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MEAN SQUARE DECAY OF FOURIER TRANSFORMS IN EUCLIDEAN AND NON EUCLIDEAN SPACES

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Abstract. We study the asymptotic behavior of the quadratic means on spheres of increasing radius of the Fourier transforms of characteristic functions of sets with smooth boundary. Beside to give different proofs of known Euclidean results, we also consider the non-Euclidean Fourier transform on the two dimensional sphere and the hyperbolic disk.

This paper is devoted to the study of Fourier transforms of characteristic functions and more generally of piecewise smooth functions, that is, functions of the form $f(x)\chi_\Omega(x)$ with $f(x)$ smooth and Ω a bounded domain with smooth boundary. The decay of these Fourier transforms is related to the geometry of the domains, in particular, Herz and Hlawka have proved that for bounded convex sets in $\boldsymbol{R}^N$ with smooth boundary of positive Gaussian curvature, $|\hat{\chi}_\Omega(\xi)| \leq c|\xi|^{-(N+1)/2}$. This agrees with the estimate for a ball, but for non convex sets or convex with flat boundary it may fail. An example is given by the Fourier transforms of characteristic functions of polyhedra which simultaneously present minimal and maximal rate of decay. Along almost all directions there is a decay $|\xi|^{-N}$, but in the directions orthogonal to the faces the decay is only $|\xi|^{-1}$. This different behavior along different directions suggests the study of an average decay, and indeed Varchenko has proved that for bounded domains with smooth boundary one has

$$\left\{\int_{\{|\sigma|=1\}}\left|\int_\Omega \exp(-2\pi i\rho\sigma\cdot x)dx\right|^2 d\sigma\right\}^{1/2} \leq c\rho^{-(N+1)/2}.$$

Observe that this quadratic estimate matches with the pointwise decay given by non-vanishing curvature, and hence it is best possible. A two dimensional result of Podkorytov replaces smoothness with convexity, but the peculiarity of Varchenko's result is that, except for smoothness, there are no geometric assumptions on the domains.

In this paper, we give a different proof of the above quadratic estimate, using a method that allows to establish analogous results for Fourier transforms on other symmetric spaces. In particular, we shall consider explicitly the sphere and the hyperbolic disk, which are models of the non-Euclidean elliptic and hyperbolic plane. In the case of the Euclidean spaces, we consider integrals of the type

$$\left\{\int_{\{|\sigma|=1\}}\left|\int_\Omega f(x)\exp(-2\pi i\rho\sigma\cdot x)dx\right|^2 d\sigma\right\}^{1/2},$$

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with $f(x)$ smooth. The Gauss-Green formula essentially reduces the integral to one on $\partial\Omega$,

$$(2\pi\rho)^{-1}\left\{\int_{\{|\sigma|=1\}}\left|\int_{\partial\Omega} f(x)\sigma\cdot n(x)\exp(-2\pi i\rho\sigma\cdot x)dx\right|^2 d\sigma\right\}^{1/2},$$

and, introducing local coordinates on $\{|\sigma|=1\}$ and on $\partial\Omega$,

$$c\rho^{-1}\left\{\int_{\boldsymbol{R}^{N-1}}\left|\int_{\boldsymbol{R}^{N-1}} F(u)A(t,u)\exp(-2\pi i\rho\Phi(t,u))du\right|^2 dt\right\}^{1/2}.$$

Hence one is led to estimate the boundedness on $L^2(\boldsymbol{R}^{N-1})$ of a Fourier integral operator with phase $\Phi(t,u)=\sigma\cdot x$. The point is that this phase degenerates, the determinant $[\partial^2\Phi(t,u)/\partial t_i\partial u_j]$ vanishes at some points and the standard techniques do not immediately apply. See, for example, [9]. Nevertheless we shall prove that there exists a constant c such that, if the support of $f(x)$ is suitably small,

$$\left\{\int_{\{|\sigma|=1\}}\left|\int_{\Omega} f(x)\exp(-2\pi i\rho\sigma\cdot x)dx\right|^2 d\sigma\right\}^{1/2} \approx c\rho^{-(N+1)/2}\left\{\int_{\partial\Omega}|f(x)|^2dx\right\}^{1/2}.$$

For functions with big supports this asymptotic result may be false, and an example is given by the Fourier transform of a ball, which vanishes on a sequence of spheres.

The exponentials $\{\exp(2\pi i\xi\cdot x)\}$ are eigenvectors of the Laplace operator on $\boldsymbol{R}^N$ with eigenvalues $\{4\pi^2|\xi|^2\}$, and the projection of the function $f(x)\chi_\Omega(x)$ to the subspace associated to eigenvalues between $4\pi^2\rho^2$ and $4\pi^2(\rho+1)^2$ is given by

$$\int_{\{\rho\le|\xi|\le\rho+1\}}\int_{\Omega} f(x)\exp(2\pi i\xi\cdot(x-y))dyd\xi\,.$$

The result of Varchenko thus implies that the square norm of this projection satisfies the estimate

$$\left\{\int_{\boldsymbol{R}^N}\left|\int_{\{\rho\le|\xi|\le\rho+1\}}\int_{\Omega} f(x)\exp(2\pi i\xi\cdot(x-y))dyd\xi\right|^2 dx\right\}^{1/2}$$
$$=\left\{\int_{\rho}^{\rho+1}\int_{\{|\sigma|=1\}}\left|\int_{\Omega} f(x)\exp(-2\pi i\rho\sigma\cdot x)dx\right|^2\rho^{N-1}d\rho d\sigma\right\}^{1/2}\le\frac{c}{\rho}.$$

We want to prove similar estimates for spherical harmonic expansions. Every square integrable function on the sphere $S=\{x\in\boldsymbol{R}^3;\,|x|=1\}$ has a spherical harmonic expansion, or Fourier-Laplace series,

$$g(x)=\sum_{k=0}^{+\infty}\left[\int_S Z_k(x\cdot y)g(y)dy\right],$$

where the zonal harmonics $\{Z_k(t)\}$ are multiples of Legendre polynomials. The terms of this series are projections of the function to the eigenspaces of the Laplace operator on the sphere corresponding to eigenvalues $\{k(k+1)\}$. It is not difficult to see that for piecewise regular functions, the terms of the series are pointwise bounded by $ck^{-1/2}$, however the mean square decay is better,

$$\left\{ \int_S \left| \int_\Omega Z_k(x \cdot y) f(y) dy \right|^2 dx \right\}^{1/2} \leq \frac{c}{k}.$$

This estimate for the norms of the spherical harmonic projections is natural. Roughly speaking, one should expect at least a decay $k^{-1/2}$, since the sum of the square of the norms of spherical harmonic projections is finite, plus an extra decay $k^{-1/2}$, because piecewise smooth functions have half a derivative in $L^2(S)$. Also, this matches with the corresponding estimate for the euclidean Fourier transform.

An important tool in the study of the decay of Fourier transforms is given by the method of stationary phase. We shall not use explicitly this method, but we shall deduce our results from an integral involving Bessel functions,

$$\lim_{\varepsilon \to 0+} \int_0^{+\infty} \psi(\varepsilon t) t^\beta J_\alpha(t) dt = 2^\beta \frac{\Gamma((\alpha+\beta+1)/2)}{\Gamma((\alpha-\beta+1)/2)} \psi(0),$$

which holds whenever $\alpha + \beta > -1$ and $\psi(t)$ is smooth with compact support.

The first section of the paper contains the easy proof of this formula and some corollaries. In the second section we prove quadratic estimates for the Euclidean Fourier transform of piecewise regular functions. In the third and fourth sections we consider analogous problems on the sphere and the hyperbolic disk. Our motivation for the study of the average decay of Fourier transform arises from problems on estimates of the number of integer points in large domains and also from problems on localization and convergence of Fourier expansions. Some of the results in this paper are part of the dissertation of A. Torlaschi at the "Università degli Studi di Milano".

1. Some integrals involving Bessel functions. In this section we consider the asymptotic behavior, when $\rho \to +\infty$, of integrals of the type

$$\int_{\mathbf{R}^N} \int_{\mathbf{R}^N} G(x,y) \frac{J_\alpha(\rho|x-y|)}{(\rho|x-y|)^\alpha} dy dx.$$

The following argument is heuristic.

$$\begin{aligned} &\int_{\mathbf{R}^N} \int_{\mathbf{R}^N} G(x,y) \frac{J_\alpha(\rho|x-y|)}{(\rho|x-y|)^\alpha} dy dx \\ &= \rho^{-N} \int_{\mathbf{R}^N} \int_S \int_0^{+\infty} G(x, x+\rho^{-1} t\vartheta) J_\alpha(t) t^{N-\alpha-1} dt d\vartheta dx \\ &\approx \rho^{-N} |S| \left(\int_0^{+\infty} t^{N-\alpha-1} J_\alpha(t) dt \right) \left(\int_{\mathbf{R}^N} G(x,x) dx \right). \end{aligned}$$

Hence we expect that the integral is dominated by ρ^{-N}, but we also expect an extra decay if by chance $\int_0^{+\infty} t^{N-\alpha-1} J_\alpha(t) dt$ vanishes. Let us consider this last integral. In books on special functions one can find Weber's formula

$$\int_0^{+\infty} t^\beta J_\alpha(t) dt = 2^\beta \frac{\Gamma((\alpha+\beta+1)/2)}{\Gamma((\alpha-\beta+1)/2)}.$$

See, for instance, [15]. The function $t^\beta J_\alpha(t)$ on the left is integrable in a neighborhood of $t = 0$ if $\alpha + \beta > -1$, and in a neighborhood of $t = +\infty$ if $\beta < -1/2$. But the quotient of Gamma functions on the right is finite for a larger set of indexes and it may vanish at the poles of the denominator. This suggests the following.

LEMMA 1.1. *If $\alpha + \beta > -1$, then there exist c and k such that for every $\psi(t)$ smooth with compact support and for every $\varepsilon > 0$,*

$$\left| \int_0^{+\infty} \psi(\varepsilon t) t^\beta J_\alpha(t) dt \right| \le c \sum_{j=0}^{k} \sup_{t \ge 0} \left| \frac{d^j}{dt^j} \psi(t) \right| .$$

Moreover,

$$\lim_{\varepsilon \to 0+} \int_0^{+\infty} \psi(\varepsilon t) t^\beta J_\alpha(t) dt = 2^\beta \frac{\Gamma((\alpha + \beta + 1)/2)}{\Gamma((\alpha - \beta + 1)/2)} \psi(0) .$$

PROOF. Since $(\partial/\partial t)(t^{\alpha+1} J_{\alpha+1}(t)) = t^{\alpha+1} J_\alpha(t)$, an integration by parts reduces an integral with parameters (α, β) for the function $\psi(\varepsilon t)$ to one with $(\alpha + 1, \beta - 1)$ for an associated function $\phi(\varepsilon t)$ with $\phi(0) = (1 + \alpha - \beta)\psi(0)$,

$$\begin{aligned} \int_0^{+\infty} \psi(\varepsilon t) t^\beta J_\alpha(t) dt &= \int_0^{+\infty} \psi(\varepsilon t) t^{\beta-\alpha-1} \frac{\partial}{\partial t} (t^{\alpha+1} J_{\alpha+1}(t)) dt \\ &= \psi(\varepsilon t) t^\beta J_{\alpha+1}(t) \Big|_0^{+\infty} - \int_0^{+\infty} t^{\alpha+1} J_{\alpha+1}(t) \frac{\partial}{\partial t} (\psi(\varepsilon t) t^{\beta-\alpha-1}) dt \\ &= \int_0^{+\infty} ((1 + \alpha - \beta)\psi(\varepsilon t) - \varepsilon t \psi'(\varepsilon t)) t^{\beta-1} J_{\alpha+1}(t) dt \\ &= \int_0^{+\infty} \phi(\varepsilon t) t^{\beta-1} J_{\alpha+1}(t) dt . \end{aligned}$$

One can iterate until $t^{\beta-n} J_{\alpha+n}(t)$ becomes absolutely integrable, then the dominated convergence theorem applies. This show that $\int_0^{+\infty} \psi(\varepsilon t) t^\beta J_\alpha(t) dt$ is uniformly bounded in $\varepsilon > 0$ by $c \sum_{j=0}^k \sup_{t \ge 0} |(d^j \psi/dt^j)(t)|$ and that $\lim_{\varepsilon \to 0+} \int_0^{+\infty} \psi(\varepsilon t) t^\beta J_\alpha(t) dt = c(\alpha, \beta)\psi(0)$. To determine the constant $c(\alpha, \beta)$ it is enough to test the distribution on a particular function. □

LEMMA 1.2. *Denote by $L = -\partial^2/\partial t^2 - (N-1)t^{-1}\partial/\partial t$ the radial part of the Laplace operator on $\mathbf{R}^N$. Then for $n = 0, 1, 2, \ldots,$*

$$t^{n-(N-2)/2} J_{(N-2)/2-n}(t) = \sum_{k=0}^{n} c(N, n, k) L(t^{k-(N-2)/2} J_{(N-2)/2-k}(t)) .$$

PROOF. We can prove this lemma by induction, using the formulas

$$\begin{aligned} &\frac{\partial}{\partial t}(t^{-\alpha} J_\alpha(t)) = -t^{-\alpha} J_{\alpha+1}(t) , \\ &t^{-\alpha} J_{\alpha+2}(t) = (2\alpha + 2)t^{-\alpha-1} J_{\alpha+1}(t) - t^{-\alpha} J_\alpha(t) . \end{aligned}$$

From these formulas it follows that

$$t^{-\alpha}J_\alpha(t)=\left(-\frac{\partial^2}{\partial t^2}-\frac{N-1}{t}\frac{\partial}{\partial t}\right)(t^{-\alpha}J_\alpha(t))+(2\alpha+2-N)t^{-\alpha-1}J_{\alpha+1}(t).$$

□

THEOREM 1.3. *Let Ω be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary and $F(x, y)$ a smooth function in $\boldsymbol{R}^N \times \boldsymbol{R}^N$. Then,*

$$\left|\int_\Omega\int_\Omega F(x,y)\frac{J_{(N-2)/2-n}(\rho|x-y|)}{(\rho|x-y|)^{(N-2)/2-n}}dxdy\right|\le c\rho^{-N-1}.$$

PROOF. The idea is that a Laplacian in front of a Bessel function gives a gain of a factor ρ^{-2} and the Gauss-Green formula reduces an integration over Ω to one over $\partial\Omega$. By the previous lemma applied twice,

$$\frac{J_{(N-2)/2-n}(\rho|x-y|)}{(\rho|x-y|)^{(N-2)/2-n}}=\rho^{-4}\sum_{k=0}^{n}d(N,n,k)\Delta_x\Delta_y\left(\frac{J_{(N-2)/2-k}(\rho|x-y|)}{(\rho|x-y|)^{(N-2)/2-k}}\right).$$

Hence we need to estimate integrals of the type

$$\begin{aligned}&\rho^{-4}\int_\Omega\int_\Omega F(x,y)\Delta_x\Delta_y\left(\frac{J_\alpha(\rho|x-y|)}{(\rho|x-y|)^\alpha}\right)dxdy\\&=\rho^{-4}\int_{\partial\Omega}\int_{\partial\Omega}F(x,y)\frac{\partial}{\partial n(x)}\frac{\partial}{\partial n(y)}\left(\frac{J_\alpha(\rho|x-y|)}{(\rho|x-y|)^\alpha}\right)dxdy+\cdots,\end{aligned}$$

with $\alpha=(N-2)/2-k$ and $0\le k\le n$. The terms omitted are zero when $F(x, y) = 1$, and in the general case a reiterated use of Gauss-Green formula shows that these terms give negligible contributions. Now observe that

$$\begin{aligned}\frac{\partial}{\partial n(x)}\frac{\partial}{\partial n(y)}\left(\frac{J_\alpha(\rho|x-y|)}{(\rho|x-y|)^\alpha}\right)&=\frac{\partial}{\partial n(x)}\left(\rho^2(x-y)\cdot n(y)\frac{J_{\alpha+1}(\rho|x-y|)}{(\rho|x-y|)^{\alpha+1}}\right)\\&=\rho^2 n(x)\cdot n(y)\frac{J_{\alpha+1}(\rho|x-y|)}{(\rho|x-y|)^{\alpha+1}}\\&\quad-\rho^4(x-y)\cdot n(x)(x-y)\cdot n(y)\frac{J_{\alpha+2}(\rho|x-y|)}{(\rho|x-y|)^{\alpha+2}}.\end{aligned}$$

Using partition of unity, we may assume that the intersection of the support of $F(x, y)$ with $\partial\Omega\times\partial\Omega$ is essentially flat. For a fixed x it is possible to introduce a sort of polar coordinates on $\partial\Omega$ centered at x, writing $y = y(x, \vartheta, t)$ with $t = |x-y|$ and $\vartheta\in S$. In order to obtain these coordinates one can consider the exponential map $y = \mathrm{Exp}[x, Y]$ from $\boldsymbol{R}^{N-1}$ in $\partial\Omega$, write in polar coordinates $Y = r\vartheta$ and then make the change of variables $t = |\mathrm{Exp}[x, r\vartheta] - x|$. In these coordinates the surface measure takes the form $dy = \varphi(x, \vartheta, t)t^{N-2}dtd\vartheta$. Also, for x and y on $\partial\Omega$, one has $|(x-y)\cdot n(x)|\le c|x-y|^2$ and

$(x-y)\cdot n(x)(x-y)\cdot n(y)=\psi(x,\vartheta,t)t^4$. Then

$$\int_{\partial\Omega}\int_{\partial\Omega} F(x,y)(x-y)\cdot n(x)(x-y)\cdot n(y)\frac{J_{\alpha+2}(\rho|x-y|)}{(\rho|x-y|)^{\alpha+2}}dxdy$$
$$=\rho^{-N-3}\int_{\partial\Omega}\int_S\int_0^{+\infty} G(x,\vartheta,t/\rho)t^{N-\alpha}J_{\alpha+2}(t)dtd\vartheta dx\,.$$

By the first lemma the inner integral in t is bounded, so that the contribution of this term is of the order of $c\rho^{-N-3}$. Finally,

$$\rho^{-2}\int_{\partial\Omega}\int_{\partial\Omega} F(x,y)n(x)\cdot n(y)\frac{J_{\alpha+1}(\rho|x-y|)}{(\rho|x-y|)^{\alpha+1}}dxdy$$
$$=\rho^{-N-1}\int_{\partial\Omega}\int_S\int_0^{+\infty} G(x,\vartheta,t/\rho)t^{N-\alpha-3}J_{\alpha+1}(t)dtd\vartheta dx\,.$$

Again by the lemma, the above quantity is bounded by $c\rho^{-N-1}$ and, when $\rho\to+\infty$, we also have the asymptotic

$$\int_{\partial\Omega}\int_S\int_0^{+\infty} G(x,\vartheta,t/\rho)t^{N-\alpha-3}J_{\alpha+1}(t)dtd\vartheta dx$$
$$\approx 2^{N-\alpha-3}\frac{\Gamma((N-1)/2)}{\Gamma((2\alpha-N+5)/2)}\int_{\partial\Omega}\int_S G(x,\vartheta,0)d\vartheta dx\,.$$

□

2. Fourier transforms in Euclidean spaces. Revisiting [14], we consider the mean square decay of the Fourier transform in $\boldsymbol{R}^N$ for functions of the type $f(x)\chi_\Omega(x)$, with $f(x)$ smooth and Ω a bounded domain with smooth boundary.

THEOREM 2.1. *Let Ω be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary and $f(x)$ a smooth function in $\boldsymbol{R}^N$. Then,*

$$\left\{\int_{\{|\sigma|=1\}}\left|\int_\Omega f(x)\exp(-2\pi i\rho\sigma\cdot x)dx\right|^2 d\sigma\right\}^{1/2}\le c\rho^{-(N+1)/2}\,.$$

Also, there exists a constant A such that, if the intersection of the support of $f(x)$ with $\partial\Omega$ is sufficiently small, then

$$\lim_{\rho\to+\infty}\left|\rho^{N+1}\int_{\{|\sigma|=1\}}\left|\int_\Omega f(x)\exp(-2\pi i\rho\sigma\cdot x)dx\right|^2 d\sigma - A\int_{\partial\Omega}|f(x)|^2dx\right|=0\,.$$

PROOF. Since

$$\int_{\{|\sigma|=1\}}\left|\int_\Omega f(x)\exp(-2\pi i\rho\sigma\cdot x)dx\right|^2 d\sigma$$
$$=\int_\Omega\int_\Omega\int_{\{|\sigma|=1\}} f(x)\overline{f(y)}\exp(2\pi i\rho\sigma\cdot(y-x))d\sigma dxdy$$
$$=(2\pi)^{N/2}\int_\Omega\int_\Omega f(x)\overline{f(y)}\frac{J_{(N-2)/2}(2\pi\rho|x-y|)}{(2\pi\rho|x-y|)^{(N-2)/2}}dxdy\,,$$

the result follows from the theorem in the previous section. □

In many respects the above theorem seems to be best possible. Indeed the example of a ball shows that in order to have an asymptotic decay of the quadratic means of Fourier transform some restrictions on the support of the function seems necessary:

$$\begin{aligned}\hat{\chi}_{\{|x|\le r\}}(\xi) &= r^N|r\xi|^{-N/2}J_{N/2}(2\pi|r\xi|)\\ &\approx \pi^{-1}r^{(N-1)/2}|\xi|^{-(N+1)/2}\cos(2\pi r|\xi| - (N+1)\pi/4)\,.\end{aligned}$$

This Fourier transform decays as $c\rho^{-(N+1)/2}$, but vanishes on a sequence of spheres centered at the origin. Also, this example reflects the general situation, since there are no domains that give a decay better than $c\rho^{-(N+1)/2}$. Assume, by way of contradiction, that for a given domain Ω and function $f(x)$,

$$\lim_{\rho\to+\infty}\rho^{(N+1)/2}\left\{\int_{\{|\sigma|=1\}}\left|\int_\Omega f(x)\exp(-2\pi i\rho\sigma\cdot x)dx\right|^2 d\sigma\right\}^{1/2} = 0\,.$$

Then for every $\varepsilon > 0$ there exists $r > 0$ such that for every $R > r$,

$$\left\{\int_{\{|\xi|>R\}}\left|\int_\Omega f(x)\exp(-2\pi i\xi\cdot x)dx\right|^2 d\xi\right\}^{1/2} < \varepsilon R^{-1/2}\,.$$

This implies a restriction on the $L^2(\boldsymbol{R}^N)$ modulus of continuity of $f(x)\chi_\Omega(x)$, see, for instance [2],

$$\left\{\int_{\boldsymbol{R}^N}|f(x+y)\chi_\Omega(x+y) - f(x)\chi_\Omega(x)|^2dx\right\}^{1/2} < c\varepsilon|y|^{1/2}\,.$$

This cannot be true unless $f(x)$ vanishes on $\partial\Omega$.

3. Spherical harmonic expansions. We start by reviewing some results on the harmonic analysis on spheres, as a reference see [10]. Let $S = \{(x_1, x_2, x_3); x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in the Euclidean space $\boldsymbol{R}^3$. We have a system of polar coordinates,

$$\begin{cases} x_1 = \sin(\varphi)\sin(\vartheta)\,, \\ x_2 = \cos(\varphi)\sin(\vartheta)\,, \quad 0 \le \vartheta \le \pi\,,\ 0 \le \varphi \le 2\pi\,. \\ x_3 = \cos(\vartheta)\,, \end{cases}$$

The restriction to the sphere of the Euclidean measure is $dx = \sin(\vartheta)d\vartheta d\varphi$ and the restriction to the sphere of the Laplace operator is

$$\Delta = -\left(\frac{\partial^2}{\partial\vartheta^2} + \frac{\cos(\vartheta)}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2}\right)\,.$$

This operator has eigenvalues $\{k(k+1); 0 \le k < +\infty\}$ and an orthonormal complete system of eigenfunctions $\{\phi_{k,j}(x); 0 \le k < +\infty, 1 \le j \le 2k+1\}$, which are restriction to the sphere of real homogeneous harmonic polynomials. Every function in $L^2(S)$ has a spherical harmonic expansion

$$\sum_{k=0}^{+\infty}\sum_{j=1}^{2k+1}\left(\int_S f(y)\phi_{k,j}(y)dy\right)\phi_{k,j}(x) = \sum_{k=0}^{+\infty}\int_S Z_k(x\cdot y)f(y)dy\,,$$

where $Z_k(x \cdot y) = \sum_{j=1}^{2k+1} \phi_{k,j}(x)\phi_{k,j}(y)$. The zonal harmonics are rotation invariant, and hence functions of the scalar product $x \cdot y$, which satisfy the relations

$$\begin{aligned} &\int_S Z_k(x \cdot y) Z_k(x \cdot z) dx = Z_k(y \cdot z)\,, \\ &\Delta_x Z_k(x \cdot y) = k(k+1) Z_k(x \cdot y)\,, \\ &Z_k(x \cdot y) = \frac{2k+1}{4\pi} P_k(x \cdot y)\,, \end{aligned}$$

where $\{P_k(t)\}_{k=0}^{+\infty}$ is the system of Legendre polynomials, and in $0 \le \vartheta \le \pi - \varepsilon$,

$$P_k(\cos(\vartheta)) = \sum_{n=0}^{m} a_n(\vartheta) \frac{J_n((k+1/2)\vartheta)}{((k+1/2)\vartheta)^n} + O(k^{-m-3/2})\,,$$

with $a_0(\vartheta) = \sqrt{\vartheta / \sin(\vartheta)}, \dots$. For this Hilb-Szegö asymptotic formula, see [11, 12].

Recall that we want to study the size of the terms of the eigenfunction expansions of piecewise regular functions. The example of the characteristic function of a spherical cap is typical,

$$\begin{aligned} \chi(x) &= \begin{cases} 1 & \text{if } x \cdot n > \cos(\alpha)\,, \\ 1/2 & \text{if } x \cdot n = \cos(\alpha)\,, \\ 0 & \text{if } x \cdot n < \cos(\alpha)\,, \end{cases} \\ &= \frac{1 - \cos(\alpha)}{2} + \sum_{k=0}^{+\infty} \frac{P_{k-1}(\cos(\alpha)) - P_{k+1}(\cos(\alpha))}{2} P_k(x \cdot n)\,. \end{aligned}$$

By the asymptotic $P_k(\cos(\vartheta)) \approx 2\cos((k+1/2)\vartheta - \pi/4)/\sqrt{\pi(2k+1)\sin(\vartheta)}$, we deduce that if $x = \pm n$, the terms of the series are of the order of $k^{-1/2}$, and if $x \ne \pm n$, then of the order of k^{-1}. The following result shows that the quadratic mean of $\int_\Omega f(y) Z_k(x \cdot y) dy$ has a decay of the order of k^{-1}.

THEOREM 3.1. *Let Ω be a domain in S with smooth boundary $\partial\Omega$ and $f(y)$ a smooth function in S. Then,*

$$\left\{ \int_S \left| \int_\Omega Z_k(x \cdot y) f(y) dy \right|^2 dx \right\}^{1/2} \le \frac{c}{k}\,.$$

PROOF. If the support of $f(y)$ is strictly contained in Ω, then the result holds, and hence, by a partition of unity argument, we may assume that $f(y)$ is supported in a small neighborhood U and that $U \cap \partial\Omega$ is suitably flat. Then, by Hilb's formula,

$$\begin{aligned}
&\int_S \left| \int_\Omega Z_k(x \cdot y) f(y) dy \right|^2 dx \\
&= \int_\Omega \int_\Omega \int_S f(y)\overline{f(z)} Z_k(x \cdot y) Z_k(x \cdot z) dx dy dz \\
&= \int_\Omega \int_\Omega f(y)\overline{f(z)} Z_k(y \cdot z) dy dz = \frac{2k+1}{4\pi} \int_\Omega \int_\Omega f(y)\overline{f(z)} P_k(y \cdot z) dy dz \\
&= \frac{2k+1}{4\pi} \int_\Omega \int_\Omega f(y)\overline{f(z)} \sqrt{\frac{\arccos(y \cdot z)}{\sin(\arccos(y \cdot z))}} J_0((k+1/2)\arccos(y \cdot z)) dy dz + \cdots .
\end{aligned}$$

A change of variables that takes $\arccos(y \cdot z)$ into $|u - v|$ transforms the integral into one of the type considered in the first section. We thus obtain the bound $c(2k+1)^{-2}$ and it can be shown that the contribution of the remainders in Hilb's formula is negligible. □

It is also possible to give pointwise estimates on the size of terms in the spherical harmonic expansions of piecewise smooth functions on the sphere, that are analogous to results in [5] and [6] on the Euclidean Fourier transform. The proofs are similar to the ones in the preceding theorem, but one has to use the method of stationary phase in the estimates of some integrals. Details can be found in [13]. These pointwise estimates suggest the convergence of $\sum_{k=0}^{+\infty} (\int_\Omega f(y) Z_k(x \cdot y) dy)$, which was proved by Weyl. In [1] there is an extension of Weyl's result to eigenfunction expansions in two dimensional compact manifolds. In dimension three the convergence may fail.

4. The non Euclidean plane. Following [3] and [4], we give a brief review of the non-Euclidean harmonic analysis in the Poincaré model of the hyperbolic plane. Let $D = \{z = x + iy; x^2 + y^2 < 1\}$ be the open unit disk in the complex plane, with Riemannian metric $ds^2 = (1 - x^2 - y^2)^{-2}(dx^2 + dy^2)$ and measure $dz = (1 - x^2 - y^2)^{-2} dx dy$. The group

$$\boldsymbol{SU}(1,1) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}; a\bar{a} - b\bar{b} = 1 \right\}$$

acts transitively on D by linear fractional transformations $z \mapsto (az + b)/(\bar{a}z + \bar{b})$, which preserves the Riemannian structure. One has the Cartan decomposition $\boldsymbol{SU}(1,1) = \boldsymbol{K}\boldsymbol{A}_+\boldsymbol{K}$, where

$$\boldsymbol{K} = \left\{ \begin{bmatrix} \exp(i\vartheta/2) & 0 \\ 0 & \exp(-i\vartheta/2) \end{bmatrix}; 0 \le \vartheta < 4\pi \right\},$$

$$\boldsymbol{A} = \left\{ \begin{bmatrix} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{bmatrix}; -\infty < t < +\infty \right\},$$

and $\boldsymbol{A}_+$ is the subset of $\boldsymbol{A}$ with $t \ge 0$. In the coordinates induced by the Cartan decomposition

$$\int_D f(z) dz = \frac{1}{16\pi} \int_0^{4\pi} \int_0^{+\infty} f(k(\vartheta)a(t) \cdot 0) \sinh(t) dt d\vartheta .$$

Let $b = \exp(i\varphi)$ be a point of the boundary $B = \partial D$. The complex powers of the Poisson kernel $\mathcal{P}^{\mu}(z, b) = ((1 - |z|^2)|z - b|^{-2})^{\mu}$ are eigenfunctions of the Laplace-Beltrami operator $\Delta = -(1 - x^2 - y^2)^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$, with eigenvalues $4\mu(1 - \mu)$. In particular,

$$\Delta \mathcal{P}^{(1+i\lambda)/2}(z, b) = (1 + \lambda^2)\mathcal{P}^{(1+i\lambda)/2}(z, b).$$

The spherical functions are defined by

$$\phi_\lambda(z) = \int_B \mathcal{P}^{(1+i\lambda)/2}(z, b)\frac{db}{2\pi},$$

and in coordinates induced by the Cartan decomposition, $\phi_\lambda(k(\vartheta)a(t) \cdot 0) = P_{-(1+i\lambda)/2}(\cosh(t))$, where $P_\mu(z)$ is the Legendre function of first kind with degree μ. When $0 \le t \le c < +\infty$, we have Hilb's formula

$$P_{-(1+i\lambda)/2}(\cosh(t)) = \sum_{n=0}^{m} a_n(t)\frac{J_n(\lambda t)}{(\lambda t)^n} + O(t^{2m+2}(\lambda t)^{-m-3/2}),$$

with $a_0(t) = \sqrt{t/\sinh(t)}, \dots$. See [11] and [8] for the asymptotic expansion of spherical functions on rank one symmetric spaces. In the sequel we shall need the formula

$$\int_B \mathcal{P}^{(1+i\lambda)/2}(z, b)\overline{\mathcal{P}^{(1+i\lambda)/2}(w, b)}\frac{db}{2\pi} = \phi_\lambda\left(\frac{z - w}{1 - z\bar{w}}\right).$$

For suitable functions on D one can define a non-Euclidean Fourier transform with an inversion formula and a Plancherel identity by

$$\hat{F}(\lambda, b) = \int_D g(z)\mathcal{P}^{(1-i\lambda)/2}(z, b)dz,$$

$$F(z) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_B \hat{F}(\lambda, b)\mathcal{P}^{(1+i\lambda)/2}(z, b)\lambda\tanh(\pi\lambda/2)dbd\lambda,$$

$$\left\{\int_D |F(z)|^2 dz\right\}^{1/2} = \left\{\frac{1}{2\pi}\int_0^{+\infty}\int_B |\hat{F}(\lambda, b)|^2\lambda\tanh(\pi\lambda/2)dbd\lambda\right\}^{1/2}.$$

As in the preceding sections, our purpose is to estimate the decay of the quadratic means $\{\int_B |\hat{F}(\lambda, b)|^2 db/2\pi\}^{1/2}$, where $F(z) = f(z)\chi_\Omega(z)$ with $f(z)$ smooth and $\chi_\Omega(z)$ the characteristic function of a bounded domain with smooth boundary.

THEOREM 4.1. *Let Ω be a bounded domain in D with smooth boundary and $f(z)$ a smooth function in D. Then,*

$$\left\{\int_B \left|\int_\Omega f(z)\mathcal{P}^{(1-i\lambda)/2}(z, b)dz\right|^2 \frac{db}{2\pi}\right\}^{1/2} \le c\lambda^{-3/2}.$$

PROOF. We have

$$\int_B \left| \int_\Omega f(z)\mathcal{P}^{(1-i\lambda)/2}(z,b)dz \right|^2 \frac{db}{2\pi}$$
$$= \int_\Omega \int_\Omega \int_B f(z)\overline{f(w)}\mathcal{P}^{(1-i\lambda)/2}(z,b)\overline{\mathcal{P}^{(1-i\lambda)/2}(w,b)}\frac{db}{2\pi}dzdw$$
$$= \int_\Omega \int_\Omega f(z)\overline{f(w)}\phi_{-\lambda}\left(\frac{z-w}{1-z\bar{w}}\right)dzdw\,.$$

If for fixed w we make the change of variables $(z-w)/(1-z\bar{w}) = u = k(\vartheta)a(t)\cdot 0$ and write $f(z)\chi_\Omega(z)\overline{f(w)}\chi_\Omega(w) = F(w,u)$, we obtain

$$\int_\Omega \int_\Omega f(z)\overline{f(w)}\phi_{-\lambda}\left(\frac{z-w}{1-z\bar{w}}\right)dzdw = \int_D \int_D F(w,u)\phi_{-\lambda}(u)dudw$$
$$= \frac{1}{16\pi}\int_D \int_0^{4\pi} \int_0^{+\infty} F(w,k(\vartheta)a(t)\cdot 0)P_{-(1+i\lambda)/2}(\cosh(t))\sinh(t)dtd\vartheta dw$$
$$= \int_D \int_0^{4\pi} \int_0^{+\infty} G(w,k(\vartheta)a(t)\cdot 0)J_0(\lambda t)tdtd\vartheta dw + \cdots$$

for an appropriate piecewise smooth function $G(w,k(\vartheta)a(t)\cdot 0)$. The integral is of the type considered in the first section, thus it is dominated by $c\lambda^{-3}$. □

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