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PRIMITIVE IDEALS OF THE RING OF DIFFERENTIAL OPERATORS ON AN AFFINE TORIC VARIETY

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Abstract. We show that the classification of A-hypergeometric systems and that of multi-graded simple modules (up to shift) over the ring of differential operators on an affine toric variety are the same. We then show that the set of multi-homogeneous primitive ideals of the ring of differential operators is finite. Furthermore, we give conditions for the algebra being simple.

Introduction. Let A be a $d \times n$ integer matrix whose column vectors generate the lattice $\mathbf{Z}^d$. Let R_A be the ring of regular functions on the affine toric variety defined by A, and $D(R_A)$ its ring of differential operators.

In this paper, we prove the following three theorems:

1. The classification of A-hypergeometric systems and that of $\mathbf{Z}^d$-graded simple $D(R_A)$-modules (up to shift) are the same (Theorem 4.10).
2. The set of $\mathbf{Z}^d$-homogeneous primitive ideals of $D(R_A)$ is finite (Theorem 6.5).
3. The algebra $D(R_A)$ is simple if and only if R_A is a scored semigroup ring, and A satisfies a certain condition (C2) (Theorem 7.25).

The ring of differential operators was introduced by Grothendieck [7] and Sweedler [22]. As for the ring of differential operators $D(R_A)$ on an affine toric variety, many recent papers such as Jones [9], Musson [13, 14], and Musson and Van den Bergh [15] describe the structure of $D(R_A)$ when R_A is normal. For general R_A, we studied the finite generation of $D(R_A)$ and its graded ring with respect to the order filtration in [18] and [19]. Moreover, in [18], we showed that the algebra $D(R_A)$ and the symmetry algebra (the algebra of contiguity operators) of A-hypergeometric systems are anti-isomorphic to each other. This paper may be considered as a continuation of [18].

The history of A-hypergeometric systems (or GKZ hypergeometric systems) goes back to Kalnins, Manocha, and Miller [10], and Hrabowski [8]. Since the papers by Gel'fand, Kapranov, and Zelevinskii (e.g., [3]–[6]), researchers in various fields have studied the systems, and established connection with representation theory, algebraic geometry, commutative ring theory, etc. See, for example, the bibliography of [17].

Associated to a parameter vector $\boldsymbol{\alpha}$ and a face τ of the cone generated by the column vectors of A, we defined a finite set $E_\tau(\boldsymbol{\alpha})$ (see (11) in Section 6) in [16], and proved that the A-hypergeometric systems are classified by those finite sets. Hence in order to show that

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the classification of A-hypergeometric systems and that of $\boldsymbol{Z}^d$-graded simple $D(R_A)$-modules (up to shift) are the same, we only need to show that $\boldsymbol{Z}^d$-graded simple $D(R_A)$-modules can be classified (up to shift) by the finite sets $E_\tau(\boldsymbol{\alpha})$ as shown in Theorem 3.4, essentially in [15] and [18]. It is however desirable to show the equivalence intrinsically. To apply one theory to the other, we need functors between them. We therefore present the second proof of the equivalence by connecting A-hypergeometric systems with certain $\boldsymbol{Z}^d$-graded $D(R_A)$-modules by functors (Corollary 4.9). This is a starting point to study the relations of the two theories.

The latter half of this paper is devoted to studying $\boldsymbol{Z}^d$-homogeneous primitive ideals and the simplicity of $D(R_A)$. The simplicity is one of the most important questions about the rings of differential operators as in [1], [12], and [20]. We give an algorithm for listing the set of $\boldsymbol{Z}^d$-homogeneous primitive ideals of $D(R_A)$, and a criterion of the simplicity of $D(R_A)$. The finite sets $E_\tau(\boldsymbol{\alpha})$ play a central role here.

Some topics discussed in this paper were treated in [15] under the conditions $(A1)$ and $(A2)$ (see [15, page 4]). In our case, $(A1)$ is always satisfied, and $(A2)$ requiring that all $\boldsymbol{Z}^d$-homogeneous components $D(R_A)_{\boldsymbol{a}}$ are singly generated $D(R_A)_{\boldsymbol{0}}$-modules is equivalent to requiring that R_A satisfies Serre's (S_2) condition (see Proposition 7.7). In this paper, we do not assume Serre's (S_2) condition.

The layout of this paper is as follows: We start with recalling the definitions and some fundamental facts about differential operators in Section 1, and about A-hypergeometric systems in Section 2. In Section 3, we recall some results by Musson and Van den Bergh [15] about the category $\mathcal{O}$, analogous to Bernstein-Gel'fand-Gel'fand's category $\mathcal{O}$, and the counterpart ${}^R\mathcal{O}$ for right $D(R_A)$-modules. Then we recall realizations $L(\boldsymbol{\alpha})$ (${}^R L(\boldsymbol{\alpha})$) of simple objects in $\mathcal{O}$ (${}^R\mathcal{O}$) from [18], and we prove a duality between $\mathcal{O}$ and ${}^R\mathcal{O}$ sending $L(\boldsymbol{\alpha})$ and ${}^R L(\boldsymbol{\alpha})$ to each other. We also show that any $\boldsymbol{Z}^d$-graded simple $D(R_A)$-module is isomorphic (up to shift) to $L(\boldsymbol{\alpha})$. These combined prove that the classifications of $L(\boldsymbol{\alpha})$, ${}^R L(\boldsymbol{\alpha})$, their projective covers $M(\boldsymbol{\alpha})$, ${}^R M(\boldsymbol{\alpha})$, and $\boldsymbol{\alpha}$ with respect to the equivalence relation determined by the finite sets $E_\tau(\boldsymbol{\alpha})$ are the same.

In Section 4, we provide two functors between the category of right $D(R_A)$-modules and the category of right $D(R)$-modules supported by the affine toric variety defined by A, where $D(R)$ is the n-th Weyl algebra. One is the direct image functor, for right D-modules, of the closed inclusion of the affine toric variety into $\boldsymbol{C}^n$, and the other is its right adjoint functor. By using these functors, we prove that ${}^R M(\boldsymbol{\alpha}) \simeq {}^R M(\boldsymbol{\beta})$ if and only if their corresponding A-hypergeometric systems are isomorphic (Theorem 4.10).

After recalling a couple of basic facts about primitive ideals in Section 5, we show in Section 6 that if we perturb a parameter $\boldsymbol{\alpha}$ properly, then the annihilator ideal $\operatorname{Ann} L(\boldsymbol{\alpha})$ remains unchanged, and in this way it is shown that the set $\operatorname{Prim} D(R_A)$ of $\boldsymbol{Z}^d$-homogeneous primitive ideals of $D(R_A)$ is finite (Theorem 6.5).

In Section 7, the simplicity of $D(R_A)$ is studied. First we consider the conditions: the scoredness and Serre's (S_2). We prove that the simplicity of $D(R_A)$ implies the scoredness, and that the conditions $(A2)$ and (S_2) are equivalent. Finally, we give a necessary and sufficient condition for the simplicity (Theorem 7.25), which is not difficult to check (see Remarks 7.14 and 7.23). In Section 8, we give an example of computation of the set Prim $D(R_A)$.

1. Ring of differential operators on an affine toric variety. In this section, we recall some fundamental facts about the rings of differential operators of semigroup algebras.

Let $A := \{ \boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n \}$ be a finite set of column vectors in $\boldsymbol{Z}^d$. Sometimes we identify A with the matrix $(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$. Let $\boldsymbol{N}A$ and $\boldsymbol{Z}A$ denote the monoid and the abelian group generated by A, respectively. Throughout this paper, we assume that $\boldsymbol{Z}A = \boldsymbol{Z}^d$ for simplicity.

Let R denote the polynomial ring $\boldsymbol{C}[x] := \boldsymbol{C}[x_1, \dots, x_n]$. The semigroup algebra $R_A := \boldsymbol{C}[\boldsymbol{N}A] = \bigoplus_{\boldsymbol{a} \in \boldsymbol{N}A} \boldsymbol{C}t^{\boldsymbol{a}}$ is the ring of regular functions on the affine toric variety defined by A, where $t^{\boldsymbol{a}} = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for $\boldsymbol{a} = {}^t(a_1, a_2, \dots, a_d)$. Then we have $R_A \simeq R/I_A(x)$, where $I_A(x)$ is the ideal of $\boldsymbol{C}[x]$ generated by all $x^{\boldsymbol{u}} - x^{\boldsymbol{v}}$ for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{N}^n$ with $A\boldsymbol{u} = A\boldsymbol{v}$.

Let M, N be R-modules. We briefly recall the module $D(M, N)$ of differential operators from M to N. For details, see [21]. For $k \in \boldsymbol{N}$, the subspaces $D^k(M, N)$ of $\mathrm{Hom}_{\boldsymbol{C}}(M, N)$ are inductively defined by

$$D^0(M, N) = \mathrm{Hom}_R(M, N)$$

and

$$D^{k+1}(M, N) = \{ P \in \mathrm{Hom}_{\boldsymbol{C}}(M, N) \, ; \, [f, P] \in D^k(M, N) \quad \text{for all } f \in R \} ,$$

where $[\ ,\ ]$ denotes the commutator. Set $D(M, N) := \bigcup_{k=0}^{\infty} D^k(M, N)$, and $D(M) := D(M, M)$. Then $D(M)$ is a $\boldsymbol{C}$-algebra, whereas $D(M, N)$ is a $(D(N), D(M))$-bimodule. Hence $D(R, R_A)$ is a $(D(R_A), D(R))$-bimodule.

The ring $D(R)$ is the n-th Weyl algebra

$$D(R) = \boldsymbol{C}\left\langle x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle ,$$

where $[\partial/\partial x_i, x_j] = \delta_{ij}$, and the other pairs of generators commute. Here δ_{ij} is 1 if $i = j$ and 0 otherwise.

Let $\boldsymbol{C}[t, t^{-1}]$ denote the Laurent polynomial ring $\boldsymbol{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$. Then its ring of differential operators $D(\boldsymbol{C}[t, t^{-1}])$ is the ring

$$\boldsymbol{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d \rangle ,$$

where $[\partial_i, t_j] = \delta_{ij}$, $[\partial_i, t_j^{-1}] = -\delta_{ij} t_j^{-2}$, and the other pairs of generators commute. The ring of differential operators $D(R_A)$ can be realized as a subring of the ring $D(\boldsymbol{C}[t, t^{-1}])$ by

$$D(R_A) = \{ P \in \boldsymbol{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d \rangle \, ; \, P(R_A) \subseteq R_A \} .$$

Put $s_j := t_j \partial_j$ for $j = 1, 2, \dots, d$. Then it is easy to see that $s_j \in D(R_A)$ for all j. We introduce a $\boldsymbol{Z}^d$-grading on the ring $D(R_A)$; for $\boldsymbol{a} = {}^t(a_1, a_2, \dots, a_d) \in \boldsymbol{Z}^d$, set

$$D(R_A)_{\boldsymbol{a}} := \{ P \in D(R_A) \, ; \, [s_j, P] = a_j P \quad \text{for } j = 1, 2, \dots, d \} .$$

Then $D(R_A) = \bigoplus_{\boldsymbol{a} \in \boldsymbol{Z}^d} D(R_A)_{\boldsymbol{a}}$.

By regarding $D(R_A)_{\boldsymbol{a}}$ as a subset of $\boldsymbol{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d \rangle$, we see that there exists an ideal I of $\boldsymbol{C}[s] := \boldsymbol{C}[s_1, \dots, s_d]$ such that $D(R_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}} I$. To describe this ideal I explicitly, we define a subset $\Omega(\boldsymbol{a})$ of the semigroup $\boldsymbol{N}A$ by

$$\Omega(\boldsymbol{a}) = \{ \boldsymbol{b} \in \boldsymbol{N}A \,;\, \boldsymbol{b} + \boldsymbol{a} \notin \boldsymbol{N}A \} = \boldsymbol{N}A \setminus (-\boldsymbol{a} + \boldsymbol{N}A) \,. \tag{1}$$

Then each $D(R_A)_{\boldsymbol{a}}$ is described as follows.

THEOREM 1.1 (Theorem 2.3 in [13]).

$$D(R_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}} \boldsymbol{I}(\Omega(\boldsymbol{a})) \quad \textit{for all } \ \boldsymbol{a} \in \boldsymbol{Z}^d,$$

where

$$\boldsymbol{I}(\Omega(\boldsymbol{a})) := \{ f(s) \in \boldsymbol{C}[s] \,;\ f \textit{ vanishes on } \Omega(\boldsymbol{a}) \} \,.$$

In particular, $D(R_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}} \boldsymbol{C}[s] = \boldsymbol{C}[s] t^{\boldsymbol{a}}$ for each $\boldsymbol{a} \in \boldsymbol{N}A$, since $\Omega(\boldsymbol{a}) = \emptyset$ in this case.

2. *A*-hypergeometric systems. Let us briefly recall the definition of an A-hypergeometric system and its classification.

Let $\boldsymbol{\alpha} = {}^t(\alpha_1, \dots, \alpha_d) \in \boldsymbol{C}^d$. The A-hypergeometric system with parameter $\boldsymbol{\alpha}$ is the left $D(R)$-module

$$H_A(\boldsymbol{\alpha}) := D(R) \Bigg/ \left(\sum_{i=1}^{d} D(R) \left(\sum_{j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_j} - \alpha_i \right) + D(R) I_A(\partial) \right),$$

where $\boldsymbol{a}_j = {}^t(a_{1j}, a_{2j}, \dots, a_{dj})$, $I_A(\partial)$ is the ideal of $\boldsymbol{C}[\partial/\partial x_1, \dots, \partial/\partial x_n]$ generated by all $\prod_{j=1}^n \left(\partial/\partial x_j\right)^{u_j} - \prod_{j=1}^n \left(\partial/\partial x_j\right)^{v_j}$ for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{N}^n$ with $A\boldsymbol{u} = A\boldsymbol{v}$.

Interchanging x_j and $\partial/\partial x_j$ for all j, we have an anti-automorphism ι of $D(R)$. Clearly, ι gives rise to a one-to-one correspondence between the left $D(R)$-modules and the right $D(R)$-modules. Thus ι induces a right $D(R)$-module

$${}^R H_A(\boldsymbol{\alpha}) := D(R) \Bigg/ \left(\sum_{i=1}^{d} \left(\sum_{j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_j} - \alpha_i \right) D(R) + I_A(x) D(R) \right).$$

Here note that $\iota(x_j \partial/\partial x_j) = \iota(\partial/\partial x_j)\iota(x_j) = x_j \partial/\partial x_j$.

In [18, Definition 4.1.1], we have introduced a partial order into the parameter space $\boldsymbol{C}^d$ (see (13)), which is equivalent by [18, Lemma 4.1.4] to

$$\boldsymbol{\alpha} \preceq \boldsymbol{\beta} \Leftrightarrow \boldsymbol{I}(\Omega(\boldsymbol{\beta} - \boldsymbol{\alpha})) \not\subseteq \mathfrak{m}_{\boldsymbol{\alpha}} \,, \tag{2}$$

where $\mathfrak{m}_{\boldsymbol{\alpha}}$ is the maximal ideal of $\boldsymbol{C}[s]$ at $\boldsymbol{\alpha}$. Note that, if $\boldsymbol{\beta} - \boldsymbol{\alpha} \notin \boldsymbol{Z}^d$, then $\Omega(\boldsymbol{\beta} - \boldsymbol{\alpha}) = \boldsymbol{N}A$, and hence $\boldsymbol{\alpha} \not\preceq \boldsymbol{\beta}$. We write $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ if $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$. This equivalence relation was introduced also by Musson and Van den Bergh (see [15, Lemma 3.1.9 (6)]).

This relation classifies A-hypergeometric systems.

THEOREM 2.1 (Theorem 2.1 in [16]). $H_A(\boldsymbol{\alpha}) \simeq H_A(\boldsymbol{\beta})$ *if and only if* $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$.

3. $D(R_A)$**-modules.** In this section, we recall some results by Musson and Van den Bergh [15] about the category $\mathcal{O}$, and the counterpart ${}^R\mathcal{O}$ for right $D(R_A)$-modules. Then we recall realizations $L(\boldsymbol{\alpha})$ (${}^RL(\boldsymbol{\alpha})$) of simple objects in $\mathcal{O}$ (${}^R\mathcal{O}$) from [18], and we prove that we have a duality between $\mathcal{O}$ and ${}^R\mathcal{O}$ sending $L(\boldsymbol{\alpha})$ and ${}^RL(\boldsymbol{\alpha})$ to each other. We also show that any $\boldsymbol{Z}^d$-graded simple $D(R_A)$-module is isomorphic (up to shift) to $L(\boldsymbol{\alpha})$. These combined prove that the classifications of $L(\boldsymbol{\alpha})$, ${}^RL(\boldsymbol{\alpha})$, their projective covers $M(\boldsymbol{\alpha})$, ${}^RM(\boldsymbol{\alpha})$, and $\boldsymbol{\alpha}$ with respect to the equivalence relation $\sim$ are the same.

3.1. Left modules. Let us recall the full subcategory $\mathcal{O}$ of the category of left $D(R_A)$-modules introduced in [15], which is an analogue of Bernstein-Gel'fand-Gel'fand's category $\mathcal{O}$ for the study of highest weight modules of semisimple Lie algebras. A left $D(R_A)$-module M is an object of $\mathcal{O}$ if M has a weight decomposition $M = \bigoplus_{\boldsymbol{\lambda}\in\boldsymbol{C}^d} M_{\boldsymbol{\lambda}}$ with each $M_{\boldsymbol{\lambda}}$ finite-dimensional, where

$$M_{\boldsymbol{\lambda}} = \{x \in M\,;\ f(s).x = f(\boldsymbol{\lambda})x \quad \text{for all } f \in \boldsymbol{C}[s]\}\,.$$

We call $\boldsymbol{\lambda}$ a weight of M if $M_{\boldsymbol{\lambda}} \neq 0$.

For $\boldsymbol{\alpha} = {}^t(\alpha_1, \dots, \alpha_d) \in \boldsymbol{C}^d$, set

$$M(\boldsymbol{\alpha}) := D(R_A)/D(R_A)(s - \boldsymbol{\alpha})\,,$$

where $D(R_A)(s-\boldsymbol{\alpha})$ means $\sum_{i=1}^d D(R_A)(s_i - \alpha_i)$. Then $M(\boldsymbol{\alpha}) = \bigoplus_{\boldsymbol{\lambda}\in\boldsymbol{\alpha}+\boldsymbol{Z}A} M(\boldsymbol{\alpha})_{\boldsymbol{\lambda}}$, and $M(\boldsymbol{\alpha}) \in \mathcal{O}$.

Among others, Musson and Van den Bergh proved the following.

PROPOSITION 3.1 (Proposition 3.1.7 in [15]).
1. $\mathrm{Hom}_{D(R_A)}(M(\boldsymbol{\alpha}), M) = M_{\boldsymbol{\alpha}}$ *for* $M \in \mathcal{O}$.
2. $M(\boldsymbol{\alpha})$ *is a projective object in* $\mathcal{O}$.
3. $M(\boldsymbol{\alpha})$ *has a unique simple quotient* (*denoted by* $L(\boldsymbol{\alpha})$).
4. *All simple objects in* $\mathcal{O}$ *are of the form* $L(\boldsymbol{\alpha})$.
5. *The natural projection* $M(\boldsymbol{\alpha}) \to L(\boldsymbol{\alpha})$ *is the projective cover.*
6. $M(\boldsymbol{\alpha}) \simeq M(\boldsymbol{\beta})$ *if and only if* $L(\boldsymbol{\alpha}) \simeq L(\boldsymbol{\beta})$.

REMARK 3.2. Musson and Van den Bergh assumed Conditions $(A1)$ and $(A2)$ (see [15, page 4]). In our case, $(A1)$ is always satisfied, and $(A2)$ requiring that all $D(R_A)_{\boldsymbol{a}}$ are singly generated $\boldsymbol{C}[s]$-modules is equivalent to requiring that R_A satisfies Serre's (S_2) condition (see Proposition 7.7). For Proposition 3.1, we do not need Condition $(A2)$.

REMARK 3.3. Let $M \in \mathcal{O}$, and let N be a left $D(R_A)$-submodule of M. Then $N \in \mathcal{O}$. Hence, if M is a simple object in $\mathcal{O}$, then M is a simple left $D(R_A)$-module.

Let $\boldsymbol{\alpha} \in \boldsymbol{C}^d$. In [18], we studied the composition factors of a $D(R_A)$-module

$$\boldsymbol{C}[t_1^{\pm1}, \dots, t_d^{\pm1}]t^{\boldsymbol{\alpha}}\,,$$

and observed that

$$\bigoplus_{\boldsymbol{\lambda}\succeq\boldsymbol{\alpha}} \boldsymbol{C}t^{\boldsymbol{\lambda}} \Big/ \bigoplus_{\boldsymbol{\lambda}\succ\boldsymbol{\alpha}} \boldsymbol{C}t^{\boldsymbol{\lambda}} \tag{3}$$

is simple [18, Theorem 4.1.6], where $\boldsymbol{\lambda} \succ \boldsymbol{\alpha}$ means $\boldsymbol{\lambda} \succeq \boldsymbol{\alpha}$ and $\boldsymbol{\lambda} \not\sim \boldsymbol{\alpha}$. The $D(R_A)$-module (3) is a simple quotient of $M(\boldsymbol{\alpha})$, and hence a realization of $L(\boldsymbol{\alpha})$. In particular, the set of weights of $L(\boldsymbol{\alpha})$ is

$$\{\boldsymbol{\lambda} \in \boldsymbol{C}^d\,;\ \boldsymbol{\lambda} \sim \boldsymbol{\alpha}\}\,.$$

THEOREM 3.4 (cf. Lemma 3.1.9 (6) in [15]). *$L(\boldsymbol{\alpha}) \simeq L(\boldsymbol{\beta})$ if and only if $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$.*

PROOF. By the realization (3), $L(\boldsymbol{\alpha}) = L(\boldsymbol{\beta})$ if $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$. If $\boldsymbol{\alpha} \not\sim \boldsymbol{\beta}$, then $L(\boldsymbol{\alpha})$ and $L(\boldsymbol{\beta})$ have different weights. Hence $L(\boldsymbol{\alpha}) \not\simeq L(\boldsymbol{\beta})$. □

PROPOSITION 3.5. *Let M be a $\boldsymbol{Z}^d$-graded simple left $D(R_A)$-module. Then M is isomorphic to $L(\boldsymbol{\alpha})$ for some $\boldsymbol{\alpha}$ as a left $D(R_A)$-module.*

PROOF. Recall that $D(R_A)_{\boldsymbol{0}} = \boldsymbol{C}[s]$. First we show that each nonzero $M_{\boldsymbol{\lambda}}$ is a simple $\boldsymbol{C}[s]$-module. Suppose that $N_{\boldsymbol{\lambda}}$ is a nontrivial $\boldsymbol{C}[s]$-submodule of $M_{\boldsymbol{\lambda}}$. Put $N := \bigoplus_{\boldsymbol{a}\in\boldsymbol{Z}^d} N_{\boldsymbol{\lambda}+\boldsymbol{a}} = \bigoplus_{\boldsymbol{a}\in\boldsymbol{Z}A} D(R_A)_{\boldsymbol{a}} N_{\boldsymbol{\lambda}}$. Then N is a nontrivial $\boldsymbol{Z}^d$-graded submodule of M, which contradicts the assumption.

Suppose that $M_{\boldsymbol{\lambda}} \neq 0$. By the first paragraph, there exists $\boldsymbol{\alpha} \in \boldsymbol{C}^d$ such that $M_{\boldsymbol{\lambda}} \simeq \boldsymbol{C}[s]/\mathfrak{m}_{\boldsymbol{\alpha}}$ as $\boldsymbol{C}[s]$-modules. Let $M[\boldsymbol{\lambda}-\boldsymbol{\alpha}]$ be the $\boldsymbol{Z}^d$-graded $D(R_A)$-module shifted by $\boldsymbol{\lambda}-\boldsymbol{\alpha}$, i.e., $M[\boldsymbol{\lambda}-\boldsymbol{\alpha}]_{\boldsymbol{\mu}} = M_{\boldsymbol{\mu}+\boldsymbol{\lambda}-\boldsymbol{\alpha}}$. Then $M[\boldsymbol{\lambda}-\boldsymbol{\alpha}] = \bigoplus_{\boldsymbol{\mu}\in\boldsymbol{\alpha}+\boldsymbol{Z}^d} M[\boldsymbol{\lambda}-\boldsymbol{\alpha}]_{\boldsymbol{\mu}}$, and $M[\boldsymbol{\lambda}-\boldsymbol{\alpha}]_{\boldsymbol{\alpha}+\boldsymbol{a}} = D(R_A)_{\boldsymbol{a}} M_{\boldsymbol{\lambda}}$. Hence $M[\boldsymbol{\lambda}-\boldsymbol{\alpha}] \in \mathcal{O}$, and $M[\boldsymbol{\lambda}-\boldsymbol{\alpha}] \simeq L(\boldsymbol{\alpha}) \in \mathcal{O}$. □

3.2. Right modules. A right $D(R_A)$-module M is an object of ${}^R\mathcal{O}$ if M has a weight decomposition $M = \bigoplus_{\boldsymbol{\lambda}\in\boldsymbol{C}^d} M_{\boldsymbol{\lambda}}$ with each $M_{\boldsymbol{\lambda}}$ finite-dimensional, where

$$M_{\boldsymbol{\lambda}} = \{x \in M\,;\ x.f(s) = f(-\boldsymbol{\lambda})x \quad \text{for all } f \in \boldsymbol{C}[s]\}\,.$$

We can make a parallel argument about the categories $\mathcal{O}$ and ${}^R\mathcal{O}$. Indeed we shall show that there exists a duality functor between them.

For $\boldsymbol{\alpha} \in \boldsymbol{C}^d$, set

$${}^RM(\boldsymbol{\alpha}) := D(R_A)/(s-\boldsymbol{\alpha})D(R_A)\,.$$

Then ${}^RM(\boldsymbol{\alpha}) = \bigoplus_{\boldsymbol{\lambda}\in-\boldsymbol{\alpha}+\boldsymbol{Z}A} {}^RM(\boldsymbol{\alpha})_{\boldsymbol{\lambda}}$, and ${}^RM(\boldsymbol{\alpha}) \in {}^R\mathcal{O}$.

The following proposition can be proved in the same way as Proposition 3.1.

PROPOSITION 3.6. 1. *$\mathrm{Hom}_{D(R_A)}({}^RM(\boldsymbol{\alpha}), M) = M_{-\boldsymbol{\alpha}}$ for $M \in {}^R\mathcal{O}$.*
2. *${}^RM(\boldsymbol{\alpha})$ is a projective object in ${}^R\mathcal{O}$.*
3. *${}^RM(\boldsymbol{\alpha})$ has a unique simple quotient (denoted by ${}^RL(\boldsymbol{\alpha})$).*
4. *All simple objects in ${}^R\mathcal{O}$ are of the form ${}^RL(\boldsymbol{\alpha})$.*
5. *The natural projection ${}^RM(\boldsymbol{\alpha}) \to {}^RL(\boldsymbol{\alpha})$ is the projective cover.*
6. *${}^RM(\boldsymbol{\alpha}) \simeq {}^RM(\boldsymbol{\beta})$ if and only if ${}^RL(\boldsymbol{\alpha}) \simeq {}^RL(\boldsymbol{\beta})$.*

The ring $D(R_A)$ is a subring of $\boldsymbol{C}[t_1^{\pm1}, \ldots, t_d^{\pm1}]\langle\partial_1, \ldots, \partial_d\rangle$, where we can take formal adjoint operators, and thus we can consider a right $D(R_A)$-module

$$\boldsymbol{C}[t_1^{\pm1}, \ldots, t_d^{\pm1}]t^{\boldsymbol{\alpha}}\frac{dt}{t}\,.$$

Here the right action of $P = \sum_{\boldsymbol{a}} t^{\boldsymbol{a}} f_{\boldsymbol{a}}(s)$ on this module is defined by

$$\left(g(t)\frac{dt}{t}\right).P := P^*(g)\frac{dt}{t} ,$$

where $P^* = \sum_{\boldsymbol{a}} f_{\boldsymbol{a}}(-s)t^{\boldsymbol{a}}$, and recall that $s_i = t_i \partial_i$ $(i = 1, \ldots, d)$.

Then

$$\bigoplus_{\boldsymbol{\beta} \preceq \boldsymbol{\alpha}} \boldsymbol{C} t^{-\boldsymbol{\beta}} \frac{dt}{t} \Big/ \bigoplus_{\boldsymbol{\beta} \prec \boldsymbol{\alpha}} \boldsymbol{C} t^{-\boldsymbol{\beta}} \frac{dt}{t} \tag{4}$$

is a realization of ${}^R L(\boldsymbol{\alpha})$.

Let $M \in \mathcal{O}$ (${}^R\mathcal{O}$). Then $\mathrm{Hom}_{\boldsymbol{C}}(M, \boldsymbol{C})$ is a right (left) $D(R_A)$-module. Define a right (left) $D(R_A)$-submodule M^* of $\mathrm{Hom}_{\boldsymbol{C}}(M, \boldsymbol{C})$ by

$$M^* := \bigoplus_{\boldsymbol{\lambda}} M^*_{\boldsymbol{\lambda}} , \quad M^*_{\boldsymbol{\lambda}} := \mathrm{Hom}_{\boldsymbol{C}}(M_{-\boldsymbol{\lambda}}, \boldsymbol{C}) .$$

Then $^* : \mathcal{O} \to {}^R\mathcal{O}$ ($^* : {}^R\mathcal{O} \to \mathcal{O}$) is a duality functor. Hence we have the following proposition.

PROPOSITION 3.7. 1. $\mathrm{Hom}_{D(R_A)}(M, {}^R M(\boldsymbol{\alpha})^*) = \mathrm{Hom}_{\boldsymbol{C}}(M_{\boldsymbol{\alpha}}, \boldsymbol{C})$ *for* $M \in \mathcal{O}$.

2. ${}^R M(\boldsymbol{\alpha})^*$ *is an injective object in* $\mathcal{O}$.
3. ${}^R M(\boldsymbol{\alpha})^*$ *has a unique simple subobject* ${}^R L(\boldsymbol{\alpha})^*$*in* $\mathcal{O}$.
4. $L(\boldsymbol{\alpha}) \simeq {}^R L(\boldsymbol{\alpha})^*$.
5. *The natural inclusion* ${}^R L(\boldsymbol{\alpha})^* \to {}^R M(\boldsymbol{\alpha})^*$ *is the injective hull.*
6. $L(\boldsymbol{\alpha}) \simeq L(\boldsymbol{\beta})$ *if and only if* ${}^R L(\boldsymbol{\alpha}) \simeq {}^R L(\boldsymbol{\beta})$.

PROOF. (4) follows from the fact that two simple modules $L(\boldsymbol{\alpha})$ and ${}^R L(\boldsymbol{\alpha})^*$ have the same weight spaces.

The other statements are clear. □

4. ***A*-hypergeometric systems and category $\mathcal{O}$.** We have proved that the classification of A-hypergeometric systems and that of simple modules $L(\boldsymbol{\alpha})$ (or ${}^R L(\boldsymbol{\alpha})$) are the same, by showing that simple modules $L(\boldsymbol{\alpha})$ are classified according to the equivalence relation $\sim$ in Theorem 3.4. In this section, we make another way to prove the coincidence of the classifications; we connect A-hypergeometric systems $H_A(\boldsymbol{\alpha})$ and right $D(R_A)$-modules ${}^R M(\boldsymbol{\alpha})$ by functors. This proof is intrinsic, and hence is more desirable.

4.1. The bimodule $D(R, R_A)$. In this subsection, we decompose the $(D(R_A), D(R))$-bimodule $D(R, R_A)$ into its $\boldsymbol{Z}^d$-graded parts in a manner similar to Theorem 1.1.

Let $\boldsymbol{C}[x, x^{-1}]$ denote the Laurent polynomial ring $\boldsymbol{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. By [2, page 31], we have

$$D(R, R_A) = \{P \in D(\boldsymbol{C}[x, x^{-1}], \boldsymbol{C}[t, t^{-1}]) ;\ P(R) \subseteq R_A\} .$$

From [21, 1.3 (e)],

$$D(\boldsymbol{C}[x, x^{-1}], \boldsymbol{C}[t, t^{-1}]) = D(\boldsymbol{C}[x, x^{-1}])/I_A(x)D(\boldsymbol{C}[x, x^{-1}]) ,$$

and hence

$$D(\boldsymbol{C}[x, x^{-1}], \boldsymbol{C}[t, t^{-1}]) = \bigoplus_{\boldsymbol{a}\in \boldsymbol{Z}^d} t^{\boldsymbol{a}} \boldsymbol{C}[\theta_1, \dots, \theta_n],$$

where $\theta_j = x_j\partial/\partial x_j$, $j = 1, \dots, n$. From [21, 1.3 (e)] again, we have

$$D(R, R_A) = D(R)/I_A(x)D(R). \tag{5}$$

Note that $D(R_A) \subseteq D(R, R_A)$ by (5). Here we identify $t^{\boldsymbol{a}_j}$ and s_i with x_j and $\sum_{j=1}^n a_{ij}\theta_j$ respectively, where $j = 1, \dots, n$, $i = 1, \dots, d$. In fact, we have

$$D(R_A) = D(R, R_A) \cap D(\boldsymbol{C}[t^{\pm 1}])$$

in $D(\boldsymbol{C}[x, x^{-1}], \boldsymbol{C}[t, t^{-1}])$.

The bimodule $D(R, R_A)$ inherits the $\boldsymbol{Z}^d$-grading from $D(\boldsymbol{C}[x, x^{-1}], \boldsymbol{C}[t, t^{-1}])$,

$$D(R, R_A)_{\boldsymbol{a}} = D(R, R_A) \cap t^{\boldsymbol{a}} \boldsymbol{C}[\theta_1, \dots, \theta_n].$$

PROPOSITION 4.1.

$$D(R, R_A) = \bigoplus_{\boldsymbol{a}\in \boldsymbol{Z}^d} t^{\boldsymbol{a}} \boldsymbol{I}(\tilde{\Omega}_A(\boldsymbol{a})),$$

where $\tilde{\Omega}_A(\boldsymbol{a}) = \boldsymbol{N}^n \cap T_A^{-1}(\Omega_A(\boldsymbol{a}))$, T_A is the linear map from $\boldsymbol{Z}^n$ to $\boldsymbol{Z}^d$ defined by A, and $\boldsymbol{I}(\tilde{\Omega}_A(\boldsymbol{a}))$ is the ideal of $\boldsymbol{C}[\theta] = \boldsymbol{C}[\theta_1, \dots, \theta_n]$ vanishing on $\tilde{\Omega}_A(\boldsymbol{a})$.

PROOF. For $p(\theta) \in \boldsymbol{C}[\theta]$, we have

$$\begin{aligned}
&t^{\boldsymbol{a}} p(\theta) \in D(R, R_A)_{\boldsymbol{a}} \\
&\quad \Leftrightarrow t^{\boldsymbol{a}} p(\theta)(x^{\boldsymbol{m}}) \in \boldsymbol{C}[\boldsymbol{N}A] \quad \text{for any } \boldsymbol{m} \in \boldsymbol{N}^n \\
&\quad \Leftrightarrow p(\boldsymbol{m}) t^{\boldsymbol{a}+A\boldsymbol{m}} \in \boldsymbol{C}[\boldsymbol{N}A] \quad \text{for any } \boldsymbol{m} \in \boldsymbol{N}^n \\
&\quad \Leftrightarrow \boldsymbol{a} + A\boldsymbol{m} \in \boldsymbol{N}A \ \text{ or } \ p(\boldsymbol{m}) = 0 \ \text{ if } \ \boldsymbol{m} \in \boldsymbol{N}^n \\
&\quad \Leftrightarrow p(\boldsymbol{m}) = 0 \quad \text{for any } \boldsymbol{m} \in \boldsymbol{N}^n \setminus T_A^{-1}(-\boldsymbol{a} + \boldsymbol{N}A).
\end{aligned}$$ □

COROLLARY 4.2. *$D(R, R_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}} \boldsymbol{C}[\theta]$ for all $\boldsymbol{a} \in \boldsymbol{N}A$.*

PROOF. In this case $\Omega(\boldsymbol{a}) = \emptyset$. Hence $\boldsymbol{I}(\tilde{\Omega}_A(\boldsymbol{a})) = \boldsymbol{C}[\theta]$. □

To close this subsection, we describe the weight decomposition of ${}^R H_A(\boldsymbol{\alpha})$. Let

$${}^R H_A(\boldsymbol{\alpha})_{\boldsymbol{\lambda}} := \left\{ x \in {}^R H_A(\boldsymbol{\alpha})\,;\ x.\left(\sum_{j=1}^n a_{ij}\theta_j + \lambda_i\right) = 0, \quad i = 1, \dots, d \right\}$$

for $\boldsymbol{\lambda} = {}^t(\lambda_1, \dots, \lambda_d)$. Note that the weight space $D(R, R_A)_{\boldsymbol{a}}$ is a $(\boldsymbol{C}[s], \boldsymbol{C}[\theta])$-bimodule, since $D(R_A)_{\boldsymbol{0}} = \boldsymbol{C}[s]$ and $D(R)_{\boldsymbol{0}} = \boldsymbol{C}[\theta]$. We have

$$\begin{aligned} {}^R H_A(\boldsymbol{\alpha}) &= \bigoplus_{\boldsymbol{a} \in \boldsymbol{Z}^d} {}^R H_A(\boldsymbol{\alpha})_{-\boldsymbol{\alpha}+\boldsymbol{a}} \\ &= \bigoplus_{\boldsymbol{a} \in \boldsymbol{Z}^d} D(R, R_A)_{\boldsymbol{a}}/(s-\boldsymbol{\alpha})D(R, R_A)_{\boldsymbol{a}} \\ &= \bigoplus_{\boldsymbol{a} \in \boldsymbol{Z}^d} D(R, R_A)_{\boldsymbol{a}}/D(R, R_A)_{\boldsymbol{a}}(A\theta + \boldsymbol{a} - \boldsymbol{\alpha}) \\ &= \bigoplus_{\boldsymbol{a} \in \boldsymbol{Z}^d} t^{\boldsymbol{a}}(\boldsymbol{I}(\widetilde{\Omega}(\boldsymbol{a}))/\boldsymbol{I}(\widetilde{\Omega}(\boldsymbol{a}))(A\theta + \boldsymbol{a} - \boldsymbol{\alpha}))\,. \end{aligned} \tag{6}$$

4.2. Functors. Let $\mathrm{Mod}^R(D(R_A))$ denote the category of right $D(R_A)$-modules, and $\mathrm{Mod}_A^R(D(R))$ the category of right $D(R)$-modules supported by the affine toric variety $\boldsymbol{V}(I_A(x))$ defined by A. A right $D(R)$-module N is said to be supported by $\boldsymbol{V}(I_A(x))$ if for every $y \in N$ there exists $m \in \boldsymbol{N}$ such that $yI_A(x)^m = 0$.

Let Φ denote the functor from $\mathrm{Mod}^R(D(R_A))$ to $\mathrm{Mod}_A^R(D(R))$ defined by

$$\Phi(M) := M \otimes_{D(R_A)} D(R, R_A)\,,$$

and Ψ the functor from $\mathrm{Mod}_A^R(D(R))$ to $\mathrm{Mod}^R(D(R_A))$ defined by

$$\begin{aligned} \Psi(N) &:= \mathrm{Hom}_{D(R)}(D(R, R_A), N) \\ &= \{y \in N\,;\ yI_A(x) = 0\}\,. \end{aligned}$$

Then Ψ is right adjoint to Φ,

$$\mathrm{Hom}_{D(R)}(\Phi(M), N) \simeq \mathrm{Hom}_{D(R_A)}(M, \Psi(N))\,.$$

REMARK 4.3. The functor Φ is the direct image functor, for right D-modules, of the closed inclusion

$$\boldsymbol{V}(I_A(x)) = \{x \in \boldsymbol{C}^n\,;\ f(x) = 0 \quad \text{for all } f \in I_A(x)\} \to \boldsymbol{C}^n\,.$$

For a closed embedding between nonsingular varieties, such a direct image functor gives an equivalence of categories, known as Kashiwara's equivalence (see, e.g., [11, Theorem 4.30]). In our case, the affine toric variety $\boldsymbol{V}(I_A(x))$ is singular whenever $n \neq d$, and the cone $\boldsymbol{R}_{\geq 0}A$ generated by A is strongly convex.

We have

$$\Phi(D(R_A)) = D(R, R_A)\,,$$

and

$$\Psi(D(R, R_A)) = \mathrm{End}_{D(R)}(D(R)/I_A(x)D(R)) = D(R_A)\,.$$

The following proposition is immediate from the definitions.

PROPOSITION 4.4. *We have*

$$\Phi({}^R M(\boldsymbol{\alpha})) = {}^R H_A(\boldsymbol{\alpha}) .$$

Hence, if ${}^R M(\boldsymbol{\alpha}) \simeq {}^R M(\boldsymbol{\beta})$, *then* ${}^R H_A(\boldsymbol{\alpha}) \simeq {}^R H_A(\boldsymbol{\beta})$.

To show the converse of Proposition 4.4, we need the following lemma.

LEMMA 4.5.

$$\mathrm{End}_{D(R)}({}^R H_A(\boldsymbol{\alpha})) = \boldsymbol{C}\,\mathrm{id} .$$

PROOF. Let $\psi \in \mathrm{End}_{D(R)}({}^R H_A(\boldsymbol{\alpha}))$. Since $\psi(\bar{1}) \in {}^R H_A(\boldsymbol{\alpha})_{-\boldsymbol{\alpha}}$, there exists a polynomial $f(\theta) \in \boldsymbol{C}[\theta]$ such that $\psi(\bar{1}) = \overline{f(\theta)}$. Here $\bar{P}$ is the element of ${}^R H_A(\boldsymbol{\alpha})$ represented by $P \in D(R)$. Let $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{N}^n$ satisfy $A\boldsymbol{u} = A\boldsymbol{v}$. Then

$$\begin{aligned} 0 &= \psi(\overline{x^{\boldsymbol{u}} - x^{\boldsymbol{v}}}) = \psi(\bar{1})(x^{\boldsymbol{u}} - x^{\boldsymbol{v}}) \\ &= \overline{f(\theta)}(x^{\boldsymbol{u}} - x^{\boldsymbol{v}}) = \overline{t^{A\boldsymbol{u}}(f(\theta + \boldsymbol{u}) - f(\theta + \boldsymbol{v}))} . \end{aligned}$$

By (6), we have

$$f(\theta + \boldsymbol{u}) - f(\theta + \boldsymbol{v}) \in (A\theta + A\boldsymbol{u} - \boldsymbol{\alpha})\boldsymbol{C}[\theta]$$

for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{N}^n$ with $A\boldsymbol{u} = A\boldsymbol{v}$. Hence $f(\theta) - f(\theta + \boldsymbol{l}) \in (A\theta - \boldsymbol{\alpha})\boldsymbol{C}[\theta]$ for all $\boldsymbol{l} \in L$, where $L = \{\boldsymbol{l} \in \boldsymbol{Z}^n\,;\ A\boldsymbol{l} = \boldsymbol{0}\}$. Letting $A\boldsymbol{\gamma} = \boldsymbol{\alpha}$ ($\boldsymbol{\gamma} \in \boldsymbol{C}^n$), we have $f(\boldsymbol{\gamma}) - f(\boldsymbol{\gamma} + \boldsymbol{l}) = 0$ for all $\boldsymbol{l} \in L$. Thus $f(\boldsymbol{\gamma} + \theta) \in f(\boldsymbol{\gamma}) + (A\theta)\boldsymbol{C}[\theta]$, or equivalently

$$f(\theta) \in f(\boldsymbol{\gamma}) + (A\theta - \boldsymbol{\alpha})\boldsymbol{C}[\theta] .$$

Hence $\psi(\bar{1}) = \overline{f(\theta)} = f(\boldsymbol{\gamma})$. Therefore $\psi = f(\boldsymbol{\gamma})\mathrm{id}$. □

Let $\boldsymbol{\beta} - \boldsymbol{\alpha} \in \boldsymbol{Z}^d$, and $Q \in D(R_A)_{\boldsymbol{\beta}-\boldsymbol{\alpha}}$. Then

$$\phi_Q : {}^R M(\boldsymbol{\alpha}) \ni \bar{P} \mapsto \overline{QP} \in {}^R M(\boldsymbol{\beta}) \tag{7}$$

$$\psi_Q : {}^R H_A(\boldsymbol{\alpha}) \ni \bar{P} \mapsto \overline{QP} \in {}^R H_A(\boldsymbol{\beta}) \tag{8}$$

are well-defined morphisms. Clearly, $\psi_Q = \Phi(\phi_Q)$.

LEMMA 4.6. *The natural map*

$$D(R_A)_{\boldsymbol{\beta}-\boldsymbol{\alpha}} \ni Q \mapsto \phi_Q \in \mathrm{Hom}_{D(R_A)}({}^R M(\boldsymbol{\alpha}), {}^R M(\boldsymbol{\beta}))$$

is surjective.

PROOF. Let $\phi \in \mathrm{Hom}_{D(R_A)}({}^R M(\boldsymbol{\alpha}), {}^R M(\boldsymbol{\beta}))$. Since $\phi(\bar{1}) \in {}^R M(\boldsymbol{\beta})_{-\boldsymbol{\alpha}}$, there exists $Q \in D(R_A)_{\boldsymbol{\beta}-\boldsymbol{\alpha}}$ such that $\phi(\bar{1}) = \bar{Q}$. Then $\phi = \phi_Q$. □

As to $\mathrm{Hom}_{D(R)}({}^R H_A(\boldsymbol{\alpha}), {}^R H_A(\boldsymbol{\beta}))$, we have the following by Lemma 4.5.

COROLLARY 4.7. *Suppose that* ${}^R H_A(\boldsymbol{\alpha}) \simeq {}^R H_A(\boldsymbol{\beta})$. *Then*

$$\dim_{\boldsymbol{C}} \mathrm{Hom}_{D(R)}({}^R H_A(\boldsymbol{\alpha}), {}^R H_A(\boldsymbol{\beta})) = 1 ,$$

and the natural map $D(R_A)_{\boldsymbol{\beta}-\boldsymbol{\alpha}} \to \mathrm{Hom}_{D(R)}({}^R H_A(\boldsymbol{\alpha}), {}^R H_A(\boldsymbol{\beta}))$ *is surjective.*

PROOF. The first statement is immediate from Lemma 4.5. The second follows from the fact that in this case the image of the map is not zero by [16]. □

Now we are in a position of proving the converse of Proposition 4.4. Note that there exists a natural morphism $M \to \Psi(\Phi(M))$ for $M \in \mathrm{Mod}^R(D(R_A))$.

PROPOSITION 4.8. *Suppose that* ${}^R H_A(\boldsymbol{\alpha}) \simeq {}^R H_A(\boldsymbol{\beta})$. *Then*

$$\mathrm{Hom}_{D(R_A)}({}^R M(\boldsymbol{\alpha}), {}^R M(\boldsymbol{\beta})) \simeq \mathrm{Hom}_{D(R)}({}^R H_A(\boldsymbol{\alpha}), {}^R H_A(\boldsymbol{\beta}))\,.$$

PROOF. Corollary 4.7 states that there exist $Q, R \in D(R_A)$ such that

$$\psi_Q : {}^R H_A(\boldsymbol{\alpha}) \ni \bar{P} \mapsto \overline{QP} \in {}^R H_A(\boldsymbol{\beta})\,,$$
$$\psi_R : {}^R H_A(\boldsymbol{\beta}) \ni \bar{P} \mapsto \overline{RP} \in {}^R H_A(\boldsymbol{\alpha})$$

satisfy $\psi_R \circ \psi_Q = \mathrm{id}_{{}^R H_A(\boldsymbol{\alpha})}$ and $\psi_Q \circ \psi_R = \mathrm{id}_{{}^R H_A(\boldsymbol{\beta})}$.

The image of the natural morphism ${}^R M(\boldsymbol{\alpha}) \to \Psi(\Phi({}^R M(\boldsymbol{\alpha}))) = \Psi({}^R H_A(\boldsymbol{\alpha}))$ equals $1_{\boldsymbol{\alpha}} D(R_A)$, where

$$\Psi({}^R H_A(\boldsymbol{\alpha})) = \mathrm{Hom}_{D(R)}(D(R, R_A), {}^R H_A(\boldsymbol{\alpha})) \ni 1_{\boldsymbol{\alpha}} : \bar{P} \mapsto \bar{P}\,.$$

The isomorphism ψ_Q induces an isomorphism $\Psi(\psi_Q)$:

$$\mathrm{Hom}_{D(R)}(D(R, R_A), {}^R H_A(\boldsymbol{\alpha})) \to \mathrm{Hom}_{D(R)}(D(R, R_A), {}^R H_A(\boldsymbol{\beta}))\,.$$

We show that the restriction of $\Psi(\psi_Q)$ to $1_{\boldsymbol{\alpha}} D(R_A)$ gives an isomorphism $1_{\boldsymbol{\alpha}} D(R_A) \simeq 1_{\boldsymbol{\beta}} D(R_A)$. By definition $\Psi(\psi_Q)(1_{\boldsymbol{\alpha}}) = 1_{\boldsymbol{\beta}} Q \in 1_{\boldsymbol{\beta}} D(R_A)$. Hence $\Psi(\psi_Q)(1_{\boldsymbol{\alpha}} D(R_A)) \subseteq 1_{\boldsymbol{\beta}} D(R_A)$. In addition, we have

$$1_{\boldsymbol{\beta}} = 1_{\boldsymbol{\beta}} QR = \Psi(\psi_Q)(1_{\boldsymbol{\alpha}} R)\,.$$

Hence $\Psi(\psi_Q)(1_{\boldsymbol{\alpha}} D(R_A)) = 1_{\boldsymbol{\beta}} D(R_A)$. We have thus proved that ψ_Q induces an isomorphism $1_{\boldsymbol{\alpha}} D(R_A) \simeq 1_{\boldsymbol{\beta}} D(R_A)$. It is lifted to an isomorphism between their projective covers in ${}^R\mathcal{O}$, ${}^R M(\boldsymbol{\alpha}) \simeq {}^R M(\boldsymbol{\beta})$, which is clearly ϕ_Q. Thus ϕ_Q and ψ_Q correspond to each other. □

Combining Propositions 4.4 and 4.8, we have the following

COROLLARY 4.9. ${}^R M(\boldsymbol{\alpha}) \simeq {}^R M(\boldsymbol{\beta})$ *if and only if* ${}^R H_A(\boldsymbol{\alpha}) \simeq {}^R H_A(\boldsymbol{\beta})$.

We summarize the classification as follows.

THEOREM 4.10. *The following are equivalent*:
1. $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$.
2. $H_A(\boldsymbol{\alpha}) \simeq H_A(\boldsymbol{\beta})$.
3. ${}^R H_A(\boldsymbol{\alpha}) \simeq {}^R H_A(\boldsymbol{\beta})$.
4. $M(\boldsymbol{\alpha}) \simeq M(\boldsymbol{\beta})$.
5. ${}^R M(\boldsymbol{\alpha}) \simeq {}^R M(\boldsymbol{\beta})$.
6. $L(\boldsymbol{\alpha}) \simeq L(\boldsymbol{\beta})$.
7. ${}^R L(\boldsymbol{\alpha}) \simeq {}^R L(\boldsymbol{\beta})$.

5. Primitive ideals. In general, a left primitive ideal is not necessarily a right primitive ideal. In our case, however, we have the following proposition thanks to the duality.

PROPOSITION 5.1. *For each* $\boldsymbol{\alpha} \in \boldsymbol{C}^d$, *the annihilators* $\mathrm{Ann}(L(\boldsymbol{\alpha}))$ *and* $\mathrm{Ann}({}^R L(\boldsymbol{\alpha}))$ *coincide*, *i.e.*,

$$\mathrm{Ann}(L(\boldsymbol{\alpha})) = \mathrm{Ann}({}^R L(\boldsymbol{\alpha}))\,.$$

PROOF. By definition, it is clear that

$$\mathrm{Ann}\, M \subseteq \mathrm{Ann}\, M^*$$

for $M \in \mathcal{O}\,({}^R\mathcal{O})$. Since ${}^R L(\boldsymbol{\alpha})^* = L(\boldsymbol{\alpha})$ and $L(\boldsymbol{\alpha})^* = {}^R L(\boldsymbol{\alpha})$, the statement follows. □

Hence the set of annihilators of $\boldsymbol{Z}^d$-graded simple left $D(R_A)$-modules and that of $\boldsymbol{Z}^d$-graded simple right $D(R_A)$-modules are the same. We denote it by $\mathrm{Prim}(D(R_A))$,

$$\mathrm{Prim}(D(R_A)) = \{\mathrm{Ann}(L(\boldsymbol{\alpha}))\,;\ \boldsymbol{\alpha} \in \boldsymbol{C}^d\}\,.$$

Next we describe the graded components of the annihilator ideal $\mathrm{Ann}\, L(\boldsymbol{\alpha})$. For $\boldsymbol{\alpha} \in \boldsymbol{C}^d$ and $\boldsymbol{a} \in \boldsymbol{Z}^d$, we define a subset $\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a})$ of $\boldsymbol{\alpha} + \boldsymbol{Z}^d$ by

$$\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}) := \{\boldsymbol{\beta} \in \boldsymbol{\alpha} + \boldsymbol{Z}^d\,;\ \boldsymbol{\beta} \sim \boldsymbol{\alpha},\ \boldsymbol{\beta} + \boldsymbol{a} \sim \boldsymbol{\alpha}\}\,. \tag{9}$$

Clearly, $\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{0})$ is the equivalence class $[\boldsymbol{\alpha}]$.

The following proposition is the key in Sections 6 and 7. See (1) to recall the set $\Omega(\boldsymbol{a})$.

PROPOSITION 5.2 (Proposition 3.2.2 in [15]). *Let*

$$(\mathrm{Ann}\, L(\boldsymbol{\alpha}))_{\boldsymbol{a}} = \mathrm{Ann}\, L(\boldsymbol{\alpha}) \cap D(R_A)_{\boldsymbol{a}}$$

for $\boldsymbol{a} \in \boldsymbol{Z}^d$. *Then*

$$(\mathrm{Ann}\, L(\boldsymbol{\alpha}))_{\boldsymbol{a}} = t^{\boldsymbol{a}} \boldsymbol{I}(\Omega(\boldsymbol{a}) \cup \Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}))$$

for all $\boldsymbol{a}$.

PROOF. Let $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$, and let $0 \neq v_{\boldsymbol{\beta}} \in L(\boldsymbol{\alpha})_{\boldsymbol{\beta}}$. Then

$$\mathrm{Ann}(v_{\boldsymbol{\beta}})_{\boldsymbol{a}} = \begin{cases} D(R_A)_{\boldsymbol{a}} & (\boldsymbol{\beta} + \boldsymbol{a} \not\sim \boldsymbol{\alpha}) \\ D(R_A)_{\boldsymbol{a}} \cap t^{\boldsymbol{a}}(s - \boldsymbol{\beta}) & (\boldsymbol{\beta} + \boldsymbol{a} \sim \boldsymbol{\alpha})\,. \end{cases}$$

Hence

$$(\mathrm{Ann}\, L(\boldsymbol{\alpha}))_{\boldsymbol{a}} = \bigcap_{\boldsymbol{\beta} \sim \boldsymbol{\alpha},\, \boldsymbol{\beta} + \boldsymbol{a} \sim \boldsymbol{\alpha}} D(R_A)_{\boldsymbol{a}} \cap t^{\boldsymbol{a}}(s - \boldsymbol{\beta})\,.$$

We have thus proved the assertion. □

6. Finiteness. In this section, we prove that the set $\mathrm{Prim}(D(R_A))$ is finite. If $\boldsymbol{\alpha} - \boldsymbol{\beta} \notin \boldsymbol{Z}^d$, then $\boldsymbol{\alpha} \not\sim \boldsymbol{\beta}$, and hence there exist infinitely many isomorphism classes of simple objects $L(\boldsymbol{\alpha})$. We however show that if we perturb a parameter $\boldsymbol{\alpha}$ properly, then the annihilator ideal $\mathrm{Ann}\, L(\boldsymbol{\alpha})$ remains unchanged.

First we recall the primitive integral support function of a facet (maximal proper face) of the cone $\boldsymbol{R}_{\geq 0}A$. We denote by $\mathcal{F}$ the set of facets of the cone $\boldsymbol{R}_{\geq 0}A$. Given $\sigma \in \mathcal{F}$, we denote by F_σ the primitive integral support function of σ, i.e., F_σ is a uniquely determined linear form on $\boldsymbol{R}^d$ satisfying

1. $F_\sigma(\boldsymbol{R}_{\geq 0}A) \geq 0$,

2. $F_\sigma(\sigma) = 0$,
3. $F_\sigma(\mathbf{Z}^d) = \mathbf{Z}$.

Given $\boldsymbol{\alpha} \in \boldsymbol{C}^d$, put

$$\mathcal{F}(\boldsymbol{\alpha}) := \{\sigma \in \mathcal{F}\,;\; F_\sigma(\boldsymbol{\alpha}) \in \boldsymbol{Z}\}\,,$$
$$V(\boldsymbol{\alpha}) := \bigcap_{\sigma \in \mathcal{F}(\boldsymbol{\alpha})} (F_\sigma = 0)\,.$$

Clearly, $\boldsymbol{\mu} \in V(\boldsymbol{\alpha})$ implies $\mathcal{F}(\boldsymbol{\alpha}) \subseteq \mathcal{F}(\boldsymbol{\alpha} + \boldsymbol{\mu})$. Note that $V(\boldsymbol{\alpha})$ may not be a linearization of a face.

EXAMPLE 6.1. Let

$$A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

There are four facets: $\sigma_{23} := \boldsymbol{R}_{\geq 0}\boldsymbol{a}_2 + \boldsymbol{R}_{\geq 0}\boldsymbol{a}_3$, $\sigma_{13} := \boldsymbol{R}_{\geq 0}\boldsymbol{a}_1 + \boldsymbol{R}_{\geq 0}\boldsymbol{a}_3$, $\sigma_{24} := \boldsymbol{R}_{\geq 0}\boldsymbol{a}_2 + \boldsymbol{R}_{\geq 0}\boldsymbol{a}_4$, $\sigma_{14} := \boldsymbol{R}_{\geq 0}\boldsymbol{a}_1 + \boldsymbol{R}_{\geq 0}\boldsymbol{a}_4$. Their primitive integral support functions are respectively $F_{\sigma_{23}}(s) = s_1$, $F_{\sigma_{13}}(s) = s_2$, $F_{\sigma_{24}}(s) = s_1 + s_3$, $F_{\sigma_{14}}(s) = s_2 + s_3$. Let $\boldsymbol{\alpha} = {}^t(\sqrt{2}, 0, -\sqrt{2})$. Then $\mathcal{F}(\boldsymbol{\alpha}) = \{\sigma_{13}, \sigma_{24}\}$, and $V(\boldsymbol{\alpha}) = \{{}^t(x, 0, -x)\,;\; x \in \boldsymbol{C}\}$, which is not a linearization of any face of $\boldsymbol{R}_{\geq 0}A$.

Translation of parameters by $\boldsymbol{\mu} \in V(\boldsymbol{\alpha})$ satisfying $\mathcal{F}(\boldsymbol{\alpha} + \boldsymbol{\mu}) = \mathcal{F}(\boldsymbol{\alpha})$ does not vary the annihilator ideals.

PROPOSITION 6.2. *Suppose that $\boldsymbol{\mu} \in V(\boldsymbol{\alpha})$ and $\mathcal{F}(\boldsymbol{\alpha} + \boldsymbol{\mu}) = \mathcal{F}(\boldsymbol{\alpha})$. Then*

$$\operatorname{Ann}(L(\boldsymbol{\alpha})) = \operatorname{Ann}(L(\boldsymbol{\alpha} + \boldsymbol{\mu}))\,. \tag{10}$$

Before proceeding to the proof of Proposition 6.2, we show that it leads us to the finiteness of $\operatorname{Prim}(D(R_A))$ with the aid of two other finiteness properties.

The first one is that, for any $\boldsymbol{\alpha} \in \boldsymbol{C}^d$, $\boldsymbol{\alpha} + \boldsymbol{Z}^d$ has only finitely many equivalence classes with respect to $\sim$. To explain this, we briefly review the finite sets $E_\tau(\boldsymbol{\alpha})$ defined in [16]. Associated to a parameter vector $\boldsymbol{\alpha} \in \boldsymbol{C}^d$ and a face τ of the cone $\boldsymbol{R}_{\geq 0}A$, $E_\tau(\boldsymbol{\alpha})$ was defined by

$$E_\tau(\boldsymbol{\alpha}) = \{\boldsymbol{\lambda} \in \boldsymbol{C}(A \cap \tau)/\boldsymbol{Z}(A \cap \tau)\,;\; \boldsymbol{\alpha} - \boldsymbol{\lambda} \in \boldsymbol{N}A + \boldsymbol{Z}(A \cap \tau)\}\,. \tag{11}$$

The set $E_\tau(\boldsymbol{\alpha})$ has at most $[\boldsymbol{Q}(A \cap \tau) \cap \boldsymbol{Z}^d : \boldsymbol{Z}(A \cap \tau)]$ elements. Indeed, suppose that $\boldsymbol{\lambda} \in \boldsymbol{C}(A \cap \tau)$ satisfies $\boldsymbol{\alpha} - \boldsymbol{\lambda} \in \boldsymbol{Z}^d$. Then $E_\tau(\boldsymbol{\alpha})$ is a subset of

$$(\boldsymbol{\lambda} + \boldsymbol{Q}(A \cap \tau) \cap \boldsymbol{Z}^d)/\boldsymbol{Z}(A \cap \tau) \tag{12}$$

(see [16, Proposition 2.3]). For a facet σ, $E_\sigma(\boldsymbol{\alpha}) \neq \emptyset$ if and only if $F_\sigma(\boldsymbol{\alpha}) \in F_\sigma(\boldsymbol{N}A)$, and, for faces $\tau \preceq \tau'$, $E_{\tau'}(\boldsymbol{\alpha}) = \emptyset$ implies $E_\tau(\boldsymbol{\alpha}) = \emptyset$ (see [16, Proposition 2.2]).

We have already introduced a partial ordering $\preceq$ into the parameter space $\boldsymbol{C}^d$ in (2). The following is its original definition in [18, Definition 4.1.1]: For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{C}^d$, we write

$$\boldsymbol{\alpha} \preceq \boldsymbol{\beta} \quad \text{if} \quad E_\tau(\boldsymbol{\alpha}) \subseteq E_\tau(\boldsymbol{\beta}) \quad \text{for all faces } \tau \,. \tag{13}$$

Hence $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ if and only if $E_\tau(\boldsymbol{\alpha}) = E_\tau(\boldsymbol{\beta})$ for all faces τ. Therefore $\boldsymbol{\alpha} + \boldsymbol{Z}^d$ has only finitely many equivalence classes, since $E_\tau(\boldsymbol{\alpha})$ and $E_\tau(\boldsymbol{\alpha} + \boldsymbol{a})$ ($\boldsymbol{a} \in \boldsymbol{Z}^d$) are subsets of the finite set (12) for $\boldsymbol{\lambda} \in \boldsymbol{C}(A \cap \tau)$ with $\boldsymbol{\alpha} - \boldsymbol{\lambda} \in \boldsymbol{Z}^d$.

REMARK 6.3. We may characterize the set $E_\tau(\boldsymbol{\alpha})$ as follows: Let $\boldsymbol{l} \in \boldsymbol{C}^\tau$, i.e., $\boldsymbol{l} = (l_1, \dots, l_n) \in \boldsymbol{C}^n$ and $l_j = 0$ for j with $\boldsymbol{a}_j \notin \tau$. Then

$$R_{\tau,\boldsymbol{l}} := \prod_{\boldsymbol{m} \in \boldsymbol{N}^{\tau^c} \times \boldsymbol{Z}^\tau} \boldsymbol{C} x^{\boldsymbol{l}+\boldsymbol{m}}$$

is a left $D(R)$-module, where

$$\boldsymbol{N}^{\tau^c} \times \boldsymbol{Z}^\tau = \{\boldsymbol{m} \in \boldsymbol{Z}^n \,;\, m_j \geq 0 \quad \text{for } j \text{ with } \boldsymbol{a}_j \notin \tau\}\,.$$

Then

$$A\boldsymbol{l} \in E_\tau(\boldsymbol{\alpha}) \ \text{ if and only if } \ \mathrm{Hom}_{D(R)}(H_A(\boldsymbol{\alpha}), R_{\tau,\boldsymbol{l}}) \neq 0\,.$$

For details, see [16].

The second finiteness property needed to prove the finiteness of $\mathrm{Prim}(D(R_A))$ is the following.

LEMMA 6.4. *Let $\mathcal{F}'$ be a set of facets of $\boldsymbol{R}_{\geq 0}A$, i.e., $\mathcal{F}' \subseteq \mathcal{F}$, and let $V(\mathcal{F}')$ be the intersection $\bigcap_{\sigma \in \mathcal{F}'}(F_\sigma = 0)$. Then*

$$\{\boldsymbol{\alpha} \,;\, \mathcal{F}(\boldsymbol{\alpha}) \supseteq \mathcal{F}'\}/(\boldsymbol{Z}^d + V(\mathcal{F}')) \tag{14}$$

is finite.

PROOF. The $\mathbf{Z}$-module $\boldsymbol{Z}^d + V(\mathcal{F}')/V(\mathcal{F}')$ is a $\mathbf{Z}$-submodule of $\{\boldsymbol{\alpha}; \mathcal{F}(\boldsymbol{\alpha}) \supseteq \mathcal{F}'\}/V(\mathcal{F}')$. Both of them are free of rank $\dim \boldsymbol{C}^d/V(\mathcal{F}')$. Hence the index is finite. □

Now we are ready to prove the finiteness of $\mathrm{Prim}(D(R_A))$.

THEOREM 6.5. *The set* $\mathrm{Prim}(D(R_A))$ *is finite.*

PROOF. It suffices to show that the set

$$\mathrm{Prim}_{\mathcal{F}'} := \{\mathrm{Ann}\, L(\boldsymbol{\alpha}) \,;\, \mathcal{F}(\boldsymbol{\alpha}) = \mathcal{F}'\}$$

is finite for each $\mathcal{F}' \subseteq \mathcal{F}$.

Let $\mathcal{F}(\boldsymbol{\alpha}) = \mathcal{F}'$. Let $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_k$ be a complete set of representatives of $\{\boldsymbol{\beta} \,;\, \mathcal{F}(\boldsymbol{\beta}) \supseteq \mathcal{F}'\}/(\boldsymbol{Z}^d + V(\mathcal{F}'))$. We take the representatives so that if a coset $\boldsymbol{\alpha}_j + \boldsymbol{Z}^d + V(\mathcal{F}')$ has an element $\boldsymbol{\beta}$ with $\mathcal{F}(\boldsymbol{\beta}) = \mathcal{F}'$, then $\mathcal{F}(\boldsymbol{\alpha}_j) = \mathcal{F}'$. Then there exist j, $\boldsymbol{a} \in \boldsymbol{Z}^d$, and $\boldsymbol{\mu} \in V(\mathcal{F}')$ such that $\boldsymbol{\alpha} = \boldsymbol{\alpha}_j + \boldsymbol{a} + \boldsymbol{\mu}$. We have $\mathcal{F}' = \mathcal{F}(\boldsymbol{\alpha}_j) = \mathcal{F}(\boldsymbol{\alpha}_j + \boldsymbol{a})$. By Proposition 6.2, $\mathrm{Ann}\, L(\boldsymbol{\alpha}) = \mathrm{Ann}\, L(\boldsymbol{\alpha}_j + \boldsymbol{a})$. Since each $\boldsymbol{\alpha}_j + \boldsymbol{Z}^d$ has only finitely many equivalence classes, $\mathrm{Prim}_{\mathcal{F}'}$ is finite. □

The above proof gives an algorithm for $\mathrm{Prim}(D(R_A))$. We exhibit a computation of $\mathrm{Prim}(D(R_A))$ in Example 8.1.

COROLLARY 6.6. *The set* $\{\mathrm{ZC}[\boldsymbol{\alpha}] : \boldsymbol{\alpha} \in \boldsymbol{C}^d\}$ *is finite, where* ZC *stands for Zariski closure.*

PROOF. This follows from Theorem 6.5 and the fact that $\mathrm{ZC}[\boldsymbol{\alpha}] = \mathrm{ZC}[\boldsymbol{\beta}]$ if and only if $\mathrm{Ann}\, L(\boldsymbol{\alpha})_{\boldsymbol{0}} = \mathrm{Ann}\, L(\boldsymbol{\beta})_{\boldsymbol{0}}$ by Proposition 5.2. □

The proof of Proposition 6.2 occupies the remainder of this section. We start the proof with giving a relation between $E_\tau(\boldsymbol{\alpha})$ and $E_\tau(\boldsymbol{\alpha}+\boldsymbol{\mu})$ when $\boldsymbol{\mu} \in V(\boldsymbol{\alpha})$ and $\mathcal{F}(\boldsymbol{\alpha}+\boldsymbol{\mu}) = \mathcal{F}(\boldsymbol{\alpha})$.

LEMMA 6.7. *Suppose that* $\boldsymbol{\mu} \in V(\boldsymbol{\alpha})$ *and* $\mathcal{F}(\boldsymbol{\alpha}+\boldsymbol{\mu}) = \mathcal{F}(\boldsymbol{\alpha})$. *Let* τ *be a face of the cone* $\boldsymbol{R}_{\geq 0}A$. *Then* $E_\tau(\boldsymbol{\alpha}+\boldsymbol{\mu}) = \emptyset$ *if and only if* $E_\tau(\boldsymbol{\alpha}) = \emptyset$. *Moreover, if* $E_\tau(\boldsymbol{\alpha}) \neq \emptyset$, *then*

$$E_\tau(\boldsymbol{\alpha}+\boldsymbol{\mu}) = \boldsymbol{\mu} + E_\tau(\boldsymbol{\alpha}). \tag{15}$$

PROOF. By symmetry, it suffices to prove that if $\boldsymbol{\lambda} \in E_\tau(\boldsymbol{\alpha})$, then $\boldsymbol{\mu} \in \boldsymbol{C}(A\cap\tau)$ and $\boldsymbol{\lambda}+\boldsymbol{\mu} \in E_\tau(\boldsymbol{\alpha}+\boldsymbol{\mu})$.

Suppose that $\boldsymbol{\lambda} \in E_\tau(\boldsymbol{\alpha})$. Then $F_\sigma(\boldsymbol{\alpha}) = F_\sigma(\boldsymbol{\alpha}-\boldsymbol{\lambda}) \in F_\sigma(\boldsymbol{N}A) \subseteq \boldsymbol{N}$ for all facets $\sigma \succeq \tau$. Hence $\boldsymbol{C}(A\cap\tau)$ is the intersection of a subset of $\mathcal{F}(\boldsymbol{\alpha})$, and thus $\boldsymbol{C}(A\cap\tau) \supseteq V(\boldsymbol{\alpha})$. This proves $\boldsymbol{\mu} \in \boldsymbol{C}(A\cap\tau)$. Since $\boldsymbol{\alpha}+\boldsymbol{\mu}-(\boldsymbol{\lambda}+\boldsymbol{\mu}) = \boldsymbol{\alpha}-\boldsymbol{\lambda} \in \boldsymbol{N}A + \boldsymbol{Z}(A\cap\tau)$, we see $\boldsymbol{\lambda}+\boldsymbol{\mu} \in E_\tau(\boldsymbol{\alpha}+\boldsymbol{\mu})$. □

COROLLARY 6.8. *Suppose that* $\boldsymbol{\mu} \in V(\boldsymbol{\alpha})$ *and* $\mathcal{F}(\boldsymbol{\alpha}+\boldsymbol{\mu}) = \mathcal{F}(\boldsymbol{\alpha})$. *Then* $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ *if and only if* $\boldsymbol{\alpha}+\boldsymbol{\mu} \sim \boldsymbol{\beta}+\boldsymbol{\mu}$.

PROOF. If $\boldsymbol{\alpha}-\boldsymbol{\beta} \notin \boldsymbol{Z}A$, then $\boldsymbol{\alpha} \not\sim \boldsymbol{\beta}$ and $\boldsymbol{\alpha}+\boldsymbol{\mu} \not\sim \boldsymbol{\beta}+\boldsymbol{\mu}$. Suppose that $\boldsymbol{\alpha}-\boldsymbol{\beta} \in \boldsymbol{Z}A$. Then $\mathcal{F}(\boldsymbol{\alpha}) = \mathcal{F}(\boldsymbol{\beta})$, and $\mathcal{F}(\boldsymbol{\alpha}+\boldsymbol{\mu}) = \mathcal{F}(\boldsymbol{\beta}+\boldsymbol{\mu})$. The assertion follows from Lemma 6.7. □

PROOF OF PROPOSITION 6.2. Recall Proposition 5.2,

$$(\mathrm{Ann}\, L(\boldsymbol{\alpha}))_{\boldsymbol{a}} = t^{\boldsymbol{a}}\boldsymbol{I}(\Omega(\boldsymbol{a}) \cup \Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}))\,,$$

where $\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}) = \{\,\boldsymbol{\beta}\,;\ \boldsymbol{\beta} \sim \boldsymbol{\beta}+\boldsymbol{a} \sim \boldsymbol{\alpha}\,\}$. We are going to prove that

$$\mathrm{ZC}(\Lambda_{[\boldsymbol{\alpha}+\boldsymbol{\mu}]}(\boldsymbol{a})) = \mathrm{ZC}(\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}))\,.$$

Since $\Lambda_{[\boldsymbol{\alpha}+\boldsymbol{\mu}]}(\boldsymbol{a}) = \emptyset$ if and only if $\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}) = \emptyset$ by Corollary 6.8, we suppose that they are not empty. Then again by Corollary 6.8

$$\Lambda_{[\boldsymbol{\alpha}+\boldsymbol{\mu}]}(\boldsymbol{a}) = \boldsymbol{\mu} + \Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a})\,.$$

Hence for the proof it suffices to show that

$$V(\boldsymbol{\alpha}) + \mathrm{ZC}(\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a})) = \mathrm{ZC}(\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}))\,. \tag{16}$$

Let $\boldsymbol{\beta} \in \Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a})$, and $\boldsymbol{\mu}' \in V(\boldsymbol{\alpha}) \cap \boldsymbol{Z}^d$. Put $\upsilon := \prod_{\boldsymbol{C}\tau \supseteq V(\boldsymbol{\alpha})} [\boldsymbol{Z}^d \cap \boldsymbol{Q}\tau : \boldsymbol{Z}(A\cap\tau)]$. We will show that $\boldsymbol{\beta} + \upsilon\boldsymbol{\mu}' \in \Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a})$.

Suppose that $\boldsymbol{C}\tau \not\supseteq V(\boldsymbol{\alpha})$. Then there exists a facet $\sigma \succeq \tau$ such that $\boldsymbol{C}\sigma \not\supseteq V(\boldsymbol{\alpha})$. Thus $F_\sigma(\boldsymbol{\alpha}) \notin \boldsymbol{Z}$. Note that $\upsilon\boldsymbol{\mu}', \boldsymbol{a}, \boldsymbol{\alpha}-\boldsymbol{\beta} \in \boldsymbol{Z}^d$, since $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$. Hence $F_\sigma(\boldsymbol{\beta})$, $F_\sigma(\boldsymbol{\beta}+\upsilon\boldsymbol{\mu}')$, $F_\sigma(\boldsymbol{\beta}+$

$\boldsymbol{a} + v\boldsymbol{\mu}') \notin \boldsymbol{Z}$. This implies $E_\sigma(\boldsymbol{\beta}) = E_\sigma(\boldsymbol{\beta} + v\boldsymbol{\mu}') = E_\sigma(\boldsymbol{\beta} + \boldsymbol{a} + v\boldsymbol{\mu}') = \emptyset = E_\sigma(\boldsymbol{\alpha})$, and $E_\tau(\boldsymbol{\beta}) = E_\tau(\boldsymbol{\beta} + v\boldsymbol{\mu}') = E_\tau(\boldsymbol{\beta} + \boldsymbol{a} + v\boldsymbol{\mu}') = \emptyset = E_\tau(\boldsymbol{\alpha})$ by [16, Proposition 2.2].

Next suppose that $\boldsymbol{C}\tau \supseteq V(\boldsymbol{\alpha})$. Then $E_\tau(\boldsymbol{\beta} + v\boldsymbol{\mu}') = E_\tau(\boldsymbol{\beta})$, and $E_\tau(\boldsymbol{\beta} + \boldsymbol{a} + v\boldsymbol{\mu}') = E_\tau(\boldsymbol{\beta} + \boldsymbol{a})$, since $v\boldsymbol{\mu}' \in \boldsymbol{Z}(A \cap \tau)$.

Hence $\boldsymbol{\beta} + v\boldsymbol{\mu}' \in \Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a})$. We have thus proved that

$$\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}) + v(V(\boldsymbol{\alpha}) \cap \boldsymbol{Z}A) \subseteq \Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}).$$

Taking the Zariski closures, we see that

$$\mathrm{ZC}(\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a})) + V(\boldsymbol{\alpha}) \subseteq \mathrm{ZC}(\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a})).$$

The other inclusion is trivial. □

7. Simplicity. In this section, we discuss the simplicity of $D(R_A)$. In the first subsection, we consider the conditions: the scoredness and Serre's (S_2). We prove that the simplicity of $D(R_A)$ implies the scoredness, and that all $D(R_A)_{\boldsymbol{a}}$ are singly generated $\boldsymbol{C}[s]$-modules if and only if Condition (S_2) is satisfied. In the second subsection, we give a necessary and sufficient condition for the simplicity (Theorem 7.25), which is not difficult to check (see Remarks 7.14 and 7.23).

We start this section by noting the $\boldsymbol{Z}^d$-graded version of a well-known fact.

LEMMA 7.1. *The ring $D(R_A)$ is simple if and only if* $\mathrm{Ann}\, L(\boldsymbol{\alpha}) = \{0\}$ *for all $\boldsymbol{\alpha} \in \boldsymbol{C}^d$.*

PROOF. First note that any two-sided ideal of $D(R_A)$ is $\boldsymbol{Z}^d$-homogeneous. (An ideal I is said to be $\boldsymbol{Z}^d$-homogeneous if $I = \bigoplus_{\boldsymbol{a} \in \boldsymbol{Z}^d} I \cap D(R_A)_{\boldsymbol{a}}$.)

It is enough to show that any maximal ideal of $D(R_A)$ is the annihilator of a simple $\boldsymbol{Z}^d$-graded module. Let I be a maximal ideal of $D(R_A)$. Let J be a maximal $\boldsymbol{Z}^d$-homogeneous left ideal containing I. Then $D(R_A)/J$ is a simple $\boldsymbol{Z}^d$-graded $D(R_A)$-module, and $\mathrm{Ann}(D(R_A)/J)$ contains I. Since I is maximal, we obtain $I = \mathrm{Ann}(D(R_A)/J)$. □

7.1. Scored semigroups. We recall the definition of a scored semigroup [18]. The semigroup $\boldsymbol{N}A$ is said to be *scored* if

$$\boldsymbol{N}A = \bigcap_{\sigma \in \mathcal{F}} \{\boldsymbol{a} \in \boldsymbol{Z}^d\,;\ F_\sigma(\boldsymbol{a}) \in F_\sigma(\boldsymbol{N}A)\}.$$

We know that $E_\sigma(\boldsymbol{a}) \neq \emptyset$ if and only if $F_\sigma(\boldsymbol{a}) \in F_\sigma(\boldsymbol{N}A)$ [16, Proposition 2.2]. Hence a semigroup $\boldsymbol{N}A$ is scored if and only if

$$\boldsymbol{N}A = \{\boldsymbol{a} \in \boldsymbol{Z}^d\,;\ E_\sigma(\boldsymbol{a}) \neq \emptyset \quad \text{for all } \sigma \in \mathcal{F}\}.$$

In the following lemma, we characterize the subset $\boldsymbol{N}A$ of $\boldsymbol{Z}^d$ in terms of the finite sets $E_\tau(\boldsymbol{a})$.

LEMMA 7.2.

$$\boldsymbol{N}A = \{\boldsymbol{a} \in \boldsymbol{Z}^d\,;\ \boldsymbol{0} \in E_\tau(\boldsymbol{a}) \ \textit{for all faces } \tau\}.$$

PROOF. Let τ_0 be the minimal face of $\boldsymbol{R}_{\geq 0}A$. We have

$$\begin{aligned}&\{\boldsymbol{a} \in \boldsymbol{Z}^d\,;\ \boldsymbol{0} \in E_\tau(\boldsymbol{a}) \ \text{ for all faces } \tau\}\\ &\quad = \bigcap_\tau (\boldsymbol{N}A + \boldsymbol{Z}(A \cap \tau)) = \boldsymbol{N}A + \boldsymbol{Z}(A \cap \tau_0)\,.\end{aligned}$$

If $\tau_0 = \{\boldsymbol{0}\}$, or if the cone $\boldsymbol{R}_{\geq 0}A$ is strongly convex, then clearly $\boldsymbol{N}A + \boldsymbol{Z}(A\cap\tau_0) = \boldsymbol{N}A$. Next suppose $\tau_0 \neq \{\boldsymbol{0}\}$. Then τ_0 is a linear subspace of $\boldsymbol{R}^d$. We have $-\boldsymbol{a}_j \in \boldsymbol{R}_{\geq 0}(A\cap\tau_0)\cap\boldsymbol{Z}(A\cap\tau_0)$ for $\boldsymbol{a}_j \in A \cap \tau_0$. Hence there exist nonzero $m_j \in \boldsymbol{N}$ and $c_{jk} \in \boldsymbol{N}$ such that $-m_j\boldsymbol{a}_j = \sum_{\boldsymbol{a}_k\in\tau_0} c_{jk}\boldsymbol{a}_k$. Let $\boldsymbol{b} \in \boldsymbol{Z}(A \cap \tau_0)$ and $\boldsymbol{b} = \sum_{\boldsymbol{a}_k\in\tau_0} d_k\boldsymbol{a}_k$ with $d_k \in \boldsymbol{Z}$. Take $N \in \boldsymbol{N}$ so that $Nm_j + d_j > 0$. Then $\boldsymbol{b} = \sum_{\boldsymbol{a}_k\in\tau_0} d_k\boldsymbol{a}_k + N(m_j\boldsymbol{a}_j + \sum_{\boldsymbol{a}_k\in\tau_0} c_{jk}\boldsymbol{a}_k)$ has a positive integer as the coefficient of $\boldsymbol{a}_j$. Do the same for each $\boldsymbol{a}_j \in \tau_0$ to see $\boldsymbol{b} \in \boldsymbol{N}(A \cap \tau_0)$. Hence $\boldsymbol{N}A + \boldsymbol{Z}(A \cap \tau_0) = \boldsymbol{N}A$. □

We introduce two subsets of $\boldsymbol{Z}^d$, which respectively characterize the properties of $\boldsymbol{N}A$: the scoredness and Serre's condition (S_2) (see Remark 7.3).

$$S_1 := \{\boldsymbol{a} \in \boldsymbol{Z}^d\,;\ E_\sigma(\boldsymbol{a}) \neq \emptyset \quad \text{for all } \ \sigma \in \mathcal{F}\}\,, \tag{17}$$

$$S_2 := \{\boldsymbol{a} \in \boldsymbol{Z}^d\,;\ E_\sigma(\boldsymbol{a}) \ni \boldsymbol{0} \quad \text{for all } \ \sigma \in \mathcal{F}\}\,. \tag{18}$$

Then $S_2 = \bigcap_{\sigma\in\mathcal{F}}(\boldsymbol{N}A + \boldsymbol{Z}(A \cap \sigma))$, and

$$\boldsymbol{N}A \subset S_2 \subset S_1$$

by (17), (18), and Lemma 7.2.

REMARK 7.3. 1. The semigroup $\boldsymbol{N}A$ is scored if and only if $\boldsymbol{N}A = S_1$.

2. Serre's condition (S_2) is the equality $\boldsymbol{N}A = S_2$.
3. By the proof of Lemma 7.2, we have

$$\boldsymbol{N}A = \{\boldsymbol{a} \in \boldsymbol{Z}^d\,;\ E_{\tau_0}(\boldsymbol{a}) \ni \boldsymbol{0}\}\,,$$

where τ_0 is the minimal face.

Our first aim in this subsection is to show that the simplicity of $D(R_A)$ implies the scoredness of $\boldsymbol{N}A$. We use the following lemma for the proof.

LEMMA 7.4.

$$\dim \mathrm{ZC}(\Omega(\boldsymbol{a})) < d \qquad \textit{for all } \ \boldsymbol{a} \in \boldsymbol{Z}^d\,.$$

PROOF. Take M so that

$$\{\boldsymbol{a} \in \boldsymbol{Z}^d\,;\ F_\sigma(\boldsymbol{a}) \geq M\} \subseteq \boldsymbol{N}A$$

(see, e.g., [19, Lemma 3.6]). Then

$$\mathrm{ZC}\,(\Omega(\boldsymbol{a})) \subseteq \bigcup_{\sigma\in\mathcal{F},\ F_\sigma(\boldsymbol{a})<M}\ \bigcup_{m=0}^{M-F_\sigma(\boldsymbol{a})-1} (F_\sigma = m)\,.$$ □

PROPOSITION 7.5. *If $D(R_A)$ is simple, then $\boldsymbol{N}A$ is scored.*

PROOF. We know

$$S_1 = \bigcap_{\sigma \in \mathcal{F}} \{\boldsymbol{a} \in \boldsymbol{Z}^d \; ; \; F_\sigma(\boldsymbol{a}) \in F_\sigma(\boldsymbol{N}A)\} \, .$$

Take M as in the proof of Lemma 7.4. Then we have

$$\mathrm{ZC}\,(S_1 \setminus \boldsymbol{N}A) \subseteq \bigcup_{\sigma \in \mathcal{F}} \bigcup_{m=0}^{M-1} (F_\sigma = m) \, . \tag{19}$$

Suppose that $\boldsymbol{N}A$ is not scored, and $\boldsymbol{\alpha} \in S_1 \setminus \boldsymbol{N}A$. Since $S_1 \setminus \boldsymbol{N}A$ is a union of some equivalence classes, we have $\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{b}) \subseteq S_1 \setminus \boldsymbol{N}A$ for all $\boldsymbol{b} \in \boldsymbol{Z}A = \boldsymbol{Z}^d$. (See (9) to recall the set $\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{b})$.) Hence, by (19), $\dim \mathrm{ZC}(\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{b})) < d$ for all $\boldsymbol{b}$. By Lemma 7.4, we have $\dim \mathrm{ZC}(\Omega(\boldsymbol{b}) \cup \Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{b})) < d$ for all $\boldsymbol{b}$. Hence $\mathrm{Ann}\, L(\boldsymbol{\alpha}) \neq 0$ by Proposition 5.2. □

REMARK 7.6. By Van den Bergh [24, Theorem 6.2.5], if $D(R_A)$ is simple, then R_A is Cohen-Macaulay. $\boldsymbol{N}A$ being scored and R_A being Cohen-Macaulay are not enough for the simplicity of $D(R_A)$.

Now we prove the fact announced in Remark 3.2.

PROPOSITION 7.7. *The $\boldsymbol{C}[s]$-modules $D(R_A)_{\boldsymbol{a}}$ are singly generated for all $\boldsymbol{a} \in \boldsymbol{Z}^d$ if and only if the semigroup $\boldsymbol{N}A$ satisfies (S_2).*

PROOF. First we paraphrase Condition (S_2). In [19, Proposition 3.4], we have shown that there exist $(\boldsymbol{b}_i, \tau_i)$, $i = 1, \dots, l$, where $\boldsymbol{b}_i \in \boldsymbol{R}_{\geq 0}A \cap \boldsymbol{Z}^d$ and τ_i is a face of the cone $\boldsymbol{R}_{\geq 0}A$, such that

$$(\boldsymbol{R}_{\geq 0}A \cap \boldsymbol{Z}^d) \setminus \boldsymbol{N}A = \bigcup_{i=1}^{l} (\boldsymbol{b}_i + \boldsymbol{Z}(A \cap \tau_i)) \cap \boldsymbol{R}_{\geq 0}A \, . \tag{20}$$

We may assume that this decomposition is irredundant. Then $\{\boldsymbol{b}_i + \boldsymbol{Z}(A \cap \tau_i) \; ; \; i = 1, \dots, l\}$ is unique. By [19, Lemma 3.6], for $\sigma \in \mathcal{F}$,

$$\boldsymbol{N}A + \boldsymbol{Z}(A \cap \sigma) = [\boldsymbol{R}_{\geq 0}A + \boldsymbol{R}(A \cap \sigma)] \cap \boldsymbol{Z}^d \setminus \bigcup_{\tau_i = \sigma} (\boldsymbol{b}_i + \boldsymbol{Z}(A \cap \tau_i)) \, .$$

Hence we obtain

$$\bigcap_{\sigma \in \mathcal{F}} (\boldsymbol{N}A + \boldsymbol{Z}(A \cap \sigma)) = \boldsymbol{R}_{\geq 0}A \cap \boldsymbol{Z}^d \setminus \bigcup_{\tau_i \in \mathcal{F}} (\boldsymbol{b}_i + \boldsymbol{Z}(A \cap \tau_i)) \, ,$$

which means that $\boldsymbol{N}A$ satisfies (S_2) if and only if each τ_i appearing in (20) is a facet.

Suppose that $\boldsymbol{N}A$ satisfies (S_2). Then, by the previous paragraph and [19, Proposition 5.1], we have

$$\mathrm{ZC}(\Omega(\boldsymbol{a})) = \Bigg(\bigcup_{F_\sigma(\boldsymbol{a}) < 0} \; \bigcup_{m < -F_\sigma(\boldsymbol{a}),\, m \in F_\sigma(\boldsymbol{N}A)} F_\sigma^{-1}(m) \Bigg) \cup \Bigg(\bigcup_{\boldsymbol{b}_i - \boldsymbol{a} \in \boldsymbol{N}A + \boldsymbol{Z}(A \cap \tau_i)} F_{\tau_i}^{-1}(\boldsymbol{b}_i - \boldsymbol{a}) \Bigg) .$$

Hence $\boldsymbol{I}(\Omega(\boldsymbol{a}))$ is singly generated.

Next suppose that $\boldsymbol{N}A$ does not satisfy (S_2). Then a face of codimension greater than one appears in the difference (20). Let τ_1 be a face of codimension greater than one, and let $\boldsymbol{b}_1 + \boldsymbol{Z}(A \cap \tau_1)$ appear in the difference. Then

$$\mathrm{ZC}(\Omega(\boldsymbol{b}_1)) = \bigcup_{\boldsymbol{b}_i - \boldsymbol{b}_1 \in \boldsymbol{N}A + \boldsymbol{Z}(A\cap\tau_i)} (\boldsymbol{b}_i - \boldsymbol{b}_1 + \boldsymbol{C}(A \cap \tau_i)) .$$

We show that $\boldsymbol{C}(A \cap \tau_1)$ is an irreducible component of $\mathrm{ZC}(\Omega(\boldsymbol{b}_1))$. Suppose the contrary. Then there exists i such that

$$\boldsymbol{b}_i - \boldsymbol{b}_1 \in \boldsymbol{N}A + \boldsymbol{Z}(A \cap \tau_i) , \tag{21}$$

$$\boldsymbol{C}(A \cap \tau_1) \subseteq \boldsymbol{b}_i - \boldsymbol{b}_1 + \boldsymbol{C}(A \cap \tau_i) . \tag{22}$$

The latter equation (22) means that $\boldsymbol{b}_i - \boldsymbol{b}_1 \in \boldsymbol{C}(A\cap\tau_i)$ and $\tau_1 \preceq \tau_i$. Combining with (21), we have $\boldsymbol{b}_i - \boldsymbol{b}_1 \in \boldsymbol{Z}(A \cap \tau_i)$, which contradicts the irredundancy of (20). We have thus proved $\boldsymbol{C}(A \cap \tau_1)$ is an irreducible component of $\mathrm{ZC}(\Omega(\boldsymbol{b}_1))$. Hence the ideal $\boldsymbol{I}(\Omega(\boldsymbol{b}_1))$ is not singly generated. □

In the rest of this subsection, we consider the simplicity of R_A and S_1. The following lemma is immediate from the definition of $E_\tau(\boldsymbol{0})$.

LEMMA 7.8.

$$E_\tau(\boldsymbol{0}) = \{\boldsymbol{0}\} \quad \textit{for all faces } \tau .$$

LEMMA 7.9. *Let* $\boldsymbol{a} \in \boldsymbol{Z}^d$. *Then there exists* $\boldsymbol{b} \in \boldsymbol{N}A$ *such that*

$$\sharp E_\tau(\boldsymbol{a} + \boldsymbol{b}) = [\boldsymbol{Q}(A \cap \tau) \cap \boldsymbol{Z}^d : \boldsymbol{Z}(A \cap \tau)] .$$

(*In this situation, we write* $E_\tau(\boldsymbol{a} + \boldsymbol{b}) = \text{full}$.)

PROOF. We may assume that $\boldsymbol{a} \in \boldsymbol{N}A$. Let $\boldsymbol{\lambda} \in \boldsymbol{Q}(A \cap \tau) \cap \boldsymbol{Z}^d/\boldsymbol{Z}(A \cap \tau)$, take its representative, and denote it by $\boldsymbol{\lambda}$ again. Write $-\boldsymbol{\lambda} = \sum_k d_k\boldsymbol{a}_k$ with $d_k \in \boldsymbol{Z}$. Then $-\sum_{d_k<0} d_k\boldsymbol{a}_k - \boldsymbol{\lambda} = \sum_{d_k\geq 0} d_k\boldsymbol{a}_k$. Hence $\boldsymbol{a} + (-\sum_{d_k<0} d_k\boldsymbol{a}_k) - \boldsymbol{\lambda} \in \boldsymbol{N}A$. Thus $\boldsymbol{\lambda} \in E_\tau(\boldsymbol{a} + (-\sum_{d_k<0} d_k\boldsymbol{a}_k))$. We repeat this argument for each pair $(\tau, \boldsymbol{\lambda})$ to prove the assertion. □

PROPOSITION 7.10. *The semigroup algebra* R_A *is a simple* $\boldsymbol{Z}^d$*-graded* $D(R_A)$*-module if and only if*

$$\boldsymbol{Q}(A \cap \tau) \cap \boldsymbol{Z}^d = \boldsymbol{Z}(A \cap \tau) \quad \textit{for all faces } \tau . \tag{23}$$

PROOF. The graded $D(R_A)$-module $R_A = \boldsymbol{C}[\boldsymbol{N}A]$ is simple if and only if all $\boldsymbol{a} \in \boldsymbol{N}A$ are equivalent to $\boldsymbol{0}$. By Lemma 7.8, it happens if and only if $E_\tau(\boldsymbol{a}) = \{\boldsymbol{0}\}$ for all faces τ and $\boldsymbol{a} \in \boldsymbol{N}A$. This is equivalent to the condition (23) by Lemma 7.9. □

LEMMA 7.11. *If the semigroup* $\boldsymbol{N}A$ *is scored, then it satisfies* (23).

PROOF. For a facet σ, take $M_\sigma \in \boldsymbol{N}$ so that M_σ is greater than any number in $\boldsymbol{N} \setminus F_\sigma(\boldsymbol{N}A)$.

Let τ be a face, and let $\boldsymbol{x} \in \boldsymbol{Q}(A \cap \tau) \cap \boldsymbol{Z}^d$. For each facet $\sigma \not\succeq \tau$, there exists $\boldsymbol{a}_\sigma \in A \cap \tau \setminus \sigma$. Take m_σ large enough to satisfy $F_\sigma(\boldsymbol{x}) + m_\sigma F_\sigma(\boldsymbol{a}_\sigma) \geq M_\sigma$. Let $\boldsymbol{y} = \boldsymbol{x} + \sum_{\sigma \not\succeq \tau} m_\sigma \boldsymbol{a}_\sigma$. Then $\boldsymbol{y} \in \boldsymbol{Q}(A \cap \tau) \cap \boldsymbol{Z}^d$, and

$$\begin{array}{ll} F_\sigma(\boldsymbol{y}) = 0 & \text{if } \sigma \succeq \tau\,, \\ F_\sigma(\boldsymbol{y}) \geq M_\sigma & \text{otherwise}\,. \end{array}$$

Since $\boldsymbol{N}A$ is scored, $\boldsymbol{y} \in \boldsymbol{N}A$. Hence $\boldsymbol{y} \in \boldsymbol{N}(A \cap \tau)$, and $\boldsymbol{x} \in \boldsymbol{Z}(A \cap \tau)$. □

COROLLARY 7.12. *If the semigroup* $\boldsymbol{N}A$ *is scored, then* R_A *is a simple* $\boldsymbol{Z}^d$*-graded* $D(R_A)$*-module.*

PROOF. This follows from Proposition 7.10 and Lemma 7.11. □

PROPOSITION 7.13. *The semigroup* $\boldsymbol{N}A$ *is scored if and only if* $\boldsymbol{C}[S_1]$ *is a simple* $\boldsymbol{Z}^d$*-graded* $D(R_A)$*-module.*

PROOF. Suppose that $\boldsymbol{N}A$ is scored. Then $S_1 = \boldsymbol{N}A$. Hence $\boldsymbol{C}[S_1]$ is a simple $\boldsymbol{Z}^d$-graded $D(R_A)$-module by Corollary 7.12.

Suppose that $\boldsymbol{C}[S_1]$ is a simple $\boldsymbol{Z}^d$-graded $D(R_A)$-module. Since R_A is a nonzero $\boldsymbol{Z}^d$-graded $D(R_A)$-submodule of $\boldsymbol{C}[S_1]$, we have $R_A = \boldsymbol{C}[S_1]$. Hence $\boldsymbol{N}A$ is scored. □

7.2. Conditions for simplicity. The aim of this subsection is to give a necessary and sufficient condition for the vanishing of a primitive ideal $\operatorname{Ann} L(\boldsymbol{\alpha})$. It leads us to a necessary and sufficient condition for the simplicity of $D(R_A)$.

We start this subsection by introducing some notation. Let $\boldsymbol{\alpha} \in \boldsymbol{C}^d$. Set

$$\mathcal{F}_+(\boldsymbol{\alpha}) := \{\sigma \in \mathcal{F}\,;\ F_\sigma(\boldsymbol{\alpha}) \in F_\sigma(\boldsymbol{N}A)\}\,,$$

$$\mathcal{F}_-(\boldsymbol{\alpha}) := \{\sigma \in \mathcal{F}\,;\ F_\sigma(\boldsymbol{\alpha}) \in \boldsymbol{Z} \setminus F_\sigma(\boldsymbol{N}A)\}\,,$$

and

$$\boldsymbol{R}_{>0}(\boldsymbol{\alpha}) := \left\{ \boldsymbol{\gamma} \in \boldsymbol{R}^d\,;\ \begin{array}{ll} F_\sigma(\boldsymbol{\gamma}) > 0 & \text{for } \sigma \in \mathcal{F}_+(\boldsymbol{\alpha})\,, \\ F_\sigma(\boldsymbol{\gamma}) < 0 & \text{for } \sigma \in \mathcal{F}_-(\boldsymbol{\alpha}) \end{array} \right\}.$$

Let $\mathrm{Face}(\boldsymbol{\alpha})$ denote the set of faces τ such that $\boldsymbol{\alpha} - \boldsymbol{\lambda} \in \boldsymbol{Z}^d$ for some $\boldsymbol{\lambda} \in \boldsymbol{C}(A \cap \tau)$, and that every facet σ containing τ belongs to $\mathcal{F}_+(\boldsymbol{\alpha})$. Let $[\boldsymbol{\alpha}]$ denote the equivalence class that $\boldsymbol{\alpha}$ belongs to. An equivalence class $[\boldsymbol{\alpha}]$ is said to be *extreme* if $E_\tau(\boldsymbol{\alpha})$ has $[\boldsymbol{Q}(A \cap \tau) \cap \boldsymbol{Z}^d : \boldsymbol{Z}(A \cap \tau)]$-many elements (i.e., $E_\tau(\boldsymbol{\alpha}) = \text{full}$) for every $\tau \in \mathrm{Face}(\boldsymbol{\alpha})$, and that $E_\tau(\boldsymbol{\alpha})$ is empty for every $\tau \notin \mathrm{Face}(\boldsymbol{\alpha})$.

We compare the conditions:

(i) An equivalence class $[\boldsymbol{\alpha}]$ is extreme.

(ii) $\boldsymbol{R}_{>0}(\boldsymbol{\alpha})$ is not empty.

(iii) $\mathrm{ZC}([\boldsymbol{\alpha}]) = \boldsymbol{C}^d$.

(iv) $\operatorname{Ann} L(\boldsymbol{\alpha}) = 0$.

REMARK 7.14. Conditions (i) and (ii) have an advantage over Condition (iii), for to check (i) and (ii) we do not need the equivalence class $[\boldsymbol{\alpha}]$, which is not easy to compute.

We need the following technical lemmas.

LEMMA 7.15. *Let τ be a face of $\boldsymbol{R}_{\geq 0}A$. Then there exists $M \in \boldsymbol{N}$ such that if $\boldsymbol{a} \in \boldsymbol{Z}^d$ satisfies $F_\sigma(\boldsymbol{a}) \geq M$ for all facets $\sigma \succeq \tau$, then $E_\tau(\boldsymbol{a}) =$* full.

PROOF. Let $\boldsymbol{\lambda} \in \boldsymbol{Q}(A \cap \tau) \cap \boldsymbol{Z}^d$. By [19, Lemma 3.6], there exists $M \in \boldsymbol{N}$ such that $\boldsymbol{c} \in \boldsymbol{N}A + \boldsymbol{Z}(A \cap \tau)$ for all $\boldsymbol{c} \in \boldsymbol{Z}^d$ satisfying $F_\sigma(\boldsymbol{c}) \geq M$ for all facets $\sigma \succeq \tau$. Hence, if $\boldsymbol{a} \in \boldsymbol{Z}^d$ satisfies $F_\sigma(\boldsymbol{a}) \geq M$ for all facets $\sigma \succeq \tau$, then $\boldsymbol{a} - \boldsymbol{\lambda} \in \boldsymbol{N}A + \boldsymbol{Z}(A \cap \tau)$, or $\boldsymbol{\lambda} \in E_\tau(\boldsymbol{a})$. □

LEMMA 7.16. *Let τ be a face of $\boldsymbol{R}_{\geq 0}A$, and let $\boldsymbol{\alpha} \in \boldsymbol{C}^d$. Assume that there exists $\boldsymbol{\lambda} \in \boldsymbol{C}(A \cap \tau)$ such that $\boldsymbol{\alpha} - \boldsymbol{\lambda}$ belongs to $\boldsymbol{Z}^d$. Then there exists $M \in \boldsymbol{N}$ such that if $\boldsymbol{\gamma} \in \boldsymbol{\alpha} + \boldsymbol{Z}^d$ satisfying $F_\sigma(\boldsymbol{\gamma}) \in \boldsymbol{Z}_{\geq M}$ for all facets $\sigma \succeq \tau$, then $E_\tau(\boldsymbol{\gamma}) =$* full.

PROOF. Apply Lemma 7.15 to $\boldsymbol{\gamma} - \boldsymbol{\lambda}$. □

PROPOSITION 7.17. *If the Zariski closure of an equivalence class $[\boldsymbol{\alpha}]$ is the whole space $\boldsymbol{C}^d$, then $\boldsymbol{R}_{>0}(\boldsymbol{\alpha})$ is not empty.*

PROOF. Since the set

$$(24) \qquad \left\{\boldsymbol{\gamma} \in \boldsymbol{\alpha} + \boldsymbol{Z}^d \,;\, \begin{array}{ll} F_\sigma(\boldsymbol{\gamma}) \in F_\sigma(\boldsymbol{N}A) & \text{for } \sigma \in \mathcal{F}_+(\boldsymbol{\alpha}), \\ F_\sigma(\boldsymbol{\gamma}) \in \boldsymbol{Z} \setminus F_\sigma(\boldsymbol{N}A) & \text{for } \sigma \in \mathcal{F}_-(\boldsymbol{\alpha}) \end{array}\right\}$$

contains $[\boldsymbol{\alpha}]$, its Zariski closure equals $\boldsymbol{C}^d$. Take a real number ε so that ε is algebraically independent over $\boldsymbol{Q}[\, F_\sigma(\mathrm{Re}(\boldsymbol{\alpha})),\ F_\sigma(\mathrm{Im}(\boldsymbol{\alpha})) \,:\, \sigma \in \mathcal{F}\,]$. Put $\boldsymbol{\alpha}_\varepsilon := \mathrm{Re}(\boldsymbol{\alpha}) + \varepsilon \mathrm{Im}(\boldsymbol{\alpha})$, where $\mathrm{Re}(\boldsymbol{\alpha})$ and $\mathrm{Im}(\boldsymbol{\alpha})$ are the vectors in $\boldsymbol{R}^d$ with $\boldsymbol{\alpha} = \mathrm{Re}(\boldsymbol{\alpha}) + \sqrt{-1}\,\mathrm{Im}(\boldsymbol{\alpha})$. Then $\mathcal{F}_+(\boldsymbol{\alpha}) = \mathcal{F}_+(\boldsymbol{\alpha}_\varepsilon)$, and $\mathcal{F}_-(\boldsymbol{\alpha}) = \mathcal{F}_-(\boldsymbol{\alpha}_\varepsilon)$, since $\sigma \in \mathcal{F}_\pm(\boldsymbol{\alpha})$ or $\mathcal{F}_\pm(\boldsymbol{\alpha}_\varepsilon)$ implies $F_\sigma(\mathrm{Im}(\boldsymbol{\alpha})) = 0$. The set

$$(25) \qquad \left\{\boldsymbol{\gamma} \in \boldsymbol{\alpha}_\varepsilon + \boldsymbol{Z}^d \,;\, \begin{array}{ll} F_\sigma(\boldsymbol{\gamma}) \in F_\sigma(\boldsymbol{N}A) & \text{for } \sigma \in \mathcal{F}_+(\boldsymbol{\alpha}), \\ F_\sigma(\boldsymbol{\gamma}) \in \boldsymbol{Z} \setminus F_\sigma(\boldsymbol{N}A) & \text{for } \sigma \in \mathcal{F}_-(\boldsymbol{\alpha}) \end{array}\right\}$$

is bijective to the set (24) under the map sending $\boldsymbol{\alpha}_\varepsilon + \boldsymbol{a}$ to $\boldsymbol{\alpha} + \boldsymbol{a}$, and hence its Zariski closure equals $\boldsymbol{C}^d$. The Zariski closure of

$$(25) \setminus \bigcup_{\sigma \in \mathcal{F}_+(\boldsymbol{\alpha})} (F_\sigma = 0) \setminus \bigcup_{\sigma \in \mathcal{F}_-(\boldsymbol{\alpha})} \bigcup_{m \in \boldsymbol{N} \setminus F_\sigma(\boldsymbol{N}A)} (F_\sigma = m)$$

also equals $\boldsymbol{C}^d$. Hence $\boldsymbol{R}_{>0}(\boldsymbol{\alpha})$ is not empty. □

PROPOSITION 7.18. *If the Zariski closure of an equivalence class $[\boldsymbol{\alpha}]$ is the whole space $\boldsymbol{C}^d$, then $[\boldsymbol{\alpha}]$ is extreme, i.e.,*

$$[\boldsymbol{\alpha}] = \left\{\boldsymbol{\gamma} \in \boldsymbol{\alpha} + \boldsymbol{Z}^d \,;\, \begin{array}{ll} E_\tau(\boldsymbol{\gamma}) = \text{full} & \textit{for } \tau \in \mathrm{Face}(\boldsymbol{\alpha}), \\ E_\tau(\boldsymbol{\gamma}) = \emptyset & \textit{for } \tau \notin \mathrm{Face}(\boldsymbol{\alpha}) \end{array}\right\}.$$

PROOF. By Lemma 7.16 the equivalence class

$$[\boldsymbol{\alpha}_1] := \left\{\boldsymbol{\gamma} \in \boldsymbol{\alpha} + \boldsymbol{Z}^d \,;\, \begin{array}{ll} E_\tau(\boldsymbol{\gamma}) = \text{full} & \text{for } \tau \in \mathrm{Face}(\boldsymbol{\alpha}), \\ E_\tau(\boldsymbol{\gamma}) = \emptyset & \text{for } \tau \notin \mathrm{Face}(\boldsymbol{\alpha}) \end{array}\right\}$$

contains

$$\left\{\boldsymbol{\gamma} \in \boldsymbol{\alpha} + \boldsymbol{Z}^d \,;\, \begin{array}{ll} F_\sigma(\boldsymbol{\gamma}) \geq M & \text{for } \sigma \in \mathcal{F}_+(\boldsymbol{\alpha}), \\ F_\sigma(\boldsymbol{\gamma}) < 0 & \text{for } \sigma \in \mathcal{F}_-(\boldsymbol{\alpha}) \end{array}\right\} \tag{26}$$

for some M sufficiently large.

Suppose that $[\boldsymbol{\alpha}] \neq [\boldsymbol{\alpha}_1]$. Then $[\boldsymbol{\alpha}]$ does not belong to the set (26). Hence we have

$$[\boldsymbol{\alpha}] \subseteq \bigcup_{\sigma \in \mathcal{F}_+(\boldsymbol{\alpha})} \bigcup_{m=0}^{M-1} \{\boldsymbol{\gamma} \in \boldsymbol{\alpha} + \boldsymbol{Z}A \,;\, F_\sigma(\boldsymbol{\gamma}) = m\}\,.$$

This contradicts the assumption that the dimension of $\mathrm{ZC}([\boldsymbol{\alpha}])$ equals d. □

PROPOSITION 7.19 (cf. Proposition 3.3.1 in [15]). *Let* $\boldsymbol{\alpha} \in \boldsymbol{C}^d$. *Then* $\mathrm{Ann}(L(\boldsymbol{\alpha})) = 0$ *if and only if* $\mathrm{ZC}([\boldsymbol{\alpha}]) = \boldsymbol{C}^d$.

PROOF. Let $I := \mathrm{Ann}(L(\boldsymbol{\alpha}))$. Recall that we have

$$I_{\boldsymbol{a}} = t^{\boldsymbol{a}} \boldsymbol{I}(\Omega(\boldsymbol{a}) \cup \Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}))\,,$$

where

$$\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}) = \{\boldsymbol{\gamma} \,;\, \boldsymbol{\gamma} \sim \boldsymbol{\alpha},\ \boldsymbol{\gamma} + \boldsymbol{a} \sim \boldsymbol{\alpha}\}\,.$$

Since $I_{\boldsymbol{0}} = \boldsymbol{I}([\boldsymbol{\alpha}])$, the vanishing of I implies that $\mathrm{ZC}([\boldsymbol{\alpha}]) = \boldsymbol{C}^d$.

Next suppose that $\mathrm{ZC}([\boldsymbol{\alpha}]) = \boldsymbol{C}^d$. As in the proof of Proposition 7.18 there exists $M \in \boldsymbol{N}$ such that

$$[\boldsymbol{\alpha}] \supseteq \left\{\boldsymbol{\gamma} \in \boldsymbol{\alpha} + \boldsymbol{Z}^d \,;\, \begin{array}{ll} F_\sigma(\boldsymbol{\gamma}) \geq M & \text{for } \sigma \in \mathcal{F}_+(\boldsymbol{\alpha})\,, \\ F_\sigma(\boldsymbol{\gamma}) < 0 & \text{for } \sigma \in \mathcal{F}_-(\boldsymbol{\alpha}) \end{array}\right\}.$$

Hence

$$\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}) \supseteq \left\{\boldsymbol{\gamma} \in \boldsymbol{\alpha} + \boldsymbol{Z}^d \,;\, \begin{array}{ll} F_\sigma(\boldsymbol{\gamma}) \geq \max\{M, -F_\sigma(\boldsymbol{a})\} & \text{for } \sigma \in \mathcal{F}_+(\boldsymbol{\alpha})\,, \\ F_\sigma(\boldsymbol{\gamma}) < \min\{0, -F_\sigma(\boldsymbol{a})\} & \text{for } \sigma \in \mathcal{F}_-(\boldsymbol{\alpha}) \end{array}\right\}.$$

Since the right hand side is d-dimensional by Proposition 7.17, the Zariski closure $\mathrm{ZC}(\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}))$ is also d-dimensional. Hence $I_{\boldsymbol{a}} = 0$ for all $\boldsymbol{a} \in \boldsymbol{Z}^d$. □

PROPOSITION 7.20. *If* $[\boldsymbol{\alpha}]$ *is extreme, and* $\boldsymbol{R}_{>0}(\boldsymbol{\alpha})$ *is not empty, then* $\mathrm{ZC}([\boldsymbol{\alpha}]) = \boldsymbol{C}^d$.

PROOF. As in the proof of Proposition 7.18, $[\boldsymbol{\alpha}]$ contains

$$\left\{\boldsymbol{\gamma} \in \boldsymbol{\alpha} + \boldsymbol{Z}^d \,;\, \begin{array}{ll} F_\sigma(\boldsymbol{\gamma}) > M & \text{for } \sigma \in \mathcal{F}_+(\boldsymbol{\alpha}), \\ F_\sigma(\boldsymbol{\gamma}) < 0 & \text{for } \sigma \in \mathcal{F}_-(\boldsymbol{\alpha}) \end{array}\right\}$$

for some M sufficiently large. By the assumption, the dimension of

$$\left\{\boldsymbol{a} \in \boldsymbol{Z}^d \,;\, \begin{array}{ll} F_\sigma(\boldsymbol{a}) > M - F_\sigma(\boldsymbol{\alpha}) & \text{for } \sigma \in \mathcal{F}_+(\boldsymbol{\alpha}), \\ F_\sigma(\boldsymbol{a}) < -F_\sigma(\boldsymbol{\alpha}) & \text{for } \sigma \in \mathcal{F}_-(\boldsymbol{\alpha}) \end{array}\right\}.$$

equals d. Hence the proposition follows. □

THEOREM 7.21. *Let* $\boldsymbol{\alpha} \in \boldsymbol{C}^d$. *Then* $\mathrm{Ann}(L(\boldsymbol{\alpha})) = 0$ *if and only if* $[\boldsymbol{\alpha}]$ *is extreme, and* $\boldsymbol{R}_{>0}(\boldsymbol{\alpha})$ *is not empty.*

PROOF. This follows from Propositions 7.17, 7.18, and 7.20. □

THEOREM 7.22. *The algebra $D(R_A)$ is simple if and only if the following conditions are satisfied*:

(C1) *Any equivalence class is extreme.*

(C2) *For any $\boldsymbol{\alpha}$, $\boldsymbol{R}_{>0}(\boldsymbol{\alpha})$ is not empty.*

PROOF. This follows from Lemma 7.1 and Theorem 7.21. □

REMARK 7.23. To know whether $D(R_A)$ is simple or not, by Theorem 6.5, we need to check (C1) and (C2) only for finitely many $\boldsymbol{\alpha}$.

PROPOSITION 7.24. *If the semigroup NA is scored, then it satisfies Condition* (C1).

PROOF. First note that Condition (23) is satisfied in the scored case by Lemma 7.11.

Let $\boldsymbol{\lambda} \in \boldsymbol{C}(A \cap \tau)$ and $\boldsymbol{\alpha} - \boldsymbol{\lambda} \in \boldsymbol{Z}^d$. Suppose that $\sigma \in \mathcal{F}_+(\boldsymbol{\alpha})$ for all facets σ containing τ. We need to show $\boldsymbol{\lambda} \in E_\tau(\boldsymbol{\alpha})$. If a facet σ contains τ, then $\sigma \in \mathcal{F}_+(\boldsymbol{\alpha})$, or $F_\sigma(\boldsymbol{\alpha}) \in F_\sigma(NA)$. Hence $F_\sigma(\boldsymbol{\alpha} - \boldsymbol{\lambda}) \in F_\sigma(NA)$. If a facet σ does not contain τ, then there exists $\boldsymbol{a}_j \in A \cap \tau$ such that $F_\sigma(\boldsymbol{a}_j) > 0$. Hence $F_\sigma(\boldsymbol{\alpha} - \boldsymbol{\lambda} + m\boldsymbol{a}_j) \in F_\sigma(NA)$ for $m \in \boldsymbol{N}$ sufficiently large. Hence there exists $\boldsymbol{a} \in \boldsymbol{N}(A \cap \tau)$ such that $F_\sigma(\boldsymbol{\alpha} - \boldsymbol{\lambda} + \boldsymbol{a}) \in F_\sigma(NA)$ for all $\sigma \in \mathcal{F}$. Since NA is scored, we obtain $\boldsymbol{\alpha} - \boldsymbol{\lambda} + \boldsymbol{a} \in NA$. This means $\boldsymbol{\lambda} \in E_\tau(\boldsymbol{\alpha})$. □

THEOREM 7.25. *The algebra $D(R_A)$ is simple if and only if the semigroup NA is scored and satisfies Condition* (C2).

PROOF. This immediately follows from Theorem 7.22 and Proposition 7.24. □

COROLLARY 7.26. *Assume that the cone $\boldsymbol{R}_{\geq 0}A$ is simplicial. Then the algebra $D(R_A)$ is simple if and only if NA is scored.*

PROOF. In this case the cone $\boldsymbol{R}_{\geq 0}A$ has exactly d facets. Since the d F_σ's are linearly independent, Condition (C2) is satisfied. □

8. Examples.

EXAMPLE 8.1. Let

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 \end{pmatrix}.$$

Then NA is the set of black dots in Figure 1, and $\boldsymbol{R}_{\geq 0}A$ has two facets: $\sigma_2 := \boldsymbol{R}_{\geq 0}\boldsymbol{a}_2$ and $\sigma_3 := \boldsymbol{R}_{\geq 0}\boldsymbol{a}_3$. Their primitive integral support functions are $F_{\sigma_2}(s) = 2s_1 - s_2$ and $F_{\sigma_3}(s) = s_2$ respectively.

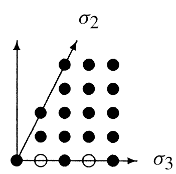

FIGURE 1. The semigroup NA.

Condition (C2) is satisfied, since $\boldsymbol{R}_{\geq 0}A$ is simplicial. However $D(R_A)$ is not simple, since $\boldsymbol{N}A$ is not scored. We have

$$\begin{aligned}
\{\boldsymbol{\alpha}\,;\ F_{\sigma_2}(\boldsymbol{\alpha}),\,F_{\sigma_3}(\boldsymbol{\alpha}) \in \boldsymbol{Z}\}/\boldsymbol{Z}^2 &= \{\boldsymbol{0},\,{}^t(1/2,0)\}\,,\\
\{\boldsymbol{\alpha}\,;\ F_{\sigma_2}(\boldsymbol{\alpha}) \in \boldsymbol{Z}\}/(\boldsymbol{Z}^2 + (F_{\sigma_2}=0)) &= \{{}^t(1/\sqrt{2},\sqrt{2})\}\,,\\
\{\boldsymbol{\alpha}\,;\ F_{\sigma_3}(\boldsymbol{\alpha}) \in \boldsymbol{Z}\}/(\boldsymbol{Z}^2 + (F_{\sigma_3}=0)) &= \{{}^t(\sqrt{2},0)\}\,,\\
\{\boldsymbol{\alpha}\,;\ \mathcal{F}(\boldsymbol{\alpha}) \supseteq \emptyset\}/\boldsymbol{C}^2 &= \{{}^t(\sqrt{2},\sqrt{3})\}\,.
\end{aligned}$$

Here we took a complete set of representatives as in the proof of Theorem 6.5. For instance, in $\{\boldsymbol{\alpha}\,;\ \mathcal{F}(\boldsymbol{\alpha}) \supseteq \emptyset\}/\boldsymbol{C}^2$, any $\boldsymbol{\alpha} \in \boldsymbol{C}^2$ represents the same element as $\boldsymbol{0}$ does. Since there exist $\boldsymbol{\alpha}$ such that $\mathcal{F}(\boldsymbol{\alpha}) = \emptyset$, we took such an element ${}^t(\sqrt{2},\sqrt{3})$ as a representative rather than $\boldsymbol{0}$.

First we classify $\boldsymbol{Z}^2$. Let $\boldsymbol{\alpha} \in \boldsymbol{Z}^2$. Then we see

- $E_{\boldsymbol{R}_{\geq 0}}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\}$,
- $E_{\sigma_2}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\} \Leftrightarrow 2\alpha_1 - \alpha_2 \geq 0$,
- $E_{\sigma_3}(\boldsymbol{\alpha}) = \{\boldsymbol{0},\,{}^t(1,0)\} \Leftrightarrow \alpha_2 \geq 1$,
- $E_{\sigma_3}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\} \Leftrightarrow \alpha_2 = 0, \alpha_1 \in 2\boldsymbol{Z}$,
- $E_{\sigma_3}(\boldsymbol{\alpha}) = \{{}^t(1,0)\} \Leftrightarrow \alpha_2 = 0, \alpha_1 \in 2\boldsymbol{Z}+1$,
- $E_{\{\boldsymbol{0}\}}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\} \Leftrightarrow \boldsymbol{\alpha} \in \boldsymbol{N}A$.

There are eight classes in $\boldsymbol{Z}^2$:

(a) $$\begin{aligned}&\{\boldsymbol{\alpha} \in \boldsymbol{Z}^2\,;\ E_{\sigma_2}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\},\ E_{\sigma_3}(\boldsymbol{\alpha}) = \{\boldsymbol{0},\,{}^t(1,0)\},\ E_{\{\boldsymbol{0}\}}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\}\}\\ &\quad = \{{}^t(\alpha_1,\alpha_2) \in \boldsymbol{Z}^2\,;\ \alpha_2 \geq 1,\ 2\alpha_1 - \alpha_2 \geq 0\}\,.\end{aligned}$$

(b) $$\begin{aligned}&\{\boldsymbol{\alpha} \in \boldsymbol{Z}^2\,;\ E_{\sigma_2}(\boldsymbol{\alpha}) = \emptyset,\ E_{\sigma_3}(\boldsymbol{\alpha}) = \{\boldsymbol{0},\,{}^t(1,0)\},\ E_{\{\boldsymbol{0}\}}(\boldsymbol{\alpha}) = \emptyset\}\\ &\quad = \{{}^t(\alpha_1,\alpha_2) \in \boldsymbol{Z}^2\,;\ \alpha_2 \geq 1,\ 2\alpha_1 - \alpha_2 < 0\}\,.\end{aligned}$$

(c) $$\begin{aligned}&\{\boldsymbol{\alpha} \in \boldsymbol{Z}^2\,;\ E_{\sigma_2}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\},\ E_{\sigma_3}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\},\ E_{\{\boldsymbol{0}\}}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\}\}\\ &\quad = \{{}^t(\alpha_1,\alpha_2) \in \boldsymbol{Z}^2\,;\ \alpha_2 = 0,\ \alpha_1 \in 2\boldsymbol{N}\}\,.\end{aligned}$$

(d) $$\begin{aligned}&\{\boldsymbol{\alpha} \in \boldsymbol{Z}^2\,;\ E_{\sigma_2}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\},\ E_{\sigma_3}(\boldsymbol{\alpha}) = \{{}^t(1,0)\},\ E_{\{\boldsymbol{0}\}}(\boldsymbol{\alpha}) = \emptyset\}\\ &\quad = \{{}^t(\alpha_1,\alpha_2) \in \boldsymbol{Z}^2\,;\ \alpha_2 = 0,\ \alpha_1 \in 2\boldsymbol{N}+1\}\,.\end{aligned}$$

(e) $$\begin{aligned}&\{\boldsymbol{\alpha} \in \boldsymbol{Z}^2\,;\ E_{\sigma_2}(\boldsymbol{\alpha}) = \emptyset,\ E_{\sigma_3}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\},\ E_{\{\boldsymbol{0}\}}(\boldsymbol{\alpha}) = \emptyset\}\\ &\quad = \{{}^t(\alpha_1,\alpha_2) \in \boldsymbol{Z}^2\,;\ \alpha_2 = 0,\ \alpha_1 \in 2(-\boldsymbol{N}-1)\}\,.\end{aligned}$$

(f) $$\begin{aligned}&\{\boldsymbol{\alpha} \in \boldsymbol{Z}^2\,;\ E_{\sigma_2}(\boldsymbol{\alpha}) = \emptyset,\ E_{\sigma_3}(\boldsymbol{\alpha}) = \{{}^t(1,0)\},\ E_{\{\boldsymbol{0}\}}(\boldsymbol{\alpha}) = \emptyset\}\\ &\quad = \{{}^t(\alpha_1,\alpha_2) \in \boldsymbol{Z}^2\,;\ \alpha_2 = 0,\ \alpha_1 \in -2\boldsymbol{N}-1\}\,.\end{aligned}$$

(g) $$\begin{aligned}&\{\boldsymbol{\alpha} \in \boldsymbol{Z}^2\,;\ E_{\sigma_2}(\boldsymbol{\alpha}) = \{\boldsymbol{0}\},\ E_{\sigma_3}(\boldsymbol{\alpha}) = \emptyset,\ E_{\{\boldsymbol{0}\}}(\boldsymbol{\alpha}) = \emptyset\}\\ &\quad = \{{}^t(\alpha_1,\alpha_2) \in \boldsymbol{Z}^2\,;\ \alpha_2 < 0,\ 2\alpha_1 - \alpha_2 \geq 0\}\,.\end{aligned}$$

(h) $$\{\boldsymbol{\alpha} \in \boldsymbol{Z}^2\,;\ E_{\sigma_2}(\boldsymbol{\alpha}) = \emptyset,\ E_{\sigma_3}(\boldsymbol{\alpha}) = \emptyset,\ E_{\{\boldsymbol{0}\}}(\boldsymbol{\alpha}) = \emptyset\} \\ = \{{}^t(\alpha_1, \alpha_2) \in \boldsymbol{Z}^2\,;\ \alpha_2 < 0,\ 2\alpha_1 - \alpha_2 < 0\}\,.$$

Let $\boldsymbol{\alpha}$ be not extreme, i.e., let $\boldsymbol{\alpha}$ belong to (c), (d), (e), or (f). Then $\mathrm{ZC}(\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a})) = \{\boldsymbol{\mu}\,;\ \mu_2 = 0\}$ if $a_2 = 0$ and $a_1 \in 2\boldsymbol{Z}$, and $\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a}) = \emptyset$ otherwise.

Let $\boldsymbol{\alpha}$ be extreme, i.e., let $\boldsymbol{\alpha}$ belong to (a), (b), (g), or (h). Then $\mathrm{ZC}(\Lambda_{[\boldsymbol{\alpha}]}(\boldsymbol{a})) = \boldsymbol{C}^2$ for all $\boldsymbol{a} \in \boldsymbol{Z}^2$.

Similarly ${}^t(\sqrt{2}, \sqrt{3}) + \boldsymbol{Z}^2$, ${}^t(1/\sqrt{2}, \sqrt{2}) + \boldsymbol{Z}^2$, ${}^t(\sqrt{2}, 0) + \boldsymbol{Z}^2$, and ${}^t(1/2, 0) + \boldsymbol{Z}^2$ have one, two, four, and eight equivalence classes, respectively. We see that $\boldsymbol{\alpha} = {}^t(\alpha_1, \alpha_2)$ is not extreme if and only if $\alpha_2 = 0$, and hence if and only if $\mathrm{Ann}\, L(\boldsymbol{\alpha}) = \mathrm{Ann}\, L(\boldsymbol{0})$.

Hence

$$\mathrm{Prim}(D(R_A)) = \{(0),\ \mathrm{Ann}\, L(\boldsymbol{0})\}\,.$$

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