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FUJITA'S VERY AMPLENESS CONJECTURE FOR SINGULAR TORIC VARIETIES

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Abstract. We present a self-contained combinatorial approach to Fujita's conjectures in the toric case. Our main new result is a generalization of Fujita's very ampleness conjecture for toric varieties with arbitrary singularities. In an appendix, we use similar methods to give a new proof of an analogous toric generalization of Fujita's freeness conjecture due to Fujino.

1. Introduction. Given an ample divisor D and any other Cartier divisor D' on an algebraic variety, we can choose t sufficiently large so that $tD + D'$ is basepoint free or very ample. In either case, it is not easy to say how large we must choose t in general. However, for the case where D' is in the canonical class K_X, Fujita made the following conjectures.

FUJITA'S CONJECTURES. *Let X be an n-dimensional projective algebraic variety, smooth or with mild singularities, D an ample divisor on X.*

(i) *For $t \geq n+1$, $tD + K_X$ is basepoint free.*
(ii) *For $t \geq n+2$, $tD + K_X$ is very ample.*

The case where X is $\boldsymbol{P}^n$ and D is a hyperplane shows that Fujita's conjectured bounds are best possible.

For smooth varieties, the corresponding statements with "basepoint free" and "very ample" replaced by "nef" and "ample", respectively, are consequences of Mori's Cone Theorem [Fuj]. For divisors on smooth *toric* varieties, nefness and ampleness are equivalent to freeness and very ampleness, respectively, so Fujita's conjectures follow immediately for smooth toric varieties. One can also deduce Fujita's conjectures for smooth toric varieties by general (non-toric) cohomological arguments of Ein and Lazarsfeld in characteristic zero [EL], and Smith in positive characteristic [Sm1, Sm2], again using the fact that ample divisors on smooth toric varieties are very ample.

For toric varieties with arbitrary singularities, a strong generalization of Fujita's freeness conjecture was proved by Fujino [Fu]. We follow the usual toric convention fixing $K_X = -\sum D_i$, the sum of the T-invariant prime divisors each with coefficient -1, as a convenient representative of the canonical class.

FUJINO'S THEOREM. *Let X be a projective n-dimensional toric variety not isomorphic to $\boldsymbol{P}^n$. Let D and D' be $\boldsymbol{Q}$-Cartier divisors such that $0 \geq D' \geq K_X$, $D + D'$ is Cartier, and $D \cdot C \geq n$ for all T-curves C. Then $D + D'$ is basepoint free.*

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Fujita's freeness conjecture for toric varieties is the special case of Fujino's Theorem when X is Gorenstein, $D' = K_X$, and $D = tL$ for some ample Cartier divisor L and some integer $t \geq n+1$. Fujino's Theorem shows that, for toric varieties, Fujita's conjectured bound can be improved by excluding the extremal case $X \cong \boldsymbol{P}^n$.

Of course, the case of $\boldsymbol{P}^n$ can be analyzed separately. The canonical divisor on $\boldsymbol{P}^n$ is linearly equivalent to $(-n-1)H$, so $D' \sim sH$ for some $0 \geq s \geq (-n-1)$. Any $\boldsymbol{Q}$-Cartier divisor D on $\boldsymbol{P}^n$ is linearly equivalent to tH for some $t \in \boldsymbol{Q}$. Then $D + D'$ is Cartier exactly when $t+s$ is an integer, and basepoint free exactly when $t+s$ is a nonnegative integer.

The main purpose of this paper is to prove an analogous generalization of Fujita's very ampleness conjecture for toric varieties with arbitrary singularities.

THEOREM 1. *Let X be a projective n-dimensional toric variety not isomorphic to $\boldsymbol{P}^n$. Let D and D' be $\boldsymbol{Q}$-Cartier divisors such that $0 \geq D' \geq K_X$, $D + D'$ is Cartier, and $D \cdot C \geq n+1$ for all T-curves C. Then $D + D'$ is very ample.*

The statement of Fujino's Theorem can be strengthened by removing the assumption that $D + D'$ is Cartier. A Cartier divisor on a toric variety is basepoint free if and only if it is nef, i.e., if and only if D intersects every curve nonnegatively [La, Proposition 1.5]. Without the hypothesis that $D + D'$ is Cartier, the sharper statement of Fujino's Theorem, which one may deduce from [Fu, Theorem 0.1], is then:

FUJINO'S THEOREM$^+$. *Let X be a projective n-dimensional toric variety not isomorphic to $\boldsymbol{P}^n$. Let D and D' be $\boldsymbol{Q}$-Cartier divisors such that $0 \geq D' \geq K_X$ and $D \cdot C \geq n$ for all T-curves C. Then $D + D'$ is nef.*

Similarly, the statement of Theorem 1 can be strengthened by using a toric characterization of very ampleness to remove the hypothesis that $D + D'$ is Cartier. Since every divisor on a toric variety is linearly equivalent to a T-invariant divisor, we may assume D and D' are $\boldsymbol{Q}$-linear combinations of T-invariant divisors. To state the stronger theorem, we need some notation from toric geometry.

Let $D = \sum d_i D_i$ be a T-$\boldsymbol{Q}$-Cartier divisor on a complete toric variety X. Let M the character lattice of T, and let $M_{\boldsymbol{Q}} = M \otimes \boldsymbol{Q}$. For each maximal cone σ in the fan defining X, we have a point $u_\sigma \in M_{\boldsymbol{Q}}$ determined by the conditions $\langle u_\sigma, v_i \rangle = -d_i$ for each of the primitive generators v_i of the rays of σ. When D and D' denote T-$\boldsymbol{Q}$-Cartier divisors, we will write u_σ and u'_σ for the points of $M_{\boldsymbol{Q}}$ associated to D and D', respectively. The association $D \rightsquigarrow u_\sigma$ is linear, i.e., $tD \rightsquigarrow tu_\sigma$ and $D + D' \rightsquigarrow u_\sigma + u'_\sigma$. A T-$\boldsymbol{Q}$-Cartier divisor D is Cartier if and only if $u_\sigma \in M$ for all maximal cones σ. Also associated to D is a polytope $P_D \subset M_{\boldsymbol{Q}}$ cut out by the inequalities $\langle u, v_i \rangle \geq -d_i$ for all of the primitive generators v_i of the rays of the fan. When D is T-$\boldsymbol{Q}$-Cartier and nef, the $\{u_\sigma\}$ are the vertices of P_D. If we translate P_D so that the vertex u_σ is at the origin, then all of P_D sits inside the dual cone $\sigma^\vee$. Write P_D^σ for this translation, i.e., $P_D^\sigma := P_D - u_\sigma$. In case D is Cartier, then D is very ample if and only if $P_D^\sigma \cap M$ generates the semigroup $\sigma^\vee \cap M$ for all maximal cones σ. Without the hypothesis that $D + D'$ is Cartier, the stronger version of Theorem 1 we will prove is then:

THEOREM 2. *Let X be a projective n-dimensional toric variety not isomorphic to $\boldsymbol{P}^n$. Let D and D' be T-$\boldsymbol{Q}$-Cartier divisors such that $0 \geq D' \geq K_X$ and $D \cdot C \geq n+1$ for all T-curves C. Then $P^{\sigma}_{D+D'} \cap M$ generates $\sigma^{\vee} \cap M$ for all maximal cones σ.*

Theorem 1 is the special case of Theorem 2 when $D + D'$ is Cartier.

Our approach starts from an observation made by Laterveer in [La]: if D is ample, then the lattice length of the edge of P_D corresponding to a T-curve C is precisely $D \cdot C$. This fact can be seen as a consequence of Riemann-Roch for toric varieties [Ful, p. 112]. Adding D' corresponds to moving the faces of P_D inward at most a unit distance with respect to the dual lattice. When all of the edges of P_D have lattice length at least $n+1$, we show that $P^{\sigma}_{D+D'}$ contains an explicit generating set for $\sigma^{\vee} \cap M$ for all maximal cones σ. The computations are straightforward in the simplicial case, as can be seen in the example at the end of the introduction.

In the proof of Fujino's Theorem, there is a simple reduction to the simplicial case (see [La, Lemma 2.4] or [Fu, 1.12, Step 2]). That reduction works via a partial projective resolution of singularities corresponding to a regular triangulation of the fan defining X. This type of reduction seems not to work for very ampleness. Instead, for each nonsimplicial maximal cone σ, we make a canonical subdivision of the dual cone $\sigma^{\vee}$.

The earliest results on Fujita's freeness conjecture for singular toric varieties of which the author is aware are due to Laterveer. In [La], Laterveer proved Fujino's Theorem for $\boldsymbol{Q}$-Gorenstein toric varieties when $D' = K_X$ using toric Mori theory, as developed in [Re]. Our statement of Theorem 1, like the statement of Fujino's Theorem, is influenced by Mustaţă's formulations in [Mu]. In particular, Mustaţă stated and proved Fujino's Theorem and Theorem 1 for smooth toric varieties when D and D' are Cartier as consequences of a characteristic-free vanishing theorem for toric varieties. For proofs of Fujino's Theorem that do not use vanishing theorems or toric Mori theory, see also [Lin] or the Appendix.

The only previous result on Fujita's very ampleness conjecture for singular toric varieties of which the author is aware is due to Lin,* who proved the conjecture for simplicial Gorenstein toric varieties in dimension ≤ 6 [Lin]. In [La], Laterveer also claimed to prove a generalization of Fujita's very ampleness conjecture for arbitrary $\boldsymbol{Q}$-Gorenstein toric varieties. As noted by Lin, there is an error in the proof of this claim. In particular, it is not true in general that P_{tD+K_X} contains $P_{(t-1)D}$. The case where X is $\boldsymbol{P}^n$ and D is a hyperplane is a counterexample. Nevertheless, Laterveer's approach to Fujita's very ampleness conjecture for singular toric varieties contains fruitful insights, in particular, the realization that $\boldsymbol{P}^n$ is the only toric extremal case and the characterization of the intersection numbers $D \cdot C$, for T-curves C, as the lattice lengths of the edges of P_D. The results we prove here are strong enough to imply all of the very ampleness results claimed in [La].

* Another effective very ampleness result for singular toric varieties, due to Ewald and Wessels [EW], may be stated as follows: let X be an n-dimensional projective toric variety and D a T-$\boldsymbol{Q}$-Cartier divisor on X such that $D \cdot C \geq n-1$ for all T-curves C. Then $P^{\sigma}_{D} \cap M$ generates $\sigma^{\vee} \cap M$ for all maximal cones σ. In particular, if D is Cartier, then D is very ample.

EXAMPLE. We illustrate the essential techniques of this paper in a concrete simplicial example. Let $u_1, \dots, u_n$ be linearly independent primitive vectors in M. Let P be the simplex with vertices $\{0, u_1, \dots, u_n\}$. Associated to P, there is a projective toric variety X_P with an ample divisor D such that $P = P_D$ [Ful, Section 1.5]. The vertex 0 of P corresponds to a maximal cone σ of the fan defining X_P whose dual cone $\sigma^\vee$ is spanned by $\{u_1, \dots, u_n\}$. Let $D_1, \dots, D_n$ be the divisors corresponding to the rays of σ, and let $D' = -D_1 - \cdots - D_n$. We will show that, for $t \geq n+1$, $P^\sigma_{tD+D'} \cap M$ generates $\sigma^\vee \cap M$.

Every point in M can be written uniquely as an integer linear combination of the $\{u_i\}$ plus a fractional part. So the semigroup $\sigma^\vee \cap M$ is generated by $\{0, u_1, \dots, u_n\}$ together with $\{(a_1u_1 + \cdots + a_nu_n) \in M \mid 0 \leq a_i < 1\}$. For $t \geq n+1$, we will show that $P^\sigma_{tD+D'}$ contains this generating set.

Define a linear function λ on $M_{\mathbf{Q}}$ by

$$\lambda(a_1u_1 + \cdots + a_nu_n) = a_1 + \cdots + a_n\,.$$

Note that $P_{tD} = \{u \in \sigma^\vee | \lambda(u) \leq t\}$. In other words, if $\{v_i\}$ are the primitive generators of the rays of σ, then P_{tD} is cut out by the conditions $\langle u, v_i\rangle \geq 0$ and the condition $\lambda(u) \leq t$. Similarly, $P_{tD+D'} = \{u'_\sigma + u \mid u \in \sigma^\vee, \lambda(u'_\sigma + u) \leq t\}$, i.e., $P_{tD+D'}$ is cut out by the conditions $\langle u, v_i\rangle \geq \langle u'_\sigma, v_i\rangle = 1$ and the condition $\lambda(u) \leq t$. It follows that any lattice point in P_{tD} that is in the interior of $\sigma^\vee$ is contained in $P_{tD+D'}$. Indeed, if u is in P_{tD}, then $\lambda(u) \leq t$, and if u is a lattice point in the interior of $\sigma^\vee$, then $\langle u, v_i\rangle$ is a positive integer.

Suppose $t \geq n+1$. Then $u_1 + \cdots + u_n$ is a lattice point in P_{tD} that is in the interior of $\sigma^\vee$, so $u_1 + \cdots + u_n$ is contained in $P_{tD+D'}$. Note that u'_σ is the point of $P_{tD+D'}$ for which λ achieves its minimum. In particular, $\lambda(u'_\sigma) \leq \lambda(u_1 + \cdots + u_n) = n$.

For each u_i, we have $\lambda(u'_\sigma + u_i) = \lambda(u'_\sigma) + 1 \leq n+1 \leq t$. Therefore $u'_\sigma + u_i \in P_{tD+D'}$, i.e., $u_i \in P^\sigma_{tD+D'}$. Given a lattice point p of the form $p = a_1u_1 + \cdots + a_nu_n$ with $0 \leq a_i < 1$, we have another lattice point $p' = (1-a_1)u_1 + \cdots + (1-a_n)u_n$ in P_{tD} that is in the interior of $\sigma^\vee$. So p' is contained in $P_{tD+D'}$, and therefore

$$\lambda(u'_\sigma) \leq \lambda(p') = n - \lambda(p)\,.$$

So $\lambda(u'_\sigma + p) \leq n < t$, and hence $p \in P^\sigma_{tD+D'}$, as required.

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2. Preliminaries. As a first step to proving Theorem 2, we have:

LEMMA 1. *Let X, D, and D' satisfy the hypotheses of Theorem 2. Let σ be a maximal cone, and let $\{u_1, \dots, u_s\}$ be the primitive generators of the rays of $\sigma^\vee$. Then $P^\sigma_{D+D'}$ contains $\{0, u_1, \dots, u_s\}$.*

PROOF. By Fujino's Theorem$^+$, $(n/(n+1))D + D'$ is nef. Therefore, for any T-curve C,

$$(D + D') \cdot C = \frac{1}{n+1}(D \cdot C) + \left(\frac{n}{n+1}D + D'\right) \cdot C \geq 1 .$$

By Laterveer's observation, this means that every edge of $P_{D+D'}$ has lattice length at least 1. Translating the vertex u_σ to the origin, it follows that $P^\sigma_{D+D'}$ contains 0 and the primitive generators of each of the rays of $\sigma^\vee$. □

If σ is regular, then $s = n$ and $\{0, u_1, \ldots, u_n\}$ generates $\sigma^\vee \cap M$, so the conclusion of Theorem 2, i.e., the fact that $P^\sigma_{D+D'} \cap M$ generates $\sigma^\vee \cap M$, follows immediately. In general, if we let $\Delta = \operatorname{conv}\{0, u_1, \ldots, u_s\}$, then Lemma 1 says that $P^\sigma_{D+D'}$ contains Δ. If σ is not regular, then Δ may not contain a generating set for $\sigma^\vee \cap M$. The following example, due to Ewald and Wessels [EW], illustrates this possibility.

EXAMPLE. Let $M = \mathbf{Z}^3$; $u_1 = (1, 0, 0)$, $u_2 = (0, 1, 0)$, and $u_3 = (1, 1, 2)$. Let $\sigma^\vee$ be the cone spanned by $\{u_1, u_2, u_3\}$, so $\Delta = \operatorname{conv}\{0, u_1, u_2, u_3\}$. Then $\Delta \cap M = \{0, u_1, u_2, u_3\}$, so the semigroup generated by $\Delta \cap M$ only contains lattice points whose third coordinate is even. In particular, the lattice point $(1, 1, 1) = (1/2)(u_1 + u_2 + u_3)$ is in $\sigma^\vee$, but not in the semigroup generated by $\Delta \cap M$.

Although Δ may not contain a generating set for $\sigma^\vee \cap M$, we will show that $P^\sigma_{D+D'}$ contains a dilation of Δ that does contain a generating set. Let $m = \min\{(D+D') \cdot V(\sigma \cap \tau)\}$, where τ varies over all maximal cones adjacent to σ, so that m is the minimum of the lattice lengths of the edges of $P^\sigma_{D+D'}$ incident to the vertex 0. Note that $m\Delta$ is the largest rational dilation of Δ contained in $P^\sigma_{D+D'}$. We will show that $m\Delta$ does contain a generating set for $\sigma^\vee \cap M$.

In preparation for proving this, we develop a few preliminaries. First, we generalize Laterveer's observation on the lattice lengths of the edges of P_D to the case where D is not necessarily ample.

LEMMA 2. *Let X be a complete toric variety and D a T-$\mathbf{Q}$-Cartier divisor on X. Let σ, τ be adjacent maximal cones in the fan defining X, and let u be the primitive generator of the ray of $\sigma^\vee$ perpendicular to $\sigma \cap \tau$. Then*

$$u_\tau \;=\; u_\sigma + (D \cdot V(\sigma \cap \tau))u \,.$$

PROOF. Since u_σ and u_τ agree on $\sigma \cap \tau$, their difference must vanish on $\sigma \cap \tau$, i.e., $u_\tau - u_\sigma = ku$ for some rational number k. By the toric intersection formulas in [Ful, Section 5.1], for v_j the primitive generator of any ray of τ not contained in σ,

$$D \cdot V(\sigma \cap \tau) = \frac{\langle u_\sigma - u_\tau, v_j \rangle}{-\langle u, v_j \rangle} \,.$$

Therefore,

$$D \cdot V(\sigma \cap \tau) = \frac{\langle ku, v_j \rangle}{\langle u, v_j \rangle} = k \,.$$ □

Now we develop some tools for working with rational cones. Let $\sigma^\vee$ be a strictly convex n-dimensional rational cone, and let $u_1, \ldots, u_s$ be the primitive generators of the rays of $\sigma^\vee$. Define a function $\lambda^{\min}$ on $\sigma^\vee$ by

$$\lambda^{\min}(u) = \min\{(a_1 + \cdots + a_s) \mid a_1 u_1 + \cdots + a_s u_s = u, a_i \geq 0\}\,.$$

Define $\lambda^{\max}$ similarly. A few combinatorial properties of $\lambda^{\min}$ and $\lambda^{\max}$, all of which are immediate from the definitions, will be useful in what follows.

First, $\lambda^{\min}$ and $\lambda^{\max}$ are anticonvex and convex, respectively. In other words, for any $u, u' \in \sigma^\vee$,

$$\lambda^{\min}(u + u') \leq \lambda^{\min}(u) + \lambda^{\min}(u')\,,$$

and similarly $\lambda^{\max}(u + u') \geq \lambda^{\max}(u) + \lambda^{\max}(u')$.

Second, suppose the restriction of $D' = \sum d'_i D_i$ to the affine open U_σ is minus-effective, i.e., for each of the primitive generators v_i of the rays of σ, $d'_i \leq 0$. Then $\langle u'_\sigma, v_i\rangle = -d'_i \geq 0$. So u'_σ is in the dual cone $\sigma^\vee$. In particular, $\lambda^{\min}(u'_\sigma)$ and $\lambda^{\max}(u'_\sigma)$ are well-defined.

Finally, with $\Delta = \mathrm{conv}\{0, u_1, \ldots, u_s\}$, note that

$$m\Delta \,=\, \{u \in \sigma^\vee \mid \lambda^{\min}(u) \leq m\}\,.$$

The distinction between $\lambda^{\min}$ and $\lambda^{\max}$ is meaningful only in the nonsimplicial case; when σ is simplicial, then the primitive generators of the rays of $\sigma^\vee$ are linearly independent, so the expression $u = a_1 u_1 + \cdots + a_n u_n$ is unique.

In order to show that $m\Delta$ contains a generating set for $\sigma^\vee \cap M$, one seeks lower bounds for m. To get a rough idea of how one might get such bounds, imagine that P_D is very large, as it will be under the hypotheses of Theorem 2. When we add a small, minus-effective divisor D' to D, we get $P_{D+D'}$ by moving the faces of P_D in a small distance. The main idea is to control the decrease in the lengths of the edges as the faces move in. After the faces containing u_σ move in a small distance, the new vertex $u_\sigma + u'_\sigma$ of $P_{D+D'}$ will be inside P_D and a small distance from the old vertex u_σ of P_D. We can measure this distance by $\lambda^{\min}(u'_\sigma)$. Suppose that P_D contains $u_\sigma + t\Delta$ for some large positive t. Looking out from the new vertex $u_\sigma + u'_\sigma$ in the direction of the ray spanned by u_i, we see that P_D contains $u_\sigma + u'_\sigma + bu_i$ for $0 \leq b \leq t - \lambda^{\min}(u'_\sigma)$. Now we want to know what portion of this segment is actually contained in $P_{D+D'}$. This will depend on how far the faces cutting off the other end of the edge move in. If these faces move in a distance r with respect to the dual lattice, then the resulting edge of $P_{D+D'}$ will have length at least $t - \lambda^{\min}(u'_\sigma) - r$. The key to giving lower bounds for m will be the following proposition, which makes the essence of this discussion precise.

PROPOSITION. *Let X be a complete toric variety, and σ a maximal cone in the fan defining X. Let D and D' be T-$\mathbf{Q}$-Cartier divisors such that D is nef and $0 \geq D' \geq K_X$. Let $t = \min\{D \cdot V(\sigma \cap \tau)\}$ and $m = \min\{(D + D') \cdot V(\sigma \cap \tau)\}$, where τ varies over all maximal cones adjacent to σ. Suppose $t \geq \lambda^{\min}(u'_\sigma)$. Then*

$$m \geq t - \lambda^{\min}(u'_\sigma) - 1\,.$$

PROOF. Although we will only use the proposition as stated, we will prove somewhat more. We replace the global condition $0 \geq D' \geq K_X$ by the following "local" conditions near σ:

1. The restriction of D' to U_σ is minus-effective, i.e., $d'_i \leq 0$ for each primitive generator v_i of a ray of σ.

2. There is a positive rational number r and, for each maximal cone τ adjacent to σ, a primitive generator v_j of a ray in $\tau \setminus \sigma$ such that $d'_j \geq -r$.

Under these revised hypotheses, we will show that $m \geq t - \lambda^{\min}(u'_\sigma) - r$. In the case where $0 \geq D' \geq K_X$, the conditions hold for $r = 1$, so the proposition as stated will follow.

Let τ be a maximal cone adjacent to σ. Let u be the primitive generator of the ray of $\sigma^\vee$ perpendicular to $\sigma \cap \tau$, and let v_j be the primitive generator of a ray in $\tau \setminus \sigma$ such that $d'_j \geq -r$. Let $c = t - \lambda^{\min}(u'_\sigma)$, and let $k = (D + D') \cdot V(\sigma \cap \tau)$. We aim to show $c - k \leq r$.

First, we claim that $u_\sigma + u'_\sigma + cu$ is in P_D. Since D is nef, P_D contains $u_\sigma + t\Delta$, so it will suffice to show $\lambda^{\min}(u'_\sigma + cu) \leq t$. Now, cu is in $\sigma^\vee$ and $\lambda^{\min}(cu) = c$, so

$$\lambda^{\min}(u'_\sigma + cu) \leq \lambda^{\min}(u'_\sigma) + c = t\,.$$

This proves that $u_\sigma + u'_\sigma + cu$ is in P_D. Therefore, we have

$$\langle u_\sigma + u'_\sigma + cu, v_j\rangle \geq -d_j\,. \tag{1}$$

Next, recall that $u_\sigma + u'_\sigma$ and $u_\tau + u'_\tau$ are the points of $M_{\boldsymbol{Q}}$ associated to $D + D'$ for σ and τ, respectively. By Lemma 2,

$$u_\tau + u'_\tau = u_\sigma + u'_\sigma + ku\,.$$

So,

$$\langle u_\sigma + u'_\sigma + ku, v_j\rangle = \langle u_\tau + u'_\tau, v_j\rangle = -d_j - d'_j \leq -d_j + r\,. \tag{2}$$

Subtracting (2) from (1), we have $(c - k)\langle u, v_j\rangle \geq -r$. Since $\langle u, v_j\rangle$ is a negative integer, it follows that $c - k \leq r$. □

REMARK. The conclusion of the proposition is false in general if $t < \lambda^{\min}(u'_\sigma)$. Consider, for example, the complete toric surface X whose fan is spanned by three rays, the primitive generators of which satisfy $v_1 + v_2 + 2v_3 = 0$. (X is isomorphic to the weighted projective plane $\boldsymbol{P}(1, 1, 2)$.) Let σ be the cone spanned by v_2 and v_3. Taking T-$\boldsymbol{Q}$-Cartier divisors $D = D_1$ and $D' = K_X$, one computes $t = D \cdot D_2 = (1/2)$, $\lambda^{\min}(u'_\sigma) = 2$, and $(D + D') \cdot D_2 = -3$, which is strictly less than $(1/2) - 2 - 1$.

The proposition gives good lower bounds for m, provided we can give good upper bounds for $\lambda^{\min}(u'_\sigma)$. We will get sufficient bounds indirectly by using convexity to bound $\lambda^{\max}(u'_\sigma)$.

LEMMA 3. *Let D' be T-$\boldsymbol{Q}$-Cartier, with $0 \geq D' \geq K_X$. Then $\lambda^{\max}(u'_\sigma) \leq \lambda^{\max}(u)$ for any lattice point u in the interior of $\sigma^\vee$.*

PROOF. For any lattice point u in the interior of $\sigma^\vee$, and for the primitive generator v_j of any ray of σ, $\langle u, v_j\rangle$ is a positive integer. Now $\langle u'_\sigma, v_j\rangle = -d'_j$, which, since $D' \geq K_X$, is at most 1. Therefore $u - u'_\sigma$ is in $\sigma^\vee$. Since $\lambda^{\max}$ is convex and nonnegative on $\sigma^\vee$, it follows that $\lambda^{\max}(u'_\sigma) \le \lambda^{\max}(u)$. □

3. Proof of Theorem 2. Let X, D, and D' satisfy the hypotheses of Theorem 2, and let σ be a maximal cone in the fan defining X. Let $m = \min\{(D + D') \cdot V(\sigma \cap \tau)\}$, where τ varies over all maximal cones adjacent to σ. Let $\{u_1, \dots, u_s\}$ be the primitive generators of the rays of $\sigma^\vee$, and let $\Delta = \mathrm{conv}\{0, u_1, \dots, u_s\}$. To prove Theorem 2, it will suffice to show that $m\Delta$ contains a generating set for $\sigma^\vee \cap M$. To prove this, we will give a canonical subdivision of $\sigma^\vee$ and show that, for each maximal cone γ of the subdivision, $\gamma \cap m\Delta$ contains a generating set for $\gamma \cap M$.

We claim that $\lambda^{\max}$ is piecewise-linear and therefore defines a canonical subdivision of $\sigma^\vee$: the subdivision whose maximal cones are the maximal subcones of $\sigma^\vee$ on which $\lambda^{\max}$ is linear. This subdivision can also be realized by looking at $Q = \mathrm{conv}\{u_1, \dots, u_s\}$ and taking the cones over the "lower faces" of Q, i.e., the faces of Q visible from the vertex 0 of $\sigma^\vee$. Indeed, for any $t > 0$, tQ is the set of u in $\sigma^\vee$ that can be written $u = a_1u_1 + \cdots + a_su_s$ with $a_i \ge 0$ and $a_1 + \cdots + a_s = t$. Now the points in the lower faces of Q are precisely those points that are not contained in tQ for any $t > 1$. So the restriction of $\lambda^{\max}$ to the lower faces of Q is identically 1. Since $\lambda^{\max}(cu) = c\lambda^{\max}(u)$ for any $c \ge 0$, it follows that $\lambda^{\max}$ is linear precisely on the cones over the lower faces of Q.

Let γ be the cone over a maximal lower face of Q, and let $\gamma^{(1)} \subset \{u_1, \dots, u_s\}$ denote the set of primitive generators of the rays of γ. We must show that $\gamma \cap m\Delta$ contains a generating set for $\gamma \cap M$. Every point of γ can be written as a nonnegative linear combination:

$$u = a_1u_{i_1} + \cdots + a_nu_{i_n}\,,$$

where $a_j \ge 0$, and $\{u_{i_j}\} \subset \gamma^{(1)}$ is linearly independent. This expression can be decomposed as a nonnegative integer combination of the $\{u_{i_j}\}$ plus a nonnegative fractional part. So $\gamma \cap M$ is generated by 0 and $\gamma^{(1)}$ together with $\{(a_1u_{i_1} + \cdots + a_nu_{i_n}) \in M \mid 0 \le a_j < 1, \{u_{i_j}\} \subset \gamma^{(1)}$ linearly independent$\}$. By Lemma 1, $m\Delta$ contains 0 and $\gamma^{(1)}$. It will therefore suffice to show that any lattice point p that is a nonnegative fractional linear combination of some independent set $\{u_{i_j}\} \subset \gamma^{(1)}$ is contained in $m\Delta$. For this, it will suffice to show that $m \ge \lambda^{\max}(p)$.

Suppose $p = a_1u_{i_1} + \cdots + a_nu_{i_n} \in M$, where $0 \le a_j < 1$, and $\{u_{i_j}\} \subset \gamma^{(1)}$ is linearly independent. Then $p' = (1 - a_1)u_{i_1} + \cdots + (1 - a_n)u_{i_n}$ is a lattice point in the interior of $\sigma^\vee$. By Lemma 3, $\lambda^{\max}(u'_\sigma) \le \lambda^{\max}(p')$, and since $\lambda^{\max}$ is linear on γ and $\lambda^{\max}(u_{i_j}) = 1$, we have

$$\lambda^{\max}(p') = (1 - a_1) + \cdots + (1 - a_n) = n - \lambda^{\max}(p)\,.$$

Therefore,

$$\lambda^{\max}(u'_\sigma) \ \le n - \lambda^{\max}(p)\,.$$

Let t be as in the Proposition, i.e., $t = \min\{D \cdot V(\sigma \cap \tau)\}$, where τ varies over all maximal cones adjacent to σ. Then $t \ge n + 1 > \lambda^{\min}(u'_\sigma)$, so we can apply the Proposition with $r = 1$

to obtain

$$
\begin{aligned}
m &\geq n + 1 - \lambda^{\min}(u'_\sigma) - 1\,. \\
&\geq n - \lambda^{\max}(u'_\sigma)\,. \\
&\geq \lambda^{\max}(p)\,.
\end{aligned}
\qquad \square
$$

REMARK. The collection of cones over the lower faces of Q is an example of what is called a "regular subdivision". In general, a regular subdivision of a cone is constructed by choosing a nonzero point on each of the rays of the cone, and perhaps specifying some additional rays inside the cone with nonzero points on them as well. One looks at the convex hull of all of these points and then takes the cones over all of the lower faces. For more details on regular subdivisions of convex polytopes, see [Lee] and [Zi]. The translation from polytopes to cones is straightforward.

In the toric literature, regular subdivisions have generally been applied to the fan defining a toric variety, and sometimes to the polytope defining an ample line bundle. See, for instance, [OP], [GKZ, Chapter 7], and [KKMS, §I.2]. The regular subdivisions that we have used in this paper are of the dual cones $\{\sigma^\vee\}$. The author is not aware of any significant geometric interpretation for these subdivisions.

The subdivisions of a fan Σ correspond naturally and bijectively to the proper birational toric morphisms $\tilde{X} \to X(\Sigma)$ [Ful, Section 2.5], and the regular subdivisions of Σ are precisely those for which the corresponding morphism is projective. A regular subdivision of Σ is obtained by specifying a continuous function Ψ on the support of Σ that is convex and piecewise-linear on each cone. By subdividing each cone of Σ into the maximal subcones on which Ψ is linear, we get a projective birational morphism for which Ψ is the piecewise-linear function associated to a relatively ample T-$\boldsymbol{Q}$-Cartier divisor on $\tilde{X}$. In particular, if $D = \sum d_i D_i$ is a T-$\boldsymbol{Q}$-Weil divisor on X, and if we define $\Psi_D^{\max}$ on each maximal cone σ by

$$
\Psi_D^{\max}(v) = \max\left\{ \sum_{v_i \in \sigma} -a_i d_i \;\middle|\; \sum_{v_i \in \sigma} a_i v_i = v,\, a_i \geq 0 \right\},
$$

then we get the unique projective birational morphism $\pi : \tilde{X} \to X$ such that the proper transform of D is $\boldsymbol{Q}$-Cartier and relatively ample, and π is an isomorphism in codimension 1. If D is effective (resp. minus effective) then, for each maximal cone σ, the same subdivision is obtained by looking at $\operatorname{conv}\{(1/d_i)v_i \mid v_i \in \sigma\}$ (resp. $\operatorname{conv}\{-(1/d_i)v_i, \mid v_i \in \sigma\}$) and taking the cones over the upper faces (resp. lower faces). Note that, for a subdivision of a fan to be regular, it is not enough for the subdivision to be regular on each cone. This is the toric manifestation of the fact that quasiprojectivity is not local on the base (see [EGA, II.5.3]).

4. Appendix: Proof of Fujino's Theorem^{+}. The ideas and techniques of the main part of this paper also give a new proof of Fujino's Theorem^{+}. This yields a unified combinatorial approach to Fujita's conjectures for toric varieties with arbitrary singularities, which is independent of vanishing theorems and toric Mori theory.

From the Proposition, we can immediately deduce a generalization of Fujita's freeness conjecture for toric varieties with arbitrary singularities. The statement we get in this way is similar to Fujino's Theorem$^+$, but without the improved bound obtained by excluding the extremal case where X is $\boldsymbol{P}^n$.

COROLLARY TO PROPOSITION. *Let X be a projective n-dimensional toric variety. Let D and D' be $\boldsymbol{Q}$-Cartier divisors such that $0 \geq D' \geq K_X$ and $D \cdot C \geq n+1$ for all T-curves C. Then $D + D'$ is nef.*

PROOF. We may assume that D and D' are T-$\boldsymbol{Q}$-Cartier. By the Proposition, it suffices to show that $\lambda^{\min}(u'_\sigma) \leq n$. Write $u'_\sigma = a_1u_1 + \cdots + a_nu_n$, where $a_i \geq 0$ and the u_i are linearly independent primitive generators of rays of $\sigma^\vee$. The condition $D' \geq K_X$ implies that each $a_i \leq 1$. So $\lambda^{\min}(u'_\sigma) \leq a_1 + \cdots + a_n \leq n$. □

To prove Fujino's Theorem$^+$, it remains to show that the bound on the intersection numbers can be improved by one by excluding the case where $X \cong \boldsymbol{P}^n$.

Using the Proposition, we can work "locally," considering one maximal cone σ at a time. The following lemma allows us to reduce to the case where σ is regular.

LEMMA 4. *Let D' be a T-$\boldsymbol{Q}$-Cartier divisor, $0 \geq D' \geq K_X$. If σ is a maximal cone in the fan defining X that is not regular, then $\lambda^{\min}(u'_\sigma) \leq n-1$.*

PROOF. By Lemma 3, it will suffice to show that there is a lattice point p in the interior of $\sigma^\vee$ such that $\lambda^{\max}(p) \leq n-1$. Since σ is not regular, $\sigma^\vee$ is not regular either. Consider two cases, according to whether $\sigma^\vee$ is simplicial.

Suppose $\sigma^\vee$ is simplicial, and let $u_1, \ldots, u_n$ be the primitive generators of the rays of $\sigma^\vee$. Since $\sigma^\vee$ is not regular, after possibly renumbering the u_i, there is a lattice point $u = a_1u_1 + \cdots + a_ru_r$ in M, where $0 < a_i < 1$ and $r \geq 2$. Then $p = u_1 + \cdots + u_n - u$ and $p' = u + u_{r+1} + \cdots + u_n$ are lattice points in the interior of $\sigma^\vee$. Now $\lambda^{\max}(p) + \lambda^{\max}(p') = 2n - r \leq 2n - 2$. So $\min\{\lambda^{\max}(p), \lambda^{\max}(p')\} \leq n-1$, as required.

Suppose $\sigma^\vee$ is not simplicial. Let $\{u_1, \ldots, u_n\}$ be a set of linearly independent primitive generators of rays in some subcone of $\sigma^\vee$ on which $\lambda^{\max}$ is linear. Let γ be the cone spanned by $\{u_1, \ldots, u_n\}$. Since $\gamma \subsetneq \sigma^\vee$, at least one of the facets of γ is not contained in a face of $\sigma^\vee$. Say τ, spanned by $\{u_1, \ldots, u_{n-1}\}$, is not contained in a face of $\sigma^\vee$. Then the relative interior of τ is contained in the interior of $\sigma^\vee$. In particular, $p = u_1 + \cdots + u_{n-1}$ is a lattice point in the interior of $\sigma^\vee$. Since $\lambda^{\max}$ is linear on γ and $\lambda^{\max}(u_i) = 1$, it follows that $\lambda^{\max}(p) = n-1$. □

PROOF OF FUJINO'S THEOREM$^+$. Let X, D, and D' satisfy the hypotheses of Fujino's Theorem$^+$. Let σ be a maximal cone in the fan defining X. Let $t = \min\{D \cdot V(\sigma \cap \tau)\}$ and $m = \min\{(D + D') \cdot V(\sigma \cap \tau)\}$, where τ varies over all maximal cones adjacent to σ. It will suffice to show that $m \geq 0$. If σ is not regular, then, by Lemma 4, $\lambda^{\min}(u'_\sigma) \leq n-1$. Applying the Proposition, we have $m \geq t - n \geq 0$, as required.

We may therefore assume that σ is regular. Let $v_1, \dots, v_n$ be the primitive generators of the rays of σ, and let $u_1, \dots, u_n$ be the dual basis. In particular, $u_1, \dots, u_n$ are the primitive generators of the rays of $\sigma^\vee$. By adding to D the numerically trivial divisor $\sum \langle u_\sigma, v_i \rangle D_i$, we may assume that $u_\sigma = 0$, and hence $P_D^\sigma = P_D$. Consider two cases, according to whether P_D is a simplex.

Case 1: *P_D is a simplex.* In this case, $\mathrm{Pic}(X) \cong \boldsymbol{Z}$. We may linearly order $\mathrm{Pic} X \otimes \boldsymbol{Q}$ so that $[D]$ is positive and the nef divisor classes are exactly those that are greater than or equal to zero. We aim to show that $[D + D'] \geq 0$. It will suffice to show that $P_{D+D'}$ is nonempty. In fact, we will show that $p = u_1 + \cdots + u_n$ is in $P_{D+D'}$. Furthermore, since $[D] \geq [(n/t)D]$, it will suffice to prove this in the case where $t = n$.

Let v_0 be the primitive generator of the unique ray of the fan defining X that is not in σ. Now $P_{D+D'}$ is cut out by the inequalities $\langle u, v_i \rangle \geq -d_i'$ for $1 \leq i \leq n$, and $\langle u, v_0 \rangle \geq -d_0 - d_0'$. For $1 \leq i \leq n$, we have $\langle p, v_i \rangle = 1 \geq -d_i'$. So it will suffice to show that $\langle p, v_0 \rangle \geq -d_0 - d_0'$.

Write $P_D = \mathrm{conv}\{0, a_1 u_1, \dots, a_n u_n\}$. After possibly renumbering, we may assume $a_1 = \min\{a_i\} = n$. Furthermore, one of the a_i must be strictly greater than n (otherwise P_D would be a regular simplex and so X would be isomorphic to $\boldsymbol{P}^n$). The ray spanned by v_0, which is perpendicular to the face of P_D not containing 0, is also spanned by

$$v = -a_2 \cdots a_n v_1 - \cdots - a_1 \cdots \hat{a_i} \cdots a_n v_i - \cdots - a_1 \cdots a_{n-1} v_n .$$

So $v_0 = bv$ for some positive rational number b. Since v_0 is a lattice point, $ba_1 \cdots \hat{a_i} \cdots a_n$ must be an integer for each i. In particular, $ba_2 \cdots a_n$ is an integer. Therefore,

$$d_0 = -\langle a_1 u_1, v_0 \rangle = ba_1 \cdots a_n = nba_2 \cdots a_n$$

is an integer.

Now, since $a_1 = \min\{a_i\} = n$ and some $a_i > n$, we have

$$\langle p, v_0 \rangle = b \langle p, v \rangle > -nba_2 \cdots a_n = -d_0 .$$

Since both $\langle p, v_0 \rangle$ and $-d_0$ are integers, their difference must be at least 1. So $\langle p, v_0 \rangle \geq -d_0 + 1 \geq -d_0 - d_0'$.

Case 2: *P_D is not a simplex.* Since σ is simplicial but P_D is not a simplex, P_D has a vertex u_0 that is not adjacent to $u_\sigma = 0$. Define a piecewise linear function λ on $\sigma^\vee$ by

$$\lambda(u) = \min\{(na_0 + a_1 + \cdots + a_n) \mid a_0 u_0 + \cdots + a_n u_n = u, a_i \geq 0\} .$$

Now P_D contains $\mathrm{conv}\{0, u_0, nu_1, \dots, nu_n\} = \{u \in \sigma^\vee \mid \lambda(u) \leq n\}$. An argument identical to the proof of the Proposition shows that $m \geq n - \lambda(u_\sigma') - 1$. It will therefore suffice to show that $\lambda(u_\sigma') \leq n - 1$.

After possibly renumbering, we may write $u_0 = b_1 u_1 + \cdots + b_r u_r$, where $b_i > 0$ and $r \geq 2$. We claim that $b_i \geq n$. Indeed, the rays along the edges of P_D coming out from u_0 span a translated cone containing P_D, and hence containing 0. Since $\langle u_0, v_i \rangle = b_i > 0$, there must be some vertex u of P_D adjacent to u_0 such that $\langle u, v_i \rangle < \langle u_0, v_i \rangle$. Let C be the T-curve corresponding to the edge connecting u and u_0. Since $u \in P_D \subset \sigma^\vee$, we have

$b_i = \langle u_0, v_i\rangle \geq \langle u_0 - u, v_i\rangle$. Let u' be the primitive generator of the ray spanned by $u_0 - u$. By Lemma 2, we have

$$\langle u_0 - u, v_i\rangle \geq (D \cdot C)\langle u', v_i\rangle \geq D \cdot C \geq n\,.$$

This proves the claim.

Let c_0 be the largest rational number such that $u'_\sigma - c_0 u_0$ is in $\sigma^\vee$, i.e.,

$$c_0 = \min\{-d'_i/b_i \mid 1 \leq i \leq r\}\,.$$

So we may write

$$u'_\sigma = c_0 u_0 + \cdots + c_n u_n\,,$$

where $c_i \geq 0$ and some $c_i = 0$ for $i \geq 1$. Say $c_1 = 0$. We claim that $nc_0 + c_2 \leq 1$. Indeed,

$$nc_0 + c_2 \leq b_2 c_0 + c_2 = -d'_2 \leq 1\,.$$

Therefore,

$$\lambda(u'_\sigma) \leq nc_0 + c_2 + \cdots + c_n \leq 1 - d'_3 - \cdots - d'_n \leq n - 1\,. \qquad \square$$

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