilon^2 t).$$

2.4. The following simple lemma will be used in Sections 4 through 8.

LEMMA 1. *Let S or T be manifolds by glueing two copies (x,y)-space and (u,v)-space or (z,w)-space of $\mathbf{C}^2$ via the identification $x=1/u,\ y=u(a-uv)$ or $x=z+a/w,\ y=w$, where a is a complex constant. Then S is isomorphic to T provided $a\neq 0$.*

3. Main results. We define manifolds $\mathcal{E}_{J\to K}$'s each of which describes the confluence process from the defining manifold E_J to that E_K. The manifold $\mathcal{E}_{J\to K}$ is by definition a complex analytic family of complex manifolds:

$$\mathcal{E}_{J\to K}=\bigcup_{\varepsilon\in\mathbf{C}}\mathcal{E}_{J\to K}(\varepsilon)\times\varepsilon.$$

Recall that the manifold E_J depends on some parameters. Let us denote by $E_J(\varepsilon)$ for which the papameters are chosen as in Subsection 2.3 depending on ε, for example, $E_{VI}(\varepsilon)$ is a manifold given in 2.2.1, where only the parameters are replaced as in 2.3.1. The manifold $\mathcal{E}_{J\to K}$ is constructed so that $\mathcal{E}_{J\to K}(\varepsilon)$ for each $\varepsilon\neq 0$ is isomorphic to $E_J(\varepsilon)$ and $\mathcal{E}_{J\to K}(0)$ is isomorphic to E_K. Although the latter assertion is easy to see, the former is not trivial. Therefore we will verify it in Sections 4 through 8.

Every fiber $\mathcal{E}_{J\to K}(\varepsilon, t)$ of $\mathcal{E}_{J\to K}(\varepsilon)$ over t is a disjoint union of a complex plane $\boldsymbol{C}^2$ and several complex lines $\boldsymbol{C}$'s. Each confluence process is understood as a collision of two complex lines. In the following theorems, we use (u, v) as a coordinate system of a special chart appropriate to see the collision. In the identification of (u, v) with another coordinate system, we observe a phenomenon such as a generation of a pole of order $m+n$ by a collision or confluence of two poles of order m and n.

We notice that these confluence processes are compatible with those given in Section 2.

THEOREM 1. *Let $\eta \neq 0$. Then $\mathcal{E}_{VI\to V}(\varepsilon)$ is obtained by glueing five copies of $\boldsymbol{C}^2 \times B_{VI\to V}(\varepsilon)$ via the following symplectic identifications*:

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty)), \quad y(00) = 1/y(0\infty),$$

$$\begin{aligned} x(00) = 1 + u, \quad y(00) &= \frac{\eta\varepsilon^{-1} + \kappa_t + 1}{u} + \frac{-\eta\varepsilon^{-1}}{u - \varepsilon t} + v \\ &= -\frac{(\eta + (\kappa_t + 1)\varepsilon)t}{u(u - \varepsilon t)} + \frac{\kappa_t + 1}{u - \varepsilon t} + v, \end{aligned}$$

$$x(00) = 1/x(\infty 0+), \quad y(00) = x(\infty 0+)(\kappa(+) - x(\infty 0+)y(\infty 0+)),$$

$$x(\infty 0+) = y(\infty 0-)(\kappa_\infty - x(\infty 0-)y(\infty 0-)), \quad y(\infty 0+) = 1/y(\infty 0-),$$

where

$$\kappa(+) = (\kappa_0 + \kappa_t + \kappa_\infty)/2, \quad B_{VI\to V}(\varepsilon) = \boldsymbol{C} - \{0, -\varepsilon^{-1}\},$$

and $(x(00), y(00))$ is the coordinate system of the original chart on which the Hamiltonian function $H_{VI\to V}(\varepsilon)$ (given in 2.3.1) is defined.

THEOREM 2. *$\mathcal{E}_{V\to IV}(\varepsilon)$ is obtained by glueing four copies of $\boldsymbol{C}^2 \times B_{V\to IV}(\varepsilon)$ via the following symplectic identifications*:

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty)), \quad y(00) = 1/y(0\infty),$$

$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)(\kappa_\infty - x(\infty 0)y(\infty 0)),$$

$$\begin{aligned} x(\infty 0) = u, \quad y(\infty 0) &= -\frac{\varepsilon^{-2}}{u} - \frac{\varepsilon^{-1}/\sqrt{2} + t}{(u - \varepsilon/\sqrt{2})^2} + \frac{\varepsilon^{-2} + 2\kappa_\infty - \kappa_0 + 1}{u - \varepsilon/\sqrt{2}} + v \\ &= -\frac{1/2}{u(u - \varepsilon/\sqrt{2})^2} - \frac{t}{(u - \varepsilon/\sqrt{2})^2} + \frac{2\kappa_\infty - \kappa_0 + 1}{u - \varepsilon/\sqrt{2}} + v, \end{aligned}$$

where

$$B_{V\to IV}(\varepsilon) = \boldsymbol{C} - \{-\varepsilon^{-1}/\sqrt{2}\},$$

and $(x(00), y(00))$ is the coordinate system of the original chart on which the Hamiltonian function $H_{V\to IV}(\varepsilon)$ (given in 2.3.2) is defined.

THEOREM 3. *Let $\eta_0\eta_\infty \neq 0$. Then $\mathcal{E}_{V\to III'}(\varepsilon)$ is obtained by glueing four copies of $\boldsymbol{C}^2 \times B_{V\to III'}(\varepsilon)$ via the following symplectic identifications*:

$$x(00) = x(0\infty), \quad y(00) = -\frac{\eta_0 t}{x(0\infty)^2} + \frac{\kappa_0 + 1}{x(0\infty)} + y(0\infty),$$

$$x(00) = 1/x(\infty 0)\,, \quad y(00) = x(\infty 0)((\kappa_0 + \kappa_\infty)/2 - x(\infty 0)y(\infty 0))\,,$$

$$\begin{aligned} x(\infty 0) = u\,, \quad y(\infty 0) &= \frac{-\eta_\infty \varepsilon^{-1} + \kappa_\infty}{u} + \frac{\eta_\infty \varepsilon^{-1}}{u+\varepsilon} + v \\ &= \frac{-\eta_\infty + \kappa_\infty \varepsilon}{u(u+\varepsilon)} + \frac{\kappa_\infty}{u+\varepsilon} + v\,, \end{aligned}$$

where

$$B_{V\to III'}(\varepsilon) = \boldsymbol{C} - \{0\}\,,$$

and $(x(00), y(00))$ *is the coordinate system of the original chart on which the Hamiltonian function* $H_{V\to III'}(\varepsilon)$ *(given in* 2.3.3) *is defined.*

THEOREM 4. *$\mathcal{E}_{IV\to II}(\varepsilon)$ is obtained by glueing three copies of $\boldsymbol{C}^2 \times B_{IV\to II}(\varepsilon)$ via the following symplectic identifications*:

$$x(00) = 1/x(\infty 0)\,, \quad y(00) = x(\infty 0)(-\alpha - 1/2 - x(\infty 0)y(\infty 0))\,,$$

$$\begin{aligned} x(\infty 0) = u\,, \quad y(\infty 0) &= -\frac{2^{1/3}\varepsilon^{-2}}{u^3} + \frac{2^{-1/3}\varepsilon^{-4} - t}{u^2} - \frac{2\alpha + \varepsilon^{-6}/2}{u} + \frac{\varepsilon^{-6}/2}{u + 2^{2/3}\varepsilon^2} + v \\ &= -\frac{2}{u^3(u + 2^{2/3}\varepsilon^2)} - \frac{t}{u^2} - \frac{2\alpha}{u} + v\,, \end{aligned}$$

where

$$B_{IV\to II}(\varepsilon) = \boldsymbol{C}\,,$$

and $(x(00), y(00))$ *is the coordinate system of the original chart on which the Hamiltonian function* $H_{IV\to II}(\varepsilon)$ *(given in* 2.3.4) *is defined.*

THEOREM 5. *$\mathcal{E}_{III\to II}(\varepsilon)$ is obtained by glueing three copies of $\boldsymbol{C}^2 \times B_{III\to II}(\varepsilon)$ via the following symplectic identifications*:

$$x(00) = 1/x(\infty 0)\,, \quad y(00) = x(\infty 0)(-\alpha - 1/2 - x(\infty 0)y(\infty 0))\,,$$

$$\begin{aligned} x(\infty 0) = u\,, \quad y(\infty 0) &= -\frac{(\varepsilon^{-2} + t)/2}{u^2} + \frac{\varepsilon^{-3}/2}{u} - \frac{(\varepsilon^{-2} + t)/2}{(u+2\varepsilon)^2} - \frac{\varepsilon^{-3}/2 + 2\alpha}{u + 2\varepsilon} + v\,, \\ &= -\frac{2}{u^2(u+2\varepsilon)^2} - \frac{t/2}{(u+2\varepsilon)^2} - \frac{t/2}{u^2} - \frac{2\alpha}{u+2\varepsilon} + v\,, \end{aligned}$$

where

$$B_{III\to II}(\varepsilon) = \boldsymbol{C} - \{-\varepsilon^{-2}\}\,,$$

and $(x(00), y(00))$ *is the coordinate system of the original chart on which the Hamiltonian function* $H_{III\to II}(\varepsilon)$ *(given in* 2.3.5) *is defined.*

The following theorem is verified by calculation.

THEOREM 6. *Each Hamiltonian function $H_{J\to K}(\varepsilon, *)$ on each $*$-chart of $\mathcal{E}_{J\to K}(\varepsilon)$ is a polynomial of $x(*)$ and $y(*)$, and it tends to the Hamiltonian function $H_K(*)$ defined on the $*$-chart of E_K as $\varepsilon \to 0$.*

4. Proof of Theorem 1. Let $E_{VI}(\varepsilon)$ be a manifold given in 2.2.1, where only the parameters are changed as in 2.3.1. The purpose of this section is to show that $E_{VI}(\varepsilon)$ is isomorphic to $\mathcal{E}_{VI\to V}(\varepsilon)$ for each sufficiently small $\varepsilon \neq 0$.

Consider a change of time variable t: $t \to 1 + \varepsilon t$, according to 2.3.1. Then $E_{VI}(\varepsilon)$ is described as a patching of six copies of $\boldsymbol{C}^2 \times B_{VI\to V}(\varepsilon)$ by the identifications:

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty)), \quad y(00) = 1/y(0\infty), \tag{4.1}$$

$$x(00) = 1 + y(1\infty)(\eta\varepsilon^{-1} + \kappa_t + 1 - x(1\infty)y(1\infty)), \quad y(00) = 1/y(1\infty), \tag{4.2}$$

$$x(00) = 1 + \varepsilon t + y^*(-\eta\varepsilon^{-1} - x^*y^*), \quad y(00) = 1/y^*, \tag{4.3}$$

$$x(00) = 1/x(\infty 0+), \quad y(00) = x(\infty 0+)((\kappa_0 + \kappa_t + \kappa_\infty)/2 - x(\infty 0+)y(\infty 0+)), \tag{4.4}$$

$$x(\infty 0+) = y(\infty 0-)(\kappa_\infty - x(\infty 0-)y(\infty 0-)), \quad y(\infty 0+) = 1/y(\infty 0-). \tag{4.5}$$

Here (x^*, y^*) is used in place of $(x(t\infty), y(t\infty))$.

The fiber $E_{VI}(\varepsilon, t)$ of $E_{VI}(\varepsilon)$ over $t \in B_{VI}(\varepsilon)$ is a disjoint union of $\boldsymbol{C}^2 \ni (x(00), y(00))$ and five complex lines:

$$E_{VI}(\varepsilon, t) = \boldsymbol{C}^2 \cup D_{0\infty}(t) \cup D_{1\infty}(t) \cup D_{1+\varepsilon t\infty}(t) \cup D_{\infty 0+}(t) \cup D_{\infty 0-}(t),$$

where $D_{0\infty}(t) := \{y(0\infty) = 0\}$, $D_{1\infty}(t) := \{y(1\infty) = 0\}$, $D_{1+\varepsilon t\infty}(t) := \{y^* = 0\}$, $D_{\infty 0+}(t) := \{x(\infty 0+) = 0\}$ and $D_{\infty 0-}(t) := \{y(\infty 0-) = 0\}$. As $\varepsilon \to 0$, the divisor $D_{1+\varepsilon t\infty}(t)$ is going to collide with the divisor $D_{1\infty}(t)$. In the following, we choose an appropriate coordinate system to observe the collision.

Since $\eta \neq 0$ by assumption, $\eta\varepsilon^{-1} + \kappa_t + 1 \neq 0$ for sufficiently small $\varepsilon \neq 0$, and hence a space obtained by patching two copies of $\boldsymbol{C}^2 \times t$, $t \in B_{VI\to V}(\varepsilon)$, via (4.2) is isomorphic to the one obtained by the patching

$$x(00) = 1 + u', \quad y(00) = (\eta\varepsilon^{-1} + \kappa_t + 1)/u' + v' \tag{4.6}$$

on account of Lemma 1. We see that $D_{1\infty}(t) = \{y(1\infty) = 0\} = \{u' = 0\}$, and $(x(1\infty), 0)$ and $(0, v')$ represent the same point on $D_{1\infty}(t)$ if $x(1\infty) = -(\eta\varepsilon^{-1} + \kappa_t + 1)v'$.

We also see that a space obtained by patching two copies of $\boldsymbol{C}^2 \times t$ via (4.3) is isomorphic to the one obtained by the patching

$$x(00) = 1 + \varepsilon t + u'', \quad y(00) = -\eta\varepsilon^{-1}/u'' + v''. \tag{4.7}$$

It is easy to see that $D_{1+\varepsilon t\infty}(t) = \{y^* = 0\} = \{u'' = 0\}$, and $(x^*, 0)$ and $(0, v'')$ represent the same point on $D_{1+\varepsilon t\infty}(t)$ if $x^* = \eta\varepsilon^{-1}v''$.

By observing (4.6) and (4.7), we find that a space obtained by patching three copies of $\boldsymbol{C}^2 \times t$ via (4.2) and (4.3) is isomorphic to a space obtained by patching two copies of $\boldsymbol{C}^2 \times t$ via a relation

$$x(00) = 1 + u, \quad y(00) = \frac{\eta\varepsilon^{-1} + \kappa_t + 1}{u} + \frac{-\eta\varepsilon^{-1}}{u - \varepsilon t} + v. \tag{4.8}$$

It should be noticed that $D_{1\infty}(t) = \{u = 0\}$ and $D_{1+\varepsilon t\infty}(t) = \{u = \varepsilon t\}$.

5. Proof of Theorem 2. We prove that $E_V(\varepsilon)$ is isomorphic to $\mathcal{E}_{V\to IV}(\varepsilon)$ for each $\varepsilon \neq 0$.

Let us replace the variables as

$$x(00) \to (\varepsilon/\sqrt{2})x(00)\,, \quad y(00) \to (\sqrt{2}/\varepsilon)y(00)\,, \quad t \to 1+\sqrt{2}\varepsilon t\,,$$

according to the known confluence process given in 2.3.2. Corresponding to this, we further make the following replacements:

$$\begin{aligned} x(0\infty) &\to (\sqrt{2}/\varepsilon)x(0\infty)\,, \quad y(0\infty) \to (\varepsilon/\sqrt{2})y(0\infty)\,,\\ x(\infty0+) &\to (\sqrt{2}/\varepsilon)x(\infty0)\,, \quad y(\infty0+) \to (\varepsilon/\sqrt{2})y(\infty0)\,,\\ x(\infty0-) &\to (\varepsilon/\sqrt{2})x(\infty0-)\,, \quad y(\infty0-) \to (\sqrt{2}/\varepsilon)y(\infty0-)\,. \end{aligned}$$

Then we have another description of $E_V(\varepsilon)$ as a patching of five copies of $\boldsymbol{C}^2 \times B_{V\to IV}(\varepsilon)$ by

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty))\,, \quad y(00) = 1/y(0\infty)\,, \tag{5.1}$$

$$\begin{aligned} &(\varepsilon/\sqrt{2})x(00) = 1 + x(1\infty)\,,\\ &\sqrt{2}\varepsilon^{-1}y(00) = \frac{(1+\sqrt{2}\varepsilon t)\varepsilon^{-2}}{x(1\infty)^2} + \frac{\varepsilon^{-2}+2\kappa_\infty-\kappa_0+1}{x(1\infty)} + y(1\infty)\,, \end{aligned} \tag{5.2}$$

$$x(00) = 1/x(\infty0)\,, \quad y(00) = x(\infty0)(\kappa_\infty - x(\infty0)y(\infty0))\,, \tag{5.3}$$

$$x(\infty0) = y(\infty0-)(-\varepsilon^{-2} - x(\infty0-)y(\infty0-))\,, \quad y(\infty0) = 1/y(\infty0-)\,. \tag{5.4}$$

The fiber $E_V(\varepsilon, t)$ of $E_V(\varepsilon)$ over $t \in B_V(\varepsilon)$ is a disjoint union of $\boldsymbol{C}^2 \ni (x(00), y(00))$ and four complex lines:

$$E_V(\varepsilon,t) = \boldsymbol{C}^2 \cup D_{0\infty}(t) \cup D_{\sqrt{2}\varepsilon^{-1}\infty}(t) \cup D_{\infty0} \cup D_{\infty0}(t) \cup D_{\infty0-}(t)\,,$$

where $D_{0\infty}(t) := \{y(0\infty)=0\}$, $D_{\sqrt{2}\varepsilon^{-1}\infty}(t) := \{x(1\infty)=0\}$, $D_{\infty0}(t) := \{x(\infty0)=0\}$ and $D_{\infty0-}(t) := \{y(\infty0-)=0\}$. To see that the divisors $D_{\sqrt{2}\varepsilon^{-1}\infty}(t)$ and $D_{\infty0-}(t)$ collide with each other as $\varepsilon \to 0$, we look for a coordinate system which is suitable to see the collision.

We first note that a space obtained by patching two copies of $\boldsymbol{C}^2 \times t$ via (5.4) is isomorphic to the one via a relation

$$x(\infty0) = u'\,, \quad y(\infty0) = -\varepsilon^{-2}/u' + v'\,, \tag{5.5}$$

by Lemma 1, and that $D_{\infty0-}(t) = \{y(\infty0-)=0\} = \{u'=0\}$, and $(x(\infty0-), 0)$ and $(0, v')$ are the same point on $D_{\infty0-}(t)$ if $x(\infty0-) = v'/\varepsilon^2$.

We next study how one can choose an appropriate coordinate system near the divisor $D_{\sqrt{2}\varepsilon^{-1}\infty}(t)$ so that the identification of the system with $(x(\infty0), y(\infty0))$ is of simple form. From (5.2) and (5.3), it follows that

$$x(\infty0) = (\varepsilon/\sqrt{2})/(1+x(1\infty))\,.$$

Considering $x(\infty 0)=\varepsilon/\sqrt{2}$ when $x(1\infty)=0$, we introduce a coodinate u'' by

$$x(\infty 0)=\varepsilon/\sqrt{2}+u''. \tag{5.6}$$

Let us obtain the expression of $y(\infty 0)$ in terms of u'' and $y(1\infty)$ by (5.2), (5.3), and (5.6). By a careful calculation, we have

$$\begin{aligned} y(\infty 0)=&-\frac{\varepsilon^{-1}/\sqrt{2}+t}{u''^2}+\frac{\varepsilon^{-2}+2\kappa_\infty-\kappa_0+1}{u''}\\ &-\frac{\varepsilon^{-2}+\kappa_\infty-\kappa_0+1}{\varepsilon/\sqrt{2}+u''}-\frac{\varepsilon}{\sqrt{2}(\varepsilon/\sqrt{2}+u'')^2}y(1\infty), \end{aligned}$$

and then take a coordinate v'' by

$$v''=-\frac{\varepsilon^{-2}+\kappa_\infty-\kappa_0+1}{\varepsilon/\sqrt{2}+u''}-\frac{\varepsilon}{\sqrt{2}(\varepsilon/\sqrt{2}+u'')^2}y(1\infty),$$

which is equivalent to

$$y(1\infty)=-\sqrt{2}\varepsilon^{-1}(\varepsilon/\sqrt{2}+u'')^2v''-\sqrt{2}\varepsilon^{-1}(\varepsilon^{-2}+\kappa_\infty-\kappa_0+1)(\varepsilon/\sqrt{2}+u'').$$

The relation between $(x(\infty 0), y(\infty 0))$ and (u'', v'') is

$$\begin{aligned} x(\infty 0)&=\varepsilon/\sqrt{2}+u'',\\ y(\infty 0)&=-\frac{\varepsilon^{-1}/\sqrt{2}+t}{u''^2}+\frac{\varepsilon^{-2}+2\kappa_\infty-\kappa_0+1}{u''}+v''. \end{aligned} \tag{5.7}$$

From these relations, it follows that a space obtained by patching $(x(\infty 0), y(\infty 0))$-space and $(x(1\infty), y(1\infty))$-space via the relation derived from (5.2) and (5.3) is isomorphic to the one obtained by patching $(x(\infty 0), y(\infty 0))$-space and (u'', v'')-space via (5.7). We can verify that $D_{\sqrt{2}\varepsilon^{-1}\infty}(t)=\{x(1\infty)=0\}=\{u''=0\}$ and that $(0, y(1\infty))$ and $(0, v'')$ with $y(1\infty)=-(\varepsilon/\sqrt{2})v''-(\varepsilon^{-2}+\kappa_\infty-\kappa_0+1)$ are the same point on $D_{\sqrt{2}\varepsilon^{-1}\infty}(t)$.

By observing (5.5) and (5.7), we introduce a coordinate system (u, v) by

$$\begin{aligned} x(\infty 0)&=u,\\ y(\infty 0)&=-\frac{\varepsilon^{-2}}{u}-\frac{\varepsilon^{-1}/\sqrt{2}+t}{(u-\varepsilon/\sqrt{2})^2}+\frac{\varepsilon^{-2}+2\kappa_\infty-\kappa_0+1}{u-\varepsilon/\sqrt{2}}+v. \end{aligned} \tag{5.8}$$

We can verify that the space obtained by patching $(x(\infty 0), y(\infty))$, $(x(\infty 0-), y(\infty 0-))$, and $(x(1\infty), y(1\infty))$-spaces is isomorphic to that obtained by patching $(x(\infty 0), y(\infty 0))$ and (u, v)-spaces via (5.8), and that $D_{\infty 0-}(t)=\{u=0\}$, $D_{\sqrt{2}\varepsilon^{-1}\infty}(t)=\{u=\varepsilon/\sqrt{2}\}$.

6. Proof of Theorem 3. We prove that $E_V(\varepsilon)$ is isomorphic to $\mathcal{E}_{V\to III'}(\varepsilon)$ for sufficiently small $\varepsilon\neq 0$. Notice that this $E_V(\varepsilon)$ is different from that in Section 5.

Since $\eta_\infty\varepsilon^{-1}$, $-\eta_0\varepsilon^{-1}+\kappa_\infty \neq 0$ for sufficiently small $\varepsilon \neq 0$ by assumption $\eta_0\eta_\infty \neq 0$, $E_V(\varepsilon)$ is described as a patching of five copies of $\boldsymbol{C}^2 \times B_{V\to III'}(\varepsilon)$ via

$$\begin{aligned}
&x(00) = z', \quad y(00) = \eta_\infty\varepsilon^{-1}/z' + w',\\
&x(00) = 1 + x(1\infty), \quad y(00) = -\frac{\eta_0\varepsilon t}{x(1\infty)^2} + \frac{\kappa_0+1}{x(1\infty)} + y(1\infty),\\
&x(00) = 1/x(\infty0+), \quad y(00) = x(\infty0+)((\kappa_0+\kappa_\infty)/2 - x(\infty0+)y(\infty0+)),\\
&x(\infty0+) = z'', \quad y(\infty0+) = (-\eta_0\varepsilon^{-1}+\kappa_\infty)/z'' + w''
\end{aligned}$$

by Lemma 1.

In accordance with the process in 2.3.3, we consider the replacement

$$x(00) \to 1 + \varepsilon x(00), \quad y(00) \to \varepsilon^{-1}y(00),$$

and coresponding to this, we make the following replacements:

$$\begin{aligned}
&x(1\infty) \to \varepsilon x(0\infty), \quad y(1\infty) \to \varepsilon^{-1}y(0\infty),\\
&x(\infty0+) \to x(\infty0)/(\varepsilon + x(\infty0)),\\
&y(\infty0+) \to (1+\varepsilon^{-1}x(\infty0))[-(\kappa_0+\kappa_\infty)/2 + (\varepsilon + x(\infty0))y(\infty0)].
\end{aligned}$$

Then we have another description of $E_V(\varepsilon)$ as a patching of five copies via

(6.1) $$1 + \varepsilon x(00) = z', \quad \varepsilon^{-1}y(00) = \eta_\infty\varepsilon^{-1}/z' + w',$$

(6.2) $$x(00) = x(0\infty), \quad y(00) = -\frac{\eta_0 t}{x(0\infty)^2} + \frac{\kappa_0+1}{x(0\infty)} + y(0\infty),$$

(6.3) $$x(00) = 1/x(\infty0), \quad y(00) = x(\infty0)((\kappa_0+\kappa_\infty)/2 - x(\infty0)y(\infty0)),$$

(6.4) $$\begin{aligned}
&x(\infty0)/(\varepsilon + x(\infty0)) = z'',\\
&(1+\varepsilon^{-1}x(\infty0))\left[-\frac{\kappa_0+\kappa_\infty}{2} + (\varepsilon + x(\infty0))y(\infty0)\right] = \frac{-\eta_0\varepsilon^{-1}+\kappa_\infty}{z''} + w''.
\end{aligned}$$

The fiber $E_V(\varepsilon, t)$ of $E_V(\varepsilon)$ over t is a disjoint union of $\boldsymbol{C}^2 \ni (x(00), y(00))$ and four complex lines

$$E_V(\varepsilon, t) = \boldsymbol{C}^2 \cup D_{-\varepsilon^{-1}\infty}(t) \cup D_{0\infty}(t) \cup D_{\infty0}(t) \cup D_{\infty0-}(t),$$

where $D_{-\varepsilon^{-1}\infty}(t) := \{z' = 0\}$, $D_{0\infty}(t) := \{x(0\infty) = 0\}$, $D_{\infty0}(t) := \{x(\infty0) = 0\}$ and $D_{\infty0-}(t) := \{z'' = 0\}$. As $\varepsilon \to 0$, the divisors $D_{-\varepsilon^{-1}\infty}(t)$ and $D_{\infty0-}(t)$ are going to collide with each other, and thus we must look for a coodinate system suitable to see the collision.

We first seek a coordinate system for $D_{-\varepsilon^{-1}\infty}(t)$ simply related to the system $(x(\infty0), y(\infty0))$. By (6.1) and (6.3), we have

$$x(\infty0) = \varepsilon/(-1+z'),$$

and $x(\infty0) = -\varepsilon$ if $z' = 0$. Therefore we introduce u' by

(6.5) $$x(\infty0) = -\varepsilon + u'.$$

A careful calculation by means of (6.1),(6.3) and (6.5) shows

$$y(\infty 0)=\frac{\eta_\infty \varepsilon^{-1}}{u'}+\frac{-\eta_\infty\varepsilon^{-1}+(\kappa_0+\kappa_\infty)/2}{-\varepsilon+u'}-\frac{\varepsilon}{(-\varepsilon+u')^2}w'.$$

Then, introducing v' by

$$v'=\frac{-\eta_\infty\varepsilon^{-1}+(\kappa_0+\kappa_\infty)/2}{-\varepsilon+u'}-\frac{\varepsilon}{(-\varepsilon+u')^2}w',$$

we have

$$x(\infty 0)=-\varepsilon+u',\quad y(\infty 0)=\frac{\eta_\infty\varepsilon^{-1}}{u'}+v'. \tag{6.6}$$

Note that $D_{-\varepsilon^{-1}\infty}(t)=\{z'=0\}=\{u'=0\}$, and that $(0,w')$ and $(0,v')$ related by

$$w'+\varepsilon v'=\eta_\infty\varepsilon^{-1}+(\kappa_0+\kappa_\infty)/2$$

represent the same point on $D_{-\varepsilon^{-1}\infty}(t)$.

We next choose a suitable coordinate system for the divisor $D_{\infty 0-}(t)=\{z''=0\}$ which is simply related to the coordinate system $(x(\infty 0), y(\infty 0))$. By (6.4), we have

$$x(\infty 0)=\varepsilon z''/(1-z''),$$

and $x(\infty 0)=0$ if $z''=0$. Therefore we introduce u'' by

$$x(\infty 0)=u''. \tag{6.7}$$

By making use of (6.4) and (6.7), we have

$$y(\infty 0)=\frac{-\eta_\infty\varepsilon^{-1}+\kappa_\infty}{u''}+\frac{\eta_\infty\varepsilon^{-1}+(\kappa_0-\kappa_\infty)/2}{\varepsilon+u''}+\frac{\varepsilon}{(\varepsilon+u'')^2}w''.$$

Therefore, introducing v'' by

$$v''=\frac{\eta_\infty\varepsilon^{-1}+(\kappa_0-\kappa_\infty)/2}{\varepsilon+u''}+\frac{\varepsilon}{(\varepsilon+u'')^2}w'',$$

we obtain a coordinate system (u'', v''), which is related to $(x(\infty 0), y(\infty 0))$ by

$$x(\infty 0)=u'',\quad y(\infty 0)=\frac{-\eta_\infty\varepsilon^{-1}+\kappa_\infty}{u''}+v''. \tag{6.8}$$

We can verify that $D_{\infty 0-}(t)=\{z''=0\}=\{u''=0\}$, and that $(0,w'')$ and $(0,v'')$ related by

$$-w''+\varepsilon v''=\eta_\infty\varepsilon^{-1}+(\kappa_0+\kappa_\infty)/2$$

represent the same point on $D_{\infty 0-}(t)$.

Observing (6.5) and (6.8), we introduce a coordinate system (u, v) by

$$x(\infty 0)=u,\quad y(\infty 0)=\frac{-\eta_\infty\varepsilon^{-1}+\kappa_\infty}{u}+\frac{\eta_\infty\varepsilon^{-1}}{u+\varepsilon}+v. \tag{6.9}$$

It is verified that $D_{-\varepsilon^{-1}\infty}(t)=\{u=-\varepsilon\}$ and $D_{\infty 0-}(t)=\{u=0\}$.

7. Proof of Theorem 4. In this section, we show that $E_{IV}(\varepsilon)$ is isomorphic to $\mathcal{E}_{IV\to II}(\varepsilon)$ for each $\varepsilon\neq 0$.

By Lemma 1, $E_{IV}(\varepsilon)$ is described as a patching of four copies of $\boldsymbol{C}^2 \times B_{IV\to II}(\varepsilon)$ via

$$x(00) = z', \quad y(00) = (\varepsilon^{-6}/2)/z' + w',$$
$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)(-\alpha - 1/2 - x(\infty 0)y(\infty 0)),$$
$$x(\infty 0) = x(\infty\infty),$$
$$y(\infty 0) = -\frac{1/2}{x(\infty\infty)^3} - \frac{-\varepsilon^{-3} + 2^{-2/3}\varepsilon t}{x(\infty\infty)^2} - \frac{\varepsilon^{-6}/2 + 2\alpha}{x(\infty\infty)} + y(\infty\infty).$$

In accordance with the process in 2.3.4, we make the replacement

$$x(00) \to \varepsilon^{-3}(1 + 2^{2/3}\varepsilon^2 x(00)), \quad y(00) \to 2^{-2/3}\varepsilon y(00),$$

and corresponding to this,

$$x(\infty 0) \to \frac{\varepsilon^3 x(\infty 0)}{2^{2/3}\varepsilon^2 + x(\infty 0)},$$
$$y(\infty 0) \to 2^{-2/3}\varepsilon^{-5}(2^{2/3}\varepsilon^2 + x(\infty 0))\{\alpha + 1/2 + (2^{2/3}\varepsilon^2 + x(\infty 0))y(\infty 0)\}.$$

Then we have another description of $E_{IV}(\varepsilon)$ by patching four copies of $\boldsymbol{C}^2 \times B_{IV\to II}(\varepsilon)$ via

$$\varepsilon^{-3}(1 + 2^{2/3}\varepsilon^2 x(00)) = z', \quad 2^{-2/3}\varepsilon y(00) = (\varepsilon^{-6}/2)/z' + w', \tag{7.1}$$
$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)(-\alpha - 1/2 - x(\infty 0)y(\infty 0)), \tag{7.2}$$
$$\frac{\varepsilon^3 x(\infty 0)}{2^{2/3}\varepsilon^2 + x(\infty 0)} = x(\infty\infty),$$
$$\begin{aligned} &2^{-2/3}\varepsilon^{-5}(2^{2/3}\varepsilon^2 + x(\infty 0))\{\alpha + 1/2 + (2^{2/3}\varepsilon^2 + x(\infty 0))y(\infty 0)\} \\ &= -\frac{1/2}{x(\infty\infty)^3} - \frac{-\varepsilon^{-3} + 2^{-2/3}\varepsilon t}{x(\infty\infty)^2} - \frac{\varepsilon^{-6}/2 + 2\alpha}{x(\infty\infty)} + y(\infty\infty). \end{aligned} \tag{7.3}$$

We see that the fiber $E_{IV}(\varepsilon, t)$ of $E_{IV}(\varepsilon)$ over t is a disjoint union of $\boldsymbol{C}^2 \ni (x(00), y(00))$ and three complex lines:

$$E_{IV}(\varepsilon, t) = \boldsymbol{C}^2 \cup D_{-2^{-2/3}\varepsilon^{-2}\infty}(t) \cup D_{\infty 0}(t) \cup D_{\infty\infty}(t),$$

where $D_{-2^{-2/3}\varepsilon^{-2}\infty}(t) := \{z' = 0\}$, $D_{\infty 0}(t) := \{x(\infty 0) = 0\}$ and $D_{\infty\infty}(t) := \{x(\infty\infty) = 0\}$. In order to see that two divisors $D_{-2^{-2/3}\varepsilon^{-2}\infty}(t)$ and $D_{\infty\infty}(t)$ collide with each other as $\varepsilon \to 0$, we are going to choose an appropriate coordinate system to describe the collision.

First, we seek a coordinate system to describe the divisor $D_{-2^{-2/3}\varepsilon^{-2}\infty}(t)$ so that it is simply related to $(x(\infty 0), y(\infty 0))$. From (7.1) and (7.2), it follows that

$$x(\infty 0) = 2^{2/3}\varepsilon^2/(-1 + \varepsilon^3 z'),$$

and $x(\infty 0) = -2^{2/3}\varepsilon^2$ if in particular $z' = 0$. Hence we take u' as

$$x(\infty 0) = -2^{2/3}\varepsilon^2 + u'. \tag{7.4}$$

From (7.1), (7.2) and (7.4), it follows that

$$y(\infty 0) = \frac{\varepsilon^{-6}/2}{u'} - \frac{\varepsilon^{-6}/2 + \alpha + 1/2}{-2^{2/3}\varepsilon^2 + u'} - \frac{2^{2/3}\varepsilon^{-1}}{(-2^{2/3}\varepsilon^2 + u')^2}w'.$$

Therefore, introducing v' as

$$v' = -\frac{\varepsilon^{-6}/2 + \alpha + 1/2}{-2^{2/3}\varepsilon^2 + u'} - \frac{2^{2/3}\varepsilon^{-1}}{(-2^{2/3}\varepsilon^2 + u')^2} w',$$

we have

$$x(\infty 0) = -2^{2/3}\varepsilon^2 + u', \quad y(\infty 0) = \frac{\varepsilon^{-6}/2}{u'} + v'. \tag{7.5}$$

We can verify that $D_{-2^{-2/3}\varepsilon^{-2}\infty}(t) = \{z' = 0\} = \{u' = 0\}$, and that the points $(0, w')$ and $(0, v')$ related by

$$w' + 2^{2/3}\varepsilon^5 v' = \varepsilon^3(\varepsilon^{-6}/2 + \alpha + 1/2)$$

represent the same point on $D_{-2^{-2/3}\varepsilon^{-2}\infty}(t)$.

Secondly, we choose a coordinate system for the divisor $D_{\infty\infty\infty}(t)$ simply related to $(x(\infty 0), y(\infty 0))$. Taking the first equation of (7.3) into account, we choose a variable u'' as

$$x(\infty 0) = u''. \tag{7.6}$$

Notice that $x(\infty\infty) = 0$ corresponds to $u'' = 0$. By making use of (7.3) and (7.6), we obtain

$$\begin{aligned} y(\infty 0) = &-\frac{2^{1/3}\varepsilon^{-2}}{u''^3} + \frac{2^{-1/3}\varepsilon^{-4} - t}{u''^2} - \frac{\varepsilon^{-6}/2 + 2\alpha}{u''} \\ &+ \frac{\varepsilon^{-6}/2 + \alpha - 1/2}{2^{2/3}\varepsilon^2 + u''} + \frac{2^{2/3}\varepsilon^5}{(2^{2/3}\varepsilon^2 + u'')^2} y(\infty\infty). \end{aligned}$$

Therefore, by introducing v'' by

$$v'' = \frac{\varepsilon^{-6}/2 + \alpha - 1/2}{2^{2/3}\varepsilon^2 + u''} + \frac{2^{2/3}\varepsilon^5}{(2^{2/3}\varepsilon^2 + u'')^2} y(\infty\infty),$$

we have

$$x(\infty 0) = u'', \quad y(\infty 0) = -\frac{2^{1/3}\varepsilon^{-2}}{u''^3} + \frac{2^{-1/3}\varepsilon^{-4} - t}{u''^2} - \frac{\varepsilon^{-6}/2 + 2\alpha}{u''} + v''. \tag{7.7}$$

It is verified that $D_{\infty\infty\infty}(t) = \{x(\infty\infty) = 0\} = \{u'' = 0\}$, and the points $(0, y(\infty\infty))$ and $(0, v'')$ related by

$$y(\infty\infty) - 2^{2/3}\varepsilon^{-1} v'' = -\varepsilon^{-3}(\varepsilon^{-6}/2 + \alpha - 1/2)$$

represent the same point on $D_{\infty\infty\infty}(t)$.

Lastly, by observing (7.5) and (7.7), we introduce a coordinate system (u, v) by

$$\begin{aligned} &x(\infty 0) = u, \\ &y(\infty 0) = -\frac{2^{1/3}\varepsilon^{-2}}{u^3} + \frac{2^{-1/3}\varepsilon^{-4} - t}{u^2} - \frac{\varepsilon^{-6}/2 + 2\alpha}{u} + \frac{\varepsilon^{-6}/2}{u + 2^{2/3}\varepsilon^2} + v. \end{aligned} \tag{7.8}$$

We can verify that $D_{-2^{-2/3}\varepsilon^{-2}\infty}(t) = \{u = -2^{2/3}\varepsilon^2\}$ and $D_{\infty\infty\infty}(t) = \{u = 0\}$.

8. Proof of Theorem 5. We show that $E_{III}(\varepsilon)$ is isomorphic to $\mathcal{E}_{III\to II}(\varepsilon)$ for each $\varepsilon \neq 0$.

The manifold $E_{III}(\varepsilon)$ is described as a patching of four copies of $\boldsymbol{C}^2 \times B_{III}(\varepsilon)$ via

$$x(00) = x(0\infty)\,, \quad y(00) = \frac{\varepsilon^{-3}t/4}{x(0\infty)^2} - \frac{\varepsilon^{-3}/2 + 2\alpha}{x(0\infty)} + y(0\infty)\,,$$

$$x(00) = 1/x(\infty 0)\,, \quad y(00) = x(\infty 0)(-\alpha - 1/2 - x(\infty 0)y(\infty 0))\,,$$

$$x(\infty 0) = x(\infty\eta_\infty t)\,, \quad y(\infty 0) = -\frac{\varepsilon^{-3}t/4}{x(\infty\eta_\infty t)^2} + \frac{\varepsilon^{-3}/2}{x(\infty\eta_\infty t)} + y(\infty\eta_\infty t)\,.$$

Acoording to the confluence process given in 2.3.5, we make the replacements

$$x(00) \to 1 + 2\varepsilon x(00)\,, \quad y(00) \to y(00)/(2\varepsilon)\,, \quad t \to 1 + \varepsilon^2 t$$

and corresponding to this,

$$x(\infty 0) \to \frac{x(\infty 0)}{2\varepsilon + x(\infty 0)}\,, \quad y(\infty 0) \to \frac{2\varepsilon + x(\infty 0)}{2\varepsilon}[\alpha + 1/2 + (2\varepsilon + x(\infty 0))y(\infty 0)]\,.$$

Then we have another description of $E_{III}(\varepsilon)$ by patching of four copies of $\boldsymbol{C}^2 \times B_{III\to II}(\varepsilon)$ via

$$1 + 2\varepsilon x(00) = x(0\infty)\,,$$

$$\text{(8.1)} \qquad y(00)/(2\varepsilon) = \frac{\varepsilon^{-3}(1+\varepsilon^2 t)/4}{x(0\infty)^2} - \frac{\varepsilon^{-3}/2 + 2\alpha}{x(0\infty)} + y(0\infty)\,,$$

$$\text{(8.2)} \qquad x(00) = 1/x(\infty 0)\,, \quad y(00) = x(\infty 0)(-\alpha - 1/2 - x(\infty 0)y(\infty 0))\,,$$

$$\text{(8.3)} \qquad \frac{x(\infty 0)}{2\varepsilon + x(\infty 0)} = x^*\,, \quad \frac{2\varepsilon + x(\infty 0)}{2\varepsilon}[\alpha + 1/2 + (2\varepsilon + x(\infty 0))y(\infty 0)] = -\frac{\varepsilon^{-3}(1+\varepsilon^2 t)/4}{x^{*2}} + \frac{\varepsilon^{-3}/2}{x^*} + y^*\,.$$

Here $(x^*, y^*) := (x(\infty\eta_\infty t), y(\infty\eta_\infty t))$.

We see that the fiber $E_{III}(\varepsilon, t)$ of $E_{III}(\varepsilon)$ over t is a disjoint union of $\boldsymbol{C}^2 \ni (x(00), y(00))$ and three complex lines:

$$E_{III}(\varepsilon, t) = \boldsymbol{C}^2 \cup D_{-\epsilon^{-1}/2\infty}(t) \cup D_{\infty 0}(t) \cup D_{\infty\varepsilon^{-3}t/4}(t)\,,$$

where $D_{-\varepsilon^{-1}/2\infty}(t) := \{x(0\infty) = 0\}$, $D_{\infty 0}(t) := \{x(\infty 0) = 0\}$ and $D_{\infty\varepsilon^{-3}t/4}(t) := \{x^* = 0\}$. As ε tends to 0, the divisors $D_{-\varepsilon^{-1}/2\infty}(t)$ and $D_{\infty\varepsilon^{-3}t/4}(t)$ collide with each other. In the following, we look for an appropriate coodinate system which is suitable to see the collision of $D_{-\varepsilon^{-1}/2\infty}(t)$ and $D_{\infty\varepsilon^{-3}t/4}(t)$.

We first obtain an coodinate system of a neighborhood of $D_{-\varepsilon^{-1}/2\infty}(t)$, so that it is related with $(x(\infty 0), y(\infty 0))$ in simple form. Since the right-hand side of

$$x(\infty 0) = 2\varepsilon/(-1 + x(0\infty))$$

is -2ε if $x(0\infty) = 0$, we introduce a variable u' by

$$\text{(8.4)} \qquad x(\infty 0) = -2\varepsilon + u'.$$

We see that

$$y(\infty 0) = -\frac{(\varepsilon^{-2}+t)/2}{u'^2} - \frac{\varepsilon^{-3}/2 + 2\alpha}{u'} + \frac{\varepsilon^{-3}/2 + \alpha - 1/2}{-2\varepsilon + u'} - \frac{2\varepsilon}{(-2\varepsilon + u')^2} y(0\infty),$$

and then introduce a variable v' by

$$v' = \frac{\varepsilon^{-3}/2 + \alpha - 1/2}{-2\varepsilon + u'} - \frac{2\varepsilon}{(-2\varepsilon + u')^2} y(0\infty).$$

The relation between $(x(\infty 0), y(\infty 0))$ and (u', v') is given by

(8.5) $$x(\infty 0) = -2\varepsilon + u', \quad y(\infty 0) = -\frac{(\varepsilon^{-2}+t)/2}{u'^2} - \frac{\varepsilon^{-3}/2 + 2\alpha}{u'} + v',$$

and $D_{-\varepsilon^{-1}/2\infty}(t) = \{x(0\infty) = 0\} = \{u' = 0\}$.

We next obtain an appropriate coodinate system of a neighborhood of $D_{\infty\varepsilon^{-3}t/4}(t)$. Since the right-hand side of

$$x(\infty 0) = 2\varepsilon x^*/(1 - x^*)$$

is 0 if $x^* = 0$, we introduce a variable u'' by

(8.6) $$x(\infty 0) = u''.$$

Then we have

$$y(\infty 0) = -\frac{(\varepsilon^{-2}+t)/2}{u''^2} + \frac{\varepsilon^{-3}/2}{u''} - \frac{\varepsilon^{-3}/2 + \alpha + 1/2}{2\varepsilon + u''} + \frac{2\varepsilon}{(2\varepsilon + u'')^2} y^*.$$

Therefore, introducing a variable v'' by

$$v'' = -\frac{\varepsilon^{-3}/2 + \alpha + 1/2}{2\varepsilon + u''} + \frac{2\varepsilon}{(2\varepsilon + u'')^2} y^*,$$

we have

(8.7) $$x(\infty 0) = u'', \quad y(\infty 0) = -\frac{(\varepsilon^{-2}+t)/2}{u''^2} + \frac{\varepsilon^{-3}/2}{u''} + v''.$$

Now, observing (8.5) and (8.7), we take a coordinate system (u, v) defined by

(8.8) $$x(\infty 0) = u, \quad y(\infty 0) = -\frac{(\varepsilon^{-2}+t)/2}{u^2} + \frac{\varepsilon^{-3}/2}{u} - \frac{(\varepsilon^{-2}+t)/2}{(u+2\varepsilon)^2} - \frac{\varepsilon^{-3}/2 + 2\alpha}{u + 2\varepsilon} + v.$$

Then we can verify

$$D_{-\varepsilon^{-1}/2\infty}(t) = \{u = -2\varepsilon\}, \quad D_{\infty\varepsilon^{-3}t/4}(t) = \{u = 0\}.$$

REFERENCES

[1] K. IWASAKI, H. KIMURA, S. SHIMOMURA AND M. YOSHIDA, From Gauss to Painlevé, Aspects of Mathematics 16, Vieweg, Braunschweig, 1991.

[2] H. KIMURA, Uniform foliation associated with the Hamiltonian system $\mathcal{H}_n$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 20 (1993), 1–60.

[3] K. KODAIRA, Complex manifolds and deformation of complex structures, Grundlehren Math. Wiss. 283, Springer-Verlag, New York-Berlin, 1986.

[4] K. OKAMOTO, Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé, Espaces des conditions initiales, Japan. J. Math. 5 (1979), 1–79.

[5] P. PAINLEVE, Œuvres t. I, II, III, Centre National de la Recherche Scientifique, Paris, 1976.

[6] T. SHIODA AND K. TAKANO, On some Hamiltonian structures of Painlevé systems I, Funkcial. Ekvac. 40 (1997), 271–291.

[7] T. MATANO, A. MATUMIYA AND K. TAKANO, On some Hamiltonian structures of Painlevé systems II, J. Math. Soc. Japan 51 (1999), 843–866.

[8] A. MATUMIYA, On some Hamiltonian structures of Painlevé systems III, Kumamoto J. Math. 10 (1997), 45–73.

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