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LOCAL SPLITTING FAMILIES OF HYPERELLIPTIC PENCILS, I

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Abstract. We construct local splitting families of hyperelliptic pencils so that the original complicated degenerate fiber decomposes into several simple degenerate fibers. In some sense, our trial is a generalization to hyperelliptic curves of arbitrary genus of Moishezon's construction for families of elliptic curves. Moreover, we study certain invariants of degenerate fiber germs.

Introduction. The aim of this paper is to construct splitting families of degenerations of hyperelliptic curves so that the original complicated singular fiber decomposes into several simple singular fibers by these local deformations. The historical background of our study is as follows:

In 1977, Moishezon [Moi] showed that any degenerate elliptic curve splits by local deformation into several singular fibers of only one or two types. One is a rational curve with a node (type I_1 in Kodaira's table [Kod1]) and the other is a multiple fiber whose reduced scheme is a nonsingular elliptic curve (type ${}_m\mathrm{I}_0$). Matsumoto [Ma1] and Ue [Ue1] studied a similar problem for good torus fibrations which are topological analogs of elliptic fibrations. These results are used in studying diffeomorphism classes of global elliptic surfaces or torus fibrations ([FM], [Ma2], [Ue2], etc.).

In 1988, Horikawa [Ho4] showed that any degenerate genus two curve splits into several fibers of type I_1 in his table [Ho2] (a stable curve with two elliptic components with a node) and several fibers of type 0 (the fibers "arising from rational double points", which include many topologically different types of fibers). At the same time, he remarked that the sum of a certain invariant (the Horikawa index for genus 2 in our terminology) is conserved in his splitting families.

In 1990, Xiao and Reid [Re] proposed this type of problem—Morsification for fiber germs in their terminology—from the viewpoint of the study of relative canonical algebra for pencils of curves as in Mendes-Lopes [Men]. Especially they pointed out the importance of looking for "atomic fibers", i.e., the atoms of degenerations.

On the other hand, Nakayama [Na] studied certain two-parameter splitting degenerations of elliptic curves from the viewpoint of threefold minimal model theory.

Now in this paper, we show that any degenerate hyperelliptic curve splits into several singular fibers belonging to very small classes called type 0_0, class I and class II defined in Section 1 step by step by preserving the relative hyperelliptic involution of pencils (Theorem

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3.3). A fiber of type 0_0 is an irreducible stable curve with one node. A fiber in class I is a stable curve with two components with two nodes. The fiber in class II is defined by a certain strong condition for the germs of singularities of the branch curve arising from its relative hyperelliptic involution. We only classify them for small genera, and obtain one, three and three types for genus 2, 3 and 4, respectively (Proposition 1.7).

The method of construction of our splitting families is as follows: We first construct a one-parameter family of normal surfaces which is a double cover over a family of local projective line bundles, and then we resolve their singularities simultaneously. A bad singular fiber corresponds to bad singularities of the branch curve of the double cover which passes through the projective line over the origin. Therefore we first construct a splitting family of branch curve singularities so that bad singularities decompose into several simpler singularities, and then we lift it to a family of local hyperelliptic pencils.

The idea of our construction of local analytic splitting family of this curve singularity essentially comes from the well-known method for the Morsification of singularity, the so-called A'Campo-Gusein Zade theory [Ac], [G], that is, a perturbation method along the way of its resolution process.

The method of lifting to the family of pencils is to write down the equation itself case by case according to the situation of the family of branch curves. Our method is completely explicit so that we can describe which types of singular fibers appear in our splitting families.

In the last section, we define two invariants for the singular fiber of a hyperelliptic pencil, i.e., the Horikawa index and the local signature. Note that the sum of these invariants are conserved in the splitting families which we construct here (Proposition 4.11).

The Horikawa index is defined to be the fractional number which measures the contribution of the fiber to the distance from the geographical lower bound of existence of hyperellitic pencils of fixed genus. From the viewpoint of singularity theory, it measures the total badness of singularities of the branch curve compared to simple (ADE) singularities.

The local signature is defined directly from the Horikawa index and the topological Euler contribution, and the usual global signature of a compact surface with hyperelliptic pencil is the sum of these local signatures (Proposition 4.7). In the genus 2 case, Matsumoto [Ma4] observed this fact by using the speciality of Meyer's signature cocycle [Mey]. (Note that the general pencil does not satisfy this localized property of the signature, cf. [At2], [Kod2], [Hi2], etc.) Matsumoto also calculated the local signature of two types of singular fibers of Lefschetz fibrations, which coincide with the "atomic fibers" of genus 2 by [Ho4] and a special case of Corollary 4.12. Therefore the global signature of genus 2 fibration is written as a simple linear combination of two terms as in Corollary 4.14, which gives a precise "third" solution to the the negative signature problem posed by Persson [P2]. (Note that Xiao [X1] and Ueno [U] solved this problem.) Corollary 4.14 gives us similar information in the genus 3 and 4 cases.

We note that the topological approach to hyperelliptic pencils by using [Mey] was recently developed by Endo [E] and Morifuji [Mor].

For further discussions, see our forthcoming paper [AA].

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1. Special classes of germs of singular fiber. In this section, we define very special classes of degenerations of hyperelliptic curves. We will show in Section 3 that any degeneration of hyperelliptic curves splits by local deformations into several fibers in these classes step by step.

1.1. Let S be a 2-dimensional complex manifold and $\Delta_\varepsilon = \{t \in \mathbf{C}\,;\, |t| < \varepsilon\}$ a complex one-dimensional open disk with radius ε. Let $f : S \to \Delta_\varepsilon$ be a relatively minimal proper surjective holomorphic map. If f satisfies the conditions

(i) for any $t \in \Delta_\varepsilon \setminus \{0\}$, the fiber $f^{-1}(t)$ is a nonsingular hyperelliptic curve of genus $g \geq 2$,

(ii) $F = f^{-1}(0)$ is a singular curve,

we call f a degeneration of hyperelliptic curves of genus g, and F the singular fiber of f.

Now let $f' : S' \to \Delta_{\varepsilon'}$ be another degeneration of hyperelliptic curves of genus g. If there is a positive real number ε'' with $\varepsilon'' \leq \varepsilon$ and $\varepsilon'' \leq \varepsilon'$ such that the restrictions $f|_{\Delta_{\varepsilon''}} : S|_{\Delta_{\varepsilon''}} \to \Delta_{\varepsilon''}$ and $f'|_{\Delta_{\varepsilon''}} : S|_{\Delta_{\varepsilon''}} \to \Delta_{\varepsilon''}$ are isomorphic to each other as analytic fiber spaces, then we say f and f' to be equivalent. The equivalence class of f, which we write $[f, F]$, is called the germ of the singular fiber associated with f.

The study of the germ $[f, F]$ is translated into the study of the singularities of the branch curve of the double cover arising from the relative hyperelliptic involution in the following way: Let $\pi : W = \boldsymbol{P}^1 \times \Delta_\varepsilon \to \Delta_\varepsilon$ be the trivial $\boldsymbol{P}^1$-bundle and set $\Gamma_t = \pi^{-1}(t)$ for $t \in \Delta_\varepsilon$. Let B be a reduced divisor on W which satisfies

i) $B\Gamma_t = 2g + 2$ $(t \in \Delta_\varepsilon)$,

ii) The *bad points* of B are at most on Γ_0, where a bad point P of B is a point such that the intersection multiplicity of B and $\Gamma_{\pi(P)}$ at P satisfies $I_P(B, \Gamma_{\pi(P)}) \geq 2$ (i.e., a singular point of B or a point tangential to $\Gamma_{\pi(P)}$).

Let B' be another divisor on $W' = \boldsymbol{P}^1 \times \Delta_{\varepsilon'}$ which satisfies the same conditions as in the above i) and ii). We say (W, B) and (W', B') to be equivalent if there is a positive real number ε'' with $\varepsilon'' \leq \varepsilon$ and $\varepsilon'' \leq \varepsilon'$ such that the restrictions $(W, B)|_{\Delta_{\varepsilon''}}$ and $(W', B')|_{\Delta_{\varepsilon''}}$ coincide with each other modulo a finite succession of elementary transformations in the sense of [Ho2, §2]. We denote by $[W, B]$ the equivalence class of (W, B) and call it the *germ of branch curve* of a degeneration of hyperelliptic curves. It is well-known that there exists a one-to-one correspondence between the set of germs of singular fibers $\{[f, F]\}$ and those of branch curves $\{[W, B]\}$ (cf. [Ho2]).

We remark that there exist examples such that $[f, F]$ and $[f', F']$ are mutually different as germs of singular fiber but F and F' have the same weighted dual graph. For instance, compare $2\mathrm{I}_{0-m}$ with II^*_{n-0} in Namikawa-Ueno's table [NU, p. 159, 172] in the $g = 2$ case. We also refer to Matsumoto-Montesinos [MM, §8] for its topological meaning. Therefore in this paper, we always express the germ of fiber by a suitable representative of the corresponding branch curve.

We introduce some notation for a germ of branch curve (W, B). We set

$$B_{\mathrm{hor}} = B - \Gamma_0 \ \text{ if } \Gamma_0 \subset B\,, \quad \text{and} \quad B_{\mathrm{hor}} = B \quad \text{otherwise}\,,$$

and call B_{hor} the *horizontal part* of B. Let

$$W = W_0 \xleftarrow{\tau_1} W_1 \xleftarrow{\tau_2} \cdots \xleftarrow{\tau_r} W_r \tag{1.1.1}$$

be a succession of blow-ups such that the center $P_{i-1} \in W_{i-1}$ of τ_i $(1 \le i \le r)$ is infinitely near to a bad point of B. Let $E_{i,i}$ be the exceptional curve for τ_i. For $0 \le i < j \le r$, let $E_{i,j}$ be the proper transform of $E_{i,i}$ by the map $\tau_{i,j} = \tau_{i+1} \circ \cdots \circ \tau_j : W_j \to W_i$. We also denote by $E_{0,j}$ the proper transform of Γ_0 by $\tau_{0,j}$. Then the reduced scheme of the total transform of Γ_0 by $\tau_{0,j}$ is written as $\boldsymbol{E}_j = \sum_{i=0}^{j} E_{i,j}$. We put $B_0 = B$ and let B_i $(1 \le i \le r)$ be the *even* proper transform of B_{i-1} in the sense of [Ho1], [P1], that is, $B_i = \tau_i^* B_{i-1} - 2[m_i/2]E_{i,i}$ where m_i is the multiplicity of B_{i-1} at P_{i-1}.

In choosing one representative from the equivalence class of it in a normalized form in some sense, the following lemma is useful:

LEMMA 1.2 ([X3, Lemma 5.1.2], [Ho3], [P2]). *Let $[f, F]$ be a germ of singular fiber of a degeneration of hyperelliptic curves of genus g. Then there exists a representative (W, B) of the branch curve of $[f, F]$ which satisfies the following condition* $(*)$;
$(*)$ *For any bad point P of B, the multiplicity at P of the horizontal part* $\mathrm{mult}_P(B_{hor})$ *is not greater than $g + 1$.*
Moreover such (W, B) is uniquely determined by $[f, F]$ except in the following case $(**)$:
$(**)$ *There exists just two bad points P_1, P_2 of B such that* $\mathrm{mult}_{P_1}(B_{hor}) = \mathrm{mult}_{P_2}(B_{hor}) = g + 1$.

We call (W, B) which satisfies $(*)$ the germ of the *normalized branch curve.* Note that the branch curve is not necessarily normalized in this paper, unless otherwise stated.

1.3. We use the following terminology. Assume two reduced (and may reducible) curves C_1 and C_2 on a nonsingular surface meet at a point Q. For a natural number n, we say C_1 is *n-tangential* to C_2 at Q if the following condition hold: Blow-up n times successively at infinitely near points of Q. Then there exist a local analytic component C_1' of C_1 at Q and a local analytic component C_2' of C_2 at Q so that the proper transforms of C_1' and C_2' by the composition of these blow-ups still meet, namely, C_1' and C_2' are contact at Q of order at least $n + 1$. Note that by definition, for two natural numbers n and m with $n \le m$, m-tangential is always n-tangential. We simply call tangential if it is 1-tangential.

We say C_1 is n-tangential to C_2 if there exists a point Q such that C_1 is n-tangential to C_2 at Q.

We go back to the previous situation. We define very special classes of germs of singular fiber.

1.3.1. Assume B is smooth on W and meets Γ_0 transversally except at one point P where the order of contact is two. Then the associated singular fiber F is an irreducible curve of genus $g-1$ with one ordinary double point. (The genus of a component means that of its normalization.) If the normalized branch curve (W, B) has the above property, we call $[f, F]$ a fiber germ of *type* 0_0.

1.3.2. Assume B does not contain Γ_0 and meets Γ_0 transversally except at one point P which is an ordinary singularity of B of multiplicity $2g' + 2$, where g' is an integer with $1 \leq g' \leq [(g-1)/2]$. Then F has an irreducible decomposition $F' + F''$ where F' and F'' are nonsingular curves of genera g' and $g - g' - 1$, respectively, meeting each other transversally at two points, whose self-intersection numbers in S are $(F')^2 = (F'')^2 = -2$. If the normalized branch curve (W, B) has the above property, we say that $[f, F]$ belongs to *class I*. In fact, this class contains $[(g-1)/2]$ types according to the number g'.

1.3.3. Assume B contains Γ_0. We can produce the succession of blow-ups (1.1.1) such that any singularity of the even proper transform B_r is ordinary. We take r the minimal number which enjoys the above property. Note that such a process is uniquely determined by B up to the order of the choice of the centers of blow-ups. Assume the following conditions (a)–(e) hold:

(a) B_r contains $\boldsymbol{E}_r$,

(b) Any singularity of B_r has even multiplicity,

(c) For any $0 \leq i \leq r$, any singularity on B_i with even multiplicity is ordinary.

(d) Assume B meets Γ_0 at P so that B is not tangential to Γ_0 at P, i.e., B_1 does not pass through the double point of $\boldsymbol{E}_1$ after blowing up at P. Then $B_1 - \boldsymbol{E}_1$ is 3-tangential to $\boldsymbol{E}_1$.

(e) If B_{hor} meets Γ_0 transversally at a smooth point P of B_{hor} (i.e., P is an ordinary double point of B), then there exits another point R such that B_{hor} is 3-tangential to Γ_0 at R.

We say a germ of fiber $[f, F]$ belongs to *class II* if at least one representative of the branch curve (W, B) of $[f, F]$ satisfy $B \supset \Gamma_0$ and moreover *any* representative of the branch curve (W, B) of $[f, F]$ with $B \supset \Gamma_0$ has the above conditions (a)–(e).

1.4. Let (C, P) be an analytic germ of a plane curve singularity. If (C, P) satisfies the following condition for some $m \geq 2$ and $n \geq 2$, we call it an m^n *point*: C has m irreducible local analytic components $C^{(1)}, \ldots, C^{(m)}$ such that $C^{(i)}$ is nonsingular and $I_P(C^{(i)}, C^{(j)}) = n$ for any $1 \leq i < j \leq m$. (A typical equation is $x^m + y^{mn} = 0$.)

LEMMA 1.5. *Let P be a bad point of the branch curve B which contains Γ_0 and defines a singular fiber in class* II.

(i) *If* $\mathrm{mult}_P B$ *is even, then* (B, P) *is an ordinary singularity.*

(ii) *If* $\mathrm{mult}_P B = 3$, *then* (B, P) *is a* 3^2 *point.*

(iii) *Assume* $\mathrm{mult}_P B = 5$ *and* B_{hor} *is not* 3*-tangencial to* Γ_0. *Then* (B, P) *is a* 5^2 *point.*

(iv) *Assume* $\mathrm{mult}_P B \leq 6$ *and* $I_P(B_{\mathrm{hor}}, \Gamma_0) \leq 10$. *Then* B_{hor} *is not* 3*-tangencial to* Γ_0 *at* P.

PROOF. The assertion (i) is clear by definition.

Let $B^{(1)}, \dots, B^{(k)}$ be the set of local analytic components of B_{hor} at P. Let $\tau_1 : W_1 \to W$ be the blow-up at P and we put $P_1 = E_{0,1} \cap E_{1,1}$. Let $\tilde{B}^{(i)}$ $(1 \le i \le k)$ be the proper transform of $B^{(i)}$ by τ_1.

Assume $\mathrm{mult}_P B = 3$. We first suppose $k = 1$. If $B^{(1)}$ is not tangential to Γ_0, then B_r has ordinary triple point. This contradict to the condition (b) in 1.3.3. Therefore $B^{(1)}$ is tangential to Γ_0. Then (B_1, P_1) is a triple or a quadruple point. However (B_1, P_1) is not an ordinary quadruple point, and so it is a triple point by the condition (c). Then B_r has an ordinary triple point, a contradiction to the condition (b). Suppose $k = 2$. If one of $B^{(i)}$ $(i = 1, 2)$ is not tangential to Γ_0, then B_r also has an ordinary triple point. Hence both of $B^{(i)}$ $(i = 1, 2)$ are tangential, and so P_1 is a quadruple point, which must be ordinary. Hence we have the assertion (ii).

Assume $\mathrm{mult}_P B = 5$. We first assume that at least one of the components, say $B^{(1)}$, is not tangential to Γ_0. Then $\tilde{B}^{(1)}$ passes through a point $Q \in E_{1,1} \backslash E_{0,1}$. Since $\mathrm{mult}_Q B_1 \le 4$ by the condition (d), it follows from (i) and (ii) that (B_1, Q) is an ordinary quadruple point or a 3^2 point or an ordinary double point. In the first case, B_1 has a triple point at P_1 which is not a 3^2 point. This is impossible. The second case contradicts the condition (d), because $B_1 - \boldsymbol{E}_1$ does not pass through P_1 and $B_1 - \boldsymbol{E}_1$ is not 3-tangential to $\boldsymbol{E}_1$. In the third case, we apply the elementary transformation so that the image of $E_{1,1}$ becomes a new fiber. Then the condition (e) is not satisfied.

Therefore all of $B^{(i)}$ $(1 \le i \le k)$ are tangential to Γ_0. If $k = 4$, then the multiplicity at P_1 of B_1 is 6, and so (B_1, P_1) is ordinary. Therefore (B, P) is a 5^2 point. We will show that $1 \le k \le 3$ is impossible.

Assume $k = 3$. We may suppose $\mathrm{mult}_P B^{(1)} = 2$ and the other two are nonsingular. Then $\tilde{B}^{(1)}$ is nonsingular, for otherwise (B_1, P_1) is a non-ordinary 6-ple point. Therefore $(B^{(1)}, P)$ is a simple cusp, i.e. $t^2 + x^3 = 0$. Then after the blow-up at P_1, B_3 has an ordinary triple point at $P_2 = E_{1,2} \cap E_{2,2}$. This is impossible.

Assume $k = 2$. We first consider the case where $\mathrm{mult}_P B^{(1)} = 3$ and $B^{(2)}$ is nonsingular. Since P_1 cannot be a 6-ple point, we have $\mathrm{mult}_{P_1} \tilde{B}^{(1)} \le 2$. Then $(B^{(1)}, P)$ is either $t^3 + x^4 = 0$ or $t^3 + x^5 = 0$. In each case, after two blow-ups, (B_2, P_2) has a triple point which is not a 3^2 point. This is impossible. When $\mathrm{mult}_P B^{(1)} = \mathrm{mult}_P B^{(2)} = 2$, the argument is similar. $k = 1$ is also impossible by a similar argument. Therefore we have the assertion (iii).

It remains to prove (iv). Suppose B_{hor} is 3-tangential to Γ_0 at P. If $\mathrm{mult}_P B_1$ is even, then (B_1, P_1) must be an ordinary singularity. This contradicts to the 3-tangentiality at P. If $\mathrm{mult}_P B_1 = 7$, then all of $B^{(1)}, \dots, B^{(k)}$ are tangential to Γ_0. Then (B, P) is a 5^2 point by the argument in (iii), which is not 3-tangential. We also have $\mathrm{mult}_P B_1 \ne 3$ by the same argument as in (ii). Therefore we have $\mathrm{mult}_P B_1 = 5$, that is, $\mathrm{mult}_{P_1}(B_1 - \boldsymbol{E}_1) = 3$.

Blow up at P_1, and set $P_2 = E_{0,2} \cap E_{2,2}$. By the same argument as above, we have $\mathrm{mult}_{P_2}(B_2 - \boldsymbol{E}_2) = 3$. By the 3-tangentiality at P, $B_2 - \boldsymbol{E}_2$ is tangential to $E_{0,2}$ at P_2. Therefore

$$I_P(B_{\mathrm{hor}}, \Gamma_0) > 4 + 3 + 3 = 10\,.$$

This contradicts the assumption. Q.E.D.

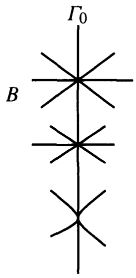

FIGURE 1. Branch curve of type $\{3, 3, 2^2\}$.

1.6. In low genus cases $2 \leq g \leq 4$, we have the classification of fibers in class II. We introduce some notation for writing them. We express the information of the set of bad points of a representative of the branch curve B inside the bracket $\{\ \}$ by writing that of B_{hor}. $\{m, \dots\}$ means that B has an ordinary $(m+1)$-ple point. $\{m^n, \dots\}$ means that B has an $(m+1)^n$ point. We express the weighted dual graph of the minimal singular fiber as follows: The component of the singular fiber is always nonsingular in our list. The bold face numbers $\mathbf{O}, \mathbf{I}, \mathbf{II} \cdots$ mean that the genus of the components are $0, 1, 2, \dots$ respectively. The coefficient means the multiplicity of the component. The suffix means the self-intersection number of the component, but we omit it if it is -2. The symbol "—" means "intersects transversally".

For instance, $\{3, 3, 2^2\}$ means that the bad points of B are two ordinary quadruple points and one 3^2 point. (See Figure 1.) This really appears as (g=4, iii) in Proposition 1.7. In this case, the even resolution consists of four blow-ups, and the minimal singular fiber is $F = 2F_0 + F_1 + F_2 + 2F_3$ where $g(F_0) = 0$, $g(F_1) = g(F_2) = g(F_3) = 1$, $F_0F_1 = F_0F_2 = F_0F_3 = 1$, $F_0^2 = F_1^2 = F_2^2 = -2$ and $F_3^2 = -1$. This is easily seen by the usual double covering method (cf. [P1]).

PROPOSITION 1.7. *The classification of germs of singular fiber of class* II *for genus* $2 \leq g \leq 4$ *are the following: We have one, three and three types for* $g = 2, 3$ *and* 4 *respectively.*

($g = 2$) (i) $\{3, 3\}$ $\mathbf{I}_{-1} - \mathbf{I}_{-1}$

($g = 3$) (i) $\{3, 5\}$ $\mathbf{I}_{-1} - \mathbf{II}_{-1}$ (ii) $\{2^2, 2^2\}$ $2\mathbf{I}_{-1} - 2\mathbf{O} - 2\mathbf{I}_{-1}$ (iii) $\{4^2\}$ $2\mathbf{II}_0$

($g = 4$) (i) $\{3, 7\}$ $\mathbf{I}_{-1} - \mathbf{III}_{-1}$ (ii) $\{5, 5\}$ $\boldsymbol{II}_{-1} - \mathbf{II}_{-1}$ (iii) $\{3, 3, 2^2\}$

$$\begin{array}{l} \mathbf{I} \\ | \\ 2\mathbf{O} - 2\mathbf{I}_{-1} \\ | \\ \mathbf{I} \end{array}$$

PROOF. Let (W, B) be the normalized branch curve of the germ of fiber. If B contains Γ_0, Lemma 1.5 easily implies the list in the assertion.

Assume B does not contain Γ_0. We set $B \cap \Gamma_0 = \{P_1, P_2, \dots, P_k\}$. We may assume that (B, P_1) is a triple or a 5-ple point and B intersects Γ_0 transversally at $P_2, \dots, P_k$. Otherwise, we choose another branch curve (W', B') such that B' contains the fiber by elementary transformations. Then the condition (c) in 1.3.3 is not satisfied for (W', B').

Now let (W'', B'') be the branch curve obtained by the elementary transformation at P_1. Then B'' contains the new fiber Γ_0''. Let Q be the point on Γ_0'' which is the image of $E_{0,1}$ by the contraction map. Since B does not contain Γ_0, the multiplicity of (B'', Q) is even. Therefore (B'', Q) is an ordinary singularity. Other bad points of B'' have their multiplicity at most 6. Then we apply Lemma 1.5, and obtain the assertion.

2. Fisson of plane curve singularities. Let C be a normal analytic curve defined on the open set $V_0 = \{(X, Y) \in \mathbf{C}^2 ; |X| < \varepsilon, |Y| < \varepsilon, \}$ for a sufficiently small ε such that $P = (0, 0)$ is the unique singular point of C. In this section, we explicitly construct certain local analytic deformations $\{C_u\}_{u \in \Delta_\delta}$ of C such that C_u $(u \neq 0)$ has several isolated singular points. The idea is essentially the same as that in A'Campo [Ac].

2.1. Let $C = \sum_{j=1}^{s} C^{(j)}$ be the local analytic irreducible decomposition of C at P. Let $V_0 \xleftarrow{\tau_1} V_1 \xleftarrow{\tau_2} \cdots \xleftarrow{\tau_r} V_r$ be the succession of blow-ups whose centers are infinitely near to P and let $\tilde{C}_r$ (resp. $\tilde{C}_r^{(j)}$, $1 \le j \le s$) be the proper transform of C (resp. $C^{(j)}$) by $\tau_{0,r} := \tau_1 \tau_2 \cdots \tau_r$. Let Q be an isolated singular point of $\tilde{C}_r$.

We first assume that Q is a smooth point of the (reduced) exceptional set $\mathcal{E}_r = \sum_{j=1}^{r} E_j$. Assume Q is on E_{j_0} $(1 \le j_0 \le r)$. Let $U(x, y)$ be the open coordinate neighborhood in the classical topology on V_r containing Q so that E_{j_0} is defined by $y = 0$ and Q is defined by $(x, y) = (\alpha, 0)$ $(\alpha \in \mathbf{C})$. Let $\tilde{C}_r^{(1)}, \dots \tilde{C}_r^{(k_0)}$ $(k_0 \le s)$ be the components of $\tilde{C}_r$ which pass through Q (by changing the order of them if necessary). Since ε is sufficiently small, we may assume that the support of the divisor $\sum_{j=0}^{k_0} \tilde{C}_r^{(j)}$ on V_r is contained in U. Let

$$g(x - \alpha, y) = \prod_{j=1}^{k_0} g^{(j)}(x - \alpha, y) = 0$$

be the equation of $\sum_{j=0}^{k_0} \tilde{C}_r^{(j)}$ on U. Now we define a divisor $\mathcal{D}$ on $U \times \Delta_\delta$ by

$$g(x - \alpha - \varphi(u), y - \psi(u)) = 0 \tag{2.1.1}$$

where u is the coordinate of Δ_δ and $\varphi(u)$, $\psi(u)$ are generic holomorphic functions of u which satisfy $\varphi(0) = \psi(0) = 0$.

Since ε and δ are sufficiently small, we can choose a classical open covering $V_r \times \Delta_\delta = \bigcup_{i=0}^{l} \mathcal{U}_i$ so that $\mathcal{U}_0$ coincides with $U \times \Delta_\delta$, and the locus $(\bigcup_{i=1}^{l} \mathcal{U}_i \cap \mathcal{U}_0)$ does not contain the support of $\mathcal{D}$. Therefore the analytic closure of $\mathcal{D}$ in $V_r \times \Delta_\delta$ coincides with $\mathcal{D}$ itself. The divisor $\mathcal{D}$ on $V_r \times \Delta_\delta$ has the following properties:

(i) $\mathcal{D}$ does not meet the divisor $\mathcal{D}' = \sum_{j=k_0+1}^{s} \tilde{C}_r^{(j)} \times \Delta_\delta$ on $V_r \times \Delta_\delta$.

(ii) For any $u \in \Delta_\delta$, let $\mathcal{D}_u$ be the restriction of $\mathcal{D}$ to $V_r \times \{u\}$ and we regard it as a curve on V_r. Then $\mathcal{D}_u$ intersects $\mathcal{E}_r$ transversally at $I_Q(\tilde{C}_r, E_{j_0})$ mutually distinct points.

Moreover, the point $Q_u = \{(x, y) = (\alpha + \varphi(u), \psi(u))\}$ on U is the unique singular point of $\mathcal{D}_u$.

Now we define a divisor M on $V_0 \times \Delta_\delta$ by

$$M = (\tau_{0,r})_* \mathcal{D} + \sum_{j=k_0+1}^{s} \tilde{C}^{(j)} \times \Delta_\delta.$$

The natural fibration $\boldsymbol{\pi} : M \to \Delta_\delta$ has the following properties:

(a) $M_0 = \boldsymbol{\pi}^{-1}(0)$ coincides with C.

(b) For any $u \in \Delta_\delta \setminus \{0\}$, the fibers M_u has just two isolated singular points at P and $R_u = \tau_{0,r}(Q_u)$. Moreover the germ of singularity (M_u, R_u) coincides with $(\tilde{C}_r, Q)$.

We call (C, P) *has a fisson into* (M_u, P) *and* (M_u, R_u) *by the perturbation at the infinitely near point* Q, *and we call the fibration* $\boldsymbol{\pi}$ *the fisson of* (C, P) *of type A*.

EXAMPLE 2.2. The singularity $x^3 + y^n = 0$ $(0 < 3k < n)$ has a fisson into $x^3 + y^{3k} = 0$ and $x^3 + y^{n-3k} = 0$ by the perturbation at the infinitely near point which appears after k blow-ups.

2.3. Assume Q is a double point of $\mathcal{E}_r$. We construct local deformations of (C, p) in the following two different ways:

(1) We assume $Q = E_{j_0} \cap E_{j_1}$ and $x = 0$ is the equation of E_{j_1} on U in the same situation as in 2.1. Then we can construct a fibration $\boldsymbol{\pi} : M \to \Delta_\delta$ such that the only one condition (ii) is replaced by the following (ii$'$):

(ii$'$) For any $u \in \Delta_\delta$, $\mathcal{D}_u$ intersects $\mathcal{E}_r$ only at nonsingular points of $E_{j_0} + E_{j_1}$. $\mathcal{D}_u$ intersects E_{j_k} $(k = 0, 1)$ transversally at $I_Q(\tilde{C}_r, E_{j_k})$ mutually distinct points. Moreover the point $Q_u = \{(x, y) = (\varphi(u), \psi(u))\}$ on U is the unique singular point of $\mathcal{D}_u$.

We also call $\boldsymbol{\pi}$ a fisson of *type A*.

(2) In the equation (2.1.1), we put $\psi(u) \equiv 0$. Then $\mathcal{D}_u$ $(u \neq 0)$ has a singularity at $Q_u = \{(x, y) = (\psi(u), 0)\}$ on E_{j_0} and intersects E_{j_1} at $I_Q(\tilde{C}_r, E_j)$ distinct points transversally. We construct $\boldsymbol{\pi} : M \to \Delta_\delta$ in the same way. Then M_u $(u \neq 0)$ has one isolated singular point at P such that the germ (M_u, P) is different from (C, P), because their embedded resolution processes are distinct. We call π a fisson of *type B*.

EXAMPLE 2.4. $x^5 + y^3 = 0$ has a fisson into $(x^4 + y^2)(x + y) = 0$ as follows: After two blow-ups, the proper transform becomes nonsingular and meets E_1 transversally and E_2 with contact of order 2. We perturb it to the direction of the generic point of E_1, and then blow-down twice.

REMARK 2.5. The above deformations are not necessarily well-defined in the algebraic category, because the local analytic reducibility does not imply algebraic reducibility. (This was pointed out to the authors by Professor Masayoshi Miyanishi.)

3. Splitting families of degenerations. In this section, we construct splitting families of degenerations of hyperelliptic curves. Our aim is not only to prove the existence of splitting families but also to describe explicitly the germs of singular fibers in our families.

DEFINITION 3.1. Let $[f, F]$ be a germ of singular fiber of a degeneration of hyperelliptic curves of genus g. Assume there exist positive real numbers ε and δ, a 3-dimensional complex manifold Z and a flat surjective holomorphic map $h : Z \to \Delta_\varepsilon \times \Delta_\delta$ such that

(i) the fibration $h_0 : Z_0 \to \Delta_\varepsilon \times \{0\}$ arising from the restriction of h over $0 \in \Delta_\delta$ has a unique singular fiber over $0 \in \Delta_\varepsilon$ such that the germ of fiber $[h_0, h_0^{-1}(0)]$ coincides with $[f, F]$,

(ii) for any $u \in \Delta_\delta \setminus \{0\}$, the fibration $h_u : Z_u \to \Delta_\varepsilon \times \{u\}$ has l $(l \geq 2)$ singular fibers. The number l is independent of u.

Then we call h a splitting family of $[f, F]$.

DEFINITION 3.2. Let $\boldsymbol{G}$ be a certain subset of the set of all germs of singular fiber of degenerations of hyperelliptic curves of genus g. We say the germ $[f, F]$ is reduced to $\boldsymbol{G}$ via several splitting families if the following conditions are satisfied:

First $[f, F]$ has a splitting family $h : Z \to \Delta_\varepsilon \times \Delta_\delta$. If any germ of singular fiber $[h_u, h_u^{-1}(t)]$ $(t \in \Delta_\varepsilon \times \{u\})$ of any general fibration $h_u : Z_u \to \Delta_\varepsilon \times \{u\}$ $(u \in \Delta_\delta \setminus \{0\})$ belongs to $\boldsymbol{G}$, then we stop our reducing process. Assume some of them do not belong to $\boldsymbol{G}$. Let $[h_u, h_u^{-1}(t_0)]$ $(t_0 \in \Delta_\varepsilon \times \{u\})$ be any germ of singular fiber of h_u which does not belong to $\boldsymbol{G}$. Then $[h_u, h_u^{-1}(t_0)]$ has a splitting family $h' : Z' \to \Delta_{\varepsilon'} \times \Delta_{\delta'}$. If any germ of singular fiber of any general fibration of h' belongs to $\boldsymbol{G}$, then we stop our process. If not, the germ which does not belong to $\boldsymbol{G}$ also has a splitting family.

Then this process terminates after finitely many steps, that is, any germ of singular fiber of any general fibration of the terminating splitting families belong to $\boldsymbol{G}$.

The main result of this paper is the following:

THEOREM 3.3. *Let $\boldsymbol{G}$ be the union of the sets of the germs of singular fiber of type* 0_0, *classes* I *and* II *defined in Section* 1. *Then any germ of singular fiber of a degeneration of hyperelliptic curves is reduced to $\boldsymbol{G}$ via several splitting families.*

COROLLARY 3.4. *Any degeneration of hyperelliptic curves of genus two* (*resp. three, four*) *is reduced via several splitting families to germs of singular fibers of two types* (*resp. five types, five types*) *whose tables are listed in* 1.3.1, 1.3.2 *and Proposition* 1.7.

For the proof of Theorem 3.3, we start with some lemmas. Let (W, B) be a representative of the branch curve of the germ of fiber $[f, F]$, where $W = \boldsymbol{P}^1 \times \Delta_\varepsilon$ for sufficiently small ε. Let P be a bad point of B. Let $W = W_0 \xleftarrow{\tau_1} W_1 \xleftarrow{\tau_2} \cdots \xleftarrow{\tau_r} W_r$ be a succession of blow-ups at infinitely near points of P. We use the same notation as in 1.1.

LEMMA 3.5. *Assume there is a component $E_{j_1,r}$ $(0 \leq j_1 \leq r)$ and a point Q on $E_{j_1,r}$ such that* (i) *Q is not a double point of $\boldsymbol{E}_r$,* (ii) *$E_{j_1,r}$ is not contained in B_r and* (iii) *B_r has a singularity at Q or B_r is tangential to $E_{j,r}$ at Q. Then $[f, F]$ has a splitting family.*

PROOF. Let $B^\sharp$ be the sum of components of B which pass through P. Let V_0 be a sufficiently small classical open neighborhood of P on W, and let

$$V_0 \times \Delta_\delta \supset \boldsymbol{B}^\sharp \xrightarrow{\pi} \Delta_\delta$$

be the fisson of type A of the germ $(B^\sharp, P)$ arising from the perturbation at Q constructed in 2.1. Let $\overline{\boldsymbol{B}^\sharp}$ be the analytic closure of $\boldsymbol{B}^\sharp$ in $W_0 \times \Delta_\delta$. If the support of B does not contain the fiber Γ_0, then $\overline{\boldsymbol{B}^\sharp}$ coincides with $\boldsymbol{B}^\sharp$. If the support of B contains Γ_0, then $\overline{\boldsymbol{B}^\sharp}$ coincides with $(\boldsymbol{B}^\sharp)_{hor} + \Gamma_0$ where $(\boldsymbol{B}^\sharp)_{hor}$ is the horizontal part of $\boldsymbol{B}^\sharp$.

Now we define a divisor $\boldsymbol{B}$ on $\boldsymbol{W} = W \times \Delta_\delta$ by

$$\boldsymbol{B} := \overline{\boldsymbol{B}^\sharp} + (B - B^\sharp) \times \Delta_\delta\,.$$

Let $\rho : \boldsymbol{B} \to \Delta_\delta$ be the natural morphism. We consider the fiber $\boldsymbol{B}_u = \rho^{-1}(u)$ as a divisor on $W \cong W \times \{u\}$. Then $\boldsymbol{B}_0$ coincides with B and $\boldsymbol{B}_u$ $(u \neq 0)$ has the following properties:

i) $\boldsymbol{B}_u \Gamma_t = 2g+2$ $(t \in \Delta_\varepsilon)$,

ii) $\boldsymbol{B}_u$ has two bad points P on Γ_0 and R_u on $\Gamma_{\pi(R_u)}$, where the point R_u is as described in 2.1.

Since the problem is local with respect to the parameter space, we may assume that there is a line bundle $\boldsymbol{L}$ on $\boldsymbol{W}$ which satisfies $[\boldsymbol{B}] \cong 2\boldsymbol{L}$. On the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{W}} \oplus \mathcal{O}_{\boldsymbol{W}}(\boldsymbol{L}))$, we construct a double cover $\mu : \mathcal{S} \longrightarrow \boldsymbol{W}$ branched along $\boldsymbol{B}$. In order to resolve the singularities on $\mathcal{S}$, we construct a *relative even resolution* for the family ρ of branch curves in the following way:

By the construction of ρ and Tomari's lemma [As, Lemma 3.8], the multiplicity of the center of i-th blow up $(1 \leq i \leq r)$ of the even resolution of the germ $(\boldsymbol{B}_u, P)$ coincides with m_i for any $u \in \Delta_\delta$. Now let

$$\boldsymbol{W} = \boldsymbol{W}_0 \xleftarrow{\boldsymbol{\tau}_1} \boldsymbol{W}_1 \xleftarrow{\boldsymbol{\tau}_2} \cdots \xleftarrow{\boldsymbol{\tau}_r} \boldsymbol{W}_r$$

be the succession of blow-ups such that the center of $\boldsymbol{\tau}_i$ $(1 \leq i \leq r)$ is $\{P_{i-1}\} \times \Delta_\delta$. Let $\boldsymbol{B}_i = \boldsymbol{\tau}_i^* \boldsymbol{B}_{i-1} - 2[m_i/2]\mathcal{C}_i$ $(1 \leq i \leq r)$ be the even proper transform of $\boldsymbol{B}_{i-1}$, where $\boldsymbol{B}_0 = \boldsymbol{B}$ and $\mathcal{C}_i$ is the exceptional set for $\boldsymbol{\tau}_i$. Note that $\boldsymbol{B}_r$ does not contain $E_{j_1,r} \times \Delta_\delta$ by our assumption. Therefore on the coordinate open neighborhood $U \times \Delta_\delta = \{(x, y, z)\}$ of $W \times \Delta_\delta$ described in 2.1, the equation of $\boldsymbol{B}_r$ is written as $g(x - \alpha - \varphi(u), y - \psi(u)) = 0$. Now let $\boldsymbol{Q}$ be the curve on $W_r \times \Delta_\delta$ defined by the image of the holomorphic map

$$\Delta_\delta \to U \times \Delta_\delta \subset W_r \times \Delta_\delta\,, \quad u \mapsto (x, y, u) = (\varphi(u), \psi(u), u)\,.$$

Then $\boldsymbol{B}_r$ is equisingular along the locus $\boldsymbol{Q}$ in the sense that the restricted germs of singularity $((\boldsymbol{B}_r)_{|\pi_r^{-1}(u)}, \boldsymbol{Q}_{|\pi_r^{-1}(u)})$ have the same even resolution process for any $u \in \Delta_\delta$, where $\pi_r : \boldsymbol{B}_r \to \Delta_\delta$ is the natural map. Therefore we have a similar process for $\boldsymbol{Q}$. Lastly the singularities of $(B - B^\sharp) \times \Delta_\delta$ are simultaneously resolved and we complete the relative even resolution for the family ρ. Namely, by the succession of blow-ups $\boldsymbol{\tau}^\flat : \boldsymbol{W}^\flat \to \boldsymbol{W}$, the even proper transform $\boldsymbol{B}^\flat$ of $\boldsymbol{B}$ becomes nonsingular and the restriction of $\boldsymbol{\tau}^\flat$ over any $u \in \Delta_\delta$ coincides with the even resolution of $\boldsymbol{B}_u$. Since $\boldsymbol{B}^\flat$ is an even divisor on $\boldsymbol{W}^\flat$, we can construct a nonsingular double covering $\boldsymbol{\pi} : \boldsymbol{M}^\flat \to \boldsymbol{W}^\flat$ which branches along $\boldsymbol{B}^\flat$ and obtain natural morphisms $\boldsymbol{M}^\flat \xrightarrow{\rho_1} \Delta_\varepsilon \times \Delta_\delta \xrightarrow{\rho_2} \Delta_\delta$.

By our construction, any (-1)-curve contained in a fiber of $\boldsymbol{B}^\flat_{|(\rho_1\rho_2)^{-1}(u)} \to \Delta_\varepsilon \times \{u\}$ for any $u \in \Delta_\delta$ is stable without shrinking the parameter space Δ_δ. Therefore any (-1)-curve contained in a fiber of $M^{\flat}_{|(\rho_1\rho_2)^{-1}(u)} \to \Delta_\varepsilon \times \{u\}$ is stable over Δ_δ, and is simultaneously contracted by Fujiki-Nakano [FN]. Repeating this process, we reach a family of relatively minimal fibrations $\boldsymbol{M} \to \Delta_\varepsilon \times \Delta_\delta \to \Delta_\delta$. This family has the following properties:

(i) $f_0 : \boldsymbol{M}_0 \to \Delta_\varepsilon \times \{0\}$ has a unique singular fiber over $0 \in \Delta_\varepsilon$ whose fiber germ coincides with $[f, F]$.

(ii) For any $u \in \Delta_\delta \setminus \{0\}, f_u : \boldsymbol{M}_u \to \Delta_\varepsilon \times \{u\}$ has just two singular fibers $f_u^{-1}(0)$ and $f_u^{-1}(\pi(R_u))$. Q.E.D.

EXAMPLE 3.6. Assume $B \not\supset \Gamma_0$, $B\Gamma_0 = 6$ and B has a triple point $x^3 + t^{6n} = 0$ at $P \in \Gamma_0$ with $I_P(B, \Gamma_0) = 3$ and meets Γ_0 transversally at the other three points. By $2k$ blowing-ups ($1 \le k \le n-1$), the infinitely near point Q of P becomes $x^3 + t'^{6(n-k)} = 0$ and B_{2k} does not contain $E_{2k,2k}$. Then we use the method in Lemma 3.5. $\boldsymbol{B}_u$ has two bad points $x^3 + t^{6k} = 0$ and $x^3 + t^{6(n-k)} = 0$ at two mutually distinct points as in Example 2.2. (In Figure 2, the solid line is a component of the branch locus while the dotted line is not.) The associated family of genus two fibrations $f : M \to \Delta_\delta$ has the following properties:
f_0 has a unique singular fiber

$$\mathbf{I}_{-1} - \overbrace{\mathbf{O} - \cdots - \mathbf{O}}^{n} - \mathbf{I}_{-1}$$

and f_u ($u \neq 0$) has two singular fibers

$$\mathbf{I}_{-1} - \overbrace{\mathbf{O} - \cdots - \mathbf{O}}^{k} - \mathbf{I}_{-1}\,, \quad \mathbf{I}_{-1} - \overbrace{\mathbf{O} - \cdots - \mathbf{O}}^{n-k} - \mathbf{I}_{-1}\,.$$

LEMMA 3.7. *Under the same conditions* ii), iii) *as in Lemma* 3.5, *we further assume* $Q = E_{j_0,r} \cap E_{j_1,r}$ *for some* $0 \le j_0 < j_1$. *Then* $[f, F]$ *has a splitting family.*

PROOF. If $E_{j_0,r} \not\subset B$, then we can construct a splitting family as in Lemma 3.5 by using the fisson of type A for (B, P) defined in 2.3, (1).

Assume $E_{j_0,r} \subset B$. Then we use the fisson of type B for (B, P) defined in 2.3, (2) arising from the perturbation of the branch curve at Q into the generic point of $E_{j_0,r}$, and construct a family $\rho : \boldsymbol{B} \to \Delta_\delta$ of branch curves as in Lemma 3.5. ρ has a relative even resolution and the associated family of relatively minimal hyperelliptic fibrations $f : \boldsymbol{M} \to \Delta_\varepsilon \times \Delta_\delta \to \Delta_\delta$ has the following properties:

(i) $f_0 : \boldsymbol{M}_0 \to \Delta_\varepsilon \times \{0\}$ has a unique singular fiber over $0 \in \Delta_\varepsilon$ whose fiber germ coincides with $[f, F]$,

(ii) for any $u \in \Delta_\delta \setminus \{0\}$, the fibration $f_u : \boldsymbol{M}_u \to \Delta_\varepsilon \times \{u\}$ has $k+1$ singular fibers, where $k = I_Q(B_r - E_{j_0,r}, E_{j_1,r})$. One of them is $f_u^{-1}(0)$ and the others are fibers of type 0_0.

Indeed, we denote by $(\boldsymbol{B}_u)_i$ ($1 \le i \le r, u \neq 0$) the even proper transform by $\tau_{1,i}$ of $\boldsymbol{B}_u = \rho^{-1}(u)$ on $W \cong W \times \{u\}$. Since $E_{j_1,r}$ is contracted to $Q' = \tau_{j_1,r}(Q)$ by the map $\tau_{j_1,r} : W_r \to W_{j_1-1}$, the images by $\tau_{j_1,r}$ of the components of $(\boldsymbol{B}_u)_r$ which intersect $E_{j_1,r}$ pass through Q'. Since the family ρ arises from the perturbation of type B at $Q = E_{j_0,r} \cap E_{j_1,r}$, $(\boldsymbol{B}_u)_{j_1-1}$ has at least k nonsingular local components $C^{(1)}, \ldots, C^{(k)}$ at Q' which transversally intersect one another at this point. Moreover $C^{(j)}$ ($1 \le j \le k$) also

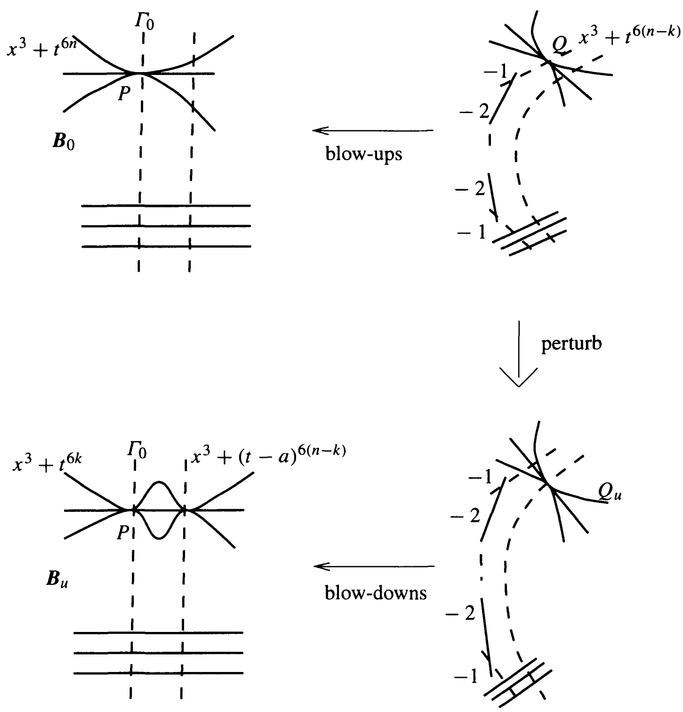

FIGURE 2. Construction of the branch curve in Example 3.6.

pass through the point $Q_u' = \tau_{j_1,r}(Q_u)$ where Q_u is as defined in 2.3, (2). Now let U' be a small classical open set of W_{j_1-1} containing Q' and Q_u', and we consider the restricted fibration $\pi_{j-1}|_U : W_{j-1}|_U \to \Delta_\varepsilon$. Then it is easy to see that each $C^{(j)}$ $(1 \leq j \leq k)$ has contact of order 2 at a point of certain mutually distinct fibers $\pi|_U^{-1}(t_j)$ $(t_j \neq 0)$. Therefore $\boldsymbol{B}_u$ also has contact of order 2 at just k points of fibers Γ_{t_j} $(j = 1, \dots, k)$ of $\pi_0 : W \to \Delta_\delta$, and transversally intersects the fiber of π_0 at the other points. Hence the fibration $\boldsymbol{f}$ has the above properties. Q.E.D.

EXAMPLE 3.8. Assume $B \supset \Gamma_0$, $B_{\mathrm{hor}}\Gamma_0 = 8$, and that B_{hor} has a triple point $x^6+t^3 = 0$ at P with $I_P(B_{\mathrm{hor}}, \Gamma_0) = 6$ and transversally intersects Γ_0 at the other two points. Let $\tau_1 : W_1 \to W$ be the blow-up at P. Then $B_1 \not\supset E_{1,1}$, $B_1 \supset E_{0,1}$ and $\tilde{B}_1 = B_1 - E_{0,1}$ has an ordinary triple point at $Q = E_{0,1} \cap E_{1,1}$. By the method in Lemma 3.7, $\boldsymbol{B}_u$ $(u \neq 0)$ has two ordinary triple points on Γ_0 and three tangential points on three mutually distinct fibers (see Figure 3). The associated family of genus three fibrations $\boldsymbol{f} : \boldsymbol{M} \to \Delta_\delta$ has the following properties: f_0 has an unique singular fiber

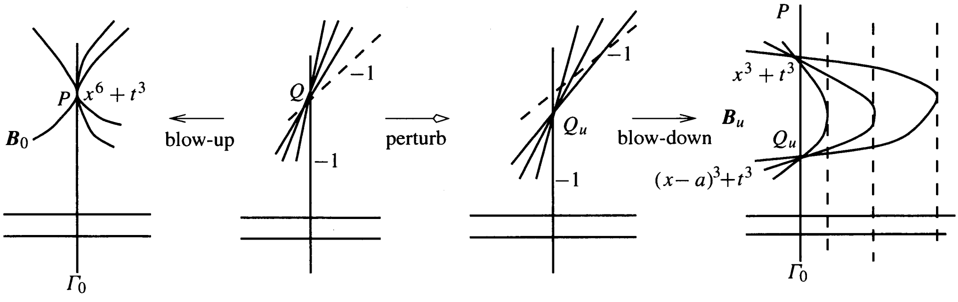

FIGURE 3. Construction of the branch curve in Example 3.8.

$$\begin{array}{ccccccc} \mathbf{O} & - & 2\mathbf{O} & - & 2\mathbf{I} & - & \mathbf{O} \\ & & | & & | & & \\ & & \mathbf{O} & & \mathbf{O} & & \end{array}$$

and f_u $(u \neq 0)$ has four singular fibers. One of them is

$$\begin{array}{ccccc} & & \mathbf{O} & & \\ & & | & & \\ \mathbf{O} & - & 2\mathbf{O} & - & \mathbf{I} \\ & & | & & \\ & & \mathbf{I} & & \end{array}$$

and the other three are of type 0_0.

LEMMA 3.9. *Assume $B \not\supset \Gamma_0$ and B has at least two bad points on Γ_0. Then* $[f, F]$ *has a splitting family.*

PROOF. Let $P_1, \ldots, P_s$ be bad points of B. Let $h_i : \Delta_\varepsilon \to W$ $(i = 1, 2)$ be sections of $\pi : W \to \Delta_\varepsilon$ such that the image $h_i(\Delta_\varepsilon)$ does not pass through any of $P_1, \ldots, P_s$. Then the pair $(h_1 : h_2)$ defines a homogeneous fiber coordinate of the $\boldsymbol{P}^1$-bundle π. Let (x, t) be a coordinate of the open set $U := W - h_2(\Delta_\varepsilon) \cong \Delta_\varepsilon \times \boldsymbol{C}$, where $x = h_1/h_2$. We may assume that the support of the divisor B on W is contained in U. We set $P_1 = \{(x, t) = (\alpha, 0)\}$ $(\alpha \in \boldsymbol{C})$. Then the equation of B in U is written as $f_1(x - \alpha, t) f_2(x, t) = 0$, where f_1, f_2 are holomorphic functions of (x, t) which satisfy $f_1(0, 0) = 0$ and $f_2(\alpha, 0) \neq 0$. For a sufficiently small δ, we define a divisor $\boldsymbol{B}'$ on $\boldsymbol{W} = W \times \Delta_\delta$ by

$$f_1(x - \alpha, t - cu) f_2(x, t) = 0,$$

where c is a complex number whose absolute value is sufficiently small. Then the analytic closure $\boldsymbol{B}$ of $\boldsymbol{B}'$ in $\boldsymbol{W}$ coincides with $\boldsymbol{B}'$ itself. Let $\boldsymbol{\varphi} : \Delta_\delta \to \boldsymbol{W}$ be the map defined by $u \mapsto (x, t, u) = (\alpha, cu, u)$. Then the family of branch curves $\boldsymbol{B} \to \Delta_\delta$ on $\boldsymbol{W}$ has a relative even resolution by the blow-ups whose centers are infinitely near to the locus $\boldsymbol{\varphi}(\Delta_\delta)$ and $\{P_i\} \times \Delta_\delta$ $(2 \leq i \leq s)$. Then we obtain a family of hyperelliptic fibrations $\boldsymbol{M} \to \Delta_\varepsilon \times \Delta_\delta$

such that $\boldsymbol{M}_u \to \Delta_\varepsilon \times \{u\}$ ($u \neq 0$) has two singular fibers. One of them has one bad point on its branch curve, and the other has $s-1$ bad points. Q.E.D.

LEMMA 3.10. *We assume* (i) $B \supset \Gamma_0$, (ii) *the even proper transform* B_1 *of* B *by* $\tau_1 : W_1 \to W$ *contains* $\boldsymbol{E}_1 = E_{0,1} + E_{1,1}$, (iii) $\tilde{B}_1 = B_1 - \boldsymbol{E}_1$ *does not pass through* $P_1 = E_{0,1} \cap E_{1,1}$, (iv) $\tilde{B}_1$ *intersects both* $E_{0,1}$ *and* $E_{1,1}$ *and* (v) $\tilde{B}_1$ *is not* 3*-tangential to* $\boldsymbol{E}_1$. *Then* $[f, F]$ *has a splitting family.*

PROOF. We take a coordinate (t_1, x) on the open set $U \cong \Delta_\varepsilon \times \boldsymbol{C}$ of W_1 so that the support of the divisor $\tilde{B}_1$ on W_1 is contained in U, the equation of $E_{0,1}$ and $E_{1,1}$ on U are $t_1 = 0$ and $x = 0$ respectively and the map $\pi\tau_1 : W_1|_U \to \Delta_\varepsilon$ is given by $(t_1, x) \mapsto t = t_1 x$. Let $P_i' = \{(t_1, x) = (\alpha_i, 0)\}$ $(1 \le i \le s)$, $P_i'' = \{(t_1, x) = (0, \beta_j)\}$ $(1 \le j \le s')$ be the intersection points of $\tilde{B}_1$ with $\boldsymbol{E}_1$ ($\alpha_i \neq 0, \beta_j \neq 0$). Then the equation of B_1 on U is written as

$$t_1 x \prod_{i=1}^{s} f_i(t_1 - \alpha_i, x) \prod_{j=1}^{s'} f_{s+j}(t_1, x - \beta_j) = 0\,.$$

Now we define a divisor $\boldsymbol{B}'$ on the open coordinate neighborhood $U \times \Delta_\delta = \{(t_1, x, u)\}$ by

$$\begin{aligned}(t_1 x - u^2) \prod_{i=1}^{s} f_i\left(t_1 - \alpha_i, x - \frac{u^2}{\alpha_i} + \frac{u^2}{\alpha_i^2}(t_1 - \alpha_i)\right) \\ \times \prod_{j=1}^{s'} f_{s+j}\left(t_1 - \frac{u^2}{\beta_j} + \frac{u^2}{\beta_j^2}(x - \beta_j), x - \beta_j\right) = 0\,.\end{aligned}$$

$\boldsymbol{B}'$ passes through the point $\tilde{P}_i' = (\alpha_i, u^2/\alpha_i, u)$ $(1 \le i \le s)$ and $\tilde{P}_i'' = (u^2/\beta_j, \beta_j, u)$ $(1 \le j \le s')$. Since the tangent line of the curve $t_1 x - u^2 = 0$ at $\tilde{P}_i'$ coincides with $t_1 - u^2/\alpha_i + (u^2/\alpha_i^2)(x - \alpha_i) = 0$ (with u fixed) and this curve has contact of order 2 with this tangent line, the germs of singularities (B, P_i') and $(\boldsymbol{B}_u', \tilde{P}_i')$ have the same even resolution process by the assumption (v). The germs (B, P_j'') and $(\boldsymbol{B}_u', \tilde{P}_j'')$ also have the same property.

Let $\boldsymbol{B}^1$ be the closure of $\boldsymbol{B}'$ in $W_1 \times \Delta_\delta$. Let $\boldsymbol{\varphi}_i$ $(1 \le i \le s)$, $\boldsymbol{\psi}_j$ $(1 \le j \le s')$ be the maps from Δ_δ to $U \times \Delta_\delta \subset W_1 \times \Delta_\delta$ defined by

$$\boldsymbol{\varphi}_i : u \mapsto \left(\alpha_i, \frac{cu}{\alpha_i}, u\right), \quad \boldsymbol{\psi}_j : u \mapsto \left(\frac{cu}{\beta_j}, \beta_j, u\right).$$

Then by the succession of blow-ups at the infinitely near loci of $\boldsymbol{\varphi}_i(\Delta_\delta)$ and $\boldsymbol{\psi}_i(\Delta_\delta)$, we have a relative even resolution $\tilde{\boldsymbol{B}} \to \boldsymbol{B}^1$ of the family of branch curves $W_1 \times \Delta_\delta \supset \boldsymbol{B}^1 \to \Delta_\delta$ except at the point P_1. Let $\tilde{\boldsymbol{M}} \to W \times \Delta_\delta$ be the double cover branched along $\tilde{\boldsymbol{B}}$. Then $\tilde{\boldsymbol{M}}$ is smooth except at one point $\tilde{P}_1$, which is a 3-dimensional A_1-singularity. We can resolve the singularity at $\tilde{P}_1$ without affecting the general fiber of $\tilde{\boldsymbol{M}} \to \Delta_\delta$ by the method of Atiyah [At1] (see also [B]). Starting from $\tilde{\boldsymbol{M}}$, we obtain a family of relatively minimal hyperelliptic fibrations $\boldsymbol{M} \to \Delta_\varepsilon \times \Delta_\delta \to \Delta_\delta$ by the same argument as in the proof of Lemma 3.5. Then $\boldsymbol{M}_u \to \Delta_\varepsilon \times \{u\}$ has two singular fibers $\boldsymbol{f}_u^{-1}(0)$ and $\boldsymbol{f}_u^{-1}(u)$ for any $u \neq 0$. Q.E.D.

LEMMA 3.11. *Assume* (i) $B \supset \Gamma_0$, (ii) $B_{\mathrm{hor}} = B - \Gamma_0$ *transversally intersects* Γ_0 *at a point* P *and* (iii) B_{hor} *is not* 3*-tangential to* Γ_0. *Then* $[f, F]$ *has a splitting family.*

PROOF. We set $B_{\mathrm{hor}} \cap \Gamma_0 = \{P, P_1, \dots, P_s\}$. Let $U \cong \boldsymbol{C} \times \Delta_\varepsilon = \{(x, t)\}$ be an open coordinate neighborhood on W such that $B_{hor} \cap \Gamma_0$ is contained in U, the map π is given by $(x, t) \mapsto t$ and the local branch of B_{hor} passing through P is given by $x = 0$. Then the equation of B on U is written as $txf_1(x - \alpha_1, t) \cdots f_s(x - \alpha_s, t) = 0$, where $f_i(x - \alpha_i, t)$ is the equation of $\tilde{B}$ passing through $P_i = \{(x, t) = (\alpha_i, 0)\}$ $(1 \le i \le s)$. Now we define a divisor $\boldsymbol{B}'$ on $U \times \Delta_\delta$ by the equation

$$(tx - u^2) \prod_{i=1}^{s} f_i\left(x - \alpha_i, t - \frac{u^2}{\alpha_i} + \frac{u^2}{\alpha_i^2}(x - \alpha_i)\right) = 0 .$$

Let $\boldsymbol{B}$ be the closure of $\boldsymbol{B}'$ in $W \times \Delta_\delta$. Then $\boldsymbol{B}$ coincides with $\boldsymbol{B}' \cup \{\infty\}$ where ∞ is a certain point on Γ_0. The family $W \times \Delta_\delta \supset \boldsymbol{B} \to \Delta_\delta$ has a relative even resolution except at P, and the A_1-singularity over P on its double cover is resolved by [At1]. The induced splitting family of hyperelliptic fibrations has s singular fibers on the general fibration. Q.E.D.

EXAMPLE 3.12. In the situation of Lemma 3.11, we assume that B_{hor} is given by $x(x^{2g+1} - 1) = 0$. Then the divisor $\boldsymbol{B}'$ is written as $(tx - u^2)(x^{2g+1} - 1) = 0$. Note that if $u \neq 0$, then $\boldsymbol{B}'_u$ has $2g + 1$ ordinary double points at $(x, t) = (\zeta^\nu, u^2\zeta^{-\nu})$ $(0 \le \nu \le 2g)$, where ζ is a primitive $(2g + 1)$-th root of unity. (See Figure 4.) The associated family of genus g fibrations $\boldsymbol{f} : \boldsymbol{M} \to \Delta_\delta$ has the following properties:

f_0 has a unique singular fiber of the form $F = 2E_0 + E_1 + \cdots + E_{2g+2}$ where E_j's are nonsingular rational curves with $E_0^2 = -g - 1$, $E_j^2 = -2$, $E_0E_j = 1$ $(1 \le j \le 2g + 2)$ and $E_iE_j = 0$ $(1 \le i < j \le 2g + 2)$.

f_u $(u \neq 0)$ has $2g + 1$ singular fibers of the following form: $F' = E_1 + E_2$ where E_1 is a nonsingular curve of genus $g - 1$ and E_2 is a nonsingular rational curve, $E_1^2 = E_2^2 = -2$ and $E_1E_2 = 2$.

LEMMA 3.13. *Assume* (i) $B \not\supset \Gamma_0$ *and* (ii) B *has an ordinary double point at a point* $P \in \Gamma_0$. *Then* $[f, F]$ *has a splitting family.*

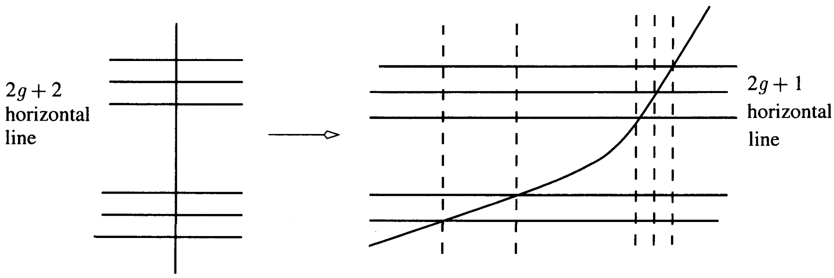

FIGURE 4. Construction of the branch curve in Example 3.12.

PROOF. Let $C^{(i)}$ $(i = 1, 2)$ be the components of B passing through P. Then the equation of B on an open set U of W is written as $f_1(x,t)f_2(x,t)g(x,t) = 0$ where $f_i(x,t)$ $(i = 1, 2)$ is the equation of $C^{(i)}$. Let $\boldsymbol{B}'$ be the divisor on $U \times \Delta_\delta$ defined by

$$(f_1(x,t)f_2(x,t) - u^2)g(x,t) = 0 .$$

Then the closure of $\boldsymbol{B}'$ in $W \times \Delta_\delta$ induces a splitting family $\boldsymbol{f}: \boldsymbol{M} \to \Delta_\varepsilon \times \Delta_\delta$ as in Lemma 3.11. The singular fibers of $\boldsymbol{f}_u$ $(u \neq 0)$ are as follows:

i) If B has bad points other than P, then $\boldsymbol{f}_u$ has three singular fibers. Moreover two of them are of type 0_0.

ii) If P is a unique bad point of B, then $\boldsymbol{f}$ has two singular fibers of type 0_0. Q.E.D.

EXAMPLE 3.14. Let $([f, F])$ be the germ given in Example 3.12. By combining Example 3.12 and Lemma 3.13, $([f, F])$ splits step by step into $4g+2$ fibers of type 0_0. (Compare [Ma4, p. 140], [Ma5].)

LEMMA 3.15. *We assume B_r has an ordinary singularity Q of odd multiplicity $k \geq 3$. Then $[f, F]$ has a splitting family.*

PROOF. We only consider the case where Q is contained in a unique $E_{j_1,r}$ $(1 \leq j_1 \leq r)$ and $E_{j_1} \subset B_r$. The other cases are similar and we omit their proof. By the coordinate $U(x, y)$ as in 2.1, B_r is written as $yf_1(x-\alpha, y)\cdots f_{k-1}(x-\alpha, y)g(x, y) = 0$, where $f_i(x-\alpha, y) = 0$ $(1 \leq i \leq k-1)$ is the equation of branch curves of $B_r - E_{j_1,r}$ passing through $P = (\alpha, 0)$ and $E_{j_1,r}$ is defined by $y = 0$. Then we define a divisor $\boldsymbol{B}'$ on $U \times \Delta_\delta$ by

$$yf_1\,(x-\alpha-\varphi(u), y-\psi(u))\prod_{i=2}^{k-1} f_i(x-\alpha, y)g(x, y) = 0$$

for generic holomorphic functions $\varphi(u)$, $\psi(u)$ with $\varphi(0) = \psi(0) = 0$. Then $\boldsymbol{B}' \subset U \times \{u\}$ has an ordinary $(k-1)$-ple point and $k-1$ ordinary double points. The analytic closure $\boldsymbol{B}$ in $W_r \times \Delta_\delta$ of $\boldsymbol{B}'$ has natural properties similar to those in 2.1.

Note that the locus $(x, y, u) = (\alpha, 0, u)$ on $U \times \Delta_\delta$ has the same multiplicity sequence as for the even resolution for any $u \in \Delta_\delta$. Therefore the family of the branch curves $W_r \times \Delta_\delta \subset \boldsymbol{B} \to \Delta_\delta$ has relative even resolution except at the $k-1$ ordinary double points, which can also be resolved by [At1]. Therefore we have a splitting family $\boldsymbol{f} : \boldsymbol{M} \to \Delta_\delta$ such that $\boldsymbol{f}_u$ $(u \neq 0)$ has $k-1$ singular fibers and $k-2$ of them are of the following type: the branch curve does not contain the fiber and the bad point is an ordinary double point. Therefore these $k-2$ fibers are written as $F' = A + B$ where A is a nonsingular curve of genus $g-1$ and B is a nonsingular rational curve, and $A^2 = B^2 = -2$, $AB = 2$. Q.E.D.

PROOF OF THEOREM 3.3. Let (W, B) be a representative of the branch curve of $[f, F]$ which has no splitting family.

Step 1. Assume $B \supset \Gamma_0$. Then we will show that the germ $[f, F]$ belongs to class II. Lemma 3.15 implies the condition (b) in 1.3.3 while Lemmas 3.5 and 3.7 imply (c) and (a). Lemma 3.11 implies (e). Lemma 3.10 implies (d).

Step 2. Assume $B \not\supset \Gamma_0$. Then the bad point P of B is unique by Lemma 3.9. If $m = \mathrm{multi}_P B \geq 1$ is odd, then the problem is reduced to Step 1 by the elementary transformation at P.

Assume m is even. Then P is an ordinary singularity by Lemmas 3.5 and 3.7.

If $m = 2$, then $[f, F]$ splits into two singular fibers of type 0_0 by Lemma 3.13. If $m \geq 4$, then $[f, F]$ belongs to class I. Hence we proved that any germ $[f, F]$ which does not belong to classes 0_0, I or II has a splitting family.

Step 3. It remains to prove the finiteness of our reducing process to class $\boldsymbol{G}$. Let $f : \boldsymbol{M} \to \Delta_\varepsilon \times \Delta_\delta \to \Delta_\delta$ be any splitting family constructed in this section for a given (f, F), and let (f_u, F') be any singular fiber germ of the fibration f_u (for any $u \neq 0$). Then there exist some representatives of branch curves (B, W) and (B', W') of (f, F) and (f_u, F') respectively so that, for any bad point P' of B', there exists a bad point P of B such that (B', P') is a (perturbed) deformation of (B, P) in our sense. Then one of the following conditions is satisfied:

(i) The germ of the singularity (B', P') coincides with (B, P),

(ii) $\mathrm{mult}_{P'} B' < \mathrm{mult}_P B$,

(iii) $\mathrm{mult}_{P'} B' = \mathrm{mult}_P B$ and the minimal number of times of blow-ups for (B', P') such that the reduced scheme of the total transform of the horizontal component $\tilde{B}'$ of B' is normal crossing is strictly less than that of (B, P).

Note that at least one bad point satisfies (ii) or (iii). Moreover, if (B, P) is perturbed into $(B^{(1)}, P^{(1)})$, $(B^{(2)}, P^{(2)})$, $\ldots$ $(B^{(l)}, P^{(l)})$ $(l \geq 2)$, then (i) does not hold for any of $(B^{(j)}, P^{(j)})$'s.

Therefore the assertion is clear.

4. Horikawa index and signature. In this section, we define the Horikawa index and the local signature of a germ of singular fiber of a degeneration of hyperelliptic curves. By using these notions and Theorem 3.3, we study compact complex surfaces with hyperelliptic pencils, especially their signature problem.

4.1. Let $f : S \to \Delta_\varepsilon$ be a degeneration of hyperelliptic curves of genus g with a unique singular fiber F. Let (W, B) be a representative of the branch curve of the germ $[f, F]$. Let $\tau_{r,0} : W_r \to W_0 = W$ be the succession of blow-ups such that the even proper transform B_r is nonsingular. Assume the number r of blow-ups is the minimum to attain the above property. Let $m_1, \ldots, m_r$ be the multiplicity sequence of $\tau_{r,0}$ as in 1.1. Let $S' \to W$ be the normal double cover branched along B on the total space of the line bundle $\mathcal{L}$ of the square root of B, i.e., $\mathcal{O}_W(\mathcal{L}^{\otimes 2})$ is isomorphic to $\mathcal{O}_W(B)$. Let $\{Q_i\}_{1 \leq i \leq l}$ be the set of isolated singular points on S'. Let $\rho : \tilde{S} \to S'$ be the canonical resolution of S', i.e. $\tilde{S}$ is the double cover of W_r branched along B_r on the total space of the line bundle of the square root of B_r. The natural fibration $\tilde{S} \to \Delta_\varepsilon$ is not relatively minimal in general. Let $\alpha_{(W,B)}$ be the number of times of contraction maps of (-1)-curves from $\tilde{S}$ to the original relative minimal model S. Let $p_g(S', Q_i)$ and $K^2_{(S',Q_i)}$ be the geometric genus and the square of the canonical cycle for the resolution ρ of the germ of singularity (S', Q_i) $(1 \leq i \leq l)$, respectively. Note that the formula in [Ho1, Lemma 6] implies that

$$\sum_{i=1}^{l} p_g(S', Q_i) = \frac{1}{2}\sum_{i=1}^{r}\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right]-1\right), \quad \sum_{i=1}^{l} K^2_{(S',Q_i)} = -2\sum_{i=1}^{r}\left(\left[\frac{m_i}{2}\right]-1\right)^2 .$$

DEFINITION-LEMMA 4.2. *We put*

$$\begin{aligned}\mathcal{H}([f, F]) :&= \sum_{i=1}^{l}\left\{\frac{4(g-1)}{g} p_g(S', Q_i) + K^2_{(S',Q_i)}\right\} + \alpha_{(W,B)} \\ &= \frac{2}{g}\sum_{i=1}^{r}\left(\left[\frac{m_i}{2}\right]-1\right)\left(g-\left[\frac{m_i}{2}\right]\right) + \alpha_{(W,B)} .\end{aligned}$$

This is well-defined as an invariant for $[f, F]$, *i.e., is determined independently of the choice of a representative* (W, B) *of the branch curve of* $[f, F]$. *We call it* the Horikawa index (*or* the H-index, *for short*) *of* $[f, F]$.

Since this notion is essential in our discussion, we give two proofs for the well-definedness. One comes from a global compactification argument, and the other comes from a local computational argument.

THE FIRST PROOF OF LEMMA 4.2. Let $P_1^{(0)}, \dots, P_{l_0}^{(0)}$ be all the bad points of B. By Artin's theorem [Ar], we may assume that the germ of singularity $(B, P_i^{(0)})$ $(1 \le i \le l_0)$ is algebraic. We fix an inclusion $W = \boldsymbol{P}^1 \times \Delta_\varepsilon \hookrightarrow \boldsymbol{P}^1 \times \boldsymbol{P}^1 := \Sigma_0$. We denote by $\bar{\pi} : \Sigma_0 \to \boldsymbol{P}^1$ the extension of the projection $\pi : W \to \Delta_\varepsilon$ and identify the original fiber Γ_0 with $\bar{\pi}^{-1}(0)$. Then it is easy to see that there exists a reduced divisor $\bar{B}^{(0)}$ on Σ_0 such that

(i) $\bar{B}^{(0)}$ is linearly equivalent to $(2g+2)C_0 + 2k_0\Gamma_0$ for a sufficiently large integer k_0, where C_0 is a fiber for another fibration of Σ_0,

(ii) the germs of bad points of $\bar{B}^{(0)}$ on Γ_0 coincide with $(B, P_1^{(0)}), \dots, (B, P_{l_0}^{(0)})$.

Let $(W^{(1)}, B^{(1)})$ be a representative of the branch curve of $[f, F]$ obtained from (W, B) by an elementary transformation. This transformation is globalized from $(\Sigma_0, \bar{B}^{(0)})$ to $(\Sigma_1, \bar{B}^{(1)})$ where $\bar{B}^{(1)}$ is a divisor on the Hirzebruch surface Σ_1 of degree 1. Let $S^{(i)}$ be the normal double cover of Σ_i branched along $\bar{B}^{(i)}$ on the total space of the line bundle of the square root of $\bar{B}^{(i)}$ for $i = 0, 1$. By the usual double covering method (cf. [P1]), we easily have

$$\omega^2_{S^{(i)}} = \frac{4(g-1)}{g}(\chi(\mathcal{O}_{S^{(i)}}) - g - 1),$$

where $\omega_{S^{(i)}}$ is the dualizing sheaf.

Now let $Q_1^{(i)}, \dots, Q_{l_i}^{(i)}$ be the points over the bad points of the branch curve on the fiber over the origin of the fibration $S^{(i)} \to \boldsymbol{P}^1$. Let $\widetilde{S^{(i)}} \to S^{(i)}$ be the canonical resolution with respect to these points. Let $\widetilde{S^{(i)}} \to S^{*(i)}$ be the succession of contractions of (-1)-curves such that the fiber over the origin of the fibration $S^{*(i)} \to \boldsymbol{P}^1$ does not contain any (-1)-curve. Let

$\alpha^{(i)}$ be the number of times of this (-1)-contractions. Then we have

$$\chi(\mathcal{O}_{S^{*(i)}}) - \chi(\mathcal{O}_{S^{(i)}}) = -\sum_{j=1}^{l_i} p_g(S^{(i)}, Q_j^{(i)}), \quad \omega^2_{S^{*(i)}} - \omega^2_{S^{(i)}} = \sum_{j=1}^{l_i} K^2_{(S^{(i)}, Q_j^{(i)})} + \alpha^{(i)}$$

for $i = 0, 1$. On the other hand, the fibrations $S^{*(i)} \to \boldsymbol{P}^1$ for $i = 0, 1$ are isomorphic to each other over $\boldsymbol{P}^1 \setminus \{0\}$, and after resolving the singularities on these loci and contracting (-1)-curves, we obtain the same surface by the uniqueness of the relatively minimal model. Therefore we have

$$\chi(\mathcal{O}_{S^{*(0)}}) = \chi(\mathcal{O}_{S^{*(1)}}), \quad \omega^2_{S^{*(0)}} = \omega^2_{S^{*(1)}}.$$

From these, we have

$$\sum_{j=1}^{l_0} \left\{ \frac{4(g-1)}{g} p_g(S^{(0)}, Q_j^{(0)}) + K^2_{(S^{(0)}, Q_i^{(0)})} \right\} + \alpha^{(0)}$$
$$= \sum_{j=1}^{l_1} \left\{ \frac{4(g-1)}{g} p_g(S^{(1)}, Q_j^{(1)}) + K^2_{(S^{(1)}, Q_i^{(1)})} \right\} + \alpha^{(1)}. \qquad \text{Q.E.D.}$$

THE SECOND PROOF OF LEMMA 4.2. Let (W', B') be a resulting pair after an elementary transformation of (W, B) and $P \in W$ the center of this elementary transformation. Let $W'_{r'} \to W'$ be the minimal even resolution of B', and let $m'_1, \dots, m'_{r'}$ be their multiplicity sequence. We put

$$\mathcal{H} = \frac{2}{g}\sum_{i=1}^{r}\left(\left[\frac{m_i}{2}\right]-1\right)\left(g-\left[\frac{m_i}{2}\right]\right)+\alpha, \quad \mathcal{H}' = \frac{2}{g}\sum_{i=1}^{r'}\left(\left[\frac{m'_i}{2}\right]-1\right)\left(g-\left[\frac{m'_i}{2}\right]\right)+\alpha',$$

where $\alpha = \alpha_{(W,B)}$ and $\alpha' = \alpha_{(W',B')}$. It suffices to prove $\mathcal{H} = \mathcal{H}'$.

Let μ be the multiplicity of B at P. Then we have $0 \le \mu \le 2g+3$.

(i) Suppose $\mu = 0$, that is, B does not pass through P. We may assume that $m'_1 = 2g+2$, $m'_2 = m_1$, $m'_3 = m_2 \cdots$. Since there exist two more (-1)-curves over the proper transform of the fiber on the double cover of $W_{r'}$, we have $\alpha' = \alpha + 2$. Hence $\mathcal{H}' = \mathcal{H} + 2([m'_1/2]-1)(g-[m'_1/2])/g + 2 = \mathcal{H}$.

(ii) Suppose $\mu = 1$ and B does not contain Γ_0. We may assume $m'_1 = 2g+2$, $m'_2 = 2$, $m'_3 = m_1$, $m'_4 = m_2 \cdots$ and we have $\alpha' = \alpha + 2$. Hence we similarly have $\mathcal{H}' = \mathcal{H}$.

(iii) Suppose $\mu = 1$ and B contains Γ_0. We may assume $m'_1 = 2g+3$, $m'_2 = 2$, $m'_3 = m_1$, $m'_4 = m_2 \cdots$ and we have $\alpha' = \alpha + 2$. Hence $\mathcal{H}' = \mathcal{H}$.

(iv) Suppose $2 \le \mu \le 2g+1$. Then, since P is a singular point of B, we can begin the canonical resolution with the blowing up at P. In this case, we have $m_1 = \mu$ and may assume that the i-th even proper transform of B and B' coincides with each other for $i \ge 1$. Especially we have $m'_2 = m_2$, $m'_3 = m_3 \cdots$ and $\alpha' = \alpha$. Moreover we have the following equality:

$$m_1' = \begin{cases} 2g+2-m_1 & \text{if } \Gamma_0 \not\subset B \text{ and } m_1 \equiv 0 \pmod 2 \\ 2g+3-m_1 & \text{if } \Gamma_0 \not\subset B \text{ and } m_1 \equiv 1 \pmod 2 \\ 2g+3-m_1 & \text{if } \Gamma_0 \subset B \text{ and } m_1 \equiv 0 \pmod 2 \\ 2g+4-m_1 & \text{if } \Gamma_0 \subset B \text{ and } m_1 \equiv 1 \pmod 2\,. \end{cases}$$

Indeed, we first assume $\Gamma_0 \not\subset B$ and m_1 is even. Let $W \xleftarrow{\tau_1} W_1 \xrightarrow{\tau_1'} W'$ be the elementary transformation. The first even proper transform B_1 coincides with the proper transform of B by τ_1, and also coincides with the proper transform of B' by τ_1'. The proper transform $E_{0,1}$ of Γ_0 by τ_1 coincides with the exceptional curve for τ_1' and vice versa for $E_{1,1}$. Therefore we have $m_1' = B_1 E_{0,1} = (\tau_1^* B - m_1 E_{1,1})E_{0,1} = B\Gamma_0 - m_1 = 2g+2-m_1$. The other cases are similar. In any case, we have $([m_1'/2]-1)(g-[m_1'/2]) = ([m_1/2]-1)(g-[m_1/2])$, hence $\mathcal{H}' = \mathcal{H}$.

(v) Suppose $\mu = 2g+2$ or $2g+3$. Then $(W', B') \to (W, B)$ is also an elementary transformation whose center has multiplicity less than two in B'. Hence the assertion follows from (i), (ii) and (iii). Q.E.D.

REMARK 4.3. The notion of the H-index was introduced for the first time by Horikawa [Ho2] in the genus 2 case. He classified germs of singular fibers of genus 2 into six types 0, I_k, II_k, III_k, IV_k and V, and their H-indices are 0, $2k-1$, $2k$, $2k-1$, $2k$ and 1, respectively (see also Reid [Re]).

The H-index is related to the invariants of compact surfaces by the following theorem, whose proof can be essentially found in Horikawa [Ho3, Theo.2.1] and Persson [P2, Prop.2.12] (see also Xiao[X2]):

THEOREM 4.4 (Horikawa-Persson). (i) *For any germ $[f, F]$ of a degeneration of hyperelliptic curves, we have $\mathcal{H}([f, F]) \geq 0$.*

(ii) *Let $\varphi : V \to C$ be a global hyperelliptic pencil of genus g, that is, h is a surjective proper holomorphic map from a nonsingular compact surface to a nonsingular compact curve such that the general fiber is a hyperelliptic curve of genus g. Let $F_1, \dots, F_k$ be all its singular fibers. Then we have*

$$K_{V/C}^2 = \frac{4(g-1)}{g}\chi_\varphi + \sum_{i=1}^{k} \mathcal{H}([\varphi, F_i])\,,$$

where $K_{V/C}$ is the relative canonical sheaf and $\chi_\varphi = \deg \varphi_* K_{V/C}$.

EXAMPLE 4.5. The calculation of $\mathcal{H} := \mathcal{H}([f, F])$ for the germs defined in Section 1 is the following:

(1) If $[f, F]$ is of type 0_0, then $\mathcal{H} = 0$.

(2) If $[f, F]$ belongs to class I with $1 \leq g' \leq [(g-1)/2]$, then $\mathcal{H} = 2g'(g-g'-1)/g$.

(3) If $[f, F]$ belongs to class II with $2 \leq g \leq 4$, then $\mathcal{H}$ of the germs corresponding to Proposition 1.7 are as follows:

$(g=2)$ (i) $\mathcal{H}=1$,
$(g=3)$ (i) $\mathcal{H}=5/3$, (ii) $\mathcal{H}=10/3$, (iii) $\mathcal{H}=8/3$,
$(g=4)$ (i) $\mathcal{H}=2$, (ii) $\mathcal{H}=3$, (iii) $\mathcal{H}=4$.

Next we study the signature of compact surfaces with hyperelliptic pencils from the viewpoint of the H-index.

DEFINITION 4.6. We put

$$\sigma([f, F]) := \frac{1}{2g+1}\{g\mathcal{H}([f, F]) - (g+1)\mathcal{E}([f, F])\}$$

and call it the local signature of the germ $[f, F]$, where $\mathcal{E}([f, F]) := \chi_{\mathrm{top}}(F) - 2g + 2$ is the Euler contribution of F (cf. [P3]).

PROPOSITION 4.7. *Let $\varphi : V \to C$ be a global hyperelliptic pencil of genus g, and let $F_1, \dots, F_k$ be all its singular fibers. Then we have*

$$\sigma(V) = \sum_{i=1}^{k} \sigma([\varphi, F_i]),$$

where $\sigma(V)$ is the global signature of V, i.e., the signature of the intersection form of the 2-homology of V.

PROOF. We use the relative formulation of Xiao [X2], [X3]. Set $e_\varphi = \sum_{i=1}^{k} \mathcal{E}(F_i)$. Noether's formula and Hirzebruch's signature theorem [Hi1] say that

$$12\chi_\varphi = K_{V/C}^2 + e_\varphi, \quad 3\sigma(V) = K_{V/C}^2 - 2e_\varphi .$$

We set $\mathcal{H}(V) = \sum_{i=1}^{k} \mathcal{H}([\varphi, F_i])$. It follows from Theorem 4.4, (ii) and the above formulas that

$$\mathcal{H}(V) = \frac{2g+1}{3g} K_{V/C}^2 - \frac{g-1}{3g} e_\varphi = \frac{2g+1}{g}\sigma(V) + \frac{g+1}{g} e_\varphi . \qquad \text{Q.E.D.}$$

EXAMPLE 4.8. The calculation of $\sigma := \sigma([f, F])$ for the germs defined in Section 1 is the following:

(1) If $[f, F]$ is of type 0_0, then $\sigma = -(g+1)/(2g+1)$.

(2) If $[f, F]$ belongs to class I with $1 \le g' \le [(g-1)/2]$, then $\sigma = 2(gg' - g'^2 - g - g' - 1)/(2g+1)$.

(3) If $[f, F]$ belongs to class II with $2 \le g \le 4$, then σ of the germs corresponding to Proposition 1.7 are as follows:

$(g=2)$ (i) $\sigma = -1/5$,
$(g=3)$ (i) $\sigma = 1/7$, (ii) $\sigma = -6/7$, (iii) $\sigma = 0$,
$(g=4)$ (i) $\sigma = 1/3$, (ii) $\sigma = 7/9$, (iii) $\sigma = -1$.

DEFINITION 4.9. Let $X([f, F])$ be a certain invariant of the germ of singular fiber $[f, F]$. Let $h : Z \to \Delta_\varepsilon \times \Delta_\delta$ be a splitting family of $[f, F]$ in the sense of Definition 3.1. We say the family h conserves the invariant X if the following condition is satisfied: Let

$F_{1,u}, \ldots, F_{l,u}$ be all singular fibers of the fibration $h_u : Z_u \to \Delta_\varepsilon \times \{u\}$ for any $u \in \Delta_\delta$. Then we have

$$X([f, F]) = X([h_u, F_{1,u}]) + \cdots + X([h_u, F_{l,u}]).$$

LEMMA 4.10. *The H-index and the local signature are conserved in any splitting family which is constructed in Section* 3.

PROOF. *Step* 1. Let $h : Z \to \Delta_\varepsilon \times \Delta_\delta$ be any splitting family constructed in Section 3. By our construction, h is induced by the double cover over a family of local $\boldsymbol{P}^1$-bundle $\boldsymbol{W} = W \times \Delta_\delta$ whose branch locus is a family of divisors $\boldsymbol{W} \supset \boldsymbol{B} \xrightarrow{\rho} \Delta_\delta$. Let $= \{m_{1,u}, \ldots, m_{r_u,u}\}$ be the multiplicity sequence of the minimal even resolution of $(\boldsymbol{W}_u, \boldsymbol{B}_u)$ for $u \in \Delta_\delta$. We note that, bad points on several fibers of ρ_u contribute to this set for $u \neq 0$, while bad points are on a unique fiber for $u = 0$. Now we define the subsequence $\boldsymbol{T}_u$ of the sequence $\{[m_{1,u}/2], \ldots, [m_{r_u,u}/2]\}$ consisting of numbers greater than 1, and we consider it as a set, i.e., we disregard its order. (Note that the multiplicity with $[m_{u,i}/2] \leq 1$ does not contribute to the H-index of the fiber.) Then it follows from our construction that the splitting family h conserves this set, i.e., we have

$$\boldsymbol{T}_0 = \boldsymbol{T}_u$$

for any $u \in \Delta_\delta$. Moreover the number of times of blow-downs of (-1)-curves from the canonical resolution to its relatively minimal model is independent of u. Therefore h conserves the H-index.

Step 2. Next we show the conservation of the Euler contribution, which induces the conservation of the local signature by definition and Step 1.

We first consider the splitting family of Lemma 3.5. We use the notation in this lemma. By the stability of (-1)-curves, we may identify $\boldsymbol{M}^\flat$ with $\boldsymbol{M}$. Let $\tilde{E}$ be the set of components of F which dominates $E_{j,r}$ by the generically two-to-one map $h_0 : \boldsymbol{M}_0 \to W_r$. Note that $\tilde{E}$ is irreducible or consists of two irreducible components. Let $A_1, \ldots, A_s, A'_1, \ldots, A'_{s'}, D$ be all the connected components of $F - \tilde{E}$, where A_i $(1 \leq i \leq s)$ intersect $\tilde{E}$ at one point, A'_j $(1 \leq j \leq s)$ intersect $\tilde{E}$ at two points and $h_0(D) = Q$. Then we have

$$\chi_{\mathrm{top}}(F) = \chi_{\mathrm{top}}(\tilde{E}) + \sum_{i=1}^{s} \chi_{\mathrm{top}}(A_i) + \sum_{j=1}^{s'} \chi_{\mathrm{top}}(A'_j) - s - 2s' + \chi_{\mathrm{top}}(D) - \varepsilon(D),$$

where $\varepsilon(D)$ $(= 1$ or $2)$ is the number of the points of intersection of D and $\tilde{E}$.

Now we consider singular fibers $F' = \boldsymbol{f}_u^{-1}(0)$, $F'' = \boldsymbol{f}_u^{-1}(\pi(R_u))$ of $h_u : \boldsymbol{M}_u \to W_r \times \{u\}$ for $u \neq 0$. Let $\tilde{E}'$ be the set of components of F' which dominates $E_{j,r} \times \{u\}$ by h_u. Then $F' - \tilde{E}'$ has connected components which are isomorphic to $A_1, \ldots, A_s, A'_1, \ldots, A'_{s'}$, and we have $\chi_{\mathrm{top}}(F') = \chi_{\mathrm{top}}(\tilde{E}') + \sum_{i=1}^{s} \chi_{\mathrm{top}}(A_i) + \sum_{j=1}^{s'} \chi_{\mathrm{top}}(A'_j) - s - 2s'$. Moreover our method of construction implies

$$\chi_{\mathrm{top}}(\tilde{E}') = \chi_{\mathrm{top}}(\tilde{E}) - 2\left[\frac{k}{2}\right],$$

where $k = I_Q(B_r, E_{j,r})$. On the other hand, F'' consists of the irreducible component $\tilde{\Gamma}$ which dominates the fiber $\boldsymbol{P}^1$ by h_u and the connected component which is isomorphic to D. We have $\chi_{\mathrm{top}}(F'') = \chi_{\mathrm{top}}(\tilde{\Gamma}) + \chi_{\mathrm{top}}(D) - \varepsilon(D)$. Moreover our construction implies

$$\chi_{\mathrm{top}}(\tilde{\Gamma}) = 2 - 2g + 2\left[\frac{k}{2}\right].$$

Therefore we have $\mathcal{E}(F) = \mathcal{E}(F') + \mathcal{E}(F'')$. For other splitting families constructed in Section 3, we have the assertion by a similar argument. Q.E.D.

From Theorem 3.3 and Lemma 4.10, we have the following:

PROPOSITION 4.11. *Let $\boldsymbol{G}$ be the union of the sets of germs of singular fibers of type* 0_0, *in class* I *and in class* II *defined in* 1.3. *Then for any germ $[f, F]$ of a degeneration of hyperelliptic curves, there exists a reduction of $[f, F]$ to $\boldsymbol{G}$ via splitting families which conserves the H-index and the local signature, that is, these invariants are conserved in any splitting family for this reduction.*

COROLLARY 4.12. *Assume the germ $[f, F]$ has the vanishing H-index. Then $[f, F]$ is reduced to germs of type* 0_0 *via splitting families.*

PROOF. Note that the germ has the vanishing H-index if and only if it has a representative of branch curve which has at most simple singularities, in other words, whose normal double cover has at most rational double points. Especially a germ in class II has the positive H-index by definition. Therefore Example 4.5 says that a germ which belongs to $\boldsymbol{G}$ has the vanishing H-index if and only if it is of type 0_0. Thus the assertion follows from Proposition 4.11. Q.E.D.

REMARK 4.13. The surface of general type with a pencil whose slope has minimal value automatically has a hyperelliptic pencil such that all the singular fibers have the vanishing H-indices ([X2], [Kon]). Therefore, considering the above corollary, one can ask whether such a surface has a reduction to surfaces with at most 0_0 fibers by global splitting families or not. This type of problem is called the "global Morsification problem".

COROLLARY 4.14. *Let $\varphi : V \to C$ be a global hyperelliptic pencil of genus $2 \leq g \leq 4$. Then the global signature $\sigma(V)$ is written as follows*:

$(g=2)$ $\sigma(V) = -(3/5)k_{0_0} - (1/5)k_{(\mathrm{II,i})}$,

$(g=3)$ $\sigma(V) = -(4/7)k_{0_0} - (6/7)k_{(\mathrm{I},1)} + (1/7)k_{(\mathrm{II,i})} - (6/7)k_{(\mathrm{II,ii})}$,

$(g=4)$ $\sigma(V) = -(5/9)k_{0_0} - (2/3)k_{(\mathrm{I},1)} + (1/3)k_{(\mathrm{II,i})} + (7/9)k_{(\mathrm{II,ii})} - k_{(\mathrm{II,iii})}$,

where $k_{0_0}, \ldots, k_{(\mathrm{II,iii})}$ are the sum of numbers of germs of the corresponding type which are obtained by our reducing process applied to every singular fiber of φ via local splitting families. For instance, $k_{(\mathrm{I},1)}$ is the number of germs in class I *with $g' = 1$, and $k_{(\mathrm{II,i})}$ is the number of germs of type* (i) *in class* II *written in Proposition* 1.7.

PROOF. The assertion is clear by Proposition 4.7, Proposition 4.11, Proposition 1.7 and Example 4.8. Q.E.D.

REMARK 4.15. For $g = 2$, the above formula is due to Matsumoto, Horikawa and Corollary 4.12 as we explained in the introduction. For $g \geq 3$, Corollary 4.14 and Example 4.8 tell us that there are many singular fibers with positive local signature. In some sense, this fact supports the existence of surfaces with hyperelliptic pencils whose global signatures are positive, which was discovered by Xiao and Chen [C].

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