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THE CLOSURE ORDERING OF ADJOINT NILPOTENT ORBITS IN $\mathfrak{so}(p,q)$

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Abstract. Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{so}(p,q)$ under the adjoint action of the full orthogonal group $\mathrm{O}(p,q)$. Then the closure of $\mathcal{O}$ (with respect to the Euclidean topology) is a union of $\mathcal{O}$ and some nilpotent $\mathrm{O}(p,q)$-orbits of smaller dimensions. In an earlier work, the first author has determined which nilpotent $\mathrm{O}(p,q)$-orbits belong to this closure. The same problem for the action of the identity component $\mathrm{SO}(p,q)^0$ of $\mathrm{O}(p,q)$ on $\mathfrak{so}(p,q)$ is much harder and we propose a conjecture describing the closures of the nilpotent $\mathrm{SO}(p,q)^0$-orbits. The conjecture is proved when $\min(p,q) \le 7$.

Our method is indirect because we use the Kostant-Sekiguchi correspondence to translate the problem to that of describing the closures of the unstable orbits for the action of the complex group $\mathrm{SO}_p(\boldsymbol{C}) \times \mathrm{SO}_q(\boldsymbol{C})$ on the space $M_{p,q}$ of complex $p \times q$ matrices with the action given by $(a,b) \cdot x = axb^{-1}$. The fact that the Kostant-Sekiguchi correspondence preserves the closure relation has been proved recently by Barbasch and Sepanski.

Introduction. For $p, q \ge 1$, we denote by $\mathfrak{g}_0 = \mathfrak{so}(p,q)$ the Lie algebra of the orthogonal group $G_0 = \mathrm{O}(p,q)$ and let $n = p+q$. We consider the adjoint action of G_0 on $\mathfrak{g}_0$ and the G_0-orbits in $\mathfrak{g}_0$ consisting of nilpotent matrices, to which we refer as the nilpotent G_0-orbits. Since the identity component $G_0^0 = \mathrm{SO}(p,q)^0$ of G_0 has index 4 in G_0, a G_0-orbit may be just a single G_0^0-orbit or it may split into two or four G_0^0-orbits.

There are only finitely many nilpotent G_0-orbits in $\mathfrak{g}_0$. The topological concepts, such as closure and connectedness, will refer to the ordinary Euclidean topology. The G_0^0-orbits contained in a given G_0-orbit are just its connected components. The closure of a nilpotent G_0-orbit in $\mathfrak{g}_0$ is a union of this orbit and some nilpotent G_0-orbits of smaller dimensions. For the description of these closures see [5], [6]. The same problem for the nilpotent G_0^0-orbits is much harder and still unresolved. This paper deals with that problem in an indirect manner, as we are going to explain next.

Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition where $\mathfrak{k}_0 = \mathfrak{so}(p) \times \mathfrak{so}(q)$ is the Lie algebra of the maximal compact subgroup $K_0 = \mathrm{O}(p) \times \mathrm{O}(q)$ of G_0. Let $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{p}$ be the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, and $\mathfrak{p}_0$, respectively, and $G = \mathrm{O}_n(\boldsymbol{C})$ and $K = \mathrm{O}_p(\boldsymbol{C}) \times \mathrm{O}_q(\boldsymbol{C})$ the complexifications of G_0 and K_0. By restricting the adjoint action of G on $\mathfrak{g}$, we obtain an action of K on $\mathfrak{g}$, and also on $\mathfrak{k}$ and $\mathfrak{p}$. There is a one-to-one correspondence between the nilpotent G_0-orbits in $\mathfrak{g}_0$ and the nilpotent K-orbits in $\mathfrak{p}$. This is a special case of the so

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called Kostant-Sekiguchi correspondence [4], [10]. It also gives a one-to-one correspondence between the nilpotent G_0^0-orbits in $\mathfrak{g}_0$ and the nilpotent K^0-orbits in $\mathfrak{p}$. (By K^0 we denote the identity component $\mathrm{SO}_p(\boldsymbol{C}) \times \mathrm{SO}_q(\boldsymbol{C})$ of K.) For more details concerning this correspondence in this concrete case see the Appendix. It was shown recently by Vergne [12] that, in the general case, the nilpotent orbits which correspond to each other under the Kostant-Sekiguchi correspondence are diffeomorphic manifolds.

There is a natural partial order "$\geq$" on nilpotent orbits: we write $\mathcal{O}_1 \geq \mathcal{O}_2$ if $\mathcal{O}_2$ is contained in the closure of $\mathcal{O}_1$. We note that, in the case of K^0-orbits in $\mathfrak{p}$, the closure with respect to the Euclidean topology coincides with the Zariski closure. (This is not true for G_0^0-orbits in $\mathfrak{g}_0$.) Very recently it was established by Barbasch and Sepanski [2] that, in the general case, the Kostant-Sekiguchi correspondence is an isomorphism of partially ordered sets with respect to the closure ordering. (In the special case that we are concerned with, this has been shown earlier by Ohta [8] for nilpotent G_0-orbits in $\mathfrak{g}_0$ and nilpotent K-orbits in $\mathfrak{p}$.) Hence our original problem of describing the closures of the nilpotent G_0^0-orbits in $\mathfrak{g}_0$ is equivalent to that of describing the closures of the nilpotent K^0-orbits in $\mathfrak{p}$. The paper deals explicitly with the latter problem.

We mention that, as a K-module, $\mathfrak{p}$ is isomorphic to the space $M_{p,q}$ of $p \times q$ complex matrices on which $K = \mathrm{O}_p(\boldsymbol{C}) \times \mathrm{O}_q(\boldsymbol{C})$ acts by $(a, b) \cdot x = axb^{-1}$. The nilpotent K^0-orbits in $\mathfrak{p}$ correspond to those K^0-orbits in $M_{p,q}$ whose closure contains the zero matrix. (Such orbits are known in the literature as unstable orbits.)

In Section 1 we recall the parametrization of the nilpotent K-orbits in $\mathfrak{p}$ by means of the so-called ab-diagrams. We also introduce a convenient labelling for the nilpotent K^0-orbits in $\mathfrak{p}$. An ab-diagram (for the orthogonal groups) is a Young diagram with n boxes whose rows are filled with alternating letters a's and b's, where rows of even length occur in pairs of the same length with one row of the pair having a as the first letter and the other row starting with the letter b. (For the definition of ab-diagram, refer to [8], [9], for example.) Two such diagrams are equivalent if one can be obtained from the other by permutation of rows. The total number of a's (resp. b's) has to be p (resp. q). We denote the set of equivalence classes of such ab-diagrams by $\mathcal{X}(p, q)$. This set parametrizes the nilpotent K-orbits in $\mathfrak{p}$. The nilpotent K-orbit that corresponds to $X \in \mathcal{X}(p, q)$ is denoted by $\mathcal{O}_X$. The closure ordering "$\geq$" on the set of nilpotent K-orbits in $\mathfrak{p}$ corresponds to a natural combinatorially defined partial order on $\mathcal{X}(p, q)$, which we denote again by "$\geq$". The Hasse diagram of these two isomorphic partially ordered sets is denoted by $\Gamma(p, q)$.

A vertex X of $\Gamma(p, q)$, i.e., an element of $\mathcal{X}(p, q)$, is called an a-vertex (resp. b-vertex) if every row of X of odd length has the letter b (resp. a) in its middle box. If X is an a-vertex and a b-vertex, we say that it is an ab-vertex. (This means that all rows of X have even length.) An a-vertex which is not a b-vertex is called a proper a-vertex, and one defines similarly proper b-vertices. A stable vertex is a vertex which is neither an a-vertex nor a b-vertex.

If X is a stable vertex, then $\mathcal{O}_X$ is a single K^0-orbit. If X is a proper a-vertex (resp. proper b-vertex), then $\mathcal{O}_X$ splits into two K^0-orbits which we denote by ${}^{\mathrm{I}}\mathcal{O}_X$ and ${}^{\mathrm{II}}\mathcal{O}_X$ (resp.

$\mathcal{O}^{\mathrm{I}}_X$ and $\mathcal{O}^{\mathrm{II}}_X$). If X is an ab-vertex, then $\mathcal{O}_X$ is the union of four K^0-orbits. We denote these orbits by ${}^{\mathrm{I}}\mathcal{O}^{\mathrm{I}}_X$, ${}^{\mathrm{I}}\mathcal{O}^{\mathrm{II}}_X$, ${}^{\mathrm{II}}\mathcal{O}^{\mathrm{I}}_X$, and ${}^{\mathrm{II}}\mathcal{O}^{\mathrm{II}}_X$. For the exact meaning of these superscripts we refer the reader to the main text. If X is an ab-vertex, we set ${}^{\mathrm{I}}\mathcal{O}_X = {}^{\mathrm{I}}\mathcal{O}^{\mathrm{I}}_X \cup {}^{\mathrm{I}}\mathcal{O}^{\mathrm{II}}_X$, ${}^{\mathrm{II}}\mathcal{O}_X = {}^{\mathrm{II}}\mathcal{O}^{\mathrm{I}}_X \cup {}^{\mathrm{II}}\mathcal{O}^{\mathrm{II}}_X$, and we define similarly $\mathcal{O}^{\mathrm{I}}_X$ and $\mathcal{O}^{\mathrm{II}}_X$.

In Section 2 we give a purely mechanical procedure for transforming the Hasse diagram $\Gamma(p, q)$ into a new diagram $\Delta(p, q)$ (see Definition 2), whose vertices are the nilpotent K^0-orbits in $\mathfrak{p}$. We define a new partial order "$\succeq$" on the set of these orbits by postulating that $\Delta(p, q)$ is its Hasse diagram. Then our conjecture can be simply stated that the partial order "$\succeq$" and the closure order "$\geq$" are the same.

In Section 3 we prove (Theorem 1) that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are two nilpotent K^0-orbits in $\mathfrak{p}$ and $\mathcal{O}_1 \succeq \mathcal{O}_2$, then also $\mathcal{O}_1 \geq \mathcal{O}_2$. The main tool that we employ to prove this theorem is Proposition 1 which is the symmetric space analogue of a result of Kostant [4, Lemma 4.1.4].

The next three sections deal with the converse of Theorem 1. While we are not able to prove the converse in general, we prove that it holds in many special cases.

In Section 4 we introduce the concept of pure pairs of ab-diagrams, viewed as vertices in $\Gamma(p, q)$. There are two types of them: the a-pairs and the b-pairs. An ordered pair of distinct a-vertices (X, Y) is called an a-pair if $X \geq Y$ and every vertex Z such that $X \geq Z \geq Y$ is an a-vertex. The b-pairs are defined similarly. We show that an a-pair cannot also be a b-pair. A pure pair is either an a-pair or a b-pair. We also introduce the concept of splitting for pure pairs. We say that an a-pair (resp. b-pair) (X, Y) splits if the closure of ${}^{\mathrm{I}}\mathcal{O}_X$ (resp. $\mathcal{O}^{\mathrm{I}}_X$) does not contain the entire orbit $\mathcal{O}_Y$. The main result of the section is Theorem 2 which asserts that the converse of Theorem 1 (and hence the conjecture itself) is valid provided that each pure pair splits.

In Section 5 we prove that the conjecture is true if $\min(p, q) \leq 7$ (Theorem 3). For that purpose we show that several infinite families of pure pairs split. Some of the required lemmas are in Section 6 which deals with some additional families of pure pairs.

We do not know how to describe explicitly all pure pairs. A pure pair (X, Y) is said to be minimal if $X \geq Z \geq Y$ implies that $Z = X$ or $Z = Y$. Two ab-diagrams are said to be disjoint if they have no common rows. It is possible to list all disjoint minimal pure pairs. There are 10 one- or two-parameter families of minimal disjoint b-pairs. They are listed in Table 8. The main result of Section 6 is that all minimal disjoint pure pairs split (Theorem 4).

In the Appendix we construct explicitly the real form $\mathfrak{g}_0 \cong \mathfrak{so}(p, q)$ of $\mathfrak{g}$ which is θ-stable, and provide an example illustrating the Kostant-Sekiguchi correspondence in this concrete case.

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1. Labelling of orbits. Let V be an n-dimensional complex vector space, $f : V \times V \to \boldsymbol{C}$ a nondegenerate symmetric bilinear form and $G = \mathrm{O}(V, f)$ the orthogonal group of the pair (V, f). We fix an involution $\theta \in G$ ($\theta \neq 1$), and denote by V_a (resp. V_b) the $+1$-eigenspace (resp. -1-eigenspace) of θ. Let $p = \dim(V_a)$ and $q = \dim(V_b)$. Since V_a and

V_b are orthogonal to each other, the restriction f_a (resp. f_b) of f to $V_a \times V_a$ (resp. $V_b \times V_b$) is nondegenerate. We shall denote by K the centralizer of θ in G, and by K^0 its identity component. Clearly, we have $K = K_a \times K_b$ where $K_a = \mathrm{O}(V_a, f_a)$ and $K_b = \mathrm{O}(V_b, f_b)$ are the orthogonal groups, and similarly $K^0 = K_a^0 \times K_b^0$ where $K_a^0 = \mathrm{SO}(V_a, f_a)$ and $K_b^0 = \mathrm{SO}(V_b, f_b)$ are the corresponding special orthogonal groups.

We denote by $\mathfrak{g} = \mathfrak{so}(V, f)$ the Lie algebra of G. It consists of all linear operators $u : V \to V$ such that $f(u(x), y) + f(x, u(y)) = 0$ for all $x, y \in V$. The Lie algebra $\mathfrak{k}$ of K is the centralizer of θ in $\mathfrak{g}$, i.e., $\mathfrak{k} = \{u \in \mathfrak{g} : u(V_a) \subset V_a,\ u(V_b) \subset V_b\}$. Thus $\mathfrak{k} = \mathfrak{k}_a \oplus \mathfrak{k}_b$ where $\mathfrak{k}_a = \mathfrak{so}(V_a, f_a)$ and $\mathfrak{k}_b = \mathfrak{so}(V_b, f_b)$. We denote by Ad (resp. ad) the adjoint representation of G (resp. $\mathfrak{g}$) on $\mathfrak{g}$. As a K-module (under the restriction of Ad), $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} = \{u \in \mathfrak{g} : u(V_a) \subset V_b,\ u(V_b) \subset V_a\}$.

We denote by $\mathcal{N}$ the nilpotent variety in $\mathfrak{p}$, i.e., $\mathcal{N} = \{u \in \mathfrak{p} : u^n = 0\}$. There are only finitely many K-orbits in $\mathcal{N}$ and they are parametrized by the so-called ab-diagrams.

One can find the definition of an ab-diagram in the literature, for example, [8], [9]. To parametrize nilpotent orbits in the orthogonal case, it is sufficient to treat ab-diagrams in the following meaning (cf. [8]). We define an ab-*diagram* to be a Young diagram with n boxes in which every box is filled by an a or a b so that the a's and the b's alternate along each row, and the rows of even length occur in pairs which are of the same length with one of them having a in the first box and the other b in the first box. Furthermore we require that the total number of a's in such a diagram be p (and consequently the number of b's is q). We say that two such ab-diagrams are *equivalent* if we can obtain one from the other by permuting rows. The nilpotent K-orbits in $\mathfrak{p}$ are in one-to-one correspondence with the equivalence classes of the ab-diagrams. From now on we shall consider equivalent ab-diagrams as being the same, i.e., we identify an ab-diagram with its equivalence class. We mention that the trivial orbit $\{0\}$ corresponds to the ab-diagram consisting of n rows of length 1, with boxes filled with p a's and q b's.

We shall write concrete ab-diagrams as a sequence of its rows. A row of length $2k+1$ with a (resp. b) in the first box will be written as $(ab)^k a$ (resp. $(ba)^k b$). The pair of rows of even length $2k$, one starting with a and the other with b, will be written as $(ab)^k$, $(ba)^k$. For instance the ab-diagram (1)

(1)

a	b	a	b	a	b	a
a	b	a	b			
b	a	b	a			
a	b	a				

will be written as $((ab)^3a, (ab)^2, (ba)^2, aba)$. If X and Y are arbitrary ab-diagrams, then we denote by $X + Y$ the ab-diagram obtained by writing Y below X and then rearranging the rows of this extended diagram.

We shall now describe how one can determine the ab-diagram that corresponds to the nilpotent K-orbit containing a given nilpotent element $u \in \mathcal{N}$(for the details, see [9]). Let us define a *Jordan chain* for u to be a sequence of nonzero vectors $v_1, v_2, \dots, v_k$ such that $u(v_i) = v_{i+1}$ for $1 \leq i < k$ and $u(v_k) = 0$. We say that k is the *length* of this chain, and that v_1 (resp. v_k) is the *top* (resp. *bottom*) *vector* of this chain. If moreover each $v_i \in V_a \cup V_b$ then we say that this Jordan chain is *graded.* By replacing each v_i by the letter a if $v_i \in V_a$ and by b if $v_i \in V_b$, we obtain an alternating sequence of these letters to which we refer as the *type* of this graded Jordan chain. A Jordan chain for u is said to be *maximal* if it cannot be extended to a larger one. This is the case if and only if the top vector of the chain is not contained in the image of u. A *graded Jordan basis* for u is a basis of V consisting of graded Jordan chains for u (necessarily maximal). They always exist. Let us choose one of them. Then we form the Young diagram by creating a row of length k for each maximal Jordan chain of length k, say $v_1, \dots, v_k$, contained in this basis. We temporarily fill the boxes of this row (successively from the left to the right) by the vectors $v_1, \dots, v_k$. Finally we replace each of the vectors, say v, in the resulting diagram by the letter a if $v \in V_a$ and by b if $v \in V_b$. We obtain an ab-diagram which is independent (up to equivalence) of the choice of the graded Jordan basis for u.

We fix, from now on, a basis $\{e_0, e_1, \dots, e_{p-1}\}$ of V_a and a basis $\{e'_0, e'_1, \dots, e'_{q-1}\}$ of V_b such that $f(e_i, e_j) = \delta_{i+j,p-1}$ for $0 \leq i, j < p$ and $f(e'_i, e'_j) = \delta_{i+j,q-1}$ for $0 \leq i, j < q$, where δ_{ij} is Kronecker's delta and we identify linear operators on V with their matrices with respect to this basis. (By definition, $\delta_{i,j}$ is 0 if $i \neq j$ and 1 if $i = j$.) The diagonal matrices in $\mathfrak{k}_a$ form a Cartan subalgebra $\mathfrak{h}_a$. These diagonal matrices have the form $h_a = \mathrm{diag}(h_0, h_1, \dots, h_{p-1})$ where $h_i + h_{p-1-i} = 0$ for $0 \leq i < p$. The centralizer of $\mathfrak{h}_a$ in K_a^0 is the maximal torus T_a which consists of all diagonal matrices in K_a^0. We denote by N_a the normalizer of T_a (or $\mathfrak{h}_a$) in K_a. The Weyl group of $(\mathfrak{k}_a, \mathfrak{h}_a)$ is $W_a = (N_a \cap K_a^0)/T_a$. We set $W_a^* = N_a/T_a$. Clearly W_a is a normal subgroup of W_a^* and the quotient group W_a^*/W_a is trivial if p is odd, and has order 2 if p is even. We introduce the real form $(\mathfrak{h}_a)_{\boldsymbol{R}}$ of $\mathfrak{h}_a$ consisting of all matrices h_a as above with $h_i \in \boldsymbol{R}$ for all $0 \leq i < p$. We define the closed Weyl chamber $C_a \subset (\mathfrak{h}_a)_{\boldsymbol{R}}$ by the inequalities

$$h_i \geq h_{i+1}, \quad 0 \leq i \leq k-1\,, \tag{2}$$

if $p = 2k + 1$ is odd, and by

$$h_i \geq h_{i+1}, \quad 0 \leq i < k-2 \tag{3}$$

and

$$h_{k-2} \geq |h_{k-1}|\,, \tag{4}$$

if $p = 2k$ is even. If p is odd we set $C_a^* = C_a$ while for $p = 2k$ even we define $C_a^* \subset (\mathfrak{h}_a)_{\boldsymbol{R}}$ by the inequalities (3) above and

$$h_{k-2} \geq h_{k-1} \geq 0\,. \tag{5}$$

We define similarly $\mathfrak{h}_b$, T_b, etc. and we set $\mathfrak{h} = \mathfrak{h}_a \times \mathfrak{h}_b$, $T = T_a \times T_b$, etc.

Given an ab-diagram X, we denote by $\mathcal{O}_X$ the corresponding nilpotent K-orbit in $\mathcal{N}$. If $\mathcal{O}_X$ is not trivial, there exists a unique element $H_X \in C^* = C_a^* \times C_b^*$ such that $[H_X, E_X] = 2E_X$ for some nonzero element $E_X \in \mathcal{O}_X$. We shall refer to this element H_X as the *characteristic* of $\mathcal{O}_X$ (or of X). If $\mathcal{O}_X$ is the trivial orbit we define $H_X = 0$. It is well known that different K-orbits in $\mathcal{N}$ have different characteristics. We denote by $(H_X)_a$ (resp. $(H_X)_b$) the component of H_X in $\mathfrak{h}_a$ (resp. $\mathfrak{h}_b$).

The eigenvalues (i.e., the diagonal entries) of $(H_X)_a$ and $(H_X)_b$ can be easily determined. For this purpose we insert in each box of X an integer as follows: if a row has length k then we insert successively in the boxes of that row the integers

$$k-1\,, \quad k-3\,, \quad k-5, \ldots\,, \quad 5-k\,, \quad 3-k\,, \quad 1-k\,.$$

Then the integers written in all a-boxes (resp. b-boxes) are the eigenvalues of $(H_X)_a$ (resp. $(H_X)_b$). The order in which these eigenvalues are written on the diagonal is determined uniquely by the condition that $H_X \in C^* = C_a^* \times C_b^*$.

We shall refer to the K^0-orbits in $\mathcal{N} \subset \mathfrak{p}$ as the *strict nilpotent orbits*. The K^0-orbits contained in $\mathcal{O}_X$ are just the connected components of $\mathcal{O}_X$. The group W^*/W permutes transitively these components and so the number of these components is 1, 2, or 4. The element E_X, as described above, is not unique but all such elements lie in the same connected component of $\mathcal{O}_X$. If $p = 2k$ is even let $x_a \in N_a$ be the linear operator which interchanges the vectors e_{k-1} and e_k and fixes all the other e_i's. If q is even we define $x_b \in N_b$ similarly.

DEFINITION 1 (labelling of K^0-orbits in $\mathcal{N}$). We introduce the following notation for the connected components of the K-orbit $\mathcal{O}_X \subset \mathcal{N}$ by considering four possibilities:

(i) Both $(H_X)_a$ and $(H_X)_b$ have 0 eigenvalues: Then $\mathcal{O}_X$ is connected and we do not need any new notation.

(ii) $(H_X)_a$ has no 0 eigenvalue but $(H_X)_b$ does: Then p is even and $\mathcal{O}_X$ has two connected components. The component containing the element E_X will be denoted by ${}^{\mathrm{I}}\mathcal{O}_X$, and the other one by ${}^{\mathrm{II}}\mathcal{O}_X = \mathrm{Ad}(x_a)({}^{\mathrm{I}}\mathcal{O}_X)$.

(iii) $(H_X)_a$ has a 0 eigenvalue but $(H_X)_b$ does not: Then q is even and again $\mathcal{O}_X$ has two connected components. The component containing the element E_X will be denoted by $\mathcal{O}_X^{\mathrm{I}}$ and the other one by $\mathcal{O}_X^{\mathrm{II}} = \mathrm{Ad}(x_b)(\mathcal{O}_X^{\mathrm{I}})$.

(iv) H_X has no 0 eigenvalue: Then both p and q must be even and $\mathcal{O}_X$ has four connected components. The one containing the representative E_X is denoted by ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$ and the remaining three are ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}} = \mathrm{Ad}(x_a)({}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}})$, ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}} = \mathrm{Ad}(x_b)({}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}})$, and ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}} = \mathrm{Ad}(x_a x_b)({}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}})$.

Note that if p (resp. q) is odd then the left (resp. right) superscripts I, II are not used. In particular if p and q are odd then all K-orbits in $\mathcal{N}$ are connected. Let us also introduce the *characteristics* for the K^0-orbits in $\mathcal{N}$. The characteristic H_X of $\mathcal{O}_X$ is not changed in case (i), it becomes the characteristic of ${}^{\mathrm{I}}\mathcal{O}_X$ in case (ii), the characteristic of $\mathcal{O}_X^{\mathrm{I}}$ in case (iii), and the characteristic of ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$ in case (iv). In case (ii), the characteristic of ${}^{\mathrm{II}}\mathcal{O}_X$ is $\mathrm{Ad}(x_a)(H_X)$. In case (iii), the characteristic of $\mathcal{O}_X^{\mathrm{II}}$ is $\mathrm{Ad}(x_b)(H_X)$. Finally, in case (iv), the characteristics of the orbits ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}}$, ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}}$, ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}}$ are $\mathrm{Ad}(x_a)(H_X)$, $\mathrm{Ad}(x_b)(H_X)$, $\mathrm{Ad}(x_a x_b)(H_X)$, respectively. All these characteristics belong to the closed Weyl chamber $C = C_a \times C_b$, and different orbits have different characteristics.

Note also that the left (resp. right) superscripts I and II depend on the choice of the basis $\{e_i\}$ of V_a (resp. $\{e_i'\}$ of V_b). If $p = 2k$ is even then there are exactly two K_a^0-orbits of maximal isotropic subspaces of V_a and the left superscripts I, II depend on the orbit to which the subspace spanned by $\{e_0, \dots, e_{k-1}\}$ belongs. If this subspace is chosen from a different orbit, then the left superscripts I and II get interchanged. The same phenomenon occurs with the right superscripts when q is even.

We conclude this section with an illustrative example.

EXAMPLE 1. Let $Z = ((ab)^3 a, aba)$. Then $p = 6$, $q = 4$ and we find that

$$(H_Z)_a = \mathrm{diag}(6, 2, 2, -2, -2, -6), \quad (H_Z)_b = \mathrm{diag}(4, 0, 0, -4).$$

As a representative E_Z of $\mathcal{O}_Z$ satisfying $[H_Z, E_Z] = 2E_Z$, we can choose the linear operator defined by:

$$\begin{aligned} &e_0 \to 0, \quad e_1 \to -e_0', \quad e_i \to -e_{i-2}' \ (2 \le i \le 5), \\ &e_i' \to e_i \ (0 \le i \le 2), \quad e_3' \to e_3 + e_4. \end{aligned}$$

In terms of matrices we have

$$E_Z = \begin{pmatrix} 0 & & & & & & 1 & 0 & 0 & 0 \\ & 0 & & & & & 0 & 1 & 0 & 0 \\ & & 0 & & & & 0 & 0 & 1 & 0 \\ & & & 0 & & & 0 & 0 & 0 & 1 \\ & & & & 0 & & 0 & 0 & 0 & 1 \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{1} & \bar{1} & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & \bar{1} & 0 & 0 & & 0 & & \\ 0 & 0 & 0 & 0 & \bar{1} & 0 & & & 0 & \\ 0 & 0 & 0 & 0 & 0 & \bar{1} & & & & 0 \end{pmatrix},$$

where $\bar{1}$ stands for -1 and the suppressed entries are zeroes.

As the graded Jordan chains for E_Z we can take

$$\begin{aligned} &e_5 \to -e_3' \to -e_4 - e_3 \to e_2' + e_1' \to e_2 + e_1 \to -2e_0' \to -2e_0 \to 0, \\ &e_4 - e_3 \to e_1' - e_2' \to e_1 - e_2 \to 0. \end{aligned}$$

Note that these chains indeed have the types $(ab)^3 a$ and aba, respectively.

Since $(H_Z)_a$ has no 0 eigenvalue, while $(H_Z)_b$ does, the nilpotent K-orbit $\mathcal{O}_Z$ has two connected components: ${}^{\mathrm{I}}\mathcal{O}_Z$ and ${}^{\mathrm{II}}\mathcal{O}_Z$. The characteristic of ${}^{\mathrm{I}}\mathcal{O}_Z$ is H_Z (the same as the characteristic of $\mathcal{O}_Z$). The characteristic of ${}^{\mathrm{II}}\mathcal{O}_Z$ has diag$(6, 2, -2, 2, -2, -6)$ as its $\mathfrak{h}_a$-component, while its $\mathfrak{h}_b$-component is the same as that of H_Z.

The element E_Z belongs to the orbit ${}^{\mathrm{I}}\mathcal{O}_Z$. As a representative of the orbit ${}^{\mathrm{II}}\mathcal{O}_Z$ we can take the element $\mathrm{Ad}(x_a)(E_Z)$. Its action on the basis vectors is given by

$$\begin{aligned}
&e_0 \to 0, \quad e_1 \to -e_0', \quad e_2 \to -e_1', \quad e_3 \to -e_0', \quad e_4 \to -e_2', \quad e_5 \to -e_3', \\
&e_0' \to e_0, \quad e_1' \to e_1, \quad e_2' \to e_3, \quad e_3' \to e_2 + e_4,
\end{aligned}$$

or in terms of matrices

$$\mathrm{Ad}(x_a)(E_Z) = \begin{pmatrix}
0 & & & & & & 1 & 0 & 0 & 0 \\
& 0 & & & & & 0 & 1 & 0 & 0 \\
& & 0 & & & & 0 & 0 & 0 & 1 \\
& & & 0 & & & 0 & 0 & 1 & 0 \\
& & & & 0 & & 0 & 0 & 0 & 1 \\
& & & & & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{1} & 0 & \bar{1} & 0 & 0 & 0 & & & \\
0 & 0 & \bar{1} & 0 & 0 & 0 & & 0 & & \\
0 & 0 & 0 & 0 & \bar{1} & 0 & & & 0 & \\
0 & 0 & 0 & 0 & 0 & \bar{1} & & & & 0
\end{pmatrix}.$$

This representative has a graded Jordan basis consisting of two chains

$$\begin{aligned}
&e_5 \to -e_3' \to -e_4 - e_2 \to e_2' + e_1' \to e_3 + e_1 \to -2e_0' \to -2e_0 \to 0, \\
&e_4 - e_2 \to e_1' - e_2' \to e_1 - e_3 \to 0.
\end{aligned}$$

2. Closure ordering conjecture. Let $\mathcal{X} = \mathcal{X}(p, q)$ denote the set (of equivalence classes) of ab-diagrams with n $(= p + q)$ boxes, p a's, and q b's. If $X \in \mathcal{X}$, we denote by X' the diagram obtained from X by deleting the first column. We set $X^{(0)} = X$ and define recursively $X^{(k+1)} = (X^{(k)})'$ for $k \geq 0$. In particular, $X^{(1)} = X'$. For any such diagram Y we shall denote by $n_a(Y)$ (resp. $n_b(Y)$) the number of a's (resp. b's) in Y. Clearly, if $X \in \mathcal{X}$ then $n_a(X) = p$ and $n_b(X) = q$. For $X, Y \in \mathcal{X}$ we write $X \geq Y$ if $n_a(X^{(k)}) \geq n_a(Y^{(k)})$ and $n_b(X^{(k)}) \geq n_b(Y^{(k)})$ for all $k \geq 0$. The relation "$\geq$" makes $\mathcal{X}$ into a partially ordered set.

Let $\mathcal{N}/K$ (resp. $\mathcal{N}/K^0$) denote the set of K-orbits (resp. K^0-orbits) in $\mathcal{N}$. If $\mathcal{O}_1, \mathcal{O}_2$ are members of $\mathcal{N}/K$ (or $\mathcal{N}/K^0$) and $\mathcal{O}_2$ is contained in the closure of $\mathcal{O}_1$, then we shall write $\mathcal{O}_1 \geq \mathcal{O}_2$. This defines a partial order on $\mathcal{N}/K$ (resp. $\mathcal{N}/K^0$) called the *closure ordering*. It is a known fact that the partially ordered sets $(\mathcal{X}, \geq)$ and $(\mathcal{N}/K, \geq)$ are isomorphic, and that an isomorphism is provided by the map that sends X to $\mathcal{O}_X$.

The description of the closure ordering in $\mathcal{N}/K^0$ is not known at present, and our main objective is to propose a conjecture in this regard and to provide some evidence for its validity. Before stating the conjecture we need to introduce a few more definitions.

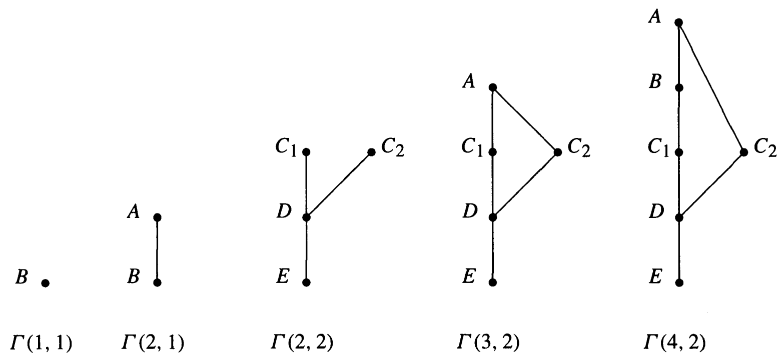


FIGURE 1.

If $X, Y \in \mathcal{X} = \mathcal{X}(p, q)$ are distinct and $X \geq Y$ then we shall write $X > Y$. We define similarly the relation ">" in the partially ordered sets $(\mathcal{N}/K, \geq)$ and $(\mathcal{N}/K^0, \geq)$. If $X, Y \in \mathcal{X}$ are such that $X > Y$ and there is no $Z \in \mathcal{X}$ such that $X > Z > Y$, then we shall write $X \to Y$. The finite partially ordered set $(\mathcal{X}, \geq)$ will be represented by its Hasse diagram $\Gamma = \Gamma(p, q)$. Each $X \in \mathcal{X}$ is represented by a node in Γ. If $X \to Y$ for some $X, Y \in \mathcal{X}$, then the node X is placed in Γ higher than the node Y and these two nodes are joined by a line. The Hasse diagram of $(\mathcal{N}/K, \geq)$ is essentially the same as Γ. We just have to replace each node $X \in \mathcal{X}$ by the corresponding node $\mathcal{O}_X \in \mathcal{N}/K$.

EXAMPLE 2. We display in Figure 1 the diagrams $\Gamma(p, 1)$ for $p = 1, 2$ and $\Gamma(p, 2)$ for $p = 2, 3, 4$.

In $\Gamma(1, 1)$ we have $B = (a, b)$ and, in $\Gamma(2, 1)$, $A = (aba)$ and $B = (a, a, b)$. For $p > 2$ the diagram $\Gamma(p, 1)$ is the same as $\Gamma(2, 1)$ except that $A = (aba, a^{p-2})$ and $B = (a^p, b)$. For simplicity we write a^k for the sequence $(a, a, \ldots, a)$ consisting of k letters a, and we shall use b^k in a similar sense. In Table 1 we list the vertices X of $\Gamma(p, 2)$, the corresponding partitions π_X, and the complex dimensions of the orbits $\mathcal{O}_X$. If E_X is a representative of the orbit $\mathcal{O}_X$, then it is known (see [4, Remark 9.5.2]) that

$$\dim_{\mathbf{C}}(\mathcal{O}_X) = \frac{1}{2}\dim_{\mathbf{C}}(G \cdot E_X).$$

The complex dimension of the orbit $G \cdot E_X$ can be computed by the formula for the dimension of the centralizer of E_X in $\mathfrak{g}$ given in [4, p. 399] (see also [4, Corollary 6.1.4]). The labels for vertices of different diagrams $\Gamma(p, 2)$ have been chosen so that $X \in \Gamma(p, 2)$ and $(X, a) \in \Gamma(p + 1, 2)$ have the same label. For $p > 4$ the diagram $\Gamma(p, 2)$ is the same as $\Gamma(4, 2)$.

EXAMPLE 3. We display in Figure 2 the diagrams $\Gamma(p, 3)$ for $p = 3, 4, 5, 6$. In Table 2 we list only the vertices X of $\Gamma(6, 3)$, the corresponding partitions π_X, and the complex dimensions of the orbits $\mathcal{O}_X$. For $p > 6$ the diagram $\Gamma(p, 3)$ is the same as $\Gamma(6, 3)$.

TABLE 1. Vertices of $\Gamma(p, 2)$.

p	label	X	π_X	dim
2	C_1	aba, b	$3 \cdot 1$	2
	C_2	bab, a	$3 \cdot 1$	2
	D	ab, ba	2^2	1
	E	a^2, b^2	1^4	0
3	A	$(ab)^2a$	5	4
	C_1	aba, a, b	$3 \cdot 1^2$	3
	C_2	bab, a, a	$3 \cdot 1^2$	3
	D	ab, ba, a	$2^2 \cdot 1$	2
	E	a^3, b^2	1^5	0
4	A	$(ab)^2a, a$	$5 \cdot 1$	6
	B	aba, aba	3^2	5
	C_1	aba, a^2, b	$3 \cdot 1^3$	4
	C_2	bab, a^3	$3 \cdot 1^3$	4
	D	ab, ba, a^2	$2^2 \cdot 1^2$	3
	E	a^4, b^2	1^6	0

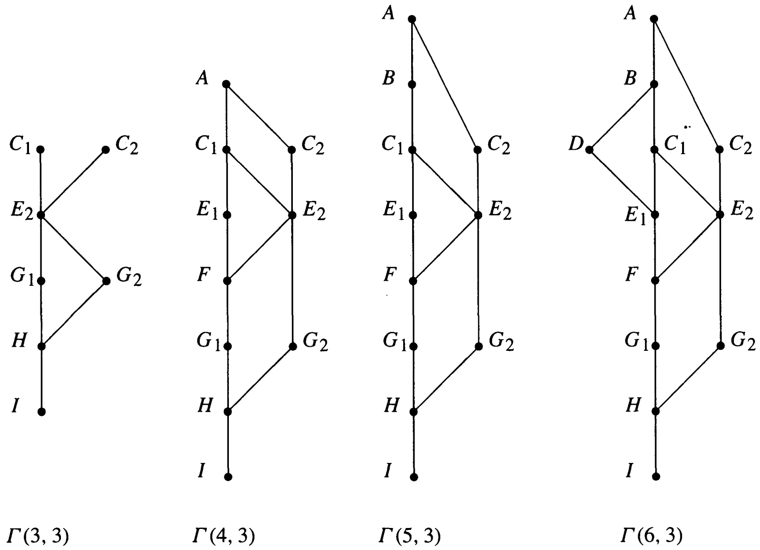


FIGURE 2.

We say that a vertex X of Γ is *stable* (resp. *unstable*) if the K-orbit $\mathcal{O}_X$ is connected (resp. disconnected). An unstable vertex X is an *a-vertex* (resp. *b-vertex*) if the linear operator $(H_X)_a$ (resp. $(H_X)_b$) is nonsingular (i.e., has no 0 eigenvalue). Equivalently, X is an a-vertex (resp. b-vertex) if the middle letter of each row of odd length (if any) in X is b (resp. a). If X is both an a-vertex and a b-vertex, then we shall say that it is an *ab-vertex*. Thus X is an ab-vertex if and only if it has no rows of odd length, i.e., the corresponding partition π_X

TABLE 2. Vertices of $\Gamma(6,3)$.

label	X	π_X	dim
A	$(ab)^3a, a^2$	$7 \cdot 1^2$	15
B	$(ab)^2a, aba, a$	$5 \cdot 3 \cdot 1$	14
C_1	$(ab)^2a, a^3, b$	$5 \cdot 1^4$	12
C_2	$(ba)^2b, a^4$	$5 \cdot 1^4$	12
D	aba, aba, aba	3^3	12
E_1	aba, aba, a^2, b	$3^2 \cdot 1^3$	11
E_2	aba, bab, a^3	$3^2 \cdot 1^3$	11
F	aba, ab, ba, a^2	$3 \cdot 2^2 \cdot 1^2$	10
G_1	aba, a^4, b^2	$3 \cdot 1^6$	7
G_2	bab, a^5, b	$3 \cdot 1^6$	7
H	ab, ba, a^4, b	$2^2 \cdot 1^5$	6
I	a^6, b^3	1^9	0

is very even (in the terminology of [4, Theorem 5.1.4]). An a-vertex that is not a b-vertex will be called a *proper a-vertex* (or a *proper vertex of type a*). One defines similarly a *proper b-vertex* (or a *proper vertex of type b*).

DEFINITION 2 (of the diagram Δ). We denote by $\Delta = \Delta(p,q)$ the diagram which is obtained from $\Gamma = \Gamma(p,q)$ by the following modifications in three steps:

Step 1: For every vertex pair (X, Y) such that $X \to Y$ and X or Y is unstable erase the line in Γ joining X to Y.

Step 2: Replace each node X by as many nodes as there are connected components in $\mathcal{O}_X$ and label them by these components.

Step 3: Insert 2 or 4 lines for each line that was erased in Step 1. For this purpose we reconsider all pairs (X, Y) from Step 1 and distinguish ten cases.

(i) X is stable and Y is unstable: Then we join $\mathcal{O}_X$ to each of the nodes corresponding to the connected components of $\mathcal{O}_Y$.

(ii) X is unstable and Y is stable: Then we join each of the nodes corresponding to the connected components of $\mathcal{O}_X$ to $\mathcal{O}_Y$.

(iii) X is a proper a-vertex and Y a proper b-vertex: Then we join each of the nodes ${}^{\mathrm{I}}\mathcal{O}_X$, ${}^{\mathrm{II}}\mathcal{O}_X$ to each of $\mathcal{O}_Y^{\mathrm{I}}$, $\mathcal{O}_Y^{\mathrm{II}}$.

(iv) X is a proper b-vertex and Y a proper a-vertex: Then we join each of the nodes $\mathcal{O}_X^{\mathrm{I}}$, $\mathcal{O}_X^{\mathrm{II}}$ to each of ${}^{\mathrm{I}}\mathcal{O}_Y$, ${}^{\mathrm{II}}\mathcal{O}_Y$.

(v) X and Y are proper a-vertices: Then we join ${}^{\mathrm{I}}\mathcal{O}_X$ to ${}^{\mathrm{I}}\mathcal{O}_Y$, and ${}^{\mathrm{II}}\mathcal{O}_X$ to ${}^{\mathrm{II}}\mathcal{O}_Y$.

(vi) X and Y are proper b-vertices: Then we join $\mathcal{O}_X^{\mathrm{I}}$ to $\mathcal{O}_Y^{\mathrm{I}}$, and $\mathcal{O}_X^{\mathrm{II}}$ to $\mathcal{O}_Y^{\mathrm{II}}$.

(vii) X is a proper a-vertex and Y an ab-vertex: Then we join ${}^{\mathrm{I}}\mathcal{O}_X$ to the nodes ${}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{I}}$, ${}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{II}}$, and ${}^{\mathrm{II}}\mathcal{O}_X$ to ${}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{I}}$, ${}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{II}}$.

(viii) X is a proper b-vertex and Y an ab-vertex: Then we join $\mathcal{O}_X^{\mathrm{I}}$ to the nodes ${}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{I}}$, ${}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{I}}$, and $\mathcal{O}_X^{\mathrm{II}}$ to ${}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{II}}$, ${}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{II}}$.

(ix) X is an ab-vertex and Y a proper a-vertex: Then we join the nodes ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$, ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}}$ to ${}^{\mathrm{I}}\mathcal{O}_Y$, and ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}}$, ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}}$ to ${}^{\mathrm{II}}\mathcal{O}_Y$.

(x) X is an ab-vertex and Y a proper b-vertex: Then we join the nodes ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$, ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}}$ to

$\mathcal{O}^{\mathrm{I}}_{Y}$, and ${}^{\mathrm{I}}\mathcal{O}^{\mathrm{II}}_{X}$, ${}^{\mathrm{II}}\mathcal{O}^{\mathrm{II}}_{X}$ to $\mathcal{O}^{\mathrm{II}}_{Y}$.

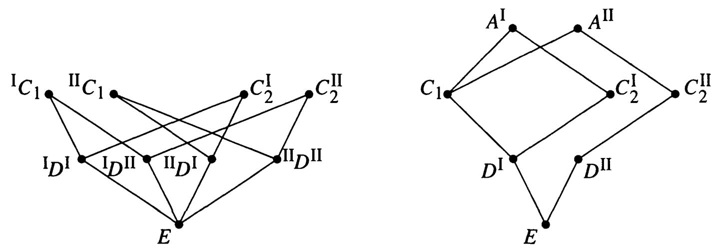
We remark that if X and Y are ab-vertices, then $X \not\to Y$ (i.e., $X \to Y$ does not hold). We can now state our main conjecture.

CONJECTURE. *The above defined diagram $\Delta = \Delta(p, q)$ is the Hasse diagram of the partially ordered set $(\mathcal{N}/K^0, \geq)$.*

In addition to the closure ordering "$\geq$" on $\mathcal{N}/K^0$, we now introduce the new partial order "$\succeq$" on the same set $\mathcal{N}/K^0$. It is defined by postulating that its Hasse diagram is $\Delta = \Delta(p, q)$. Our conjecture can be reformulated as follows: The two partial orders "$\geq$" and "$\succeq$" are the same.

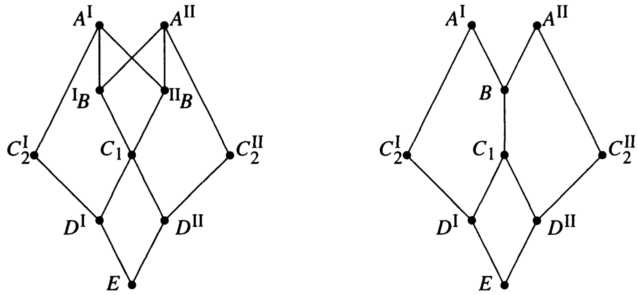
EXAMPLE 4. In order to illustrate Definition 2, we display in Figures 3 and 4 the diagrams $\Delta(p, 2)$ for $p = 2, 3, 4, 5$. For the sake of simplicity we write X instead of $\mathcal{O}_X$. For $p > 5$ the diagram $\Delta(p, 2)$ is identical to $\Delta(5, 2)$.

3. Comparison of two partial orders. Let $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{N}$ be two K^0-orbits. In order to prove our conjecture we have to show that $\mathcal{O}_1 \succeq \mathcal{O}_2$ holds true if and only if $\mathcal{O}_1 \geq \mathcal{O}_2$ does. Our objective in this section is to prove that the former condition implies the latter.



$\Delta(2, 2)$ $\Delta(3, 2)$

FIGURE 3.



$\Delta(4, 2)$ $\Delta(5, 2)$

FIGURE 4.

THEOREM 1. *If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{N}/K^0$ and $\mathcal{O}_1 \succeq \mathcal{O}_2$ then $\mathcal{O}_1 \geq \mathcal{O}_2$.*

The proof will be given later in this section.

We say that (E, H, F) is a *standard triple* if $\{E, H, F\} \subset \mathfrak{g}$, $E \neq 0$, and

$$[H, E] = 2E\,, \quad [H, F] = -2F\,, \quad [F, E] = H\,.$$

(Our definition is different from that in [4] where the last relation is replaced by $[E, F] = H$.) We refer to H as the *neutral element* of this triple. A standard triple (E, H, F) is called a *normal triple* if $H \in \mathfrak{k}$ and $E, F \in \mathfrak{p}$.

If H is the neutral element of a normal triple, we define

$$\begin{aligned} &\mathfrak{g}_i(H) = \{x \in \mathfrak{g} : [H, x] = ix\}, \quad i \in \mathbf{Z}\,, \\ &\mathfrak{k}_i(H) = \mathfrak{k} \cap \mathfrak{g}_i(H)\,, \quad \mathfrak{p}_i(H) = \mathfrak{p} \cap \mathfrak{g}_i(H)\,, \\ &\mathfrak{q}(H) = \sum_{i \geq 0} \mathfrak{k}_i(H)\,, \quad \mathfrak{s}_i(H) = \sum_{j \geq i} \mathfrak{p}_j(H)\,. \end{aligned}$$

By $Z_K(H)^0$ we denote the identity component of the centralizer of H in K. Our proof of Theorem 1 is based on the following proposition, which is a symmetric space analogue of [4, Lemma 4.1.4] due to Kostant.

PROPOSITION 1. *Let (E, H, F) be a normal triple and let Q be the parabolic subgroup of K^0 with Lie algebra $\mathfrak{q}(H)$. Then $(K^0 \cdot E) \cap \mathfrak{s}_2(H)$ is a dense open subset of $\mathfrak{s}_2(H)$ and*

$$(K \cdot E) \cap \mathfrak{s}_2(H) = (K^0 \cdot E) \cap \mathfrak{s}_2(H) = Q \cdot E = (Z_K(H)^0 \cdot E) + \mathfrak{s}_3(H)\,.$$

Consequently, $Z_K(H)^0 \cdot E$ is a dense open subset of $\mathfrak{p}_2(H)$ and

$$(K \cdot E) \cap \mathfrak{p}_2(H) = (K^0 \cdot E) \cap \mathfrak{p}_2(H) = Z_K(H)^0 \cdot E\,.$$

PROOF. In this proof we use the Zariski topology. The unipotent radical U of Q has $\mathfrak{u} := \mathfrak{k}_1(H) + \mathfrak{k}_2(H) + \cdots$ as its Lie algebra. Note that $Z_K(H)^0$ is a Levi factor of Q. As $[E, \mathfrak{u}] = \mathfrak{s}_3(H)$, we have $\dim \mathfrak{s}_3(H) = \dim \mathfrak{u} - \dim Z_{\mathfrak{u}}(E)$, where $Z_{\mathfrak{u}}(E)$ is the centralizer of E in $\mathfrak{u}$. Hence $\dim(U \cdot E) = \dim \mathfrak{s}_3(H)$. As $U \cdot E \subset E + \mathfrak{s}_3(H)$ and $U \cdot E$ is closed (see [11, Section 2.5, Proposition] or [7, Satz 4, p. 154]), we conclude that $U \cdot E = E + \mathfrak{s}_3(H)$. By acting with $Q = Z_K(H)^0 U$, we deduce that $Q \cdot E = (Z_K(H)^0 \cdot E) + \mathfrak{s}_3(H)$.

Since $[E, \mathfrak{q}(H)] = \mathfrak{s}_2(H)$, $Q \cdot E$ is a dense open subset of $\mathfrak{s}_2(H)$. Now let $x \in (K \cdot E) \cap \mathfrak{s}_2(H)$. Then $\dim Z_{\mathfrak{g}}(x) = \dim Z_{\mathfrak{g}}(E)$. As $Z_{\mathfrak{g}}(E) = Z_{\mathfrak{q}}(E)$, we have $\dim Z_{\mathfrak{q}}(x) \leq \dim Z_{\mathfrak{q}}(E)$ and consequently $\dim(Q \cdot x) \geq \dim(Q \cdot E)$. Hence $Q \cdot x$ is also a dense open subset of $\mathfrak{s}_2(H)$. It follows that $Q \cdot x = Q \cdot E$. In particular $x \in Q \cdot E$. We have shown that $(K \cdot E) \cap \mathfrak{s}_2(H) = Q \cdot E$. □

We represent a linear operator L on V by its matrix, which we also denote by L. It will be convenient to partition this matrix as follows:

$$L = \begin{pmatrix} L_a & L_{ab} \\ L_{ba} & L_b \end{pmatrix},$$

where L_a (resp. L_b) is a square block of size p (resp. q). Let S_k denote the matrix of order k whose (i, j)-th entry is $\delta_{i+j,k+1}$ for $i, j = 1, 2, \dots, k$. We have $L \in \mathfrak{p}$ if and only if $L_a = 0$, $L_b = 0$, and $L_{ba} = -S_q\,{}^t L_{ab} S_p$. (By ${}^t X$ we denote the transpose of a matrix X.) Hence a matrix $L \in \mathfrak{p}$ is uniquely determined by its ab-block L_{ab}. Equivalently, a linear operator $L \in \mathfrak{p}$ is uniquely determined by the images $L(e_i') \in V_a$ of the basis vectors e_i' of V_b.

If $L \in \mathfrak{p}$, then the subspaces V_a and V_b are L^2-invariant. Thus $(L^2)_{ab} = 0$ and $(L^2)_{ba} = 0$. We also have $(L^2)_a = L_{ab}L_{ba}$ and $(L^2)_b = L_{ba}L_{ab}$. The matrices $(L^2)_a$ and $(L^2)_b$ are symmetric with respect to the non-principal diagonal, i.e., we have $(L^2)_a = S_p{}^t(L^2)_a S_p$ and $(L^2)_b = S_q{}^t(L^2)_b S_q$. Let us prove the second assertion. As $(L^2)_b = L_{ba}L_{ab} = -S_q{}^t L_{ab} S_p L_{ab}$, we have

$$\begin{aligned} S_q{}^t(L^2)_b S_q &= S_q{}^t(-S_q{}^t L_{a,b} S_p L_{a,b}) S_q \\ &= -S_q({}^t L_{a,b} S_p L_{a,b} S_q) S_q \\ &= -S_q{}^t L_{a,b} S_p L_{a,b} = (L^2)_b. \end{aligned}$$

It follows that $(L^2)_a^k$ and $(L^2)_b^k$ have the same symmetry properties for each $k \geq 1$.

The following observation will be also useful. If $x \in K_a$, $y \in K_b$, and $L \in \mathfrak{p}$, then

$$(\mathrm{Ad}(x)(L))_{ab} = xL_{ab}, \quad (\mathrm{Ad}(y)(L))_{ab} = L_{ab}y^{-1}.$$

In other words, as a K-module, $\mathfrak{p}$ is isomorphic to the space of $p \times q$ complex matrices z on which $K = K_a \times K_b$ acts by $(x, y) \cdot z = xzy^{-1}$.

Let $X, Y, Z \in \mathcal{X}$. For convenience we write $X > Y, Z$ for the pair of statements $X > Y$ and $X > Z$, and $X, Y > Z$ for the pair of statements $X > Z$ and $Y > Z$. Similar notation will be used for orbits.

We proceed with a series of five lemmas needed for the proof of Theorem 1.

LEMMA 1. *Let* $X = ((ab)^m a, (ba)^{m-1} b)$, $Y = ((ab)^m, (ba)^m)$, $m \geq 1$. *For* m *even, we have*

$$\mathcal{O}_X^{\mathrm{I}} > {}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{I}},\ {}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{I}}; \quad \mathcal{O}_X^{\mathrm{II}} > {}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{II}},\ {}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{II}};$$

and, for m *odd,*

$${}^{\mathrm{I}}\mathcal{O}_X > {}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{I}},\ {}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{II}}; \quad {}^{\mathrm{II}}\mathcal{O}_X > {}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{I}},\ {}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{II}}.$$

PROOF. Assume that m is even. Since $\mathfrak{s}_2(H_Y) \subset \mathfrak{s}_2(H_X)$, Proposition 1 implies that $\mathcal{O}_X^{\mathrm{I}} > {}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{I}}$. Since W_a^* leaves $\mathcal{O}_X^{\mathrm{I}}$ invariant and permutes transitively the components ${}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{I}}$ and ${}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{I}}$, we also have $\mathcal{O}_X^{\mathrm{I}} > {}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{I}}$. By using the action of W_b^*, we derive now easily that $\mathcal{O}_X^{\mathrm{II}} > {}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{II}}$, ${}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{II}}$. The case of odd m can be treated similarly. □

LEMMA 2. *Let* $X = ((ab)^m, (ba)^m, (ab)^k, (ba)^k)$, $Y = ((ab)^{m-1}a, (ab)^{m-1}a, (ba)^k b, (ba)^k b)$ *where* $m > k \geq 0$ *and* $m \equiv k$ mod 2. *For* m *even, we have*

$$^{\mathrm{I}}\mathcal{O}^{\mathrm{I}}_X,\ ^{\mathrm{I}}\mathcal{O}^{\mathrm{II}}_X > {}^{\mathrm{I}}\mathcal{O}_Y\,, \quad ^{\mathrm{II}}\mathcal{O}^{\mathrm{I}}_X,\ ^{\mathrm{II}}\mathcal{O}^{\mathrm{II}}_X > {}^{\mathrm{II}}\mathcal{O}_Y\,;$$

and, for m odd,

$$^{\mathrm{I}}\mathcal{O}^{\mathrm{I}}_X,\ ^{\mathrm{II}}\mathcal{O}^{\mathrm{I}}_X > \mathcal{O}^{\mathrm{I}}_Y\,, \quad ^{\mathrm{I}}\mathcal{O}^{\mathrm{II}}_X,\ ^{\mathrm{II}}\mathcal{O}^{\mathrm{II}}_X > \mathcal{O}^{\mathrm{II}}_Y\,.$$

PROOF. We have $p = q = 2m + 2k$. Let us choose a representative $L \in \mathfrak{p}_2(H_Y)$ of ${}^{\mathrm{I}}\mathcal{O}_Y$ (resp. $\mathcal{O}^{\mathrm{I}}_Y$) if m is even (resp. odd) such that its ab-block, L_{ab}, is the $\{0, 1\}$-matrix having 1's at the positions (i, i) for $1 \leq i < m - k$ and positions $(i, i + 2)$ for $m - k \leq i \leq p - 2$. To verify this claim it suffices to check that $L \in \mathcal{O}_Y$ because then Proposition 1 shows that in fact $L \in {}^{\mathrm{I}}\mathcal{O}_Y$. Now we observe that L has a graded Jordan basis consisting of four Jordan chains: two of type $(ab)^{m-1}a$ with top vectors e_{p-1} and e_{p-2}, and another two of type $(ba)^k b$ with top vectors e'_{p-m+k} and $e'_{p-m+k-1}$. For instance if $m = 4$ and $k = 2$, these Jordan chains are:

$$\begin{aligned}
&e_{11} \to -e'_{11} \to -e_9 \to e'_7 \to e_5 \to -e'_3 \to -e_1 \to 0\,,\\
&e_{10} \to -e'_8 \to -e_6 \to e'_4 \to e_2 \to -e'_0 \to -e_0 \to 0\,,\\
&e'_{10} \to e_8 \to -e'_6 \to -e_4 \to e'_2 \to 0\,,\\
&e'_9 \to e_7 \to -e'_5 \to -e_3 \to e'_1 \to 0\,.
\end{aligned}$$

By applying the permutation

$$(1, 2, \ldots, m - k)(p, p - 1, \ldots, m + 3k + 1) \in W_b$$

to L, we obtain an element of $\mathfrak{s}_2(H_X)$. By Proposition 1 we have $L \in \overline{{}^{\mathrm{I}}\mathcal{O}^{\mathrm{I}}_X}$. Hence we deduce that ${}^{\mathrm{I}}\mathcal{O}^{\mathrm{I}}_X > {}^{\mathrm{I}}\mathcal{O}_Y$ for m even, and ${}^{\mathrm{I}}\mathcal{O}^{\mathrm{I}}_X > \mathcal{O}^{\mathrm{I}}_Y$ for m odd. The remaining assertions follow easily by using the action of W^*. □

LEMMA 3. *Let* $X = ((ab)^m a, (ba)^k b)$, $Y = ((ba)^{m-1}b, (ab)^{k+1}a)$ *where* $m - 2 > k \geq 0$ *and* $m \not\equiv k \bmod 2$. *For m even, we have*

$$\mathcal{O}^{\mathrm{I}}_X > \mathcal{O}^{\mathrm{I}}_Y\,, \quad \mathcal{O}^{\mathrm{II}}_X > \mathcal{O}^{\mathrm{II}}_Y\,;$$

and, for m odd,

$$^{\mathrm{I}}\mathcal{O}_X > {}^{\mathrm{I}}\mathcal{O}_Y\,, \quad ^{\mathrm{II}}\mathcal{O}_X > {}^{\mathrm{II}}\mathcal{O}_Y\,.$$

PROOF. Set $m + k + 1 = 2r$ and define $L \in \mathfrak{s}_2(H_X)$ by

$$\begin{aligned}
&e'_0 \to 0\,, \quad e'_i \to e_i\ (0 < i < r - k - 1)\,, \quad e'_{r-k-1} \to e_{r-k-1} - e_0\,,\\
&e'_{r-k} \to e_{r-k-1} + e_0\,, \quad e'_i \to e_{i-1}\ (r - k < i \leq m + k)\,.
\end{aligned}$$

By Proposition 1, L belongs to the closure of the strict orbit $\mathcal{O}^{\mathrm{I}}_X$ for m even, and the closure of ${}^{\mathrm{I}}\mathcal{O}_X$ if m is odd. We claim that L belongs to $\mathcal{O}^{\mathrm{I}}_Y$ for m even and to ${}^{\mathrm{I}}\mathcal{O}_Y$ for m odd. Indeed, L has a graded Jordan basis consisting of two chains. One of them has type $(ba)^{m-1}b$ and top vector e'_{m+k}, while the other one has type $(ab)^{k+1}a$ and top vector e_{m+k}. This means that $L \in \mathcal{O}_Y$. Our conditions imply that $r - k \geq 2$. If $r - k = 2$ then also $L \in \mathfrak{p}_2(H_Y)$. Otherwise we transform L by the element of W_a which acts on the basis $\{e_i\}$ by two cyclic permutations:

$$e_0 \to e_{r-k-1} \to e_{r-k-2} \to \cdots \to e_1 \to e_0\,, \quad e_{r+k} \to e_{r+k+1} \to \cdots \to e_{m+k} \to e_{r+k}\,.$$

The new element belongs to $\mathfrak{p}_2(H_Y)$. Now our claim follows from Proposition 1. We deduce that $\mathcal{O}^{\mathrm{I}}_X > \mathcal{O}^{\mathrm{I}}_Y$ for m even, and ${}^{\mathrm{I}}\mathcal{O}_X > {}^{\mathrm{I}}\mathcal{O}_Y$ for m odd. The remaining assertions follow easily by using the action of W^*. □

LEMMA 4. *Let* $X = ((ab)^k a, (ab)^m, (ba)^m)$, $Y = ((ba)^{k-1}b, (ab)^m a, (ab)^m a)$ *where* $k > m \geq 0$ *and* $k \equiv m \bmod 2$. *For m even, we have*

$$\mathcal{O}^{\mathrm{I}}_X > \mathcal{O}^{\mathrm{I}}_Y\,, \quad \mathcal{O}^{\mathrm{II}}_X > \mathcal{O}^{\mathrm{II}}_Y\,;$$

and, for m odd,

$${}^{\mathrm{I}}\mathcal{O}_X > {}^{\mathrm{I}}\mathcal{O}_Y\,, \quad {}^{\mathrm{II}}\mathcal{O}_X > {}^{\mathrm{II}}\mathcal{O}_Y\,.$$

PROOF. Set $m + k = 2r$ and define $L \in \mathcal{N}$ by specifying the action of L on the basis elements of V_b.

We must treat the case $m = 0$ separately. For $m = 0$, L is defined by $e'_0 \to 0$, $e'_i \to e_i$ $(0 < i < k)$. We have $L \in \mathfrak{p}_2(H_X)$. L has a graded Jordan basis consisting of three chains, one of type $(ba)^{k-1}b$ and the other two of type a. The respective top vectors are e'_{k-1}, e_0 and e_k. Note that, in this case, $k = 2r$. Let $z_a \in N_a \cap K^0_a$ be the element that permutes the basis vectors via two cycles

$$e_{r-1} \to e_{r-2} \to \cdots \to e_1 \to e_0 \to e_{r-1}\,, e_{r+1} \to e_{r+2} \to \cdots \to e_{k-1} \to e_k \to e_{r+1}\,.$$

Then $\mathrm{Ad}(z_a)(L) \in \mathfrak{p}$ is the element defined by

$$e'_0 \to 0\,, \quad e'_i \to e_{i-1}\ (0 < i < r)\,, \quad e'_r \to e_r\,, \quad e'_i \to e_{i+1}\ (r < i < k)\,.$$

Now $\mathrm{Ad}(z_a)(L)$ lies in the same K^0-orbit as L and one may check that $\mathrm{Ad}(z_a)(L) \in \mathfrak{p}_2(H_Y)$ so that $L \in \mathcal{O}^{\mathrm{I}}_Y$.

For $m > 0$, L is given by

$$e'_0 \to 0\,, \quad e'_i \to e_i\ (0 < i < r-m,\ 2m + r \leq i < 2m + k)\,,$$
$$e'_{r-m} \to e_0\,, \quad e'_i \to e_{i-1}\ (r - m < i < 2m + r)\,.$$

L has a graded Jordan basis consisting of three chains, one of type $(ba)^{k-1}b$ and the other two of type $(ab)^m a$. The respective top vectors are e'_{k+2m-1}, e_{k+2m}, and e_{2m+r-1}. Let $z_a \in N_a \cap K^0_a$ be the element that cyclically permutes the vectors $e_0, \ldots, e_{r-m-1}$ via $e_i \to e_{i-1}$ and $e_0 \to e_{r-m-1}$ and hence also the vectors $e_{2m+r+1}, \ldots, e_{k+2m}$ via $e_i \to e_{i+1}$ and $e_{k+2m} \to e_{2m+r+1}$. Then $\mathrm{Ad}(z_a)(L) \in \mathfrak{p}$ is the element defined by

$$e'_0 \to 0\,, \quad e'_i \to e_{i-1}\ (0 < i < 2m + r)\,, \quad e'_{2m+r} \to e_{2m+r}\,,$$
$$e'_i \to e_{i+1}\ (2m + r < i < k + 2m)\,.$$

Now $\mathrm{Ad}(z_a)(L)$ lies in the same K^0-orbit as L and one may check that $\mathrm{Ad}(z_a)(L) \in \mathfrak{p}_2(H_Y)$. We conclude that $L \in {}^{\mathrm{I}}\mathcal{O}_Y$ if m is odd and $L \in \mathcal{O}^{\mathrm{I}}_Y$ if m is even.

Hence ${}^{\mathrm{I}}\mathcal{O}_X > {}^{\mathrm{I}}\mathcal{O}_Y$ for m odd, and $\mathcal{O}_X^{\mathrm{I}} > \mathcal{O}_Y^{\mathrm{I}}$ for m even. By using the action of W^*, we obtain the remaining two assertions of the lemma. □

LEMMA 5. *Let* $X = ((ab)^k, (ba)^k, (ba)^{m-1}b)$ *and* $Y = ((ba)^{k-1}b, (ba)^{k-1}b, (ab)^m a)$, *where* $k > m \geq 1$ *and* $k \equiv m \bmod 2$. *For* m *even, we have*

$$\mathcal{O}_X^{\mathrm{I}} > \mathcal{O}_Y^{\mathrm{I}}, \quad \mathcal{O}_X^{\mathrm{II}} > \mathcal{O}_Y^{\mathrm{II}};$$

and, for m *odd,*

$${}^{\mathrm{I}}\mathcal{O}_X > {}^{\mathrm{I}}\mathcal{O}_Y, \quad {}^{\mathrm{II}}\mathcal{O}_X > {}^{\mathrm{II}}\mathcal{O}_Y.$$

PROOF. Set $m + k = 2r$ and define $L \in \mathcal{N}$ by specifying the action of L on the basis elements of V_b

$$e_0' \to 0, \quad e_1' \to 0, \quad e_{k-m+1}' \to e_0, \quad e_i' \to e_{i-1} \ (1 < i \leq k-m),$$
$$e_i' \to e_{i-2} \ (k-m+1 < i < m+2k).$$

L has a graded Jordan basis consisting of three chains, two of type $(ba)^{k-1}b$ and the other one of type $(ab)^m a$. The respective top vectors are e_{2k+m-2}', e_{2k+m-1}', and e_{2k+m-2}. Let $z_a \in N_a \cap K_a^0$ be the element that cyclically permutes the vectors $e_0, \dots, e_{k-m-1}$ via $e_i \to e_{i-1}$ and $e_0 \to e_{k-m-1}$ and hence also the vectors $e_{k+2m-1}, \dots, e_{2k+m-2}$ via $e_i \to e_{i+1}$ and $e_{2k+m-2} \to e_{k+2m-1}$. Then $\mathrm{Ad}(z_a)(L) \in \mathfrak{p}$ is the element defined by

$$e_0' \to 0, \quad e_1' \to 0, \ e_i' \to e_{i-2} \ (1 < i \leq k+2m),$$
$$e_i' \to e_{i-1} \ (k+2m < i < m+2k, \ 3r \leq i < m+2k).$$

Now $\mathrm{Ad}(z_a)(L)$ lies in the same K^0-orbit as L and one may check that $\mathrm{Ad}(z_a)(L) \in \mathfrak{p}_2(H_Y)$. We conclude that $L \in {}^{\mathrm{I}}\mathcal{O}_Y$ if m is odd and $L \in \mathcal{O}_Y^{\mathrm{I}}$ if m is even.

Hence ${}^{\mathrm{I}}\mathcal{O}_X > {}^{\mathrm{I}}\mathcal{O}_Y$ for m odd, and $\mathcal{O}_X^{\mathrm{I}} > \mathcal{O}_Y^{\mathrm{I}}$ for m even. By using the action of W^*, we obtain the remaining two assertions of the lemma. □

PROOF OF THEOREM 1. Let $P, Q \in \mathcal{X}$ be such that $\mathcal{O}_1 \subset \mathcal{O}_P$ and $\mathcal{O}_2 \subset \mathcal{O}_Q$. In view of Definition 2 and the definition of "$\succeq$", without any loss of generality, we may assume that $P \to Q$. We shall distinguish ten possibilities for the pair (P, Q) according to the cases (i)–(x) of Definition 2.

In the case (i), $\mathcal{O}_P$ is connected. Thus $\mathcal{O}_1 = \mathcal{O}_P$ and the whole orbit $\mathcal{O}_Q$ is contained in the closure of $\mathcal{O}_1$. Hence the assertion of the theorem holds.

In the case (ii), $\mathcal{O}_Q$ is connected. Then $\mathcal{O}_Q$ is contained in the closure of at least one connected component of $\mathcal{O}_P$. As W^* permutes transitively these connected components (and leaves $\mathcal{O}_Q$ invariant) we infer that the assertion of the theorem holds.

In the case (iii), W_a^* permutes transitively the two components of $\mathcal{O}_P$ and leaves invariant each connected component of $\mathcal{O}_Q$. On the other hand, W_b^* permutes transitively the two components of $\mathcal{O}_Q$ and leaves invariant each connected component of $\mathcal{O}_P$. Since each connected component of $\mathcal{O}_Q$ lies in the closure of at least one connected component of $\mathcal{O}_P$, the assertion of the theorem holds.

In the case (iv), the argument is similar to the one in the case (iii).

Now assume that $P \to Q$ belongs to one of the cases (v)–(x). By symmetry (i.e., by switching V_a and V_b, if necessary), it suffices to consider only the cases (v), (vii), and (ix). Note that in these cases $(H_P)_a$ has no 0 eigenvalue. Without any loss of generality we may assume that H_P (resp. H_Q) is the characteristic of $\mathcal{O}_1$ (resp. $\mathcal{O}_2$), i.e., that $E_P \in \mathcal{O}_1$ (resp. $E_Q \in \mathcal{O}_2$). The assertion of the theorem will be deduced from Lemmas 1–5. We can write $P = X + Z$ and $Q = Y + Z$ where Z is the ab-diagram made up of the common rows of P and Q. Then X and Y have no common rows and $X \to Y$. Such pairs (X, Y) are listed in [8, Table V, p. 182, type (BDI)] (see also [6, formulae (8.9–17)]). The entry (3) in that table has two misprints: $\bar{\eta}(= X)$ should be $((ab)^{p+1}a, (ba)^{q-1}b)$ and $\bar{\sigma}(= Y)$ should be $((ab)^p a, (ba)^q b)$. We remark that if $P \to Q$ belongs to one of the cases (vii) or (ix) then $X \to Y$ belongs to the same case. On the other hand if $P \to Q$ belongs to the case (v) then $X \to Y$ may belong to any of the cases (v), (vii), (ix).

By close inspection of Ohta's list, we deduce that the pair (X, Y) is exactly one of the pairs treated in Lemmas 1–5. If Z is empty, i.e., $P = X$ and $Q = Y$, then the assertion of the theorem follows immediately from Lemmas 1–5. Assume now that Z is not empty.

Let V_1 (resp. V_2) denote the ambient vector space of the orbit $\mathcal{O}_Z$ (resp. $\mathcal{O}_X$) and f_1 (resp. f_2) its symmetric bilinear form. We set $p_1 = n_a(Z)$, $q_1 = n_b(Z)$, $p_2 = n_a(X) = n_a(Y)$, and $q_2 = n_b(X) = n_b(Y)$. Hence $p = p_1 + p_2$ and $q = q_1 + q_2$. The basis vectors e_i and e'_i of these spaces will now be renamed $e_i(1)$ and $e'_i(1)$ for V_1 and $e_i(2)$ and $e'_i(2)$ for V_2. As $(H_P)_a$ is nonsingular, p_1 and p_2 are even.

Assume first that q_1 or q_2 is even. Then we can choose an isometry $\varphi : (V_1 \oplus V_2, f_1 \oplus f_2) \to (V, f)$ such that $\{e_i(1)\} \cup \{e_i(2)\}$ (resp $\{e'_i(1)\} \cup \{e'_i(2)\}$) is mapped bijectively onto $\{e_i\}$ (resp. $\{e'_i\}$) and the following conditions are satisfied:

(i) if $\varphi(e_i(k)) = e_j$, $i < (p_k - 1)/2$, then $j < (p - 1)/2$ and $\varphi(e_{p_k-i-1}(k)) = e_{p-j-1}$, $(k = 1, 2)$;

(ii) if $\varphi(e'_i(k)) = e'_j$, $i < (q_k - 1)/2$, then $j < (q - 1)/2$ and $\varphi(e'_{q_k-i-1}(k)) = e'_{q-j-1}$, $(k = 1, 2)$;

(iii) if λ_j (resp. μ_j) is the eigenvalue of $\varphi \circ (H_Z \oplus H_X) \circ \varphi^{-1}$ belonging to the eigenvector e_j (resp e'_j), then the λ_j's (resp. μ_j's) are non-increasing.

By identifying V_1 and V_2 with their images in V, we have $V = V_1 \oplus V_2$ and $V_1 \perp V_2$. Moreover

$$V_a = (V_1)_a \oplus (V_2)_a\,, \qquad V_b = (V_1)_b \oplus (V_2)_b\,,$$
$$H_P = H_Z \oplus H_X\,, \qquad H_Q = H_Z \oplus H_Y\,,$$

and we may assume that $E_P = E_Z \oplus E_X$ and $E_Q = E_Z \oplus E_Y$. Since $X \to Y$, Lemmas 1–5 imply that E_Y lies in the closure of the strict orbit of E_X. Consequently, E_Q lies in the closure of the strict orbit of E_P, i.e., $\mathcal{O}_1 \geq \mathcal{O}_2$.

Now let q_1 and q_2 be odd. We can choose an isometry φ so that (i) holds as well as the part of (iii) that refers to the λ_j's, and such that φ maps $(V_1)_b \oplus (V_2)_b$ onto V_b. Although now

H_P and $H_Z \oplus H_X$ are not equal, they are K_b-conjugate. Consequently, E_P and $E_Z \oplus E_X$ belong to the same K^0-orbit, i.e., to $\mathcal{O}_1$. Similarly, $E_Z \oplus E_Y$ belongs to $\mathcal{O}_2$. The rest of the argument is the same as in the previous case. $\square$

The following example illustrates the argument used in the above proof.

EXAMPLE 5. Let $X = ((ab)^3a, (ba)^2b)$ and $Y = ((ab)^3, (ba)^3)$, and let Z be as in Example 1. Set $P = X + Z$ and $Q = Y + Z$. Then $P, Q \in \mathcal{X}(p, q)$ with $p = 12$ and $q = 10$. Note that $\mathcal{O}_Y$ has four connected components, while $\mathcal{O}_Q$ has only two: ${}^{\mathrm{I}}\mathcal{O}_Q$ and ${}^{\mathrm{II}}\mathcal{O}_Q$. We find that

$$\begin{aligned}
(H_P)_a &= \operatorname{diag}(6, 6, 2, 2, 2, 2, -2, -2, -2, -2, -6, -6)\,,\\
(H_P)_b &= \operatorname{diag}(4, 4, 4, 0, 0, 0, 0, -4, -4, -4)\,,\\
(H_Q)_a &= \operatorname{diag}(6, 5, 3, 2, 2, 1, -1, -2, -2, -3, -5, -6)\,,\\
(H_Q)_b &= \operatorname{diag}(5, 4, 3, 1, 0, 0, -1, -3, -4, -5)\,.
\end{aligned}$$

Let V_1 (resp. V_2) denote the ambient vector space of the orbit $\mathcal{O}_X$ (resp. $\mathcal{O}_Z$). The basis vectors e_i and e'_i of these spaces will now be renamed $e_i(1)$ and $e'_i(1)$ for V_1 and $e_i(2)$ and $e'_i(2)$ for V_2. We embed V_1 and V_2 isometrically into V by sending

$$\begin{aligned}
&e_0(2) \to e_1\,, \quad e_1(2) \to e_4\,, \quad e_2(2) \to e_5\,, \quad e_3(2) \to e_6\,,\ e_4(2) \to e_7\,, \quad e_5(2) \to e_{10}\,;\\
&e'_0(2) \to e'_2\,, \quad e'_1(2) \to e'_4\,, \quad e'_2(2) \to e'_5\,, \quad e'_3(2) \to e'_7\,;
\end{aligned}$$

and

$$\begin{aligned}
&e_0(1) \to e_0\,, \quad e_1(1) \to e_2\,, \quad e_2(1) \to e_3\,, \quad e_3(1) \to e_8\,,\ e_4(1) \to e_9\,, \quad e_5(1) \to e_{11}\,;\\
&e'_0(1) \to e'_0,\ e'_1(1) \to e'_1\,, \quad e'_2(1) \to e'_3\,, \quad e'_3(1) \to e'_6\,,\ e'_4(1) \to e'_8\,, \quad e'_5(1) \to e'_9\,.
\end{aligned}$$

By identifying V_1 and V_2 with their images in V, we have $V = V_1 \oplus V_2$ and $V_1 \perp V_2$. Moreover

$$\begin{aligned}
V_a &= (V_1)_a \oplus (V_2)_a\,, \quad & V_b &= (V_1)_b \oplus (V_2)_b\,,\\
H_P &= H_Z \oplus H_X\,, \quad & H_Q &= H_Z \oplus H_Y\,.
\end{aligned}$$

Since $E_Z \oplus E_X \in {}^{\mathrm{I}}\mathcal{O}_P$, $E_Z \oplus E_Y \in {}^{\mathrm{I}}\mathcal{O}_Q$, and $E_Y \in \overline{K^0 \cdot E_X}$, we deduce that $E_Z \oplus E_Y \in \overline{K^0 \cdot (E_Z \oplus E_X)}$. Thus ${}^{\mathrm{I}}\mathcal{O}_P \geq {}^{\mathrm{I}}\mathcal{O}_Q$ and consequently also ${}^{\mathrm{II}}\mathcal{O}_P \geq {}^{\mathrm{II}}\mathcal{O}_Q$. $\square$

4. Simplification of the conjecture. In this section we simplify our problem and prepare the ground for the verification of the conjecture for small values of p or q, to be carried out in the next section.

We define a *path* in $\Gamma = \Gamma(p, q)$ to be a sequence of vertices $(X_0, X_1, \ldots, X_k)$ of Γ such that $X_0 \to X_1 \to \cdots \to X_k$. We also say that the *length* of this path is k and that this path *joins* X_0 to X_k. If $X \geq Y$ then there exists a path joining X to Y (by the definition of Hasse diagrams). We say that a pair (X, Y) of vertices of Γ is an *a-pair* if $X > Y$ and every vertex Z such that $X \geq Z \geq Y$ is an a-vertex. A *b-pair* is defined similarly. We say that (X, Y) is a *pure pair* if it is either an a-pair or a b-pair. We remark that an a-pair cannot be a

b-pair (see Lemma 7 (iii) below). A *maximal a-pair* (or ma-*pair* for short) is an a-pair (X, Y) such that there is no a-pair (P, Q) with $P \geq X > Y \geq Q$ and $P > X$ or $Y > Q$. We define similarly the *maximal b-pairs* (or mb-pairs). A *maximal pure pair* (or mp-pair) is an ma-pair or an mb-pair.

LEMMA 6. *Let $X, Y \in \mathcal{X}(p, q)$. If $((X, a), (Y, a))$ is a b-pair in $\Gamma(p+1, q)$, then (X, Y) is a b-pair in $\Gamma(p, q)$. The converse holds if $p \geq 2q$.*

PROOF. Assume that (X, Y) is not a b-pair. Then there exists a vertex Z such that $X \geq Z \geq Y$ and Z is either stable or a proper a-vertex. It follows that $(X, a) \geq (Z, a) \geq (Y, a)$ and (Z, a) is stable. Hence $((X, a), (Y, a))$ is not a b-pair. This proves the first assertion.

Now assume that $p \geq 2q$ and that (X, Y) is a b-pair. Let P be any vertex such that $(X, a) \geq P \geq (Y, a)$. Since $p \geq 2q$, P necessarily has the form $P = (Z, a)$. It follows that $X \geq Z \geq Y$ and so Z must be a b-vertex. Consequently, P is a proper b-vertex. The second assertion is proved. □

REMARK. The hypothesis $p \geq 2q$ in the above lemma is probably superfluous.

To break the monotony and help the reader digest the above definitions, we give two examples which will be needed in the next section.

EXAMPLE 6. Let us enumerate the unstable vertices and pure pairs in $\Gamma(p, q)$ for $p \geq q \leq 3$. (These diagrams are displayed in Figures 1 and 2.)

All vertices of $\Gamma(p, 1)$ are stable except for the proper a-vertex $A = (aba)$ when $p = 2$.

In $\Gamma(2, 2)$, C_1 is a proper a-vertex, C_2 a proper b-vertex, and D an ab-vertex. (C_1, D) is an a-pair, and (C_2, D) a b-pair. For $p > 2$, the unstable vertices of $\Gamma(p, 2)$ are A, C_2, and D (all of them proper b-vertices), and B is a proper a-vertex if $p = 4$. The b-pairs are (A, C_2) and (C_2, D) (both maximal).

We now consider the diagrams $\Gamma(p, 3)$, $p \geq 3$. If p is odd, all vertices are stable. If $p = 4$ the unstable vertices are A, E_1, and F (all of them proper a-vertices), and (E_1, F) is the only a-pair. Finally if p is even and ≥ 6 then all vertices are stable except that D is a proper a-vertex if $p = 6$.

EXAMPLE 7. For large p and q the diagram $\Gamma(p, q)$ is rather complicated. We shall describe here only the part $\Gamma_u(p, q)$ of $\Gamma(p, q)$ which consists of the unstable vertices and the lines between them. We do this only for $p \geq q = 4$. In Table 3 we list all vertices of $\Gamma(p, 4)$ for $p \geq 4$.

In the "type" columns, for each $p = 4, 5, \ldots, 9$ we indicate the type of the vertex X. The letter a means that X is a proper a-vertex, the letter b stands for a proper b-vertex, ab stands for an ab-vertex, and s for a stable vertex. The asterisk indicates that the vertex with that label does not exist for that particular value of p. If $p > 9$, the type of the vertex is the same as for $p = 9$. In the "dim" columns we list the complex dimensions of $\mathcal{O}_X$ for each value of $p = 4, 5, 6, 7, 8$.

In Figures 5 and 6, we display the subdiagrams $\Gamma_u(p, 4)$ for $p = 4, 5, 6, 7$ and list on the side all mp-pairs. The subdiagram $\Gamma_u(8, 4)$ is the same as $\Gamma_u(7, 4)$ except for an additional

TABLE 3. Vertices of $\Gamma(p, 4)$.

label	X	π_X	type						dim				
			4	5	6	7	8	9	4	5	6	7	8
A	$(ab)^4a$	$9 \cdot 1^{p-5}$	$*$	b	b	b	b	b		16	20	24	28
B	$(ab)^3a, aba$	$7 \cdot 3 \cdot 1^{p-6}$	$*$	$*$	a	s	s	s			19	23	27
C	$(ab)^2a, (ab)^2a$	$5^2 \cdot 1^{p-6}$	$*$	$*$	b	b	b	b			18	22	26
D	$(ab)^2a, aba, aba$	$5 \cdot 3^2 \cdot 1^{p-7}$	$*$	$*$	$*$	s	s	s				21	25
E_1	$(ab)^3a, b$	$7 \cdot 1^{p-3}$	a	s	s	s	s	s	12	15	18	21	24
E_2	$(ba)^3b$	$7 \cdot 1^{p-3}$	b	b	b	b	b	b	12	15	18	21	24
F_1	$(ab)^2a, aba, b$	$5 \cdot 3 \cdot 1^{p-4}$	$*$	s	s	s	s	s		14	17	20	23
F_2	$(ab)^2a, bab$	$5 \cdot 3 \cdot 1^{p-4}$	b	b	b	b	b	b	11	14	17	20	23
F_3	$(ba)^2b, aba$	$5 \cdot 3 \cdot 1^{p-4}$	a	s	s	s	s	s	11	14	17	20	23
G	aba, aba, aba, aba	$3^4 \cdot 1^{p-8}$	$*$	$*$	$*$	$*$	a	s					22
H	$(ab)^2a, ab, ba$	$5 \cdot 2^2 \cdot 1^{p-5}$	$*$	b	b	b	b	b		13	16	19	22
I	$abab, baba$	$4^2 \cdot 1^{p-4}$	ab	b	b	b	b	b	10	13	16	19	22
J_1	aba, aba, aba, b	$3^3 \cdot 1^{p-5}$	$*$	$*$	a	s	s	s			15	18	21
J_2	aba, aba, bab	$3^3 \cdot 1^{p-5}$	$*$	s	s	s	s	s		12	15	18	21
K	aba, aba, ab, ba	$3^2 \cdot 2^2 \cdot 1^{p-6}$	$*$	$*$	a	s	s	s			14	17	20
L_1	$(ab)^2a, b^2$	$5 \cdot 1^{p-1}$	s	s	s	s	s	s	10	12	14	16	18
L_2	$(ba)^2b, b$	$5 \cdot 1^{p-1}$	s	s	s	s	s	s	10	12	14	16	18
M_1	aba, aba, b^2	$3^2 \cdot 1^{p-2}$	a	s	s	s	s	s	9	11	13	15	17
M_2	aba, bab, b	$3^2 \cdot 1^{p-2}$	s	s	s	s	s	s	9	11	13	15	17
M_3	bab, bab	$3^2 \cdot 1^{p-2}$	b	b	b	b	b	b	9	11	13	15	17
N_1	aba, ab, ba, b	$3 \cdot 2^2 \cdot 1^{p-3}$	a	s	s	s	s	s	8	10	12	14	16
N_2	bab, ab, ba	$3 \cdot 2^2 \cdot 1^{p-3}$	b	b	b	b	b	b	8	10	12	14	16
O	ab, ba, ab, ba	$2^4 \cdot 1^{p-4}$	ab	b	b	b	b	b	6	8	10	12	14
P_1	aba, b^3	$3 \cdot 1^{p+1}$	s	s	s	s	s	s	6	7	8	9	10
P_2	bab, b^2	$3 \cdot 1^{p+1}$	s	s	s	s	s	s	6	7	8	9	10
Q	ab, ba, b^2	$2^2 \cdot 1^p$	s	s	s	s	s	s	5	6	7	8	9
R	b^4	1^{p+4}	s	s	s	s	s	s	0	0	0	0	0

isolated vertex, namely G. If $p > 8$, it follows easily from Lemma 6 that $\Gamma_u(p, 4)$ is the same as $\Gamma_u(7, 4)$.

The following lemma is useful for identification of pure pairs.

LEMMA 7. *Let $X, Y \in \mathcal{X}$ with $X > Y$. Then:*

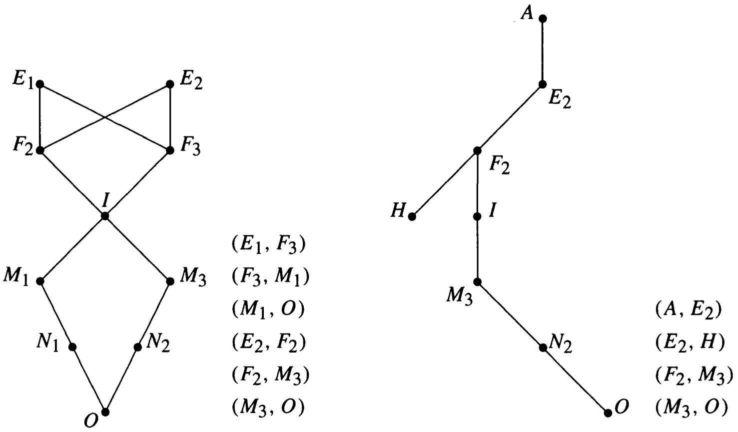
(i) *If X and Y are ab-vertices, there exists a stable vertex Z such that $X > Z > Y$.*

(ii) *If (X, Y) is a pure pair, every path joining X to Y contains at most one ab-vertex.*

(iii) *If (X, Y) is a pure pair, X or Y is not an ab-vertex.*

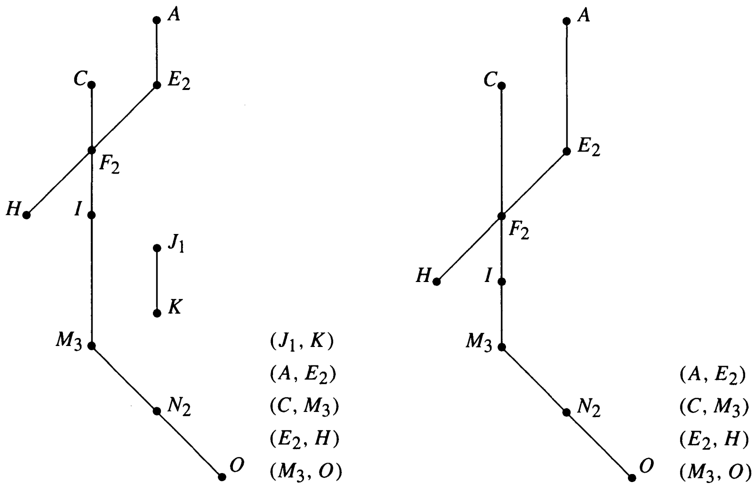
PROOF. It is clear that (i) implies (ii), and (ii) implies (iii). To prove (i), it suffices to consider the case $X = ((ab)^m, (ba)^m, (ab)^k, (ba)^k)$, $Y = ((ab)^{m-1}, (ba)^{m-1}, (ab)^{k+1}, (ba)^{k+1})$, where $m - 2 \geq k \geq 0$. In that case we can take $Z = ((ab)^{m-1}a, (ba)^{m-1}b, (ab)^k a, (ba)^k b)$. □

In the next lemma we collect some elementary facts concerning the partial order "$\succeq$".



$\Gamma_u(4,4)$ $\Gamma_u(5,4)$

FIGURE 5.



$\Gamma_u(6,4)$ $\Gamma_u(7,4)$

FIGURE 6.

LEMMA 8. *Let X, Y be vertices of Γ such that $X > Y$.*

(i) *If X is stable, then $\mathcal{O}_X \succeq \mathcal{O}_2$ for each connected component $\mathcal{O}_2$ of $\mathcal{O}_Y$.*

(ii) *If Y is stable, then $\mathcal{O}_1 \succeq \mathcal{O}_Y$ for each connected component $\mathcal{O}_1$ of $\mathcal{O}_X$.*

(iii) *If X and Y are proper a-vertices, then ${}^{\mathrm{I}}\mathcal{O}_X \succeq {}^{\mathrm{I}}\mathcal{O}_Y$ and ${}^{\mathrm{II}}\mathcal{O}_X \succeq {}^{\mathrm{II}}\mathcal{O}_Y$.*

(iv) *If X and Y are proper b-vertices, then $\mathcal{O}_X^{\mathrm{I}} \succeq \mathcal{O}_Y^{\mathrm{I}}$ and $\mathcal{O}_X^{\mathrm{II}} \succeq \mathcal{O}_Y^{\mathrm{II}}$.*

(v) *If X and Y are proper vertices of different types, then $\mathcal{O}_1 \succeq \mathcal{O}_2$ for each connected component $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) of $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$).*

(vi) *If X is a proper a-vertex and Y an ab-vertex, then ${}^{\mathrm{I}}\mathcal{O}_X \succeq {}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{I}}, {}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{II}}$ and ${}^{\mathrm{II}}\mathcal{O}_X \succeq {}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{I}}, {}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{II}}$.*

(vii) *If X is an ab-vertex and Y a proper a-vertex, then ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}, {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}} \succeq {}^{\mathrm{I}}\mathcal{O}_Y$ and ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}}, {}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}} \succeq {}^{\mathrm{II}}\mathcal{O}_Y$.*

(viii) *If X is a proper b-vertex and Y an ab-vertex, then $\mathcal{O}_X^{\mathrm{I}} \succeq {}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{I}}, {}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{I}}$ and $\mathcal{O}_X^{\mathrm{II}} \succeq {}^{\mathrm{I}}\mathcal{O}_Y^{\mathrm{II}}, {}^{\mathrm{II}}\mathcal{O}_Y^{\mathrm{II}}$.*

(ix) *If X is an ab-vertex and Y a proper b-vertex, then ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}, {}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}} \succeq \mathcal{O}_Y^{\mathrm{I}}$ and ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}}, {}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}} \succeq \mathcal{O}_Y^{\mathrm{II}}$.*

PROOF. Let $(X = X_0, X_1, \dots, X_k = Y)$ be a path joining X to Y and having the minimal length k. We prove the lemma by induction on k. If $k = 1$ all the assertions of the lemma follow immediately from Definition 2. Now let $k > 1$ and let $Z = X_{k-1}$.

To prove the assertion (iii), we consider several possibilities for Z.

Case 1: Z is stable. Then by applying the induction hypothesis (ii) to the pair (X, Z), we obtain that ${}^{\mathrm{I}}\mathcal{O}_X, {}^{\mathrm{II}}\mathcal{O}_X \succeq \mathcal{O}_Z$. Since $\mathcal{O}_Z \succeq {}^{\mathrm{I}}\mathcal{O}_Y, {}^{\mathrm{II}}\mathcal{O}_Y$ by Definition 2, it follows that (iii) holds.

Case 2: Z is a proper a-vertex. Then ${}^{\mathrm{I}}\mathcal{O}_X \succeq {}^{\mathrm{I}}\mathcal{O}_Z$ and ${}^{\mathrm{II}}\mathcal{O}_X \succeq {}^{\mathrm{II}}\mathcal{O}_Z$ by the induction hypothesis (iii), and ${}^{\mathrm{I}}\mathcal{O}_Z \succeq {}^{\mathrm{I}}\mathcal{O}_Y$ and ${}^{\mathrm{II}}\mathcal{O}_Z \succeq {}^{\mathrm{II}}\mathcal{O}_Y$ by Definition 2. Consequently (iii) holds.

Case 3: Z is a proper b-vertex. By the induction hypothesis (v) we have ${}^{\mathrm{I}}\mathcal{O}_X, {}^{\mathrm{II}}\mathcal{O}_X \succeq \mathcal{O}_Z^{\mathrm{I}}$ and, by Definition 2, $\mathcal{O}_Z^{\mathrm{I}} \succeq {}^{\mathrm{I}}\mathcal{O}_Y, {}^{\mathrm{II}}\mathcal{O}_Y$. Hence (iii) holds.

Case 4: Z is an ab-vertex. By the induction hypothesis (vii) we have ${}^{\mathrm{I}}\mathcal{O}_X \succeq {}^{\mathrm{I}}\mathcal{O}_Z^{\mathrm{I}}$ and ${}^{\mathrm{II}}\mathcal{O}_X \succeq {}^{\mathrm{II}}\mathcal{O}_Z^{\mathrm{I}}$. By Definition 2 we have ${}^{\mathrm{I}}\mathcal{O}_Z^{\mathrm{I}} \succeq {}^{\mathrm{I}}\mathcal{O}_Y$ and ${}^{\mathrm{II}}\mathcal{O}_Z^{\mathrm{I}} \succeq {}^{\mathrm{II}}\mathcal{O}_Y$. So, again (iii) holds.

We omit the routine details of the proof for the other assertions. □

We shall also need the following useful fact.

LEMMA 9. *Let X, Y be vertices of Γ such that $X > Y$. If (X, Y) is not pure, then $\mathcal{O}_1 \succeq \mathcal{O}_2$ for each connected component $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) of $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$).*

PROOF. If there exists a stable vertex Z such that $X \geq Z \geq Y$, then the assertion follows from Lemma 8 (i), (ii). We assume from now on that there are no such stable vertices. Lemma 7 implies that X or Y is not an ab-vertex.

If X and Y are proper vertices of different types, then the assertion follows from Lemma 8 (v). Assume now that X and Y are proper vertices of the same type, say type a. Since the pair (X, Y) is not pure, there exists a proper b-vertex Z such that $X > Z > Y$. By Lemma 8 (v) we have $\mathcal{O}_1 \succeq \mathcal{O}_Z^{\mathrm{I}}$ and $\mathcal{O}_Z^{\mathrm{I}} \succeq \mathcal{O}_2$. Hence the assertion follows.

Assume now that X is an ab-vertex. Then Y is necessarily a proper vertex, say a proper a-vertex. Since (X, Y) is not pure, there exists a proper b-vertex Z with $X > Z > Y$. Without any loss of generality, we may assume that $\mathcal{O}_1 = {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$. By Lemma 8 (ix) we have $\mathcal{O}_1 \succeq \mathcal{O}_Z^{\mathrm{I}}$, and by Lemma 8 (v) we have $\mathcal{O}_Z^{\mathrm{I}} \succeq \mathcal{O}_2$. The assertion follows.

The case where Y is an ab-vertex can be treated similarly. □

Let X be an ab-vertex. Recall that the K-orbit $\mathcal{O}_X$ has four connected components ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$, ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}}$, ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}}$, and ${}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}}$. In this case we define the sets ${}^{\mathrm{I}}\mathcal{O}_X$, ${}^{\mathrm{II}}\mathcal{O}_X$, $\mathcal{O}_X^{\mathrm{I}}$, and $\mathcal{O}_X^{\mathrm{II}}$ by

$$
\begin{aligned}
{}^{\mathrm{I}}\mathcal{O}_X &= {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}} \cup {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}}, \quad & {}^{\mathrm{II}}\mathcal{O}_X &= {}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}} \cup {}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}}, \\
\mathcal{O}_X^{\mathrm{I}} &= {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}} \cup {}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{I}}, \quad & \mathcal{O}_X^{\mathrm{II}} &= {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}} \cup {}^{\mathrm{II}}\mathcal{O}_X^{\mathrm{II}}.
\end{aligned}
$$

It follows from Lemma 8 that if (X, Y) is an a-pair (resp. b-pair), then

$$\overline{{}^{\mathrm{I}}\mathcal{O}_X} \cap \mathcal{O}_Y \supset {}^{\mathrm{I}}\mathcal{O}_Y \quad (\text{resp. } \overline{\mathcal{O}_X^{\mathrm{I}}} \cap \mathcal{O}_Y \supset \mathcal{O}_Y^{\mathrm{I}}),$$

where the bar denotes the closure in the Zariski topology. Clearly, the superscripts I can be replaced by II. As we shall see below, our conjecture is equivalent to the assertion that the above inclusion signs can be replaced by the equality signs. This motivates the following definition.

DEFINITION 3. We say that an a-pair (X, Y) *splits* if

$$\overline{{}^{\mathrm{I}}\mathcal{O}_X} \cap \mathcal{O}_Y \subset {}^{\mathrm{I}}\mathcal{O}_Y \quad (\text{or, equivalently, } \overline{{}^{\mathrm{II}}\mathcal{O}_X} \cap \mathcal{O}_Y \subset {}^{\mathrm{II}}\mathcal{O}_Y).$$

One defines the concept of splitting for b-pairs similarly (just move the superscripts I and II from the left to the right).

We can now state the main result of this section.

THEOREM 2. *In order to prove the conjecture, it suffices to prove that every* mp*-pair splits.*

PROOF. Assume that every mp-pair splits. This clearly implies that every pure pair splits. Let $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{N}$ be K^0-orbits such that $\mathcal{O}_1 \geq \mathcal{O}_2$. We have to show that

$$\mathcal{O}_1 \succeq \mathcal{O}_2\,. \tag{6}$$

There are unique vertices $X, Y \in \mathcal{X}$ such that $\mathcal{O}_1 \subset \mathcal{O}_X$ and $\mathcal{O}_2 \subset \mathcal{O}_Y$. As $\mathcal{O}_1 \geq \mathcal{O}_2$, we have $X \geq Y$. Without any loss of generality we may assume that $X \neq Y$, and so $X > Y$. If (X, Y) is not pure, then (6) follows from Lemma 9.

Now assume that (X, Y) is a pure pair, say an a-pair. Without any loss of generality, we may assume that $\mathcal{O}_1 \subset {}^{\mathrm{I}}\mathcal{O}_X$. Since (X, Y) splits and $\mathcal{O}_1 \geq \mathcal{O}_2$, we must have $\mathcal{O}_2 \subset {}^{\mathrm{I}}\mathcal{O}_Y$. Now (6) follows from Lemma 8 (vi), (vii). □

5. Special cases of the conjecture. In this section we shall prove that several infinite families of pure pairs split and verify our conjecture when $\min(p, q) \leq 7$.

Let B be the Borel subgroup of K^0 consisting of all upper triangular matrices in K^0. We have $B = B_a \times B_b$ where $B_a = B \cap K_a^0$ and $B_b = B \cap K_b^0$ are the Borel subgroups of K_a^0 and K_b^0, respectively.

As in Section 1, for $X \in \mathcal{X} = \mathcal{X}(p,q)$, we shall denote by E_X a representative of the orbit $\mathcal{O}_X$ such that $[H_X, E_X] = 2E_X$. If X is a proper a-vertex (resp. proper b-vertex, an ab-vertex), then $E_X \in {}^{\mathrm{I}}\mathcal{O}_X$ (resp. $E_X \in \mathcal{O}_X^{\mathrm{I}}$, $E_X \in {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$).

The following proposition and Proposition 1 play a key role in the sequel.

PROPOSITION 2. *Let (X, Y) be an a-pair in $\Gamma = \Gamma(p,q)$. If $\overline{B \cdot E_X} \cap \mathcal{O}_Y \subset {}^{\mathrm{I}}\mathcal{O}_Y$, then (X,Y) splits. The analogous assertion is valid for b-pairs.* (*We use the dot to denote the adjoint action.*)

PROOF. Assume that $\overline{B \cdot E_X} \cap \mathcal{O}_Y \subset {}^{\mathrm{I}}\mathcal{O}_Y$. By [7, Satz 2, pp. 182–183] we have $\overline{K^0 \cdot E_X} = K^0 \cdot \overline{B \cdot E_X}$. Hence if X is a proper a-vertex, then

$$\overline{{}^{\mathrm{I}}\mathcal{O}_X} \cap \mathcal{O}_Y = \overline{K^0 \cdot E_X} \cap \mathcal{O}_Y = K^0 \cdot (\overline{B \cdot E_X} \cap \mathcal{O}_Y) \subset {}^{\mathrm{I}}\mathcal{O}_Y\,,$$

i.e., (X,Y) splits. If X is an ab-vertex (and so Y is a proper a-vertex) then ${}^{\mathrm{I}}\mathcal{O}_X = {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}} \cup {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}}$. As $E_X \in {}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$, the above argument gives $\overline{{}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}} \cap \mathcal{O}_Y \subset {}^{\mathrm{I}}\mathcal{O}_Y$. As ${}^{\mathrm{I}}\mathcal{O}_Y$ is K_b-invariant, by applying a suitable element of W_b^*, we obtain that also $\overline{{}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{II}}} \cap \mathcal{O}_Y \subset {}^{\mathrm{I}}\mathcal{O}_Y$. Hence (X,Y) splits. □

We note that, by Proposition 1, $\overline{B \cdot E_X} \subset \mathfrak{s}_2(H_X) \subset \overline{K^0 \cdot E_X}$.

LEMMA 10. *Let $X, Y \in \mathcal{X}(p_1, q_1)$, $Z \in \mathcal{X}(p_2, q_2)$, and let $\tilde{X} = X + Z$, $\tilde{Y} = Y + Z$, $p = p_1 + q_1$ and $q = q_1 + q_2$. Assume that (X,Y) is a b-pair in $\Gamma(p_1, q_1)$ and $(\tilde{X}, \tilde{Y})$ a b-pair in $\Gamma(p,q)$. Then if $(\tilde{X}, \tilde{Y})$ splits so does (X,Y).*

PROOF. Let $n_1 = p_1 + q_1$, $n_2 = p_2 + q_2$, and $n = p + q$. We consider $\mathrm{O}_{n_1}(\boldsymbol{C}) \times \mathrm{O}_{n_2}(\boldsymbol{C})$ as embedded in $\mathrm{O}_n(\boldsymbol{C})$ in the standard way. We may assume that the bases of the underlying vector spaces are chosen so that $\mathcal{O}_Y^{\mathrm{I}} \times \mathcal{O}_Z^{\mathrm{I}} \subset \mathcal{O}_{\tilde{Y}}^{\mathrm{I}}$ (and consequently $\mathcal{O}_Y^{\mathrm{II}} \times \mathcal{O}_Z^{\mathrm{I}} \subset \mathcal{O}_{\tilde{Y}}^{\mathrm{II}}$). Assume that (X,Y) does not split. Then $\overline{\mathcal{O}_X^{\mathrm{I}}} \supset \mathcal{O}_Y$, and so $\overline{\mathcal{O}_X^{\mathrm{I}} \times \mathcal{O}_Z^{\mathrm{I}}} \supset \mathcal{O}_Y \times \mathcal{O}_Z^{\mathrm{I}}$. As $\mathcal{O}_Y \times \mathcal{O}_Z^{\mathrm{I}}$ meets both $\mathcal{O}_{\tilde{Y}}^{\mathrm{I}}$ and $\mathcal{O}_{\tilde{Y}}^{\mathrm{II}}$, this contradicts the hypothesis that $(\tilde{X}, \tilde{Y})$ splits. □

Recall that for $X \in \mathcal{X}$ and $k \geq 0$ we have defined the diagrams $X^{(k)}$ (see Section 2). If k is even then $X^{(k)}$ is an ab-diagram, but this may fail for odd k. When k is odd, then $n_a(X^{(k)}) = n_b(X^{(k)})$ and we introduce the parameter $r_k(X) = n_a(X^{(k)})$. When k is even, we introduce two parameters: $r_{k,a}(X) = n_a(X^{(k)})$ and $r_{k,b}(X) = n_b(X^{(k)})$. In particular we have $r_{0,a}(X) = p$ and $r_{0,b}(X) = q$ for all $X \in \mathcal{X}$. The following lemma explains the meaning of these parameters.

LEMMA 11. *If $X \in \mathcal{X}$ and $L \in \mathcal{O}_X$, then* $\operatorname{rank}(L^{2k})_a = r_{2k,a}(X)$, $\operatorname{rank}(L^{2k})_b = r_{2k,b}(X)$, *and* $\operatorname{rank}(L^{2k+1})_{ab} = r_{2k+1}(X)$.

PROOF. This follows immediately by considering a graded Jordan basis for L. □

Let us outline the procedure that we use repeatedly in our verifications below. Let, say, (X, Y) be an mb-pair. As usual, H_X denotes the characteristic of $\mathcal{O}_X$. Next let $L \in \mathfrak{s}_2(H_X) \cap \mathcal{O}_Y$ be an arbitrary element. We use Lemma 11 and Proposition 1 to prove that L must belong to $\mathcal{O}_Y^{\mathrm{I}}$. Then Proposition 2 shows that (X, Y) splits. The arguments use the vectors $v_i := L(e_i') \in V_a$, $0 \le i < q$. The matrix $S_q(L^2)_b$ is the negative of the Gram matrix of the sequence $v_0, v_1, \dots, v_{q-1}$. (Recall that the matrices S_k have been defined in Section 3.) Hence its rank $r_{2,b}(X)$ is the dimension of the quotient space of $L(V_b) = \langle v_0, v_1, \dots, v_{q-1} \rangle$ modulo its radical. Similarly, in certain cases we may use the vectors $v_i' := L(e_i) \in V_b$, $0 \le i < p$. The matrix $S_p(L^2)_a$ is the negative of the Gram matrix of these vectors. Let us illustrate this method by proving several useful lemmas. We remark that although we state these lemmas just for one kind of pure pairs, the analogous assertion is also valid for the other kind.

In the proofs of these lemmas and Theorem 3 we often use the action of K_a or K_b in order to modify the orbit representative that we are working with. If say $p = 2k$ is even, then K_a^0 contains the subgroup isomorphic to $\mathrm{GL}_k(\mathbf{C})$ which leaves invariant the maximal totally isotropic subspaces $\langle e_0, e_1, \dots, e_{k-1} \rangle$ and $\langle e_k, e_{k+1}, \dots, e_{p-1} \rangle$. Most often we use elements from this subgroup or from the intersection of it with the Borel subgroup B_a. The specific details for the choice of these "suitable elements" will be omitted.

LEMMA 12. *The a-pair $(X = ((aba)^k, b^{k+m}), Y = ((ab, ba)^k, b^m))$ splits.*

(The notation means that, in X, aba is repeated k times and, in Y, (ab, ba) is repeated k times.)

PROOF. Note that $p = 2k$ and $q = 2k + m$. Let $L \in \mathfrak{s}_2(H_X) \cap \mathcal{O}_Y$ and let $v_i' = L(e_i)$ for $0 \le i < p$. As $(H_X)_a = \mathrm{diag}(2, 2, \dots, 2, -2, \dots, -2, -2)$ and $(H_X)_b = 0$, we have $v_i' = 0$ for $i < k$. As rank $L_{ab} = r_1(Y) = k$, the vectors v_i' for $i \ge k$ form a basis of $L(V_a)$.

As rank $(L^2)_a = r_{2,a}(Y) = 0$, $L(V_a)$ is totally isotropic. By applying a suitable element of K_b and by using Witt's theorem (see, e.g., [1, p. 121, Theorem 3.9]), we may assume that $L(V_a) = \langle e_0', e_1', \dots, e_{k-1}' \rangle$. By inspecting the eigenspaces of $\mathrm{ad}(H_Y)$, we see that $L \in \mathfrak{s}_2(H_Y)$. Hence, by Proposition 1, $L \in {}^{\mathrm{I}}\mathcal{O}_Y$. By Proposition 2, (X, Y) splits. □

LEMMA 13. *If $X = ((ababa)^k, (bab)^r, a^m)$ and $Y = ((bab)^{k+r}, a^{m+2k})$, then the b-pair (X, Y) splits.*

PROOF. Note that $p = 3k + m + r$ and $q = 2k + 2r$. Let $L \in \mathfrak{s}_2(H_X) \cap \mathcal{O}_Y$ and let $v_i = L(e_i')$ for $0 \le i < q$. By inspection of the eigenspaces of $\mathrm{ad}(H_X)$, we see that $v_i \in \langle e_0, e_1, \dots, e_{k-1} \rangle$ for $0 \le i < k + r$ and $v_i \in \langle e_0, e_1, \dots, e_{p-k-1} \rangle$ for $k + r \le i < q$. It follows that the vectors v_i for $i < k + r$ belong to the radical of $L(V_b)$. Since $r_1(Y) = r_{2,b}(Y) = k+r$, the subspace $L(V_b)$ has dimension $k+r$ and is nondegenerate. Consequently, $v_i = 0$ for $i < k + r$. It follows that $L \in \mathfrak{s}_2(H_Y)$. By Proposition 1, $L \in \mathcal{O}_Y^{\mathrm{I}}$. Hence (X, Y) splits by Proposition 2. □

LEMMA 14. *If $X = ((ab)^3a, (aba)^k, b^{k+m+2})$ and $Y = ((ba)^2b, (ab, ba)^{k+1}, b^m)$, then the a-pair (X, Y) splits.*

PROOF. We have $p = 2k + 4$, $q = 2k + m + 5$, and

$$(H_X)_a = \operatorname{diag}(6, 2, \dots, 2, -2, \dots, -2, -6),$$
$$(H_X)_b = \operatorname{diag}(4, 0, 0, \dots, 0, 0, -4),$$
$$(H_Y)_a = \operatorname{diag}(2, 1, \dots, 1, -1, \dots, -1, -2),$$
$$(H_Y)_b = \operatorname{diag}(4, 1, \dots, 1, 0, 0, \dots, 0, 0, -1, \dots, -1, -4),$$

with $m + 1$ zeroes in $(H_Y)_b$. Let $L \in \mathfrak{s}_2(H_X) \cap \mathcal{O}_Y$ and $v_i' = L(e_i)$ for $0 \le i < p$. By inspecting the eigenspaces of $\operatorname{ad}(H_X)$ we see that $v_0' = 0$, $v_i' \in \langle e_0' \rangle$ for $0 < i \le k + 1$, and $v_i' \in \langle e_0', e_1', \dots, e_{q-2}' \rangle$ for $k + 1 < i < p - 1$.

Let us write $L_{ab} = (\xi_{ij})$ with $1 \le i \le p$ and $1 \le j \le q$. Since $L \in \mathfrak{s}_2(H_X)$, we must have $\xi_{i1} = 0$ for $2 \le i \le p$, $\xi_{ij} = 0$ for $k + 2 < i < p$ and $2 \le j < q$, and also $\xi_{pq} = 0$. Since $\operatorname{rank} L_{ab} = r_1(Y) = k + 3$, at least one of the entries ξ_{iq}, $k + 2 < i \le p$, is nonzero. As $\operatorname{rank}(L^2)_a = r_{2,a}(Y) = 1$ and

$$(L^2)_a = L_{ab}L_{ba} = -L_{ab}S_q{}^tL_{ab}S_p,$$

we deduce that $\xi_{11}\xi_{iq} = 0$ for $k + 2 < i < p$. Consequently, $\xi_{11} = 0$, and so $L(V_a) \subset \langle e_0', e_1', \dots, e_{q-2}' \rangle$. It follows that e_0' belongs to the radical of $L(V_a)$. The dimension of this radical is $k + 2$. By applying a suitable element of K_a^0, we may assume that $\xi_{k+3,q} \ne 0$ and $\xi_{iq} = 0$ for $i > k + 3$.

All entries of $(L^2)_b$ are 0 apart from the non-diagonal entries in the first row or last column. The $(1, j)$-th entry of this matrix is $-\xi_{k+3,q}\xi_{k+2,j}$ for $2 \le j < q$. Since this matrix is symmetric with respect to the side diagonal, it follows that all entries of $(L^4)_b$ are 0 except possibly the entry in the upper right hand corner which is equal to $\xi_{k+3,q}^2 f(v_{k+2}', v_{k+2}')$. Since $\operatorname{rank}(L^4)_b = r_{4,b}(Y) = 1$, this entry is not 0. We conclude that the vector v_{k+2}' is nonisotropic. By transforming L with a suitable element of K_a^0, we may assume that the vectors v_i' belong to the radical of $L(V_a)$ for $i > k+2$. As a side effect of this transformation, the entries $\xi_{i,q}$ for $k + 2 < i < p$ may become nonzero. By transforming L with a suitable element of K_b which fixes the vectors e_0' and e_{q-1}', we may further assume that the radical of $L(V_a)$ is $\langle e_0', e_1', \dots, e_{k+1}' \rangle$. Consequently, now $v_{k+2}' \in \langle e_0', e_1', \dots, e_{k+m+2}' \rangle$. Finally, by transforming L with the element of W_a which exchanges the vectors e_0 and e_{k+1} and also e_{k+2} and e_q, we obtain an element in $\mathfrak{s}_2(H_Y)$. By Propositions 1 and 2, (X, Y) splits. □

We can now verify the conjecture when p or q is small. In the proof we shall also use several lemmas from the next section.

THEOREM 3. *The conjecture is true if* $\min(p, q) \le 7$.

PROOF. Without any loss of generality we may assume that $p \ge q$. By Theorem 2, it suffices to show that all mp-pairs split. By Lemma 10, if (X, Y) and $(X + Z, Y + Z)$ are both b-pairs and if the latter pair splits then also the former does. We shall use this to reduce the consideration of b-pairs to large values of p by taking $Z = (a^k)$. (It suffices for us to take $p \ge 2q$.)

If $q \le 3$ then the pure pairs are listed in Example 6. If $q = 1$ there are no pure pairs.

TABLE 4. Unstable vertices of $\Gamma(p, 5)$.

p	label	X	π_X	dim
6	A	$(ab)^5a$	11	25
	B	$(ab)^3a, aba, b$	$7\cdot 3\cdot 1$	23
	C	$(ab)^3a, ab, ba$	$7\cdot 2^2$	22
	D	$(ba)^2b, (aba)^2$	$5\cdot 3^2$	21
	E	$abab, baba, aba$	$4^2\cdot 3$	20
	F	$(aba)^3, b^2$	$3^3\cdot 1^2$	18
	G	$(aba)^2, ab, ba, b$	$3^2\cdot 2^2\cdot 1$	17
	H	$aba, (ab, ba)^2$	$3\cdot 2^4$	15
8	I	$(ab)^3a, (aba)^2$	$7\cdot 3^2$	32
	J	$(aba)^4, b$	$3^4\cdot 1$	26
	K	$(aba)^3, ab, ba$	$3^3\cdot 2^2$	25
10	L	$(aba)^5$	3^5	35

Let $q = 2$. The pairs (C_2, D) and, for $p = 2$, the pair (C_1, D) split by Lemma 12. The pair (A, C_2) splits by Lemma 13.

Now let $q = 3$. Then (see Example 6) there is only one pure pair, namely the a-pair (E_1, F) when $p = 4$. One can show that this pair splits by using Lemma 12. Indeed, by that lemma, the a-pair $(X = ((aba)^2, b^2), Y = ((ab, ba)^2))$ splits. As $X > Z = (aba, ab, ba, b) > Y$, it follows that (X, Z) splits. As $X = E_1 + (b)$ and $Z = F + (b)$, (E_1, F) splits by Lemma 10.

Next let $q = 4$. The mp-pairs are exhibited in Figures 5 and 6. If $p = 4$ there are three ma-pairs and three mb-pairs and, by symmetry, it suffices to show only that the mb-pairs split. The a-pair (J_1, K), for $p = 6$, and the b-pair (M_3, O) split by Lemma 12. The b-pair (C, M_3) splits by Lemma 13, (A, E_2) by Lemma 15, and (E_2, H) by Lemma 14. In the case of (A, E_2) this may not be so obvious, so we give a few more details. By taking $k = 2$ and $r = p - 5$ in Lemma 15, we see that the b-pair (X, Y), with $X = (((ab)^4a)^2, a^{p-5})$ and $Y = (((ba)^3b)^2, a^{p-1})$, splits. Since

$$X = (A, (ab)^4a) > (E_2, (ab)^4a) > (E_2, (ba)^3b, a^2) = Y\,,$$

it follows that the pair $(A, (ab)^4a) > (E_2, (ab)^4a)$ splits. Now, by Lemma 10, the b-pair (A, E_2) splits.

Now let $q = 5$. In Table 4 we list the unstable vertices of $\Gamma(p, 5)$. They exist only for $p = 6, 8$, and 10 and they are all proper a-vertices. The notation in this table is the same as in Table 1. If $p = 6$, the ma-pairs are (B, D), (D, F), and (F, H). To prove the maximality of these pairs, it suffices to observe that

$$C > (ababa, aba, bab) > E > (abab, baba, a, a, b) > G\,.$$

The subdiagram $\Gamma_u(6, 5)$ is exhibited on Figure 7. If $p = 8$ there is only one a-pair, (J, K), and for $p = 10$ there are no a-pairs.

The pairs (F, H) and (J, K) split by Lemma 12, the pair (D, F) splits by Lemma 13, and (B, D) by Lemma 14. Perhaps the last claim needs an explanation. Let $\tilde{B} = (B, b^2)$,

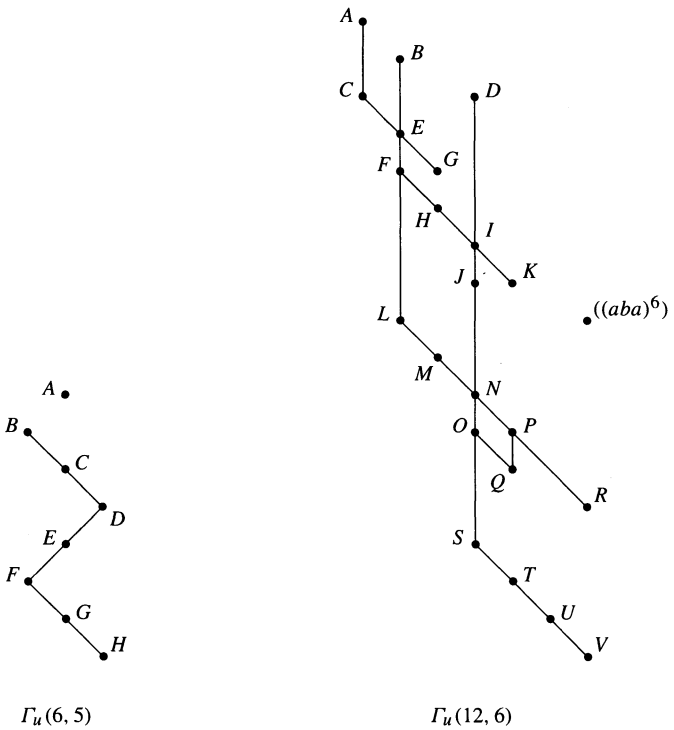


FIGURE 7.

$\tilde{D} = (D, b^2)$, and $Y = ((ba)^2b, (ab, ba)^2)$. Then the a-pair $(\tilde{B}, Y)$ in $\Gamma(6,7)$ splits by Lemma 14. Since $\tilde{B} > \tilde{D} > Y$, the a-pair $(\tilde{B}, \tilde{D})$ also splits. Consequently, (B, D) splits by Lemma 10.

Now let $q = 6$. We first list in Table 5 all a-vertices X in the diagrams $\Gamma(p, 6)$. They exist only for $p = 6, 8, 10$, and 12. We also list there the corresponding partitions π_X, the complex dimensions of the orbits $\mathcal{O}_X$, and introduce labels for the vertices, except that the isolated a-vertex in case $p = 12$ is not labelled. The b-vertices in the diagrams $\Gamma(p, 6)$ are listed separately in Table 6. We reuse the letters A–V to label the b-vertices. The orbit dimensions are now given only for $p = 12$.

When $p = 6$, because of symmetry, it suffices to prove only that the mb-pairs split, which will be dealt with later. The other ma-pairs are: (Q, S), (S, U), and (U, W) for $p = 8$, and (Y, Z) for $p = 10$. The pairs (U, W) and (Y, Z) split by Lemma 12, the pair (S, U) by Lemma 13, and (Q, S) by Lemma 14.

We consider now the mb-pairs. As explained in the beginning of this proof, we may assume that $p \geq 12$. By using Table 6, one can determine these pairs. For fixed $p \geq 12$, there are eight of them: (A, C), (B, L), (C, G), (D, S), (F, K), (L, R), (N, Q), and (S, V). (Some of them should be omitted or modified if $6 \leq p < 12$.) The maximality of these pairs follows from

TABLE 5. a-vertices of $\Gamma(p, 6)$.

p	label	X	π_X	dim
6	A	$(ab)^5a, b$	$11 \cdot 1$	30
	B	$(ba)^4b, aba$	$9 \cdot 3$	29
	C	$(ab)^3a, (ba)^2b$	$7 \cdot 5$	28
	D	$(ab)^3, (ba)^3$	6^2	27
	E	$(ab)^3a, aba, b^2$	$7 \cdot 3 \cdot 1^2$	27
	F	$(ab)^3a, ab, ba, b$	$7 \cdot 2^2 \cdot 1$	26
	G	$(ba)^2b, (aba)^2, b$	$5 \cdot 3^2 \cdot 1$	25
	H	$abab, baba, aba, b$	$4^2 \cdot 3 \cdot 1$	24
	I	$(ba)^2b, aba, ab, ba$	$5 \cdot 3 \cdot 2^2$	24
	J	$abab, baba, ab, ba$	$4^2 \cdot 2^2$	23
	K	$(aba)^3, b^3$	$3^3 \cdot 1^3$	21
	L	$(aba)^2, ab, ba, b^2$	$3^2 \cdot 2^2 \cdot 1^2$	20
	M	$aba, (ab, ba)^2, b$	$3 \cdot 2^4 \cdot 1$	18
	N	$(ab, ba)^3$	2^6	15
8	O	$(ab)^5a, aba$	$11 \cdot 3$	41
	P	$((ab)^3a)^2$	7^2	39
	Q	$(ab)^3a, (aba)^2, b$	$7 \cdot 3^2 \cdot 1$	37
	R	$(ab)^3a, aba, ab, ba$	$7 \cdot 3 \cdot 2^2$	36
	S	$(ba)^2b, (aba)^3$	$5 \cdot 3^3$	34
	T	$abab, baba, (aba)^2$	$4^2 \cdot 3^2$	33
	U	$(aba)^4, b^2$	$3^4 \cdot 1^2$	30
	V	$(aba)^3, ab, ba, b$	$3^3 \cdot 2^2 \cdot 1$	29
	W	$(aba)^2, (ab, ba)^2$	$3^2 \cdot 2^4$	27
10	X	$(ab)^3a, (aba)^3$	$7 \cdot 3^3$	48
	Y	$(aba)^5, b$	$3^5 \cdot 1$	40
	Z	$(aba)^4, ab, ba$	$3^4 \cdot 2^2$	39
12		$(aba)^6$	3^6	51

$$A > ((ab)^5a, aba, a^{p-8}) > E\,, \quad B > ((ab)^4a, (aba)^2, a^{p-9}) > G\,,$$
$$C > ((ba)^4b, aba, a^{p-6}) > F\,, \quad D > ((ababa)^2, (aba)^2, a^{p-10}) > K\,,$$
$$E > ((ab)^3a, babab, a^{p-6}) > H\,, \quad F > ((ba)^3b, (aba)^2, a^{p-7}) > M\,,$$
$$H > (ababa, babab, aba, a^{p-7}) > J > (ababa, (aba)^2, bab, a^{p-8}) > P,$$
$$M > (babab, aba, bab, a^{p-5}) > O > ((aba)^2, (bab)^2, a^{p-6}) > T\,.$$

The subdiagram $\Gamma_u(12, 6)$ is exhibited on Figure 7. It contains a single isolated a-vertex, $((aba)^6)$.

The pair (A, C) splits by Lemma 15, (D, S) by Lemma 13, (S, V) by Lemma 12, and (L, R) by Lemma 14. The pairs (C, G) and (F, K) split by Lemma 16, and (B, L) by Lemma 18 (see the next section). It remains to consider the b-pair (N, Q).

For the reader's convenience we list the relevant characteristics:

$$H_N = \mathrm{diag}(4, 0, 0, \dots, 0, 0, -4, 2, 2, 2, -2, -2, -2)\,,$$
$$H_Q = \mathrm{diag}(3, 1, 1, 0, 0, \dots, 0, 0, -1, -1, -3, 3, 1, 1, -1, -1, -3)\,.$$

TABLE 6. b-vertices of $\Gamma(p, 6)$.

label	X	π_X	dim
A	$(ab)^6a, a^{p-7}$	$13 \cdot 1^{p-7}$	66
B	$(ab)^4a, (ab)^2a, a^{p-8}$	$9 \cdot 5 \cdot 1^{p-8}$	64
C	$(ba)^5b, a^{p-5}$	$11 \cdot 1^{p-5}$	60
D	$(ababa)^3, a^{p-9}$	$5^3 \cdot 1^{p-9}$	60
E	$(ab)^4a, bab, a^{p-6}$	$9 \cdot 3 \cdot 1^{p-6}$	59
F	$(ba)^3b, (ab)^2a, a^{p-6}$	$7 \cdot 5 \cdot 1^{p-6}$	58
G	$(ab)^4a, ab, ba, a^{p-7}$	$9 \cdot 2^2 \cdot 1^{p-7}$	58
H	$(ab)^3, (ba)^3, a^{p-6}$	$6^2 \cdot 1^{p-6}$	57
I	$(ababa)^2, bab, a^{p-7}$	$5^2 \cdot 3 \cdot 1^{p-7}$	56
J	$(ab)^2a, abab, baba, a^{p-7}$	$5 \cdot 4^2 \cdot 1^{p-7}$	55
K	$(ababa)^2, ab, ba, a^{p-8}$	$5^2 \cdot 2^2 \cdot 1^{p-8}$	55
L	$(ba)^3b, bab, a^{p-4}$	$7 \cdot 3 \cdot 1^{p-4}$	51
M	$(ba)^3b, ab, ba, a^{p-5}$	$7 \cdot 2^2 \cdot 1^{p-5}$	50
N	$(ab)^2a, (bab)^2, a^{p-5}$	$5 \cdot 3^2 \cdot 1^{p-5}$	49
O	$abab, baba, bab, a^{p-5}$	$4^2 \cdot 3 \cdot 1^{p-5}$	48
P	$ababa, bab, ab, ba, a^{p-6}$	$5 \cdot 3 \cdot 2^2 \cdot 1^{p-6}$	48
Q	$abab, baba, ab, ba, a^{p-6}$	$4^2 \cdot 2^2 \cdot 1^{p-6}$	47
R	$(ab)^2a, (ab, ba)^2, a^{p-7}$	$5 \cdot 2^4 \cdot 1^{p-7}$	46
S	$(bab)^3, a^{p-3}$	$3^3 \cdot 1^{p-3}$	39
T	$(bab)^2, ab, ba, a^{p-4}$	$3^2 \cdot 2^2 \cdot 1^{p-4}$	38
U	$bab, (ab, ba)^2, a^{p-5}$	$3 \cdot 2^4 \cdot 1^{p-5}$	36
V	$(ab, ba)^3, a^{p-6}$	$2^6 \cdot 1^{p-6}$	33

We choose a representative $E_N \in \mathcal{O}_N^{\mathrm{I}}$ such that $(E_N)_{ab}$ is the $\{0, 1\}$-matrix having 1's at the positions $(1, 2)$, $(2, 4)$, $(3, 5)$, $(p-2, 5)$, and $(p-1, 6)$. Let $L \in \overline{B \cdot E_N} \cap \mathcal{O}_Q$ and let $v_i = L(e_i')$ for $0 \le i < 6$. As the Borel subgroup B consists of upper triangular matrices, we have $v_0 = 0$, $v_1, v_2 \in \langle e_0 \rangle$, $v_3 \in \langle e_0, e_1 \rangle$, $v_4 \in \langle e_0, e_1, \dots, e_{p-3} \rangle$, and $v_5 \in \langle e_0, e_1, \dots, e_{p-2} \rangle$. Assume that the vector v_4 is isotropic. By applying a suitable element of K_a, which fixes the four vectors e_0, e_1, e_{p-2}, and e_{p-1}, we may assume that $v_4 \in \langle e_0, e_1, e_2 \rangle$. By inspecting the eigenspaces of $\mathrm{ad}(H_Q)$, we see that L belongs to $\mathfrak{s}_2(H_Q)$. Hence (N, Q) splits by Propositions 1 and 2.

It remains to consider the case where v_4 is nonisotropic. Write $v_1 = \xi e_0$, $v_2 = \eta e_0$, and $v_3 = \alpha e_0 + \beta e_1$. As L_{ab} has rank 4, $\beta \ne 0$. The entry in the upper right hand corner of the matrix $(L^4)_a$ is equal to $-\xi^2 f(v_4, v_4)$. Since $L^4 = 0$, this entry must be 0. This forces $\xi = 0$. As L_{ab} has rank 4, we must have $\eta \ne 0$. The 6×6 matrix $(L^2)_b$ has all entries zero except those in the 3×3 block in the upper right hand corner which has the form:

$$-\begin{pmatrix} f(v_5, v_3) & f(v_5, v_4) & f(v_5, v_5) \\ 0 & f(v_4, v_4) & f(v_4, v_5) \\ 0 & 0 & f(v_3, v_5) \end{pmatrix}.$$

As $r_{2,b}(Q) = 2$, this block must have rank 2. Since $f(v_4, v_4) \ne 0$, we have $f(v_5, v_3) = 0$.

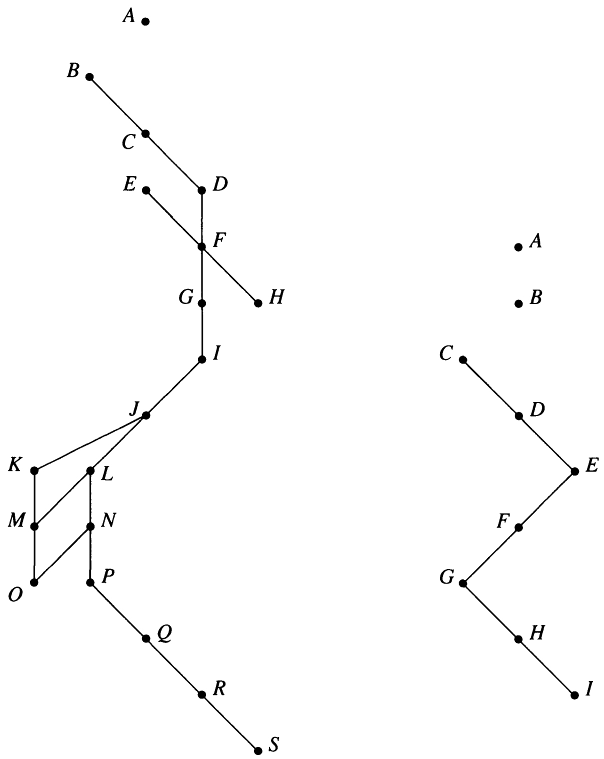
As $\beta \neq 0$, this implies that $v_5 \in \langle e_0, e_1, \dots, e_{p-3} \rangle$. By subtracting a suitable scalar multiple of v_4 from v_5, we may assume that v_5 is isotropic. By transforming L with the element of W_b that exchanges e'_0 and e'_1 and also e'_4 and e'_5, we reduce this case to the previous case where v_4 is isotropic.

This completes the proof of the assertion that the b-pair (N, Q) splits.

Finally let $q = 7$. Then every unstable vertex is necessarily a proper a-vertex. They exist only for $p = 8$, 10, 12, and 14. The subdiagrams $\Gamma_u(p, 7)$ for $p = 8$ and $p = 10$ are shown in Figure 8. If $p = 14$, then there is only one a-vertex, namely $((aba)^7)$, and so there are no a-pairs. If $p = 12$, then there are three a-vertices:

$$A = ((ab)^3 a, (aba)^4), \quad B = ((aba)^6, b), \quad C = ((aba)^5, ab, ba),$$

and only one a-pair, namely (B, C). This pair splits by Lemma 12. If $p = 10$, then there are nine a-vertices:



$\Gamma_u(8, 7)$ $\Gamma_u(10, 7)$

FIGURE 8.

TABLE 7. Unstable vertices of $\Gamma(8,7)$.

label	X	π_X	dim
A	$(ab)^7a$	15	49
B	$(ab)^5a, aba, b$	$11\cdot 3\cdot 1$	47
C	$(ab)^5a, ab, ba$	$11\cdot 2^2$	46
D	$(ba)^4b, (aba)^2$	$9\cdot 3^2$	45
E	$((ab)^3a)^2, b$	$7^2\cdot 1$	45
F	$(ab)^3a, (ba)^2b, aba$	$7\cdot 5\cdot 3$	44
G	$(ab)^3a, abab, baba$	$7\cdot 4^2$	43
H	$ababab, bababa, aba$	$6^2\cdot 3$	43
I	$(ab)^3a, (aba)^2, b^2$	$7\cdot 3^2\cdot 1^2$	42
J	$(ab)^3a, aba, ab, ba, b$	$7\cdot 3\cdot 2^2\cdot 1$	41
K	$(ab)^3a, (ab, ba)^2$	$7\cdot 2^4$	39
L	$(ba)^2b, (aba)^3, b$	$5\cdot 3^3\cdot 1$	39
M	$(ba)^2b, (aba)^2, ab, ba$	$5\cdot 3^2\cdot 2^2$	38
N	$abab, baba, (aba)^2, b$	$4^2\cdot 3^2\cdot 1$	38
O	$abab, baba, aba, ab, ba$	$4^2\cdot 3\cdot 2^2$	37
P	$(aba)^4, b^3$	$3^4\cdot 1^3$	34
Q	$(aba)^3, ab, ba, b^2$	$3^3\cdot 2^2\cdot 1^2$	33
R	$(aba)^2, (ab, ba)^2, b$	$3^2\cdot 2^4\cdot 1$	31
S	$aba, (ab, ba)^3$	$3\cdot 2^6$	28

$$
\begin{array}{lll}
A = ((ab)^5a, (aba)^2) & B = (((ab)^3a)^2, aba) & C = ((ab)^3a, (aba)^3, b) \\
D = ((ab)^3a, (aba)^2, ab, ba) & E = ((ba)^2b, (aba)^4) & F = (abab, baba, (aba)^3) \\
G = ((aba)^5, b^2) & H = ((aba)^4, ab, ba, b) & I = ((aba)^3, (ab, ba)^2)
\end{array}
$$

and three ma-pairs, namely (C, E), (E, G), and (G, I). To prove the maximality of these pairs, it suffices to observe that

$$D > (ababa, (aba)^3, bab) > F > ((aba)^4, bab, a, b) > H\,.$$

The pair (C, E) splits by Lemma 14, (E, G) by Lemma 13, and (G, I) by Lemma 12.

It remains to consider the case $p = 8$. Then there are nineteen a-vertices and we list all of them in Table 7.

There are seven ma-pairs: (B, D), (D, I), (E, H), (I, M), (L, O), (L, P), and (P, S). The maximality of these pairs follows from

$$
\begin{aligned}
& C > ((ab)^4a, aba, bab) > F\,, \quad D > ((ba)^3b, (ab)^2a, aba) > H\,, \\
& E > ((ab)^3a, (ab)^2a, bab) > G > ((ab)^3a, aba, bab, a, b) > J\,, \\
& J > ((ababa)^2, aba, ab, ba, b) > N > ((aba)^3, (bab)^2) > Q\,, \\
& K > (ababa, aba, bab, ab, ba) > O\,.
\end{aligned}
$$

The pair (P, S) splits by Lemma 12, the pair (L, P) by Lemma 13 (with a and b switched), and the pair (I, M) by Lemma 14.

We consider first the pair (B, D). We have

$$
\begin{aligned}
H_B &= \operatorname{diag}(10, 6, 2, 2, -2, -2, -6, -10, 8, 4, 0, 0, 0, -4, -8)\,, \\
H_D &= \operatorname{diag}(6, 2, 2, 2, -2, -2, -2, -6, 8, 4, 0, 0, 0, -4, -8)\,.
\end{aligned}
$$

We choose the representative $E_B \in {}^{\mathrm{I}}\mathcal{O}_B$ such that $(E_B)_{ab}$ is the $\{0, 1\}$-matrix having 1's at the positions $(4, 5)$, $(5, 6)$, and at the positions (i, i) for $i = 1, 2, 3, 6, 7$. Let $L \in \overline{B \cdot H_B} \cap \mathcal{O}_D$ and let

$$v_i' = L(e_i) = \xi_{i,0}e_0' + \xi_{i,1}e_1' + \cdots + \xi_{i,6}e_6'.$$

We have $v_0' = 0$; $\xi_{1,j} = 0$ for $j > 0$; $\xi_{i,j} = 0$ for $i = 2, 3$ and $j > 1$, $\xi_{4,j} = 0$ for $j > 2$; $\xi_{5,j} = 0$ for $j = 5, 6$; and $\xi_{6,6} = 0$. As $r_1(D) = 6$, the subspace $L(V_a)$ has dimension 6, and so $\xi_{4,2} \neq 0$. The condition $\operatorname{rank}(L^8)_b = r_{8,b}(D) = 1$ implies that $\xi_{1,0}$, $\xi_{2,1}$, and $\xi_{6,5}$ are all nonzero. As $\xi_{2,1}$ and $\xi_{6,5}$ are nonzero, we may assume that $\xi_{3,1} = \xi_{7,5} = 0$. Now the same condition implies that the vector v_5' is nonisotropic. The condition $\operatorname{rank}(L^2)_a = r_{2,a}(D) = 5$ implies that $\xi_{5,4}\xi_{7,6} = 0$. Next the condition $r_{8,a}(D) = 0$ implies that $(L^8)_a = 0$. By computing the $(1, 7)$-entry, we find that $\xi_{7,6} = 0$. By applying the permutation $(1, 2)(7, 8) \in W_a$ to L we obtain an element of $\mathfrak{s}_2(H_D)$. We now invoke Proposition 1 to conclude that $L \in {}^{\mathrm{I}}\mathcal{O}_D$. Hence (B, D) splits by Proposition 2.

Next let us consider the pair (D, I). We have

$$H_D = \operatorname{diag}(6, 2, 2, 2, -2, -2, -2, -6, 8, 4, 0, 0, 0, -4, -8),$$
$$H_I = \operatorname{diag}(6, 2, 2, 2, -2, -2, -2, -6, 4, 0, 0, 0, 0, 0, -4).$$

Let $L \in \mathfrak{s}_2(H_D) \cap \mathcal{O}_I$ and let $v_i' = L(e_i)$ for $0 \leq i < 8$. We have $v_0' \in \langle e_0' \rangle$; $v_1', v_2', v_3' \in \langle e_0', e_1' \rangle$; $v_4', v_5', v_6' \in \langle e_0', e_1', e_2', e_3', e_4' \rangle$, and $v_7' \in \langle e_0', e_1', e_2', e_3', e_4', e_5' \rangle$. As $r_1(I) = r_{2,a}(I) = 5$, the subspace $L(V_a)$ has dimension 5 and is nondegenerate. Since e_0' is orthogonal to $L(V_a)$, it follows that $v_0' = 0$. By transforming L with the element of W_b that exchanges e_0' and e_1' and also e_5' and e_6', we obtain an element in $\mathfrak{s}_2(H_I)$. Hence (D, I) splits by Propositions 1 and 2.

We consider next the pair (E, H). Let $X = (E, b)$ and $Y = ((ab)^3, (ba)^3, ab, ba)$. As $X > (H, b) > Y$, it suffices to prove that (X, Y) splits. We have

$$H_X = \operatorname{diag}(6, 6, 2, 2, -2, -2, -6, -6, 4, 4, 0, 0, 0, 0, -4, -4),$$
$$H_Y = \operatorname{diag}(5, 3, 1, 1, -1, -1, -3, -5, 5, 3, 1, 1, -1, -1, -3, -5).$$

We choose the representative $E_X \in {}^{\mathrm{I}}\mathcal{O}_X$ such that $(E_X)_{ab}$ is the $\{0, 1\}$-matrix having 1's at the positions $(1, 1)$, $(2, 2)$, $(3, 3)$, $(4, 6)$, $(5, 7)$, and $(6, 8)$. Let $L \in \overline{B \cdot H_X} \cap \mathcal{O}_Y$ and let

$$v_i' = L(e_i) = \xi_{i,0}e_0' + \xi_{i,1}e_1' + \cdots + \xi_{i,7}e_7'.$$

We have $v_0' = v_1' = 0$; $\xi_{i,j} = 0$ for $i = 2, 3, 4$ and $j > i - 2$, $\xi_{5,j} = 0$ for $j = 6, 7$; and $\xi_{6,7} = 0$. The condition $\operatorname{rank}(L^4)_b = r_{4,b}(Y) = 2$ implies that $\xi_{2,0}$, $\xi_{3,1}$, $\xi_{4,2}$, and $\xi_{5,5}$ are all nonzero. The condition $\operatorname{rank}(L^2)_a = r_{2,a}(Y) = 4$ now implies that $\xi_{6,6}\xi_{7,7} = 0$.

If $\xi_{6,6} = 0$ then $r_{4,a}(Y) = 2$ implies that $\xi_{7,6}$ or $\xi_{7,7}$ is not 0. As $\xi_{5,5} \neq 0$, we may assume that also $\xi_{6,5} = 0$. As $r_1(Y) = 6$, the vectors v_i' for $2 \leq i < 8$ form a basis of $L(V_a)$. Since $r_{2,a}(Y) = 4$, $L(V_a)$ has 2-dimensional radical. Hence the maximal totally isotropic subspaces of $L(V_a)$ have dimension 4. Let M be such a subspace containing $\langle v_2', v_3', v_4' \rangle = \langle e_0', e_1', e_2' \rangle$. Thus $M = \langle e_0', e_1', e_2', v \rangle$, where v is a linear combination of v_5', v_6', and v_7'. Since v is orthogonal to e_0' and e_1' and at least one of $\xi_{7,6}$ and $\xi_{7,7}$ is not 0, it follows that v must be a linear combination of v_5' and v_6' only. Since v is also orthogonal to e_2' and $\xi_{5,5} \neq 0$, it follows that in fact v is a scalar multiple of v_6'. This proves that v_6' is isotropic, i.e., one of the coordinates $\xi_{6,3}$ and $\xi_{6,4}$ must be 0. By using the transposition $(4,5) \in W_b^*$ (if necessary), we may assume that $\xi_{6,4} = 0$, and, consequently, $\xi_{6,3} \neq 0$. If $\xi_{7,7} \neq 0$, we may assume that $\xi_{7,6} = 0$. By applying the permutations $(2,3)(6,7) \in W_a$ and $(1,2)(7,8) \in W_b$ to L, we obtain an element of $\mathfrak{s}_2(H_Y)$.

If $\xi_{6,6} \neq 0$, then $\xi_{7,7} = 0$ and we may assume that $\xi_{7,5} = \xi_{7,6} = 0$. We now repeat the argument from the previous paragraph to show that v_7' must be isotropic. We also may assume that $\xi_{7,4} = 0$, and, consequently, $\xi_{7,3} \neq 0$. By applying the permutation $(3,2,1)(6,7,8) \in W_a$ to L we obtain an element of $\mathfrak{s}_2(H_Y)$.

By Proposition 1, this proves that (in both cases) $L \in {}^{\mathrm{I}}\mathcal{O}_Y$. Hence (X,Y) splits by Proposition 2.

Finally, we consider the pair (L, O). Let $X = (L, b)$ and $Y = (abab, baba, (ab, ba)^2)$. As $X > (O, b) > Y$, it suffices to prove that (X,Y) splits. We have

$$H_X = \operatorname{diag}(2,2,2,2,-2,-2,-2,-2,4,0,0,0,0,0,0,-4)\,,$$
$$H_Y = \operatorname{diag}(3,1,1,1,-1,-1,-1,-3,3,1,1,1,-1,-1,-1,-3)\,.$$

Let $L \in \mathfrak{s}_2(H_X) \cap \mathcal{O}_Y$ and let $v_i' = L(e_i)$ for $0 \leq i < 8$. We have $v_i' \in \langle e_0' \rangle$ for $0 \leq i < 4$ and $v_i' \in \langle e_0', e_1', \ldots, e_6' \rangle$ for $4 \leq i < 8$. As $r_1(Y) = 5$, the subspace $L(V_a)$ has dimension 5, and so at least one of the vectors v_0', v_1', v_2', v_3' is nonzero and $e_0', v_4', v_5', v_6', v_7'$ form a basis of $L(V_a)$. As $r_{2,a}(Y) = 2$, this subspace has 3-dimensional radical and e_0' obviously belongs to the radical. By applying a suitable element of K_a^0 which fixes e_0' and e_7', we may assume that $\langle e_0', v_4', v_5' \rangle$ is the radical of $L(V_a)$ and that $\langle e_0', v_4', v_5', v_6' \rangle$ is a maximal totally isotropic subspace of $L(V_a)$. Next, by applying a suitable element of K_b, we may assume that $v_i' \in \langle e_0', \ldots, e_{i-3}' \rangle$ for $i = 4,5,6$. Now the radical of $L(V_a)$ is $\langle e_0', e_1', e_2' \rangle$. Consequently, $v_7' = \xi_0 e_0' + \cdots + \xi_4 e_4'$ with $\xi_4 \neq 0$. If $v_0' = 0$, then $L \in \mathfrak{s}_2(H_Y)$. Assume now that $v_0' \neq 0$. Then we may assume that $v_1' = v_2' = v_3' = 0$. The condition $\operatorname{rank}(L^4)_b = r_{4,b}(Y) = 0$ implies that v_7' is isotropic, i.e., $\xi_3 = 0$. We now apply the permutation $(1,2)(7,8) \in W_a$ and then the permutation $(4,5) \in W_b^*$. The new element L then belongs to $\mathfrak{s}_2(H_Y)$. By Proposition 1 we conclude that $L \in {}^{\mathrm{I}}\mathcal{O}_Y$. Hence (X,Y) splits by Proposition 2.

This completes the proof of the theorem. □

6. Disjoint minimal pure pairs. We say that the ab-diagrams $X, Y \in \mathcal{X} = \mathcal{X}(p,q)$ are *disjoint* if they have no common rows. In general, if $X, Y \in \mathcal{X}$ then we have $X = P + Z$, $Y = Q + Z$, where $P, Q \in \mathcal{X}(p_1, q_1)$, $Z \in \mathcal{X}(p - p_1, q - q_1)$, and X and Y are disjoint. If (X,Y) is a pure pair in $\Gamma = \Gamma(p,q)$, then (P,Q) is also a pure pair (of the same kind). We say that a pure pair (X,Y) is *minimal* if $X \to Y$. (Recall that the last condition means that there are no vertices U in Γ such that $X > U > Y$.) If $X = P + Z$, $Y = Q + Z$, as above, and (X,Y) is a minimal pure pair in Γ, then (P,Q) is necessarily a minimal pure pair in $\Gamma(p_1,q_1)$. The converse of this statement is false as shown by the following counterexample: $P = ((ba)^3 b, a)$, $Q = ((ab)^2 a, bab)$, and $Z = ((ab)^2 a)$. Then (P,Q) is a minimal b-pair

TABLE 8. Disjoint minimal b-pairs in $\Gamma(p,q)$.

No.	X	Y	
1	$(ab)^{2k}a, (ba)^{2k-1}b$	$(ab)^{2k}, (ba)^{2k}$	$k \geq 1$
2	$(ba)^{2k+1}b, (ab)^{2k}a$	$(ab)^{2k+1}, (ba)^{2k+1}$	$k \geq 0$
3	$(ab)^{2k}, (ba)^{2k}, (ab)^{2m}, (ba)^{2m}$	$((ba)^{2k-1}b)^2, ((ab)^{2m}a)^2$	$k > m \geq 0$
4	$(ab)^{2k+1}, (ba)^{2k+1}, (ab)^{2m+1}, (ba)^{2m+1}$	$((ab)^{2k}a)^2, ((ba)^{2m+1}b)^2$	$k > m \geq 0$
5	$(ab)^{2k}a, (ba)^{2m-1}b$	$(ba)^{2k-1}b, (ab)^{2m}a$	$k > m \geq 1$
6	$(ba)^{2k+1}b, (ab)^{2m}a$	$(ab)^{2k}a, (ba)^{2m+1}b$	$k > m \geq 0$
7	$(ab)^{2k}a, (ab)^{2m}, (ba)^{2m}$	$(ba)^{2k-1}b, ((ab)^{2m}a)^2$	$k > m \geq 0$
8	$(ba)^{2k+1}b, (ab)^{2m+1}, (ba)^{2m+1}$	$(ab)^{2k}a, ((ba)^{2m+1}b)^2$	$k > m \geq 0$
9	$(ab)^{2k}, (ba)^{2k}, (ba)^{2m-1}b$	$((ba)^{2k-1}b)^2, (ab)^{2m}a$	$k > m \geq 1$
10	$(ab)^{2k+1}, (ba)^{2k+1}, (ab)^{2m}a$	$((ab)^{2k}a)^2, (ba)^{2m+1}b$	$k > m \geq 0$

in $\Gamma(4,4)$. On the other hand $(X = P + Z, Y = Q + Z)$, which is a b-pair in $\Gamma(7,6)$, is not minimal. Indeed, we have $X > U > Y$ with $U = ((ab)^3, (ba)^3, a)$.

In this section we consider the minimal pure pairs (X, Y) in Γ that are disjoint. To be specific, we shall consider only the disjoint minimal b-pairs in Γ. The list of all such pairs (X, Y), which can be extracted from [8] or [6], is given in Table 8. It consists of 10 one- or two-parameter families. The parameters are the nonnegative integers k and m.

If our conjecture is true, then all pure pairs must split. In particular all the pairs listed in Table 8 must split. In order to provide some additional evidence for the conjecture, we prove below that this is indeed the case.

THEOREM 4. *All disjoint minimal pure pairs split.*

The proof is contained in the series of lemmas that follow. Some of these lemmas are stronger than what is needed for this theorem. (Some of them have been used in the proof of Theorem 3 in the previous section.)

LEMMA 15. *If $X = (((ab)^{2k}a)^2, a^r)$ and $Y = (((ba)^{2k-1}b)^2, a^{r+4})$, then the b-pair (X, Y) splits. Consequently the b-pairs of the first family of Table 8 split.*

PROOF. We have $p = 4k + r + 2$, $q = 4k$, and

$$(H_X)_a = \mathrm{diag}(q, q, q-4, q-4, \dots, 4, 4, 0, \dots, 0, -4, -4, \dots, 4-q, 4-q, -q, -q),$$
$$(H_X)_b = \mathrm{diag}(q-2, q-2, q-6, q-6, \dots, 2, 2, -2, -2, \dots, 6-q, 6-q, 2-q, 2-q),$$
$$(H_Y)_a = \mathrm{diag}(q-4, q-4, \dots, 8, 8, 4, 4, 0, \dots, 0, -8, -8, -4, -4, \dots, 4-q, 4-q),$$

and $(H_Y)_b = (H_X)_b$. (All the eigenvalues of H_X and H_Y have multiplicity 2 except that 0 has multiplicity $r + 2$ in H_X and $r + 6$ in H_Y.)

Let $L \in \mathfrak{s}_2(H_X) \cap \mathcal{O}_Y$ and let $v_i = L(e_i')$ for $0 \leq i < q$. Let us introduce the subspaces $V_i = \langle v_0, v_1, \dots, v_i \rangle$ for $0 \leq i < q$. By inspecting the eigenspaces of $\mathrm{ad}(H_X)$, we see that $V_{2i+1} \subset \langle e_0, e_1, \dots, e_{2i+1} \rangle$ for $0 \leq i < k$ and $V_{2i+1} \subset \langle e_0, e_1, \dots, e_{2i+r+1} \rangle$ for $k \leq i < 2k$. This implies that V_{2i+1} is orthogonal to V_{2j+1} if $i + j \leq 2k$. In particular, V_1 is contained in the radical of $L(V_b) = V_{q-1}$. As $r_1(Y) = r_{2,b}(Y) = q - 2$, this subspace is nondegenerate and has dimension $q-2$. It follows that $V_1 = 0$, i.e., $v_0 = v_1 = 0$, and that the vectors $v_2, v_3, \ldots, v_{q-1}$ are linearly independent. By Witt's theorem, there exists an $x \in K_a$ such that $x(V_{2i+1}) = \langle e_0, e_1, \ldots, e_{2i-1}\rangle$ for $0 < i < k$. The above orthogonality conditions then imply that $x(V_{2i+1}) \subset \langle e_0, e_1, \ldots, e_{2i+r+3}\rangle$ for $k < i < 2k$. By inspecting the eigenspaces of $\mathrm{ad}(H_Y)$, we conclude that $\mathrm{Ad}(x)(L) \in \mathfrak{s}_2(H_Y)$. Hence $L \in \mathcal{O}_Y^{\mathrm{I}}$ by Proposition 1. So, (X, Y) splits by Proposition 2.

The b-pairs in the first family of Table 8 split because for $r = 0$ we have

$$X > ((ab)^{2k}a, (ba)^{2k-1}b, a^2) > ((ab)^{2k}, (ba)^{2k}, a^2) > Y\,. \qquad \square$$

LEMMA 16. *If $X = ((ba)^{2k+1}b, (ab)^{2k}a, a^{r+2})$ and $Y = (((ab)^{2k}a)^2, ab, ba, a^r)$, then the b-pair (X, Y) splits. Consequently, the b-pairs of the second family of Table 8 split.*

PROOF. We have $p = 4k + r + 4$, $q = 4k + 2$. The eigenvalues of $(H_X)_a$ are the integers $4i$ for $-k \le i \le k$. Each of them has multiplicity 2, except that 0 has multiplicity $r + 4$. The eigenvalues of $(H_X)_b$ are the integers $q - 4i$ for $0 \le i \le 2k + 1$. Each of them has multiplicity 2, except for $\pm q$ which are simple. Hence

$$\begin{aligned}(H_X)_a &= \mathrm{diag}(4k, 4k, 4k-4, 4k-4, \ldots, 4, 4, 0, \ldots, 0, -4, -4, \ldots, -4k, -4k)\,,\\ (H_X)_b &= \mathrm{diag}(q, q-4, q-4, \ldots, 2, 2, -2, -2, \ldots, 4-q, 4-q, -q)\,.\end{aligned}$$

We choose the representative $E_X \in \mathcal{O}_X^{\mathrm{I}}$ such that $(E_X)_{ab}$ is the $\{0,1\}$-matrix with 1's at the positions $(i, i+1)$ for $1 \le i \le 2k+1$, positions $(p-i, q-i+1)$ for $1 \le i \le 2k$, and also at the position (p, q). Let $L \in \overline{B \cdot E_X} \cap \mathcal{O}_Y$ and let

$$v_i = L(e_i') = \xi_{i,0}e_0 + \xi_{i,1}e_1 + \cdots + \xi_{i,p-1}e_{p-1}$$

for $0 \le i < q$. Since the Borel subgroup $B = B_a \times B_b$ consists of upper triangular matrices, we see that $v_0 = 0$, $\xi_{i,j} = 0$ if $i \le j$ and $1 \le i \le 2k+1$, and also that $\xi_{q-i,p-j} = 0$ if $i \ge j$ and $1 < i \le 2k$.

It is easy to check that $L^{4k}(e_i) = 0$ for $i < p-2$, $L^{4k}(e_{p-2}) = \lambda e_0$, and $L^{4k}(e_{p-1}) = \lambda e_1$, where λ is the product of the coefficients $\xi_{i,i-1}$ for $1 \le i \le 2k+1$ and $\xi_{q-i,p-i-1}$ for $1 < i \le 2k$. (The fact that these two coefficients λ are the same follows from the symmetry property of $(L^2)_a$ mentioned in Section 3.) From the fact that $\mathrm{rank}\,(L^{4k})_a = r_{4k,a}(Y) = 2$ it follows that $\lambda \ne 0$ and so all of the mentioned coefficients are nonzero. Since $r_{4k,b}(Y) = 0$, we have $(L^{4k})_b = 0$. By computing $L^{4k}(e'_{q-3})$, we find that $\xi_{q-1,p-1} = 0$. Then, by computing $L^{4k}(e'_{q-1})$, we see that also $\xi_{q-1,p-2} = 0$. By subtracting from v_{q-1} a suitable linear combination of $v_{2k+2}, v_{2k+3}, \ldots, v_{q-2}$, we may assume that also $\xi_{q-1,p-i-1} = 0$ for $1 < i \le 2k$. This means that v_{q-1} is orthogonal to v_i for $0 < i \le 2k+1$. As $\dim L(V_b) = r_1(Y) = q-1$ and $\mathrm{rank}\,(L^2)_b = r_{2,b}(Y) = q-4$, the maximal totally isotropic subspaces of $L(V_b)$ have dimension $2k+2$. Let $M' = \langle M, v\rangle$ be a maximal totally isotropic subspace of $L(V_b)$ containing the totally isotropic subspace

$$M = \langle v_1, v_2, \ldots, v_{2k+1}\rangle = \langle e_0, e_1, \ldots, e_{2k}\rangle\,.$$

We may assume that $v = \zeta_{q-1}v_{q-1}+\zeta_{q-2}v_{q-2}+\cdots+\zeta_{2k+2}v_{2k+2}$. Since v is orthogonal to M and $\xi_{q-i,p-i-1} \neq 0$ for $1 < i \leq 2k$, we must have $\zeta_{q-i} = 0$ for the same i's. Hence $\zeta_{q-1} \neq 0$ and v_{q-1} must be isotropic. By using Witt's theorem and by applying a suitable element of K_a which fixes the vectors e_i and e_{p-i-1} for $0 \leq i \leq 2k+1$, we may assume that $\xi_{q-1,i} = 0$ for $i > 2k+1$. Since $\xi_{q-1,2k+1}$ and $\xi_{2k+2,p-2k-1}$ are nonzero, we may assume that $\xi_{q-1,2k}$ and $\xi_{2k+2,p-2k-2}$ are 0. Next we apply the permutation $(2k+1, 2k+2)(p-2k-1, p-2k) \in W_a$ to L. After this modification, $\xi_{q-1,2k}$, $\xi_{2k+1,2k+1}$, and $\xi_{2k+2,p-2k-2}$ are nonzero, while $\xi_{q-1,2k+1} = \xi_{2k+2,p-2k-1} = 0$. Finally we apply to L the permutation

$$(1, 2k+1, 2k, \ldots, 2)(2k+2, 2k+3, \ldots, q) \in W_b .$$

By inspecting the eigenspaces of $\mathrm{ad}(H_Y)$, we find that the new L belongs to $\mathfrak{s}_2(H_Y)$. By Proposition 1 we conclude that $L \in \mathcal{O}_Y^{\mathrm{I}}$. Hence (X, Y) splits by Proposition 2.

For $r = 0$ we have $X = ((ba)^{2k+1}b, (ab)^{2k}a, a^2) > ((ab)^{2k+1}, (ba)^{2k+1}, a^2) > Y$. It follows that the b-pairs in the second family of Table 8 split. □

LEMMA 17. *The b-pairs of the third and fourth family of Table* 8 *split.*

PROOF. Let us switch the letters a and b in X and Y of the fourth family. We then combine the resulting a-pairs with the b-pairs of the third family. We obtain a single family of pure pairs (X, Y) with

$$X = ((ab)^k, (ba)^k, (ab)^m, (ba)^m), \quad Y = (((ba)^{k-1}b)^2, ((ab)^m a)^2),$$

where $k > m \geq 0$ and $k \equiv m \pmod 2$. Thus $p = q = 2k + 2m$ is divisible by 4, and the difference $r = k - m$ is even. The eigenvalues of $(H_X)_a$ are the odd integers $2i - 1$ for $-k < i \leq k$. Those for $-m < i \leq m$ have multiplicity 2, while the other eigenvalues are simple. As matrices, $(H_X)_b = (H_X)_a$. We choose the representative E_X of ${}^{\mathrm{I}}\mathcal{O}_X^{\mathrm{I}}$ such that $(E_X)_{ab}$ is the $\{0, 1\}$-matrix having 1's at the positions $(i, i+1)$ for $1 \leq i \leq r$ and $p - r \leq i < p$, and also at the positions $(i, i+2)$ for $r \leq i < p - r$.

Let $L \in \overline{B \cdot E_X} \cap \mathcal{O}_Y$ and let

$$v_i = L(e_i') = \xi_{i,0}e_0 + \xi_{i,1}e_1 + \cdots + \xi_{i,p-1}e_{p-1}$$

for $0 \leq i < p$. As B consists of upper triangular matrices, we have $v_0 = 0$, $\xi_{i,j} = 0$ if $j \geq i$ and $0 < i \leq r$ or $p - r \leq i < p$, and also if $j \geq i - 1$ and $r < i < p - r$. For simplicity, we shall write $\alpha_i = \xi_{i,i-1}$ for $0 < i \leq r$ or $p - r \leq i < p$, and $\beta_i = \xi_{i,i-2}$ for $r < i \leq p - r$.

As $r_1(Y) = p - 2$, it follows that $\beta_i \neq 0$ for $r + 1 < i < p - r$. Furthermore, at least one of α_r, β_{r+1} and at least one of α_{p-r}, β_{p-r} is not 0. By performing some elementary row and column operations belonging to K_a^0 and/or K_b^0, we may additionally assume that $\xi_{k+m+2i+1,k+m+2i-2} = 0$ for $-m < i < m$. One can easily check that $L^{2k-2}(e_i') = 0$ for $i < p - 2$ and that $L^{2k-2}(e_{p-2}') = \lambda' e_0'$ and $L^{2k-2}(e_{p-1}') = \lambda' e_1' + \mu' e_0'$ for some scalars λ' and μ'. As $r_{2k-2,b}(Y) = 2$, λ' is not 0. Similarly, we have $L^{2k-2}(e_{p-2}) = \lambda e_0$ for some scalar λ. As $\operatorname{rank}(L^{2k-2})_a = r_{2k-2,a}(Y) = 0$, λ must be 0.

We shall now compute the coefficient λ'. Set $V_i = \langle e_0, \dots, e_i \rangle$ and $V'_i = \langle e'_0, \dots, e'_i \rangle$. The above conditions on the structure of L show that

$$\begin{aligned} L^{i+1}(V'_i) &= 0, \quad 0 \le i \le r, \\ L^{r+1+i}(V'_{r+1+2i}) &= 0, \quad 0 \le i < 2m, \\ L^{k+m+1+i}(V'_{p-r+i}) &= 0, \quad 0 \le i < r, \end{aligned}$$

and that the same relations hold for V_i.

We will determine $L^{2k-2}(e'_{p-2})$ in stages. First, we have

$$\begin{aligned} L(e'_{p-2}) &\equiv \alpha_{p-2} e_{p-3} \pmod{V_{p-4}}, \\ L^2(e'_{p-2}) &\equiv -\alpha_{p-2}\alpha_3 e'_{p-4} \pmod{V'_{p-5}}. \end{aligned}$$

As $L^{2k-4}(V'_{p-5}) = 0$, we obtain that

$$L^{2k-2}(e'_{p-2}) = -\alpha_{p-2}\alpha_3 L^{2k-4}(e'_{p-4}).$$

By repeating this argument, we obtain that

$$L^{2k-2}(e'_{p-2}) = (-1)^t \Bigg(\prod_{1 \le j \le t} \alpha_{p-2j}\alpha_{1+2j} \Bigg) L^{2k-2-2t}(e'_{p-2-2t}), \quad 1 \le t < r/2.$$

By using the above formula for $t = r/2 - 1$, we obtain that

$$L^{2k-2}(e'_{p-2}) = (-1)^{r/2} a \cdot (\alpha_{p-r}\beta_{r+2} L^{2k-2-r}(e'_{p-r-3}) + \beta_{p-r}\beta_{r+3} L^{2k-2-r}(e'_{p-r-4})),$$

where

$$a = \prod_{1 \le j < r/2} \alpha_{p-2j}\alpha_{1+2j}.$$

For $1 \le t < m$, we have

$$\begin{aligned} L^{2k-2}(e'_{p-2}) =& (-1)^{r/2+t} a \cdot \Bigg[\alpha_{p-r}\beta_{r+2} \prod_{1 \le j \le t} \beta_{p-r+1-4j}\beta_{r+2+4j} L^{2k-2-r-2t}(e'_{p-r-3-4j}) \\ &+ \beta_{p-r}\beta_{r+3} \prod_{1 \le j \le t} \beta_{p-r-4j}\beta_{r+3+4j} L^{2k-2-r-2t}(e'_{p-r-4-4j}) \Bigg]. \end{aligned}$$

Let

$$b = \prod_{1 \le j < m} \beta_{p-r+1-4j}\beta_{r+2+4j}, \quad c = \prod_{1 \le j < m} \beta_{p-r-4j}\beta_{r+3+4j}.$$

Then, by using the above formula with $t = m - 1$, we obtain that

$$\begin{aligned} &L^{2k-2}(e'_{p-2}) \\ &= (-1)^{(k+m)/2} a \cdot [\alpha_{p-r}\beta_{r+2} \cdot b \cdot \beta_{r+1}\alpha_{p-r+1} + \beta_{p-r}\beta_{r+3} \cdot c \cdot \alpha_r \alpha_{p-r+1}] L^{k-m-2}(e'_{r-2}). \end{aligned}$$

For $1 \leq t < r/2$, we have

$$L^{k+m+2t}(e'_{p-2}) =(-1)^{(k+m)/2+t} a \cdot \Bigg[\alpha_{p-r}\beta_{r+2} \cdot b \cdot \beta_{r+1}\alpha_{p-r+1} \prod_{1\leq j\leq t} \alpha_{p-r+1+2j}\alpha_{r-2j}$$
$$+ \beta_{p-r}\beta_{r+3} \cdot c \cdot \alpha_r\alpha_{p-r+1} \prod_{1\leq i\leq t} \alpha_{p-r+1+2j}\alpha_{r-2j}\Bigg] L^{k-m-2t-2}(e'_{r-2-2t}) .$$

By taking $t = r/2 - 1$, we see that

$$\lambda' =(-1)^{k-1}\alpha_{p-1}\Bigg(\prod_{1<i<r} \alpha_i\alpha_{p-i}\Bigg)$$
$$\cdot \Bigg(\alpha_{p-r} \prod_{0\leq i<m} \beta_{r+4i+1}\beta_{r+4i+2} + \alpha_r \prod_{0\leq i<m} \beta_{r+4i+3}\beta_{r+4i+4}\Bigg) .$$

Similar calculations show that λ has the same expression as λ' except that the factor α_{p-1} is replaced by α_1.

Assume first that $\alpha_{p-r} \neq 0$. It is not hard to see that we can then assume that $\beta_{p-r} = 0$ without spoiling the zero entries which we have already claimed. Observe that $r + 4(m-1) + 4 = p - r$ and so the last product in the above formula for λ' is 0. Hence

$$\lambda' = (-1)^{k-1}\alpha_{p-1}\alpha_{p-r} \prod_{1<i<r} \alpha_i\alpha_{p-i} \cdot \prod_{0\leq i<m} \beta_{r+4i+1}\beta_{r+4i+2} .$$

Since $\lambda' \neq 0$ and $\lambda = 0$, we deduce from the expressions above that the α_i's are nonzero for $i \neq 1, r$ and that $\alpha_1 = 0$.

Assume now that $\alpha_{p-r} = 0$, and consequently $\beta_{p-r} \neq 0$. In this case the formula for λ' becomes

$$\lambda' = (-1)^{k-1} \prod_{1<i\leq r} \alpha_i\alpha_{p-i+1} \cdot \prod_{0\leq i<m} \beta_{p-r-4i}\beta_{p-r-4i-1} .$$

Recall that the coefficient λ has the same expression except that the factor α_{p-1} should be replaced by α_1. Since $\lambda' \neq 0$ and $\lambda = 0$, we deduce that α_i's are nonzero for $i \neq 1, p-r$ and that $\alpha_1 = 0$.

Hence $\alpha_1 = 0$ in both cases. If $r = 2$ then $L \in \mathfrak{s}_2(H_Y)$. Assume now that $r > 2$. Since $\alpha_i \neq 0$ for $1 < i < r$, by subtracting a linear combination of the rows with indices $2, 3, \ldots, r-1$ from the first row of L, we may assume that $\xi_{i,0} = 0$ for $i < r$. We now apply the permutation

$$(r-1, r-2, \ldots, 2, 1)(p-r+2, p-r+3, \ldots, p) \in W_a$$

to L. Then this modified L belongs to $\mathfrak{s}_2(H_Y)$. We can check this (using the fact that r is even) by listing the eigenvalues of H_Y and by inspecting the eigenspaces of $\mathrm{ad}(H_Y)$. By Proposition 1 we conclude that $L \in \mathcal{O}_Y^{\mathrm{I}}$ for k even and $L \in {}^{\mathrm{I}}\mathcal{O}_Y$ for k odd. By Proposition 2, (X, Y) splits. $\square$

LEMMA 18. *If* $X = ((ab)^{2k}a, (ab)^{2m}a, a^r)$, $Y = ((ba)^{2k-1}b, (ba)^{2m-1}b, a^{r+4})$, $r \geq 0$, *and* $k > m > 0$, *then the b-pair* (X, Y) *splits. Consequently, the b-pairs of the fifth family of Table* 8 *split.*

PROOF. We have $p = 2k + 2m + r + 2$ and $q = 2k + 2m$. The eigenvalues of $(H_X)_a$ are the integers $4i$, $-k \leq i \leq k$. Those for $-m \leq i \leq m$ have multiplicity 2 except that 0 has multiplicity $r + 2$. The other eigenvalues are simple. The eigenvalues of $(H_X)_b$ are the integers $4i - 2$, $-k < i \leq k$. Those for $-m < i \leq m$ have multiplicity 2, while the other ones are simple. Let $L \in \mathfrak{s}_2(H_X) \cap \mathcal{O}_Y$ and let $v_i = L(e_i')$, $0 \leq i < q$. Let us introduce the subspaces $V_i = \langle v_0, v_1, \dots, v_i \rangle$ for $0 \leq i < q$. Since $r_1(Y) = r_{2,b}(Y) = q - 2$, the subspace $V_{q-1} = L(V_b)$ is nondegenerate and has dimension $q - 2$. By inspecting the eigenspaces of $\mathrm{ad}(H_X)$, we see that

$$\begin{aligned}
&v_i = \xi_{i,0}e_0 + \xi_{i,1}e_1 + \cdots + \xi_{i,i}e_i\,, \quad 0 \leq i < k - m\,,\\
&V_{k+m-2i-1} \subset \langle e_0, e_1, \dots, e_{k+m-2i-1} \rangle\,, \quad 0 \leq i < m\,,\\
&V_{k+m+2i+1} \subset \langle e_0, e_1, \dots, e_{k+m+r+2i+1} \rangle\,, \quad 0 \leq i < m\,,\\
&V_{q-i-1} \subset \langle e_0, e_1, \dots, e_{p-i-2} \rangle\,, \quad 0 \leq i < k - m\,.
\end{aligned}$$

As $v_0 \in \langle e_0 \rangle$ and e_0 is obviously orthogonal to $L(V_b)$, it follows that $v_0 = 0$. As $r_{4k-2,b}(Y) = 1$, we have $(L^{4k-2})_b \neq 0$. From this fact one can deduce that $\xi_{i,i} \neq 0$ for $0 < i < k - m$. Hence the vectors v_i, $0 < i < k - m$, are linearly independent. The subspace V_{k-m+1} is orthogonal to V_{k+3m-1}. It is clear that V_{k-m+1} has dimension $k - m$ or $k - m + 1$. Hence its subspace that is orthogonal to the vectors v_i for $k + 3m \leq i < q$ is contained in the radical of $L(V_b)$, and so it is 0. It follows that V_{k-m+1} has dimension $k - m$. Consequently, V_i has dimension i for $i < k - m$ and dimension $i - 1$ for $i > k - m$.

By Witt's theorem, there exists an $x \in K_a$ such that $x(V_i) = \langle e_0, e_1, \dots, e_{i-1} \rangle$ for $0 < i < k - m$, and $x(V_{k+m-2i-1}) = \langle e_0, e_1, \dots, e_{k+m-2i-3} \rangle$ for $0 \leq i < m$. Since V_i is orthogonal to V_{q-1-i} for $0 < i < k - m$, and $V_{k+m-2i-1}$ is orthogonal to $V_{k+m+2i+1}$ for $0 \leq i < m$, it follows that $x(V_{q-1-i}) \subset \langle e_0, e_1, \dots, e_{p-i-1} \rangle$ for $0 < i < k - m$, and $x(V_{k+m+2i+1}) \subset \langle e_0, e_1, \dots, e_{k+m+r+2i+3} \rangle$ for $0 \leq i < m$.

Now the eigenvalues of $(H_Y)_a$ are the integers $4i$ for $-k < i < k$. Those for $-m < i < m$ have multiplicity 2 except that 0 has multiplicity $r + 6$, and the other ones are simple. The eigenvalues of $(H_Y)_b$ are the same as those of $(H_X)_b$ (with the same multiplicities). By inspecting the eigenspaces of $\mathrm{ad}(H_Y)$, we conclude that $\mathrm{Ad}(x)(L) \in \mathfrak{s}_2(H_Y)$. By Proposition 1, $L \in \mathcal{O}_Y^{\mathrm{I}}$. Hence (X, Y) splits by Proposition 2.

The second assertion follows from the first by taking $r = 0$ because then

$$X > ((ab)^{2k}a, (ba)^{2m-1}b, a^2) > ((ba)^{2k-1}b, (ab)^{2m}a, a^2) > Y\,.$$

□

LEMMA 19. *The b-pairs of the sixth family of Table* 8 *split.*

PROOF. We have $p = q = 2k + 2m + 2$ and let $r = k - m$ and $s = k + m + 1$. The eigenvalues of $(H_X)_a$ are the integers $4i$ for $-k \leq i \leq k$. Those for $-m \leq i \leq m$

have multiplicity 2, while the other eigenvalues are simple. The eigenvalues of $(H_X)_b$ are the integers $4i+2$ for $-k-1 \leq i \leq k$. Those for $-m \leq i < m$ have multiplicity 2 and the other ones are simple. We choose the representative $E_X \in \mathcal{O}_X^{\mathrm{I}}$ such that $(E_X)_{ab}$ is the $\{0,1\}$-matrix having 1's at the positions $(i-1,i)$ for $1 < i \leq p-r$, and the positions (i,i) for $p-r \leq i \leq p$.

Let $L \in \overline{B \cdot E_X} \cap \mathcal{O}_Y$ and let

$$v_i = L(e_i') = \xi_{i,0}e_0 + \xi_{i,1}e_1 + \cdots + \xi_{i,p-1}e_{p-1}$$

for $0 \leq i < q$. As B consists of upper triangular matrices, we have $v_0 = 0$, $\xi_{i,j} = 0$ if $j \geq i$ and $0 < i < q-r-1$, and also if $j > i$ and $q-r-1 \leq i < q$. For simplicity, we shall write $\alpha_i = \xi_{i,i-1}$ for $0 < i < q-r$ and $\beta_i = \xi_{i,i}$ for $q-r-1 \leq i < q$. As $\operatorname{rank} L_{a,b} = r_1(Y) = q-1$, we conclude that the vectors v_i for $0 < i < q$ form a basis of $L(V_b)$. It follows that $\alpha_i \neq 0$ for $0 < i < q-r-1$. By applying suitable elements from B_a and B_b, we may further assume that the coefficients $\xi_{s+2i+1,s+2i-1}$ are 0 for $-m \leq i < m$. A simple computation shows that $L^{4k}(e_i) = 0$ for $i < p-1$ and $L^{4k}(e_{p-1}) = \lambda e_0$ where

$$\lambda = 2\alpha_1^2\beta_{q-r-1}\Bigg(\prod_{1<i\leq r}\alpha_i\beta_{q-i}\Bigg)^2 \cdot \prod_{r<i<q-r}\alpha_i\,.$$

As $\operatorname{rank}(L^{4k})_a = r_{4k,a}(Y) = 1$, λ is not 0. Hence $\alpha_{q-r-1} \neq 0$ and $\beta_i \neq 0$ for $q-r-1 \leq i < q-1$. Now the condition $\operatorname{rank}(L^{4k})_b = r_{4k,b}(Y) = 0$ implies that $(L^{4k})_b = 0$. By computing the $(1,q-1)$ entry of $(L^{4k})_b$ we find that it is equal to the above expression for λ except that the factor α_1^2 should be replaced by $\alpha_1\beta_{q-1}$. As this entry must be 0, we infer that $\beta_{q-1} = 0$. Finally we apply the permutation

$$(r+1, r, \ldots, 2, 1)(q-r, q-r+1, \ldots, q) \in W_b$$

to modify L further. By listing the eigenvalues of H_Y and by inspecting the eigenspaces of $\operatorname{ad}(H_Y)$, we see that $L \in \mathfrak{s}_2(H_Y)$. Proposition 1 shows that $L \in \mathcal{O}_Y^{\mathrm{I}}$, and so (X,Y) splits by Proposition 2. □

LEMMA 20. *The b-pairs of the seventh and eighth family of Table* 8 *split.*

PROOF. Let us switch the letters a and b in X and Y of the eighth family. We then combine the resulting a-pairs with the b-pairs of the seventh family. We obtain a single family of pure pairs (X,Y) with

$$X = ((ab)^k a, (ab)^m, (ba)^m)\,, \quad Y = ((ba)^{k-1}b, ((ab)^m a)^2)\,,$$

where $k > m \geq 0$ and $k \equiv m \pmod 2$. Set $k-m = 2r$. Thus $p = k+2m+1$ and $q = k+2m$. The eigenvalues of $(H_X)_a$ are the odd integers $2i-1$ for $-m < i \leq m$ and the even integers $2k-4i$ for $0 \leq i \leq k$. Those of $(H_X)_b$ are the same odd integers and the even integers $2k-4i-2$ for $0 \leq i < k$. All these eigenvalues of H_X are simple. We choose the representative E_X of $\mathcal{O}_X^{\mathrm{I}}$ if k is even and of ${}^{\mathrm{I}}\mathcal{O}_X$ if k is odd such that $(E_X)_{ab}$ is the $\{0,1\}$-matrix having 1's at the positions (i,i) for $1 \leq i \leq r$ and $q-r < i \leq q$, and also at the positions $(i, i+1)$ for $r < i < q - r$. Let $L \in \overline{B \cdot E_X} \cap \mathcal{O}_Y$ and let

$$v_i = L(e'_i) = \xi_{i,0}e_0 + \xi_{i,1}e_1 + \cdots + \xi_{i,p-1}e_{p-1}$$

for $0 \le i < q$. As B consists of upper triangular matrices, we have $\xi_{i,j} = 0$ if $j > i$ and $0 \le i < r$ or $q - r \le i < q$, and also if $j \ge i$ and $r \le i < q - r$. For simplicity, we shall write $\alpha_i = \xi_{i,i}$ for $0 \le i < r$ or $q - r \le i < q$, and $\beta_i = \xi_{i,i-1}$ for $r \le i < q - r$.

As $r_1(Y) = q - 1$, it follows that $\beta_i \neq 0$ for $r < i < q - r$. One can easily check that $L^{2k-2}(e'_i) = 0$ for $i < q - 1$ and that $L^{2k-2}(e'_{q-1}) = \lambda' e'_0$, where

$$\lambda' = (-1)^{k-1}\alpha_{q-1}^2 \Bigg(\prod_{0<i<r} \alpha_i \alpha_{q-i-1} \cdot \prod_{0\le i<m} \beta_{r+3i+1} \Bigg)^2 .$$

As $r_{2k-2,b}(Y) = 1$, λ' is not 0. We conclude that α_i is not 0 for $i \neq 0$. Similarly, we have $L^{2k-2}(e_{p-2}) = \lambda e_0$, where the scalar λ is given by the same expression as λ' except that the factor α_{q-1}^2 should be replaced by the product $\alpha_0 \alpha_{q-1}$. As $r_{2k-2,a}(Y) = 0$, λ must be 0. It follows that $\alpha_0 = 0$.

If $r = 1$ then $L \in \mathfrak{s}_2(H_Y)$. (This can be checked by listing the eigenvalues of H_Y and inspecting the eigenspaces of $\mathrm{ad}(H_Y)$.) Assume now that $r > 1$. As $\alpha_i \neq 0$ for $0 < i \le r$, we may assume that $\xi_{i,0} = 0$ for the same i's. We now apply the permutation

$$(r, r-1, \dots, 2, 1)(p - r + 1, p - r + 2, \dots, p) \in W_a$$

to L. Again we can check that we obtain an element of $\mathfrak{s}_2(H_Y)$. By Proposition 1 we conclude that $L \in \mathcal{O}_Y^{\mathrm{I}}$ for k even and $L \in {}^{\mathrm{I}}\mathcal{O}_Y$ for k odd. By Proposition 2, (X, Y) splits. □

LEMMA 21. *The b-pairs of the last two families of Table* 8 *split.*

PROOF. Let us switch the letters a and b in X and Y of the tenth family. We then combine the resulting a-pairs with the b-pairs of the ninth family. We obtain a single family of pure pairs (X, Y) with

$$X = ((ab)^k, (ba)^k, (ba)^{m-1}b), \quad Y = (((ba)^{k-1}b)^2, (ab)^m a),$$

where $k > m \ge 1$ and $k \equiv m \pmod 2$. Thus $p = 2k + m - 1$ and $q = 2k + m$. We set $r = k - m$. The eigenvalues of $(H_X)_a$ are the odd integers $2i - 1$ for $-k < i \le k$ and the even integers $2m - 4i$ for $0 < i < m$. Those of $(H_X)_b$ are the same odd integers and the even integers $2m - 4i - 2$ for $0 \le i < m$. All these eigenvalues of H_X are simple. We choose the representative E_X of $\mathcal{O}_X^{\mathrm{I}}$ if k is even and of ${}^{\mathrm{I}}\mathcal{O}_X$ if k is odd such that $(E_X)_{ab}$ is the $\{0, 1\}$-matrix having 1's at the positions $(i, i+1)$ for $1 \le i \le r$ and the positions $(i, i+2)$ for $r < i \le p - 1$. Let $L \in \overline{B \cdot E_X} \cap \mathcal{O}_Y$ and let

$$v_i = L(e'_i) = \xi_{i,0}e_0 + \xi_{i,1}e_1 + \cdots + \xi_{i,p-1}e_{p-1}$$

for $0 \le i < q$. As B consists of upper triangular matrices, we have $v_0 = 0$, and $\xi_{i,j} = 0$ for $0 < i \le r$ and $j \ge i$ and also for $r < i < q$ and $j \ge i - 1$. For simplicity, we shall write $\alpha_i = \xi_{i,i-1}$ for $0 < i \le r$ and $\beta_i = \xi_{i,i-2}$ for $r < i < q$.

As $r_1(Y) = q-2$, it follows that $\beta_i \neq 0$'s for $r+1 < i < q$. One can easily check that $L^{2k-2}(e'_i) = 0$ for $i < q-2$ and that $L^{2k-2}(e'_{q-2}) = \lambda' e'_0$ and $L^{2k-2}(e'_{q-1}) = \lambda' e'_1 + \mu' e'_0$ for some scalars λ' and μ'. A computation gives the following formula for λ':

$$\lambda' = (-1)^{k-1} \prod_{1<i\leq r} \alpha_i \beta_{q-i+1} \cdot \prod_{0\leq i<m} \beta_{r+3i+2}\beta_{r+3i+3}\,.$$

As $r_{2k-2,b}(Y) = 2$, λ' is not 0. We conclude that α_i is not 0 for $i \neq 1$. Similarly, we have $L^{2k-2}(e_i) = 0$ for $i < p-2$ and $L^{2k-2}(e_{p-2}) = \lambda e_0$, where the scalar λ is given by the same expression as λ' except that the factor β_{q-1} should be replaced by α_1. As $r_{2k-2,a}(Y) = 0$, λ must be 0. We conclude that $\alpha_1 = 0$.

If $r = 2$ then $L \in \mathfrak{s}_2(H_Y)$. (This can be checked by listing the eigenvalues of H_Y and inspecting the eigenspaces of $\mathrm{ad}(H_Y)$.) Assume now that $r > 2$. As $\alpha_i \neq 0$ for $1 < i \leq r$, we may assume that $\xi_{i,0} = 0$ for the same i's. We now apply the permutation

$$(r-1, r-2, \ldots, 2, 1)(k+2m+1, k+2m+2, \ldots, p) \in W_a$$

to L. Again we can check that we obtain an element of $\mathfrak{s}_2(H_Y)$. By Proposition 1 we conclude that $L \in \mathcal{O}_Y^{\mathrm{I}}$ for k even and $L \in {}^{\mathrm{I}}\mathcal{O}_Y$ for k odd. By Proposition 2, (X, Y) splits. □

7. Appendix. Define three commuting real involutory symmetric matrices of order n:

$$J_1 = \begin{pmatrix} S_p & 0 \\ 0 & S_q \end{pmatrix}, \quad J_2 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad J = J_1 J_2 = \begin{pmatrix} S_p & 0 \\ 0 & -S_q \end{pmatrix},$$

where S_k is the matrix of order k defined in Section 3 and I_k is the identity matrix of order k. By viewing G as a matrix group, we have

$$G = \{x \in \mathrm{GL}_n(\boldsymbol{C}) : {}^t x J_1 x = J_1\},$$
$$\mathfrak{g} = \{X \in \mathfrak{gl}_n(\boldsymbol{C}) : {}^t X J_1 + J_1 X = 0\}.$$

If we partition X as in Section 3:

$$X = \begin{pmatrix} X_a & X_{ab} \\ X_{ba} & X_b \end{pmatrix},$$

then

$$\mathfrak{g} = \{X \in \mathfrak{gl}_n(\boldsymbol{C}) : {}^t X_a S_p + S_p X_a = 0, {}^t X_b S_q + S_q X_b = 0, {}^t X_{ab} S_p + S_q X_{ba} = 0\}.$$

The involution θ of G (see Section 1) is given by $\theta(x) = J_2 x J_2$, and its differential $d\theta$ is the involutorial automorphism of $\mathfrak{g}$ given by $d\theta(X) = J_2 X J_2$. The eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ of $d\theta$ are

$$\mathfrak{k} = \{X \in \mathfrak{g} : X_{ab} = 0, X_{ba} = 0\},$$
$$\mathfrak{p} = \{X \in \mathfrak{g} : X_a = 0, X_b = 0\}.$$

We also use the standard definitions:

$$\begin{aligned}
\mathrm{O}_n(\boldsymbol{C}) &= \{x \in \mathrm{GL}_n(\boldsymbol{C}) : {}^t x x = I_n\}\,,\\
\mathfrak{so}_n(\boldsymbol{C}) &= \{X \in \mathfrak{gl}_n(\boldsymbol{C}) : {}^t X + X = 0\}\,,\\
\mathrm{O}(p,q) &= \{x \in \mathrm{GL}_n(\boldsymbol{R}) : {}^t x J_2 x = J_2\}\,,\\
\mathfrak{so}(p,q) &= \{X \in \mathfrak{gl}_n(\boldsymbol{R}) : {}^t X J_2 + J_2 X = 0\}\,.
\end{aligned}$$

The map $\sigma : G \to G$, defined by $\sigma(x) = J\bar{x}J$, is an antiholomorphic involutory automorphism of G. Its differential $\mathrm{d}\sigma : \mathfrak{g} \to \mathfrak{g}$, given by $\mathrm{d}\sigma(X) = J\bar{X}J$, is a conjugation of $\mathfrak{g}$. We introduce the real forms of G and $\mathfrak{g}$:

$$G_0 = \{x \in G : \sigma(x) = x\}\,, \qquad \mathfrak{g}_0 = \{X \in \mathfrak{g} : \mathrm{d}\sigma(X) = X\}\,.$$

Since θ and σ commute, G_0 (resp. $\mathfrak{g}_0$) is stable under θ (resp. $\mathrm{d}\theta$). We denote by θ_0 (resp. $\mathrm{d}\theta_0$) the restriction of θ (resp. $\mathrm{d}\theta$) to G_0 (resp. $\mathfrak{g}_0$).

One can easily verify that $G_0 \cong \mathrm{O}(p,q)$, and, consequently, $\mathfrak{g}_0 \cong \mathfrak{so}(p,q)$. An explicit isomorphism can be constructed as follows. Choose a matrix P of order n which commutes with J_2 (and, consequently, also with J and J_1) and such that $P^2 = J = P^{-1}\bar{P}$. It is easy to construct such a matrix. For instance, if $p = 4$ and $q = 3$, then we can take P to be the block diagonal matrix

$$P = \begin{pmatrix}
\xi & 0 & 0 & \bar{\xi} & & & \\
0 & \xi & \bar{\xi} & 0 & & & \\
0 & \bar{\xi} & \xi & 0 & & & \\
\bar{\xi} & 0 & 0 & \xi & & & \\
 & & & & \bar{\xi} & 0 & -\xi \\
 & & & & 0 & i & 0 \\
 & & & & -\xi & 0 & \bar{\xi}
\end{pmatrix},$$

where $\xi = (1-i)/2$. It is now straightforward to check that $P^{-1}GP = \mathrm{O}_n(\boldsymbol{C})$ and $P^{-1}G_0P = \mathrm{O}(p,q)$, and, consequently, $P^{-1}\mathfrak{g}P = \mathfrak{so}_n(\boldsymbol{C})$ and $P^{-1}\mathfrak{g}_0P = \mathfrak{so}(p,q)$. Hence, as our desired isomorphism we can take the map $G_0 \to \mathrm{O}(p,q)$ sending $x \mapsto P^{-1}xP$.

It is easy to verify that

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} : X_a^* = -X_a,\ X_b^* = -X_b,\ X_{ba} = X_{ab}^*\}\,,$$

where the asterisk denotes the transpose conjugate of a matrix. As θ and σ commute, G_0 (resp. $\mathfrak{g}_0$) is θ-stable (resp. $\mathrm{d}\theta$-stable). We have the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$, where $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$ and $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$ are the eigenspaces of the restriction $\mathrm{d}\theta|_{\mathfrak{g}_0}$ for the eigenvalues $+1$ and -1, respectively. Hence

$$\begin{aligned}
\mathfrak{k}_0 &= \{X \in \mathfrak{k} : X_a^* = -X_a,\ X_b^* = -X_b\} = \{X \in \mathfrak{g}_0 : X_{ab} = 0\}\,,\\
\mathfrak{p}_0 &= \{X \in \mathfrak{p} : X_{ba} = X_{ab}^*\} = \{X \in \mathfrak{g}_0 : X_a = 0, X_b = 0\}\,.
\end{aligned}$$

The subgroup $K_0 = \{x \in G_0 : \theta(x) = x\}$ has $\mathfrak{k}_0$ as its Lie algebra. K_0 is a maximal compact subgroup of G_0 and is isomorphic to $\mathrm{O}(p) \times \mathrm{O}(q)$.

Let us also introduce the conjugation $\tau : V \to V$, i.e., an involutory real linear map satisfying $\tau(iv) = -i\tau(v)$ for all $v \in V$, by stipulating that $\tau(e_i) = e_{p-i-1}$ for $0 \le i < p$ and $\tau(e'_i) = -e'_{q-i-1}$ for $0 \le i < q$. Thus if

$$v = \xi_0 e_0 + \xi_1 e_1 + \cdots + \xi_{p-1} e_{p-1} + \eta_0 e'_0 + \eta_1 e'_1 + \cdots + \eta_{q-1} e'_{q-1}$$

then

$$\tau(v) = \bar{\xi}_{p-1} e_0 + \bar{\xi}_{p-2} e_1 + \cdots + \bar{\xi}_0 e_{p-1} - \bar{\eta}_{q-1} e'_0 - \bar{\eta}_{q-2} e'_1 - \cdots - \bar{\eta}_0 e'_{q-1}\,.$$

If we identify $v \in V$ with the column vector of its coordinates with respect to the basis $\{e_0, \dots, e_{p-1}, e'_0, \dots, e'_{q-1}\}$, then we have $\tau(v) = J\bar{v}$. We have $V = V_0 \oplus i V_0$, where $V_0 = \{v \in V : \tau(v) = v\}$ is the real form of V defined by τ. For the v given above, $v \in V_0$ holds if and only if $\xi_i = \bar{\xi}_{p-1-i}$ for $0 \le i < p$ and $\eta_i = -\bar{\eta}_{q-i-1}$ for $0 \le i < q$. The conjugation τ induces the conjugation $L \mapsto \tau \circ L \circ \tau^{-1}$ on $\mathrm{GL}(V)$. Our group G is stable under the latter conjugation, and its restriction to G coincides with σ.

By restricting the symmetric bilinear form $f : V \times V \to \boldsymbol{C}$, we obtain a nondegenerate real valued symmetric bilinear form $f_0 : V_0 \times V_0 \to \boldsymbol{R}$ of signature (p, q). Indeed, if we set $V_0^+ = V_0 \cap V_a$ and $V_0^- = V_0 \cap V_b$, then $V_0 = V_0^+ \oplus V_0^-$, V_0^+ is orthogonal to V_0^-, and the restriction of f_0 to $V_0^+ \times V_0^+$ (resp. $V_0^- \times V_0^-$) is positive (resp. negative) definite. If $p = 2k$ is even, then the vectors

$$\frac{1}{\sqrt{2}}(e_j + e_{p-j-1})\,, \quad \frac{i}{\sqrt{2}}(e_j - e_{p-j-1})\,, \quad 0 \le j < k\,,$$

form an orthonormal basis of V_0^+. If $p = 2k + 1$ is odd, then we have to add also the vector e_k. One can similarly construct an orthonormal basis of V_0^-.

We have one more interpretation of the real forms G_0 and $\mathfrak{g}_0$:

$$G_0 = \{x \in \mathrm{GL}(V_0) : f_0(x(v), x(w)) = f_0(v, w), \text{ for any } v, w \in V_0\}\,,$$
$$\mathfrak{g}_0 = \{X \in \mathfrak{gl}(V_0) : f_0(X(v), w) + f_0(v, X(w)) = 0, \text{ for any } v, w \in V_0\}\,,$$

provided that we identify a linear operator on the real vector space V_0 with its complex extension to V.

Recall that standard triples and normal triples were defined in Section 3. We say that a normal triple (E', H', F') is a *complex Cayley triple* if $d\sigma(E') = -F'$. A standard triple (E, H, F) in the real form $\mathfrak{g}_0$ is called a *real Cayley triple* if $d\theta_0(E) = F$ (and consequently $d\theta_0(F) = E$ and $d\theta_0(H) = -H$). The Cayley transformation maps the real Cayley triples (E, H, F) to the complex Cayley triples (E', H', F') according to the formulas

$$E' = \frac{1}{2}(H + iF - iE)\,, \quad H' = i(E + F)\,, \quad F' = \frac{1}{2}(-H + iF - iE)\,.$$

The inverse Cayley transformation is given by

$$E = \frac{i}{2}(-H' + E' + F')\,, \quad H = E' - F'\,, \quad F = -\frac{i}{2}(H' + E' + F')\,.$$

Assume that the complex Cayley triple (E', H', F') is the Cayley transform of the real Cayley triple (E, H, F). The Kostant-Sekiguchi correspondence associates to the nonzero nilpotent G_0^0-orbit $G_0^0 \cdot E$ in $\mathfrak{g}_0$ the nilpotent K^0-orbit $K^0 \cdot E'$ in $\mathfrak{p}$. This establishes a bijection from the set of nonzero nilpotent G_0^0-orbits in $\mathfrak{g}_0$ to the set of the nonzero nilpotent K^0-orbits in $\mathfrak{p}$.

EXAMPLE 8. Let us find a representative of the nilpotent G_0^0-orbit in $\mathfrak{g}_0$ that corresponds to the nilpotent K^0-orbit ${}^{\mathrm{I}}\mathcal{O}_X$ in $\mathfrak{p}$, where $X = (babab, aba)$. Thus $p = q = 4$ in this example. The characteristic H_X of the orbit ${}^{\mathrm{I}}\mathcal{O}_X$ is given by

$$H_X = \operatorname{diag}(2, 2, -2, -2, 4, 0, 0, -4)\,.$$

We need a nonzero element $E' \in \mathfrak{p}_2(H_X)$ such that the element F' defined by $F' = d\sigma(E') = J\overline{E'}J$ satisfies the equation $[F', E'] = H_X$. A simple computation produces such a matrix:

$$E' = \begin{pmatrix} & & & & 0 & 1 & 1 & 0 \\ & & & & 0 & \sqrt{3} & -\sqrt{3} & 0 \\ & & & & 0 & 0 & 0 & 2 \\ & & & & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & & & & \\ 0 & 0 & \sqrt{3} & -1 & & & & \\ 0 & 0 & -\sqrt{3} & -1 & & & & \\ 0 & 0 & 0 & 0 & & & & \end{pmatrix}.$$

Consequently, F' is given by:

$$F' = \begin{pmatrix} & & & & 0 & 0 & 0 & 0 \\ & & & & 2 & 0 & 0 & 0 \\ & & & & 0 & -\sqrt{3} & \sqrt{3} & 0 \\ & & & & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & & & & \\ -1 & -\sqrt{3} & 0 & 0 & & & & \\ -1 & \sqrt{3} & 0 & 0 & & & & \\ 0 & 0 & -2 & 0 & & & & \end{pmatrix}.$$

Hence we have a complex Cayley triple (E', H', F') with $H' = H_X$. By applying the inverse Cayley transformation, we find the representative

$$E = \frac{i}{2}\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 2 & \sqrt{3} & -\sqrt{3} & 0 \\ 0 & 0 & 2 & 0 & 0 & -\sqrt{3} & \sqrt{3} & 2 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & -4 & 0 & 0 & 0 \\ -1 & -\sqrt{3} & \sqrt{3} & -1 & 0 & 0 & 0 & 0 \\ -1 & \sqrt{3} & -\sqrt{3} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \in \mathfrak{g}_0$$

of the nilpotent G_0^0-orbit in $\mathfrak{g}_0$ that corresponds to ${}^{\mathrm{I}}\mathcal{O}_X$ by the Kostant-Sekiguchi bijection. □

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