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A SPLITTING THEOREM FOR PROPER COMPLEX EQUIFOCAL SUBMANIFOLDS

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Abstract. In this paper, we define the notion of the complex Coxeter group associated with a proper complex equifocal submanifold in a symmetric space of non-compact type. We prove that a proper complex equifocal submanifold is decomposed into a non-trivial (extrinsic) product of two such submanifolds if and only if its associated complex Coxeter group is decomposable. Its proof is performed by showing a splitting theorem for an infinite-dimensional proper anti-Kaehlerian isoparametric submanifold.

1. Introduction. In 1995, the notion of an equifocal submanifold in a symmetric space was defined as a submanifold with globally flat and abelian normal bundle such that the focal radii for each parallel normal vector field are constant [12]. This notion is a generalization of isoparametric submanifolds in an Euclidean space and isoparametric hypersurfaces in a sphere or a hyperbolic space. The investigation of equifocal submanifolds in a symmetric space of compact type is reduced to that of isoparametric submanifolds in a (separable) Hilbert space through a Riemannian submersion $\tilde{\phi}$ of a Hilbert space onto the symmetric space. Concretely, a submanifold M in the symmetric space is equifocal if and only if each component of $\tilde{\phi}^{-1}(M)$ is isoparametric (see [12]). For each equifocal submanifold M in a symmetric space of compact type, a Coxeter group is defined as a discrete group generated by reflections with respect to hyperplanes in the normal space $T^{\perp}_x M$ whose images under the normal exponential map constitute the focal set of (M, x) (where x is an arbitrary point of M). Similarly, a Coxeter group is defined for each isoparametric submanifold in a Hilbert space. Note that the Coxeter groups associated with the equifocal submanifold M and the isoparametric submanifold $\tilde{\phi}^{-1}(M)$ are isomorphic. In 1997, Heintze and Liu [4] showed that an isoparametric submanifold in a Hilbert space is decomposed into a non-trivial (extrinsic) product of two such submanifolds if and only if the associated Coxeter group is decomposable. In 1998, by using this splitting theorem of Heintze-Liu, Ewert [2] showed that an equifocal submanifold in a simply connected symmetric space of compact type is decomposed into a non-trivial (extrinsic) product of two such submanifolds if and only if the associated Coxeter group is decomposable.

For non-compact submanifolds in a symmetric space of non-compact type, the equifocality is a rather weak condition (see [3, 8]). So, we have recently introduced the stronger condition of complex equifocality for submanifolds in the symmetric space [7]. Note that

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isoparametric hypersurfaces in a hyperbolic space are complex equifocal. Furthermore, we defined the notion of a proper complex equifocal submanifold as a subclass of the class consisting of complex equifocal submanifolds [9]. Let G/K be a symmetric space of non-compact type, where we assume that G is a connected semi-simple Lie group admitting a faithful linear representation and that K is a maximal compact subgroup of G. As G admits a faithful linear representation, we can define the complexificaton $G^{\mathbf{c}}$ (respectively $K^{\mathbf{c}}$) of G (respectively K). Let $\tilde{G}^{\mathbf{c}}$ be the universal covering of $G^{\mathbf{c}}$ and $\tilde{K}^{\mathbf{c}}$ be the connected subgroup of $\tilde{G}^{\mathbf{c}}$ corresponding to $K^{\mathbf{c}}$. Then $(\tilde{G}^{\mathbf{c}}, \tilde{K}^{\mathbf{c}})$ is a symmetric pair and $\tilde{G}^{\mathbf{c}}/\tilde{K}^{\mathbf{c}}$ is a simply connected (pseudo-Riemannian) symmetric space. Also, $\tilde{G}^{\mathbf{c}}/\tilde{K}^{\mathbf{c}}$ is an anti-Kaehlerian manifold in a natural manner. We call this anti-Kaehlerian manifold $\tilde{G}^{\mathbf{c}}/\tilde{K}^{\mathbf{c}}$ the *anti-Kaehlerian symmetric space associated with* G/K. For simplicity, we denote $\tilde{G}^{\mathbf{c}}$ and $\tilde{K}^{\mathbf{c}}$ by $G^{\mathbf{c}}$ and $K^{\mathbf{c}}$, respectively. For a complete C^{ω}-submanifold M in G/K, we defined its extrinsic complexification $M^{\mathbf{c}}$ as an anti-Kaehlerian submanifold in $G^{\mathbf{c}}/K^{\mathbf{c}}$, where C^{ω} means real analyticity [8]. Also, we defined an anti-Kaehlerian submersion $\tilde{\phi}^{\mathbf{c}}$ of an infinite-dimensional anti-Kaehlerian space $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ onto $G^{\mathbf{c}}/K^{\mathbf{c}}$, where $\mathfrak{g}^{\mathbf{c}}$ is the Lie algebra of $G^{\mathbf{c}}$ and $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ is the space of all paths which are L^2-integrable with respect to an inner product of $\mathfrak{g}^{\mathbf{c}}$ defined in a natural manner [8]. We showed that the following three conditions are equivalent [8]:

(i) M is complex equifocal;

(ii) $M^{\mathbf{c}}$ is anti-Kaehlerian equifocal;

(iii) each component of $\tilde{\phi}^{\mathbf{c}-1}(M^{\mathbf{c}})$ is anti-Kaehlerian isoparametric;

where an anti-Kaehlerian equifocal submanifold and an anti-Kaehlerian isoparametric one are notions introduced in [8] (see Section 2 about the definitions of these notions). We defined the notion of a proper anti-Kaehlerian isoparametric submanifold as a subclass of the class consisting of anti-Kaehlerian isoparametric submanifolds. It is easy to show that M is proper complex equifocal if and only if $\tilde{\phi}^{\mathbf{c}-1}(M^{\mathbf{c}})$ is proper anti-Kaehlerian isoparametric.

Let M be a proper anti-Kaehlerian isoparametric submanifold in an infinite-dimensional anti-Kaehlerian space V. It is shown that the focal set of M at x consists of some complex hyperplanes in the normal space $T^{\perp}_x M$, where x is an arbitrary point of M (see [8, Theorem 2]). Let W be the complex reflection group generated by complex reflections of order 2 with respect to these complex hyperplanes. Note that W is independent of the choice of $x \in M$ up to isomorphism. It is shown that W is discrete (see Proposition 3.7). In that case, we call W the *complex Coxeter group associated with* M.

In the sequel, we assume that all proper complex equifocal submanifolds are complete C^{ω}-ones and that all proper anti-Kaehlerian isoparametric submanifolds are complete unless otherwise mentioned.

We first prove the following splitting theorem of Heintze-Liu type for a proper anti-Kaehlerian isoparametric submanifold.

THEOREM 1. *Let M be a proper anti-Kaehlerian isoparametric submanifold in an infinite-dimensional anti-Kaehlerian space and W be the complex Coxeter group associated* *with M. Then M is decomposed into an extrinsic product of two proper anti-Kaehlerian isoparametric submanifolds if and only if W is decomposable.*

REMARK 1.1. Let G/K be a symmetric space of non-compact type and H be a symmetric subgroup of G. Let $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}}) := \{g \in H^1([0,1], G^{\mathbf{c}}) \mid (g(0), g(1)) \in H^{\mathbf{c}} \times K^{\mathbf{c}}\}$, which acts on the path space $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ as gauge actions. It is shown that the principal orbits of this action are proper anti-Kaehlerian isoparametric submanifolds (see [8, 9]).

Let M be a proper complex equifocal submanifold in a symmetric space G/K of non-compact type and W be the complex Coxeter group associated with the proper anti-Kaehlerian isoparametric submanifold $\tilde{\phi}^{\mathbf{c}-1}(M^{\mathbf{c}})$. We call W the *complex Coxeter group associated with M*. Note that W is obtained by analyzing the complex focal normal vectors of M (without analyzing the focal set of $\tilde{\phi}^{\mathbf{c}-1}(M^{\mathbf{c}})$) (see [8, Theorem 1]). Next, by using Theorem 1, we prove the following splitting theorem of Ewert-type for a proper complex equifocal submanifold.

THEOREM 2. *Let M be a proper complex equifocal submanifold in a symmetric space G/K of non-compact type and W be the complex Coxeter group associated with M. Then M is decomposed into an extrinsic product of two proper complex equifocal submanifolds if and only if W is decomposable.*

REMARK 1.2. (i) All isoparametric submanifolds in G/K in the sense of Heintze-Liu-Olmos (see [5] for the definition) are complex equifocal (see [8, Section 11]). It is conjectured that the converse is also true.

(ii) It is shown that all principal orbits of the action of Hermann type (i.e. the action of a (not necessarily compact) symmetric subgroup of G) on a symmetric space G/K of non-compact type are curvature adapted and proper complex equifocal (see [9]). See [1] for the definition of the curvature adaptedness. Hence it is shown that those orbits are isoparametric submanifolds with flat sections in the sense of Heintze-Liu-Olmos (see [8, Section 11]).

(iii) An action H of Hermann type on a symmetric space G/K of non-compact type has the dual action H^* (by taking its conjugate action if necessary), which is a Hermann action on the compact dual G^*/K. Thus, the principal orbits of the H-action are obtained as the duals of equifocal submanifolds in G^*/K. However, it is not clear that any proper complex equifocal submanifolds in G/K are obtained as the duals of equifocal submanifolds in G^*/K. Thus, we cannot reduce the study of proper complex equifocal submanifolds in G/K to that of equifocal submanifolds in G^*/K.

Here we propose the following questions.

QUESTION 1. Are all complex equifocal submanifolds homogeneous?

According to the classification by Kollross [10] of hyperpolar actions on irreducible symmetric spaces of compact type, all homogeneous equifocal submanifolds of codimension larger than one are obtained as principal orbits of Hermann actions. From this fact, Remark 1.2(ii) and Question 1, the following question is naturally proposed.

QUESTION 2. Are all complex equifocal submanifolds of codimension larger than one curvature adapted and proper complex equifocal (hence isoparametric with flat section in the sense of Heintze-Liu-Olmos)?

In Section 2, we recall basic notions and facts. In Section 3, we define the notion of the complex Coxeter group associated with a proper anti-Kaehlerian isoparametric submanifold. In Sections 4 and 5, we prove Theorems 1 and 2, respectively.

Throughout this paper, the notation 'G/K' means that (G, K) is a symmetric pair.

2. Basic notions and facts. In this section, we first recall the notion of a proper complex equifocal submanifold. Let M be an immersed submanifold with abelian normal bundle in a symmetric space $N = G/K$ of non-compact type. Denote by A the shape tensor of M. Let $v \in T^{\perp}_x M$ and $X \in T_x M$ $(x = gK)$. Denote by γ_v the geodesic in N with $\dot{\gamma}_v(0) = v$. The Jacobi field Y along γ_v with $Y(0) = X$ and $Y'(0) = -A_v X$ is given by

$$Y(s) = (P_{\gamma_v|_{[0,s]}} \circ (D^{\mathrm{co}}_{sv} - sD^{\mathrm{si}}_{sv} \circ A_v))(X)\,,$$

where $Y'(0) = \tilde{\nabla}_v Y$, $P_{\gamma_v|_{[0,s]}}$ is the parallel translation along $\gamma_v|_{[0,s]}$,

$$D^{\mathrm{co}}_{sv} = g_* \circ \cos(\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v)) \circ g_*^{-1}$$

and

$$D^{\mathrm{si}}_{sv} = g_* \circ \frac{\sin(\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v))}{\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v)} \circ g_*^{-1}\,.$$

Here ad is the adjoint representation of the Lie algebra $\mathfrak{g}$ of G. All focal radii of M along γ_v are obtained as real numbers s_0 with $\mathrm{Ker}(D^{\mathrm{co}}_{s_0v} - s_0 D^{\mathrm{si}}_{s_0v} \circ A_v) \neq \{0\}$. So, we call a complex number z_0 with $\mathrm{Ker}(D^{\mathrm{co}}_{z_0v} - z_0 D^{\mathrm{si}}_{z_0v} \circ A^{\mathbf{c}}_v) \neq \{0\}$ a *complex focal radius of M along γ_v* and call $\dim \mathrm{Ker}(D^{\mathrm{co}}_{z_0v} - z_0 D^{\mathrm{si}}_{z_0v} \circ A^{\mathbf{c}}_v)$ the *multiplicity* of the complex focal radius z_0, where $D^{\mathrm{co}}_{z_0v}$ (respectively $D^{\mathrm{si}}_{z_0v}$) implies the complexification of a map $(g_* \circ \cos(\sqrt{-1}z_0\,\mathrm{ad}(g_*^{-1}v)) \circ g_*^{-1})|_{T_xM}$ (respectively $(g_* \circ \sin(\sqrt{-1}z_0\,\mathrm{ad}(g_*^{-1}v))/\sqrt{-1}z_0\,\mathrm{ad}(g_*^{-1}v) \circ g_*^{-1})|_{T_xM}$) from T_xM to $T_xN^{\mathbf{c}}$. Also, for a complex focal radius z_0 of M along γ_v, we call z_0v $(\in T^{\perp}_x M^{\mathbf{c}})$ a *complex focal normal vector of M at x*. Furthermore, assume that M has globally flat normal bundle. Let $\tilde{v}$ be a parallel unit normal vector field of M. Assume that the number (which may be 0 and ∞) of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$. Furthermore, assume that the number is not equal to 0. Let $\{r_{i,x} \mid i = 1, 2, \dots\}$ be the set of all complex focal radii along $\gamma_{\tilde{v}_x}$, where $|r_{i,x}| < |r_{i+1,x}|$ or '$|r_{i,x}| = |r_{i+1,x}|$ and $\mathrm{Re}\, r_{i,x} > \mathrm{Re}\, r_{i+1,x}$' or '$|r_{i,x}| = |r_{i+1,x}|$ and $\mathrm{Re}\, r_{i,x} = \mathrm{Re}\, r_{i+1,x}$ and $\mathrm{Im}\, r_{i,x} = -\mathrm{Im}\, r_{i+1,x} > 0$'. Let r_i $(i = 1, 2, \dots)$ be complex valued functions on M defined by assigning $r_{i,x}$ to each $x \in M$. We call these functions r_i $(i = 1, 2, \dots)$ *complex focal radius functions for* $\tilde{v}$. We call $r_i\tilde{v}$ a *complex focal normal vector field for* $\tilde{v}$. If, for each parallel unit normal vector field $\tilde{v}$ of M, the number of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, each complex focal radius function for $\tilde{v}$ is constant on M and it has constant multiplicity, then we call M a *complex equifocal submanifold*. Let $\phi : H^0([0, 1], \mathfrak{g}) \to G$ be the parallel transport map for G (see [7] for this definition) and $\pi : G \to G/K$ be the natural projection. It is shown that M is complex equifocal if and only if each component of $(\pi \circ \phi)^{-1}(M)$ is complex isoparametric (see [7]). In particular, if each component of $(\pi \circ \phi)^{-1}(M)$ is proper complex isoparametric (see [7] about this definition), then we call M a *proper complex equifocal submanifold*.

Next we recall the notion of an infinite-dimensional anti-Kaehlerian isoparametric submanifold. Let M be an anti-Kaehlerian Fredholm submanifold in an infinite-dimensional anti-Kaehlerian space V and A be the shape tensor of M. See [8] for the definitions of an infinite-dimensional anti-Kaehlerian space and anti-Kaehlerian Fredholm submanifold in the space. Denote by the same symbol J the complex structures of M and V. Fix a unit normal vector v of M. If there exists $X(\neq 0) \in TM$ with $A_v X = aX + bJX$, then we call the complex number $a + b\sqrt{-1}$ a *J-eigenvalue of A_v* (or a *complex principal curvature of direction v*) and call X a *J-eigenvector for $a + b\sqrt{-1}$*. Also, we call the space of all J-eigenvectors for $a + b\sqrt{-1}$ a *J-eigenspace for $a + b\sqrt{-1}$*. The J-eigenspaces are orthogonal to one another and each J-eigenspace is J-invariant. We call the set of all J-eigenvalues of A_v the *J-spectrum of A_v* and denote it by $\mathrm{Spec}_J A_v$. The set $\mathrm{Spec}_J A_v \setminus \{0\}$ is described as follows:

$$\mathrm{Spec}_J A_v \setminus \{0\} = \{\lambda_i \mid i = 1, 2, \dots\}$$

$$\left(\begin{array}{c} |\lambda_i| > |\lambda_{i+1}| \ \text{ or } \ `|\lambda_i| = |\lambda_{i+1}| \ \text{ and } \ \mathrm{Re}\,\lambda_i > \mathrm{Re}\,\lambda_{i+1}\text{'} \\ \text{or } \ `|\lambda_i| = |\lambda_{i+1}| \ \text{ and } \ \mathrm{Re}\,\lambda_i = \mathrm{Re}\,\lambda_{i+1} \ \text{ and } \ \mathrm{Im}\,\lambda_i = -\mathrm{Im}\,\lambda_{i+1} > 0\text{'} \end{array} \right).$$

Also, the J-eigenspace for each J-eigenvalue of A_v other than 0 is of finite dimension. We call the J-eigenvalue λ_i the *ith complex principal curvature of direction v*. Assume that M has globally flat normal bundle. Fix a parallel normal vector field $\tilde{v}$ of M. Assume that the number (which may be ∞) of distinct complex principal curvatures of direction $\tilde{v}_x$ is independent of the choice of $x \in M$. Then we can define functions $\tilde{\lambda}_i$ $(i = 1, 2, \dots)$ on M by assigning the ith complex principal curvature of direction $\tilde{v}_x$ to each $x \in M$. We call this function $\tilde{\lambda}_i$ the *ith complex principal curvature function of direction $\tilde{v}$*. We consider the following condition.

CONDITION (AKI). For each parallel normal vector field $\tilde{v}$, the number of distinct complex principal curvatures of direction $\tilde{v}_x$ is independent of the choice of $x \in M$, each complex principal curvature function of direction $\tilde{v}$ is constant on M and it has constant multiplicity.

If M satisfies Condition (AKI), then we call M an *anti-Kaehlerian isoparametric submanifold*. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal system of $T_x M$. If $\{e_i\}_{i=1}^{\infty} \cup \{Je_i\}_{i=1}^{\infty}$ is an orthonormal base of $T_x M$, then we call $\{e_i\}_{i=1}^{\infty}$ a *J-orthonormal base*. If there exists a J-orthonormal base consisting of J-eigenvectors of A_v, then A_v is said to *be diagonalized with respect to the J-orthonormal base*. If M is anti-Kaehlerian isoparametric and, for each $v \in T^{\perp}M$, the shape operator A_v is diagonalized with respect to a J-orthonormal base, then we call M a *proper anti-Kaehlerian isoparametric submanifold*. For arbitrary two unit normal vectors v_1 and v_2 of a proper anti-Kaehlerian isoparametric submanifold, the shape operators A_{v_1} and A_{v_2} are simultaneously diagonalized with respect to a J-orthonormal base. As stated in the introduction, we assume that all proper anti-Kaehlerian isoparametric submanifolds are properly immersed complete submanifolds. Let M be a proper anti-Kaehlerian isoparametric submanifold in an infinite-dimensional anti-Kaehlerian space V. Let $\{E_i \mid i \in I\}$ be the family of distributions on M such that, for each $x \in M$, $\{E_i(x) \mid i \in I\}$ is the set of all common J-eigenspaces of A_v ($v \in T^{\perp}_x M$). The relation $T_x M = \overline{\bigoplus_{i\in I} E_i}$ holds. Let λ_i ($i \in I$) be the section of $(T^{\perp}M)^* \otimes \boldsymbol{C}$ such that $A_v = \mathrm{Re}\,\lambda_i(v)\mathrm{id} + \mathrm{Im}\lambda_i(v)J$ on $E_i(\pi(v))$ for each $v \in T^{\perp}M$, where π is the bundle projection of $T^{\perp}M$. We call λ_i ($i \in I$) *complex principal curvatures* of M and call distributions E_i ($i \in I$) *complex curvature distributions* of M. It is shown that there uniquely exists a normal vector field v_i of M with $\lambda_i(\cdot) = \langle v_i, \cdot\rangle - \sqrt{-1}\langle Jv_i, \cdot\rangle$ (see [8, Lemma 5]). We call v_i ($i \in I$) the *complex curvature normals* of M. Note that v_i is parallel with respect to the normal connection $\nabla^{\perp}$.

3. The complex Coxeter group associated with a proper anti-Kaehlerian isoparametric submanifold. In this section, we introduce the new notion of the complex Coxeter group associated with a proper anti-Kaehlerian isoparametric submanifold. Let M be a proper anti-Kaehlerian isoparametric submanifold in an infinite-dimensional anti-Kaehlerian space V, $\{\lambda_i \mid i \in I\}$ (respectively $\{v_i \mid i \in I\}$) be the set of all complex principal curvatures (respectively the set of all complex curvature normals) of M and E_i ($i \in I$) be the complex curvature distribution for λ_i. Then we showed that the following facts (i) and (ii) hold [8].

(i) The focal set of (M, x) coincides with the sum $\bigcup_{i\in I}(\lambda_i)_x^{-1}(1)$ of the complex hyperplanes $(\lambda_i)_x^{-1}(1)$ ($i \in I$).

(ii) E_i ($i \in I$) are totally geodesic on M. If $\lambda_i \neq 0$, then the leaves of E_i are complex spheres of radius $\sqrt{\lambda_i(v_i)}/|\lambda_i(v_i)|$ (this quantity is constant over M) and the mean curvature vector of leaves of E_i is equal to v_i. Also, if $\lambda_i = 0$, then the leaves of E_i are complex affine subspaces.

Let T_i^x be the complex reflection of order 2 with respect to the complex hyperplane $l_i^x := (\lambda_i)_x^{-1}(1)$ of $T_x^{\perp}M$ (i.e. the rotation of angle π having l_i^x as the axis), which is an affine transformation of $T_x^{\perp}M$. When T_i^x is regarded as a linear transformation of $T_x^{\perp}M$, we denote it by R_i^x. Also, when l_i^x is regarded as a linear subspace of $T_x^{\perp}M$, we denote it by $\hat{l}_i^x$. Let W_x^A (respectively W_x^L) be the group generated by T_i^x (respectively R_i^x) ($i \in I$). Now we shall show the finiteness of W_x^L. For its purpose, we prepare some lemmas. Let v be a parallel normal vector field of M and define an immersion $\eta_v : M \to V$ by $\eta_v(x) = \exp^{\perp} v_x$ ($x \in M$). Denote by f the original immersion of M into V. When M is regarded as a submanifold in V immersed by η_v, we denote it by M_v. Denote by A (respectively A^v) the shape tensor of M (respectively M_v). Then we have the following relation.

LEMMA 3.1. *For each $x \in M$, we have*

$$\eta_{v*x} = f_{*x} - f_{*x} \circ A_{v_x}$$

and hence $\eta_{v*x}T_x M = f_{*x}T_x M$, where we identify $T_{f(x)}V$ and $T_{\eta_v(x)}V$ with V in the natural manner.

PROOF. Let $X \in T_x M$. Take a curve $c(: (-\varepsilon, \varepsilon) \to M)$ in M with $\dot{c}(0) = X$, where $\dot{c}(0)$ is the velocity vector of c at 0. Then we have

$$\begin{aligned}\eta_{v*x} X &= \left.\frac{d}{dt}\right|_{t=0} \eta_v(c(t)) = \left.\frac{d}{dt}\right|_{t=0} (f(c(t)) + v_{c(t)}) \\ &= f_{*x} X + (f^* \tilde{\nabla})_X v = f_{*x}(X - A_{v_x} X)\,,\end{aligned}$$

where $f^* \tilde{\nabla}$ is the connection on $f^* TV$ induced from $\tilde{\nabla}$ by f. Thus we can obtain $\eta_{v*x} = f_{*x} - f_{*x} \circ A_{v_x}$, which together with $\dim \eta_{v*x}(T_x M) = \dim f_{*x}(T_x M)$ implies that $\eta_{v*x}(T_x M) = f_{*x}(T_x M)$. □

By using this lemma, we can show the following relation.

LEMMA 3.2. *For $w \in T_x^{\perp} M = T_x^{\perp} M_v$, we have*

$$A_w^v|_{E_i(x)} = \frac{(\lambda_i)_x(w)}{1 - (\lambda_i)_x(v_x)} \mathrm{id}_{E_i(x)}\,,$$

where $\mathrm{id}_{E_i(x)}$ is the identity transformation of $E_i(x)$.

PROOF. Let $X \in E_i(x)$. Take a curve $c(: (-\varepsilon, \varepsilon) \to M)$ in M with $\dot{c}(0) = X$. Let $\tilde{w}$ be the parallel normal vector field of M with $(\tilde{w})_x = w$. This vector field $\tilde{w}$ is also regarded as a parallel normal vector field of the parallel submanifold M_v of M under the identification $T_y^{\perp} M = T_y^{\perp} M_v$ (where y is an arbitrary point of M). Then it follows from Lemma 3.1 that

$$\begin{aligned}(\eta_v^* \tilde{\nabla})_X \tilde{w} &= (f^* \tilde{\nabla})_X \tilde{w} = -f_{*x}(A_w X) \\ &= -(\lambda_i)_x(w) f_* X = \frac{(\lambda_i)_x(w)}{(\lambda_i)_x(v_x) - 1} \eta_{v*} X\,,\end{aligned}$$

where $\eta_v^* \tilde{\nabla}$ is the connection on $\eta_v^* TV$ induced from $\tilde{\nabla}$ by η_v. On the other hand, we have $(\eta_v^* \tilde{\nabla})_X \tilde{w} = -\eta_{v*} A_w^v X$. Therefore, we can obtain the desired relation. □

Let w_i be the focal vector field of leaves of E_i defined by $(w_i)_x = \overrightarrow{f(x) o_x}$ $(x \in M)$, where o_x is the center of the complex sphere $L_x^{E_i}$. Clearly we have $\eta_{2w_i}(M) = f(M)$. Define a diffeomorphism $\phi_i : M \to M$ by $f(\phi_i(x)) = \eta_{2w_i}(f(x))$ $(x \in M)$. Next we prepare the following lemma.

LEMMA 3.3. *For each $x \in M$, we have $R_i^x((v_j)_x) = (v_j)_{\phi_i(x)}$ and hence $R_i^x(\hat{l}_j^x) = \hat{l}_j^{\phi_i(x)}$, where we identify $T_x^{\perp} M$ with $T_{\phi_i(x)}^{\perp} M$.*

PROOF. Let $L_x^{E_i}$ be the leaf of E_i through x, which (precisely, $f(L_x^{E_i})$) is a complex sphere in V. Let $X \in E_i(x)$. As v_j is parallel with respect to the normal connection of M, we have

$$(f^* \tilde{\nabla})_X v_j = -(\lambda_i)_x(v_j) f_* X \in f_*(T_x L_x^{E_i})\,.$$

This fact implies that $v_j|_{L_x^{E_i}}$ is parallel with respect to the normal connection of $L_x^{E_i}$. In general, if w is a parallel normal vector field of a complex sphere S (which may not be a complex hypersurface) in a finite-dimensional anti-Kaehlerian space V_1, then for each $y \in S$, we have

$R(w_y) = w_{y^*}$, where y^* is the anti-podal point of y in S and R is the complex reflection of order two with respect to the complex hyperplane $l := o + V_1'$ in $T_y^{\perp} S$ (where o is the center of S, V_1' is the orthogonal complement of the anti-Kaehlerian subspace of V_1 containing S as a complex hypersurface). Hence, we have $R_i^x((v_j)_x) = (v_j)_{\phi_i(x)}$. This relation deduces $R_i^x(\hat{l}_j^x) = \hat{l}_j^{\phi_i(x)}$ directly. □

From $\eta_{2w_i}(M) = f(M)$ and Lemma 3.2, we have $\{E_j(x) \mid j \in I\} = \{E_j(\phi_i(x)) \mid j \in I\}$.

LEMMA 3.4. *Let $E_j(\phi_i(x)) = E_{\sigma_i(j)}(x)$ $(i, j \in I)$. Then we have*

$$\begin{aligned}&(1/2)(\langle v_i, v_i\rangle^2 + \langle Jv_i, v_i\rangle^2)(v_{\sigma_i(j)})_x \\ &\quad = \{(1/2)\langle v_i, v_i\rangle^2 + (1/2)\langle Jv_i, v_i\rangle^2 - \langle v_i, v_i\rangle\langle v_{\sigma_i(j)}, v_i\rangle - \langle Jv_i, v_i\rangle\langle Jv_{\sigma_i(j)}, v_i\rangle \\ &\qquad + (\langle v_i, v_i\rangle\langle Jv_{\sigma_i(j)}, v_i\rangle - \langle Jv_i, v_i\rangle\langle v_{\sigma_i(j)}, v_i\rangle)\sqrt{-1}\}R_i^x((v_j)_x)\,.\end{aligned}$$

PROOF. According to Lemma 3.2, we have

$$(\lambda_j)_{\phi_i(x)} = \frac{(\lambda_{\sigma_i(j)})_x}{1 - (\lambda_{\sigma_i(j)})_x(2w_i)}\,,$$

that is,

$$(v_j)_{\phi_i(x)} = \frac{(v_{\sigma_i(j)})_x}{1 - (\lambda_{\sigma_i(j)})_x(2w_i)}\,,$$

where we identify $\sqrt{-1}(\cdot)$ with $J(\cdot)$. On the other hand, according to Lemma 3.3, we have $R_i^x((v_j)_x) = (v_j)_{\phi_i(x)}$. Also, we have

$$w_i = \frac{1}{|(\lambda_i)_x((v_i)_x)|^2}(\langle v_i, v_i\rangle(v_i)_x + \langle Jv_i, v_i\rangle J(v_i)_x)\,.$$

From these relations, the desired relation follows. □

By this lemma, we can show the following fact.

LEMMA 3.5. *Each T_i^x $(i \in I)$ permutes $\{l_j^x \mid j \in I\}$.*

PROOF. Let $E_j(\phi_i(x)) = E_{\sigma_i(j)}(x)$ $(i, j \in I)$. We shall show $T_i^x(l_j^x) = l_{\sigma_i(j)}^x$ $(i, j \in I)$. From Lemma 3.4, we see that $T_i(l_j^x)$ and $l_{\sigma_i(j)}^x$ are parallel. Hence, we have only to show that these complex hyperplanes have a common point. If $(v_i)_x \in \hat{l}_j^x$, then we have $T_i^x(l_j^x) = l_j^x$ $(\sigma_i(j) = j)$. Hence, we consider the case of $(v_i)_x \notin \hat{l}_j^x$. Let Π be the complex line through the origin of $T_x^{\perp} M$ that is orthogonal to l_i^x, that is, $\Pi = \mathrm{Span}\{(v_i)_x, J(v_i)_x\}$. Denote by p_1 (respectively p_2) the intersection point of $l_{\sigma_i(j)}^x$ (respectively $T_i^x(l_j^x)$) with Π. Also, denote by q_i (respectively q_j) that of l_i^x (respectively l_j^x) with Π. By using $\overrightarrow{op_1} \in \mathrm{Span}\{(v_i)_x,\ J(v_i)_x\}$, $(\lambda_{\sigma_i(j)})_x(\overrightarrow{op_1}) = 1$, we can explicitly express $\overrightarrow{op_1}$ as a linear combination of $(v_i)_x$ and $J(v_i)_x$, where we also use $(\lambda_{\sigma_i(j)})_x(*) = \langle(v_{\sigma_i(j)})_x, *\rangle - \sqrt{-1}\langle J(v_{\sigma_i(j)})_x, *\rangle$ and $\langle R_i^x(*), \cdot\rangle = \langle *, R_i^x(\cdot)\rangle$. On the other hand, by using $\overrightarrow{op_2} \in \mathrm{Span}\{(v_i)_x,\ J(v_i)_x\}$, $\overrightarrow{op_2} = 2\overrightarrow{oq_i} - \overrightarrow{oq_j}$ and $(\lambda_i)_x(\overrightarrow{oq_i}) = (\lambda_j)_x(\overrightarrow{oq_j}) = 1$, we can explicitly express $\overrightarrow{op_2}$ as a linear combination of $(v_i)_x$ and $J(v_i)_x$. By comparing these expressions in terms of the relation in Lemma 3.4, we can show $\overrightarrow{op_1} = \overrightarrow{op_2}$, that is, $p_1 = p_2$. Therefore, we obtain $T_i^x(l_j^x) = l_{\sigma_i(j)}^x$. □

Also, we can show the following fact.

LEMMA 3.6. *We have that* $\{l_i^x \mid i \in I\}$ *is locally finite.*

PROOF. First we show that $I_w := \{i \in I \mid w \in l_i^x\}$ is finite for each $w \in T_x^\perp M$. The J-spectrum $\mathrm{Spec}_J A_w$ of A_w is given by $\{(\lambda_i)_x(w) \mid i \in I\}$. Assume that $i \in I_w$. Then we have $(\lambda_i)_x(w) = 1$, that is, $E_i(x) \subset \mathrm{Ker}(A_w - \mathrm{id}_{T_x M})$. As the multiplicity of each J-eigenvalue other than 0 of A_w is finite, we have $\dim \mathrm{Ker}(A_w - \mathrm{id}_{T_x M}) < \infty$. Hence, we see that I_w is finite. Take an arbitrary $w_0 \in T_x^\perp M$. As $\mathrm{Spec}_J A_{w_0}$ has no accumulating point other than 0, there exists $\delta > 0$ such that the δ-neighborhood $B_\delta(1)$ of 1 in $\boldsymbol{C}$ does not intersect with $\mathrm{Spec}_J A_{w_0} \setminus \{1\}$. For each $i \in I \setminus I_{w_0}$, we have $w_0 \notin (\lambda_i)_x^{-1}(B_\delta(1))$ because $1 \neq (\lambda_i)_x(w_0) \in \mathrm{Spec}_J A_{w_0}$. Fix an inner product $\langle\cdot,\cdot\rangle_0$ of $T_x^\perp M$ such that $J|_{T_x^\perp M}$ is skew-symmetric with respect to $\langle\cdot,\cdot\rangle_0$. The set $(\lambda_i)_x^{-1}(B_\delta(1))$ is a tubular neighborhood of l_i^x foliated by complex hyperplanes $(\lambda_i)_x^{-1}(z)$'s ($z \in B_\delta(1)$). As $\sup_{i \in I} |(\lambda_i)_x(w)| < \infty$ for each $w \in T_x^\perp M$, we can show $\sup_{i \in I} \langle v_i, v_i \rangle_0 < \infty$. Furthermore, we can show that there exists $i_0 \in I \setminus I_w$ such that $\langle v_{i_0}, v_{i_0} \rangle_0 = \sup_{i \in I \setminus I_{w_0}} \langle v_i, v_i \rangle_0$. Clearly there exists $\varepsilon > 0$ such that the ε-tubular neighborhood of $l_{i_0}^x$ with respect to $\langle\cdot,\cdot\rangle_0$ is contained in $(\lambda_{i_0})_x^{-1}(B_\delta(1))$. Then, for each $i \in I \setminus I_{w_0}$, it follows from $\langle v_i, v_i \rangle_0 \leq \langle v_{i_0}, v_{i_0} \rangle_0$ that the ε-tubular neighborhood of l_i^x with respect to $\langle\cdot,\cdot\rangle_0$ is contained in $(\lambda_i)_x^{-1}(B_\delta(1))$. Hence, for each $i \in I \setminus I_{w_0}$, we have $d_0(w_0, l_i^x) > \varepsilon$, that is, $B_\varepsilon(w_0) \cap l_i^x = \emptyset$, where d_0 is the Euclidean distance function associated with $\langle\cdot,\cdot\rangle_0$ and $B_\varepsilon(w_0)$ is the ε-neighborhood of w_0 with respect to $\langle\cdot,\cdot\rangle_0$. This fact together with the arbitrariness of w_0 implies that $\{l_i^x \mid i \in I\}$ is locally finite. □

From Lemmas 3.5 and 3.6, we can show the following fact by imitating the proof of the theorem in the Appendix of [11].

PROPOSITION 3.7. *The group* W_x^A *is discrete.*

It is clear that W_x^A ($x \in M$) are isomorphic to one another. Hence, we denote this discrete group by W^A. We call W^A the *complex Coxeter group associated with the proper anti-Kaehlerian isoparametric submanifold* M. For simplicity, we denote W^A by W. We have the following fact with respect to the decomposability of the complex Coxeter group in a similar manner to that of a Coxeter group.

LEMMA 3.8. *The complex Coxeter group* W *is decomposable* (*i.e. it is decomposed into a non-trivial product of two discrete complex reflection groups*) *if and only if there exist two* J*-invariant linear subspaces* P_1 $(\neq \{0\})$ *and* P_2 $(\neq \{0\})$ *of* $T_x^\perp M$ *such that* $T_x^\perp M = P_1 \oplus P_2$ (*orthogonal direct sum*), $P_1 \cup P_2$ *contains all complex curvature normals of* M *at* x *and that* P_i $(i = 1, 2)$ *contains at least one complex curvature normal of* M *at* x.

4. Proof of Theorem 1. In this section, we prove Theorem 1. Let M be a proper anti-Kaehlerian isoparametric submanifold in an infinite-dimensional anti-Kaehlerian space V.

Denote by the same symbol J the complex structures of M and V. It follows from Lemma 3.8 that, if M is decomposed into an extrinsic product of two proper anti-Kaehlerian isoparametric submanifolds, then W is decomposable. In the sequel, we prove the converse. Assume that W is decomposable. Without loss of generality, we may assume that M contains the zero element o of V. According to Lemma 3.8, there exist J-invariant linear subspaces $P_1\ (\neq 0)$ and $P_2\ (\neq 0)$ of $T_o^\perp M$ such that $T_o^\perp M = P_1 \oplus P_2$ (orthogonal direct sum), $P_1 \cup P_2$ contains all complex curvature normals of M at o and that $P_i\ (i = 1, 2)$ contain at least one complex curvature normal of M at o. Let $\tilde{P}_i\ (i = 1, 2)$ be the a $\nabla^\perp$-parallel subbundle of $T^\perp M$ with $\tilde{P}_i(o) = P_i$, where $\nabla^\perp$ is the normal connection of M. Set $V_{P_i} := \overline{\mathrm{Span}_J \bigcup_{x\in M} \tilde{P}_i(x)}$ $(i = 1, 2)$ and $V' := \overline{\mathrm{Span}_J \bigcup_{x\in M} T_x^\perp M}$, where $\tilde{P}_i(x)\ (i = 1, 2)$ and $T_x^\perp M$ are regarded as linear subspaces of V and $\mathrm{Span}_J(\cdot)$ implies the J-invariant linear subspace spanned by $(\cdot)$. Clearly we have $V_{P_1} + V_{P_2} = V'$. Set $V_0 := (V')^\perp$ and $M' := M \cap V'$, which is regarded as an immersed submanifold in V'. Denote by ι' the immersion of M' into V' and by ι that of M into V. We first prove the following fact by imitating the proof of Lemma 3.1 of [4].

PROPOSITION 4.1. (i) *There exist an isometry $\tilde{F}$ of $V' \times V_0$ onto V and an isometry F of the anti-Kaehlerian product manifold $M' \times V_0$ onto M satisfying $\tilde{F} \circ (\iota' \times \mathrm{id}_{V_0}) = \iota \circ F$, where id_{V_0} is the identity transformation of V_0.*

(ii) *M' is totally geodesic in M.*

(iii) *M' is proper anti-Kaehlerian isoparametric in V'.*

PROOF. First we shall show $V_0 \subset E_0(x)$, where x is an arbitrary point of M and $E_0(x) = \bigcap_{v\in T_x^\perp M} \mathrm{Ker} A_v$. From the definition of V_0, we have $V_0 \subset T_x M$. Let $X \in E_i(x)$ $(i \in I)$. The leaf $L_x^{E_i}$ of E_i through x is a complex sphere. Let c be the center of this complex sphere and γ be a geodesic in $L_x^{E_i}$ with $\dot{\gamma}(0) = X$. As $L_x^{E_i}$ is totally geodesic in M, we have $\gamma(t) - c \in T_{\gamma(t)}^\perp M \subset V'$ and hence $\dot{\gamma}(t) \in V'$. In particular, we have $X \in V'$. From the arbitrarinesses of X and i, we have $\bigoplus_{i\in I} E_i(x) \subset V' = V_0^\perp$. This together with $T_x^\perp M \subset V'$ deduces $V_0 \subset E_0(x)$. As $L_x^{E_0}$ is a J-invariant affine subspace of V, we have $x + V_0 \subset L_x^{E_0}$ and hence $\bigcup_{x\in M'}(x + V_0) \subset M$. It is clear that $\bigcup_{x\in M'}(x + V_0)$ is complete and open in M. Hence, we have $\bigcup_{x\in M'}(x + V_0) = M$. This implies that there exist isometries $\tilde{F}$ and F as in the statement (i). Also, the statement (ii) also follows from this fact. Next we show that M' is a proper anti-Kaehlerian isoparametric in V'. It is clear that the normal space $T_x^\perp M'$ of M' in V' coincide with the normal space $T_x^\perp M$ of M in V. Let $\tilde{v}$ be a parallel normal vector field of M. It is clear that the restriction of $\tilde{v}$ to M' is a parallel normal vector field of M'. Hence, the globally flatness of the normal bundle of M' follows from that of the normal bundle of M. Furthermore, it is easy to show that the restrictions of the complex principal curvatures of M to M' are the complex principal curvatures of M', the tangent space $T_x M'$ coincides with $\overline{\bigoplus_{i\in I} E_i(x)}(\subset T_x M)$ and that $T_x M' = \overline{\bigoplus_{i\in I} E_i(x)}$ is the common J-eigenspace decomposition of A'_v $(v \in T_x^\perp M')$, where A' is the shape tensor of M'. Thus, M' is a proper anti-Kaehlerian isoparametric submanifold in V'. □

Define a distribution D_{P_j} $(j = 1, 2)$ on M by $D_{P_j}(x) := \overline{E_0(x) \oplus (\bigoplus_{i\in I_j} E_i(x))}$ $(x \in M)$, where $I_j := \{i \in I \mid (v_i)_o \in P_j\}$ $(j = 1, 2)$. Next we prove the following fact.

PROPOSITION 4.2. *The subspace V' is the orthogonal direct sum of V_{P_1} and V_{P_2}.*

To show this fact, we prepare the following lemma.

LEMMA 4.3. *Let $\tilde{v}$ be a parallel normal vector field of M with $\tilde{v}_o \in P_j$. Then $\tilde{v}$ is parallel along $L_x^{D_{P_i}}$ $(i \neq j)$ with respect to the Levi-Civita connection $\tilde{\nabla}$ of V, where x is an arbitrary point of M.*

PROOF. Take an arbitrary $X \in T_y L_x^{D_{P_i}} (= D_{P_i}(y))$. Let $X = X_0 + \sum_{k\in I_i} X_k$, where $X_0 \in E_0(y)$ and $X_k \in E_k(y)$. Then we have

$$\begin{aligned}\tilde{\nabla}_X \tilde{v} &= -\sum_{k\in I_i} (\lambda_k)_y(\tilde{v}_y) f_* X_k \\ &= -\sum_{k\in I_i} (\langle (v_k)_y, \tilde{v}_y\rangle f_* X_k - \langle J(v_k)_y, \tilde{v}_y\rangle J f_* X_k)\,.\end{aligned}$$

As $\tilde{v}_y \in \tilde{P}_j(y)$ and $(v_k)_y \in \tilde{P}_i(y)$ $(k \in I_i)$, we have $\langle (v_k)_y, \tilde{v}_y\rangle = \langle J(v_k)_y, \tilde{v}_y\rangle = 0$. Hence, we have $\tilde{\nabla}_X \tilde{v} = 0$. Thus, the statement of this lemma follows. □

By using this lemma, we show Proposition 4.2.

Proof of Proposition 4.2. As $V' = V_{P_1} + V_{P_2}$, it suffices to show $V_{P_1} \perp V_{P_2}$. Let $\tilde{v}_i$ $(i = 1, 2)$ be a parallel normal vector field on M with $(\tilde{v}_i)_o \in P_i$. We have only to show that $(\tilde{v}_1)_{x_1} \perp (\tilde{v}_2)_{x_2}$ for arbitrary two points x_1 and x_2 of M. Set $U(x_1) := \bigcup_{x\in L_{x_1}^{D_{P_2}}} L_x^{D_{P_1}}$. It is clear that $U(x_1)$ is open in M. By using Lemma 4.3, we can show that $(\tilde{v}_1)_{x_1} \perp (\tilde{v}_2)_x$ for every $x \in U(x_1)$. Hence, as $U(x_1)$ is open and $\tilde{v}_2 : M \to V$ is real analytic, we see that $(\tilde{v}_1)_{x_1} \perp (\tilde{v}_2)_x$ for every $x \in M$. In particular, we have $(\tilde{v}_1)_{x_1} \perp (\tilde{v}_2)_{x_2}$. □

Let Δ be the interior of a fundamental domain containing o of the complex Coxeter group W_o^A of M at o, where we note that the choice of Δ is not unique. Define a map $F : M \times \Delta \to V$ by $F(x, v) := \exp^{\perp}(\tilde{v}_x)$ $((x, v) \in M \times \Delta)$, where $\tilde{v}$ is the parallel normal vector field of M with $\tilde{v}_o = v$ and $\exp^{\perp}$ is the normal exponential map of M. Set $U := F(M \times \Delta)$. This set U is a connected open dense subset of V consisting of non-focal points of M and F is a diffeomorphism of $M \times \Delta$ into V. Define a distribution $\tilde{D}_{P_j}$ on U by $\tilde{D}_{P_j}(F(x, v)) = \tilde{P}_j(x) \oplus \eta_{\tilde{v}*} D_{P_j}(x)$ $((x, v) \in M \times \Delta)$, where $\tilde{P}_j(x)$ is regarded as a subspace of $T_{F(x,v)}U$ and $\eta_{\tilde{v}}$ is a map of M into U defined by $\eta_{\tilde{v}}(x) = F(x, v)$ $(x \in M)$. We can show the following fact by imitating the proof of Proposition 2.3 of [4].

LEMMA 4.4. *The distributions D_{P_i} $(i = 1, 2)$ are totally geodesic on M.*

PROOF. Take $X, Y \in \Gamma(D_{P_1})$ and $Z \in \Gamma(D_{P_1}^{\perp})$, where $\Gamma(*)$ is the space of all sections of $*$. Let $X = X_0 + \sum_{k\in I_1} X_k$, $Y = Y_0 + \sum_{k\in I_1} Y_k$ and $Z = \sum_{k\in I_2} Z_k$, where $X_0, Y_0 \in \Gamma(E_0)$, $X_k, Y_k \in \Gamma(E_k)$ $(k \in I_1)$ and $Z_k \in \Gamma(E_k)$ $(k \in I_2)$. Denote by ∇ (respectively h) the

Levi-Civita connection (respectively the second fundamental form) of M. Also, denote by h_1 the second fundamental form of D_{P_1}. We have

$$\langle h_1(X, Y), Z\rangle = \langle \nabla_X Y, Z\rangle = \sum_{k_1\in I_1\cup\{0\}} \sum_{k_2\in I_1\cup\{0\}} \sum_{k_3\in I_2} \langle \nabla_{X_{k_1}} Y_{k_2}, Z_{k_3}\rangle\,,$$

where we note that the termwise differentiability as in Lemma 2.2 of [4] also holds on a pseudo-Riemannian Hilbert manifold. It suffices to show $\langle \nabla_{X_{k_1}} Y_{k_2}, Z_{k_3}\rangle = 0$ ($k_1, k_2 \in I_1 \cup \{0\}, k_3 \in I_2$) in order to show that D_{P_1} is totally geodesic. As $\langle X_{k_1}, Z_{k_3}\rangle = \langle X_{k_1}, JZ_{k_3}\rangle = 0$, we have

$$\langle \nabla_{Y_{k_2}} X_{k_1}, Z_{k_3}\rangle + \langle X_{k_1}, \nabla_{Y_{k_2}} Z_{k_3}\rangle = 0\,, \tag{4.1}$$

and

$$\langle \nabla_{Y_{k_2}} X_{k_1}, JZ_{k_3}\rangle + \langle JX_{k_1}, \nabla_{Y_{k_2}} Z_{k_3}\rangle = 0\,. \tag{4.2}$$

For any $u_i \in E_i,\ u_j \in E_j$ and any $v \in T^{\perp}M$, we have

$$\begin{aligned}\langle h(u_i, u_j), v\rangle &= \langle A_v u_i, u_j\rangle = \langle \lambda_i(v) u_i, u_j\rangle \\ &= \langle \langle v_i, v\rangle u_i - \langle J v_i, v\rangle J u_i, u_j\rangle \\ &= \langle \langle u_i, u_j\rangle v_i - \langle J u_i, u_j\rangle J v_i, v\rangle\end{aligned}$$

and hence

$$h(u_i, u_j) = \langle u_i, u_j\rangle v_i - \langle J u_i, u_j\rangle J v_i\,. \tag{4.3}$$

Let $\bar{\nabla} := \nabla^* \otimes \nabla^* \otimes \nabla^{\perp}$, where ∇^* is the dual connection of ∇ and $\nabla^{\perp}$ is the normal connection of M. From (4.3), we have

$$\begin{aligned}(\bar{\nabla}_{X_{k_1}} h)(Y_{k_2}, Z_{k_3}) &= \nabla^{\perp}_{X_{k_1}}(h(Y_{k_2}, Z_{k_3})) - h(\nabla_{X_{k_1}} Y_{k_2}, Z_{k_3}) - h(Y_{k_2}, \nabla_{X_{k_1}} Z_{k_3}) \\ &= \langle \nabla_{X_{k_1}} Y_{k_2}, Z_{k_3}\rangle (v_{k_2} - v_{k_3}) - \langle \nabla_{X_{k_1}} Y_{k_2}, JZ_{k_3}\rangle J(v_{k_2} - v_{k_3})\,.\end{aligned} \tag{4.4}$$

Similarly we have

$$(\bar{\nabla}_{Y_{k_2}} h)(X_{k_1}, Z_{k_3}) = \langle \nabla_{Y_{k_2}} X_{k_1}, Z_{k_3}\rangle (v_{k_1} - v_{k_3}) - \langle \nabla_{Y_{k_2}} X_{k_1}, JZ_{k_3}\rangle J(v_{k_1} - v_{k_3})\,. \tag{4.5}$$

As $\bar{\nabla} h$ is totally symmetric by the Codazzi equation, the left-hand side of (4.4) is equal to that of (4.5), that is,

$$\begin{aligned}&\langle \nabla_{X_{k_1}} Y_{k_2}, Z_{k_3}\rangle (v_{k_2} - v_{k_3}) - \langle \nabla_{X_{k_1}} Y_{k_2}, JZ_{k_3}\rangle J(v_{k_2} - v_{k_3}) \\ &\quad = \langle \nabla_{Y_{k_2}} X_{k_1}, Z_{k_3}\rangle (v_{k_1} - v_{k_3}) - \langle \nabla_{Y_{k_2}} X_{k_1}, JZ_{k_3}\rangle J(v_{k_1} - v_{k_3})\,.\end{aligned} \tag{4.6}$$

Similarly, we have

$$\begin{aligned}&\langle \nabla_{Z_{k_3}} Y_{k_2}, X_{k_1}\rangle (v_{k_2} - v_{k_1}) - \langle \nabla_{Z_{k_3}} Y_{k_2}, JX_{k_1}\rangle J(v_{k_2} - v_{k_1}) \\ &\quad = \langle \nabla_{Y_{k_2}} Z_{k_3}, X_{k_1}\rangle (v_{k_3} - v_{k_1}) - \langle \nabla_{Y_{k_2}} Z_{k_3}, JX_{k_1}\rangle J(v_{k_3} - v_{k_1})\,.\end{aligned} \tag{4.7}$$

According to (4.1) and (4.2), the right-hand sides of (4.6) and (4.7) coincide with each other. Hence, we have

$$\begin{aligned}&\langle \nabla_{X_{k_1}} Y_{k_2}, Z_{k_3}\rangle (v_{k_2} - v_{k_3}) - \langle \nabla_{X_{k_1}} Y_{k_2}, JZ_{k_3}\rangle J(v_{k_2} - v_{k_3}) \\ &\quad = \langle \nabla_{Z_{k_3}} Y_{k_2}, X_{k_1}\rangle (v_{k_2} - v_{k_1}) - \langle \nabla_{Z_{k_3}} Y_{k_2}, JX_{k_1}\rangle J(v_{k_2} - v_{k_1})\,.\end{aligned} \tag{4.8}$$

At each point of M, the left-hand side of (4.8) does not belong to $\tilde{P}_1$ or is equal to the zero vector. On the other hand, the right-hand side of (4.8) is a section of $\tilde{P}_1$. Hence, we have $\langle \nabla_{X_{k_1}} Y_{k_2}, Z_{k_3} \rangle = 0$. Thus, D_{P_1} is totally geodesic. Similarly, it is shown that D_{P_2} is totally geodesic. □

By imitating the proof of Lemma 3.2 of [4], we can show the following fact in terms of Lemma 4.4.

LEMMA 4.5. *The distributions* $\tilde{D}_{P_i}$ $(i = 1, 2)$ *are totally geodesic on* U *and hence leaves of* $\tilde{D}_{P_i}$ $(i = 1, 2)$ *are open potions of closed complex affine subspaces of* V.

PROOF. For each tangent vector field X and each $w \in \Delta$, vector fields $\hat{X}$ and $\hat{w}$ on U are defined by $\hat{X}_{F(x,v)} = X_x$ and $\hat{w}_{F(x,v)} := \tilde{w}_x$ for $(x, v) \in M \times \Delta$, where we identify $T_{F(x,v)}U$ with $T_x U$. The parallel submanifold $M_{\tilde{w}}$ of M is a proper anti-Kaehlerian isoparametric submanifold in V. Define distributions E_i^w $(i \in I \cup \{0\})$ on $M_{\tilde{w}}$ by $E_i^w(F(x, w)) := E_i(x)$ $(x \in M)$. According to Lemma 3.2, $\{E_i^w \mid i \in I \cup \{0\}\}$ is the set of all complex curvature distributions of $M_{\tilde{w}}$. Define a distribution $\tilde{D}^T_{P_i}$ (respectively $\tilde{D}^N_{P_i}$) on U by

$$\tilde{D}^T_{P_i}(F(x, v)) := \{\hat{X}_{F(x,v)} \mid X_x \in D_{P_i}(x)\}$$
$$(\text{respectively } \tilde{D}^N_{P_i}(F(x, v)) := \{\hat{w}_{F(x,v)} \mid w \in P_i \cap \Delta, x \in M\})$$

for $(x, v) \in M \times \Delta$. Then it is clear that $\tilde{D}_{P_i} = \tilde{D}^T_{P_i} \oplus \tilde{D}^N_{P_i}$ and that $\tilde{D}^N_{P_i}$ is totally geodesic (hence integrable). To show that $\tilde{D}_{P_i}$ is totally geodesic on U, we suffice to show that $\tilde{\nabla}_{\hat{X}} \hat{Y}$, $\tilde{\nabla}_{\hat{X}} \hat{w}$, $\tilde{\nabla}_{\hat{w}} \hat{Y}$ and $\tilde{\nabla}_{\hat{w}} \hat{v}$ (X, Y are tangent vector fields on M, $v, w \in \Delta$) are sections of $\tilde{D}_{P_i}$. It is clear that $\tilde{\nabla}_{\hat{w}} \hat{Y}$ and $\tilde{\nabla}_{\hat{w}} \hat{v}$ vanish, that is, they are sections of $\tilde{D}_{P_i}$. We show that $\tilde{\nabla}_{\hat{X}} \hat{Y}$ is a section of $\tilde{D}_{P_i}$. Denote by ∇^u, A^u and h_u the Levi-Civita connection, the shape tensor and the second fundamental form of $M_{\tilde{u}}$ $(u \in \Delta)$, respectively. By the Gauss equation, we have

$$(\tilde{\nabla}_{\hat{X}} \hat{Y})_{F(x,u)} = \eta_{\tilde{u}*}(\nabla^u_X Y)_x + h_u(X_x, Y_x) \quad ((x, u) \in M \times \Delta). \tag{4.9}$$

According to Lemma 4.4, $\tilde{D}^T_{P_i}$ is integrable and the leaf of $\tilde{D}^T_{P_i}$ through $F(x, u)$ is totally geodesic in $M_{\tilde{u}}$. Hence, we have

$$\eta_{\tilde{u}*}(\nabla^u_X Y)_x \in \tilde{D}^T_{P_i}(F(x, u)). \tag{4.10}$$

Let $X_x = (X_x)_0 + \sum_{k \in I_i} (X_x)_k$, where $(X_x)_0 \in E_0(x)$ and $(X_x)_k \in E_k(x)$ $(k \in I_i)$. Then, for any $\nu \in T^\perp_{F(x,u)} M_{\tilde{u}} \ominus \tilde{D}^N_{P_i}(F(x, u))$, we have

$$\begin{aligned}
\langle h_u(X_x, Y_x), \nu \rangle &= \langle h_u((X_x)_0, Y_x), \nu \rangle + \sum_{k \in I_i} \langle h_u((X_x)_k, Y_x), \nu \rangle \\
&= \langle A^u_\nu (X_x)_0, Y_x \rangle + \sum_{k \in I_i} \langle A^u_\nu (X_x)_k, Y_x \rangle \\
&= \sum_{k \in I_i} \left\langle \frac{(\lambda_k)_x(\nu)}{1 - (\lambda_k)_x(\tilde{u}_x)} (X_x)_k, Y_x \right\rangle = 0,
\end{aligned}$$

where we use Lemma 3.2 and $(\lambda_k)_x(\nu) = 0$. Thus, we have $h_u(X_x, Y_x) \in \tilde{D}^N_{P_i}(F(x,u))$. From (4.9), (4.10) and this fact, we have $(\tilde{\nabla}_{\hat{X}}\hat{Y})_{F(x,u)} \in \tilde{D}_{P_i}(F(x,u))$. Next we show that $\tilde{\nabla}_{\hat{X}}\hat{w}$ is a section of $\tilde{D}_{P_i}$. As $\hat{w}|_{M_{\tilde{u}}}$ is parallel with respect to the normal connection of $M_{\tilde{u}}$, we have

$$\begin{aligned}(\tilde{\nabla}_{\hat{X}}\hat{w})_{F(x,u)} &= -\eta_{\tilde{u}*}(A^u_{\hat{w}_{F(x,u)}}X_x)\\ &= -\sum_{k\in I_i}\frac{(\lambda_k)_x(\tilde{w}_x)}{1-(\lambda_k)_x(\tilde{u}_x)}\eta_{\tilde{u}*}(X_x)_k \in \tilde{D}^T_{P_i}(F(x,u)).\end{aligned}$$

Thus, it is shown that $\tilde{D}_{P_i}$ is totally geodesic. The rest of the statement follows from the following general fact. □

FACT 1. *Any connected totally geodesic submanifold in a pseudo-Hilbert space, whose tangent spaces are closed subspaces of the pseudo-Hilbert space, is an open potion of a closed affine subspace of the pseudo-Hilbert space, where closedness is one for the original topology of the pseudo-Hilbert space.*

Next we prepare the following lemma.

LEMMA 4.6. *The leaf $L_x^{D_{P_i}}$ of D_{P_i} through x is a proper anti-Kaehlerian isoparametric submanifold in $x+\tilde{D}_{P_i}(x)$.*

PROOF. From Lemma 4.5, we have $L_x^{D_{P_i}} \subset x+\tilde{D}_{P_i}(x)$. Let $\tilde{v}$ be a parallel normal vector field of M with $\tilde{v}_o \in P_i$. It is clear that the restriction of $\tilde{v}$ to $L_x^{D_{P_i}}$ is a parallel normal vector field of $L_x^{D_{P_i}}$ in $x+\tilde{D}_{P_i}(x)$. Also the normal space $T^\perp_y L_x^{D_{P_i}}$ of $L_x^{D_{P_i}}$ at y is equal to $\tilde{P}_i(y)$. These facts imply that $L_x^{D_{P_i}}$ has globally flat normal bundle. Furthermore, it is easy to show that the restrictions of the complex curvature normals of M belonging to $\tilde{P}_i$ to $L_x^{D_{P_i}}$ are the complex curvature normals of $L_x^{D_{P_i}}$, the tangent space $T_y L_x^{D_{P_i}}$ coincides with $\overline{E_0(y)\oplus(\bigoplus_{j\in I_i}E_j(y))}$ and that $T_y L_x^{D_{P_i}} = \overline{E_0(y)\oplus(\bigoplus_{j\in I_i}E_j(y))}$ is the common J-eigenspace decomposition of A^i_v ($v \in T^\perp_y L_x^{D_{P_i}}$), where A^i is the shape tensor of $L_x^{D_{P_i}}$. Thus, $L_x^{D_{P_i}}$ is a proper anti-Kaehlerian isoparametric submanifold in $x+\tilde{D}_{P_i}(x)$. □

For $x \in M$, we set $M_i(x) := M\cap(x+V_{P_i})$ $(i=1,2)$ and $M'(x) := M\cap(x+V')$. The set $M_i(x)$ (respectively $M'(x)$) is regarded as an immersed submanifold in $x+V_{P_i}$ (respectively $x+V'$).

PROPOSITION 4.7. *The submanifold $M_i(x)$ is a proper anti-Kaehlerian isoparametric submanifold in $x+V_{P_i}$.*

PROOF. We show this fact in the case $i=1$. According to Lemma 4.3, the subbundle $\tilde{P}_1$ of $T^\perp M$ is parallel along each leaf of D_{P_2} with respect to the Levi-Civita connection $\tilde{\nabla}$ of V. From this fact and the real analyticity of $\tilde{P}_1$, we have $V_{P_1} = \overline{\mathrm{Span}_J\bigcup_{y\in L_x^{D_{P_1}}}\tilde{P}_1(y)}$. Denote by $T^{\perp}L_x^{D_{P_1}}$ the normal bundle of $L_x^{D_{P_1}}$ in $x+\tilde{D}_{P_1}(x)$. As $T_y^{\perp}L_x^{D_{P_1}} = \tilde{P}_1(y)$ $(y \in L_x^{D_{P_1}})$, we have $V_{P_1} = \overline{\mathrm{Span}_J \bigcup_{y\in L_x^{D_{P_1}}} T_y^{\perp}L_x^{D_{P_1}}}$. Let $V_{P_1}^{\perp}$ be the orthogonal complement of V_{P_1} in $\tilde{D}_{P_1}(x)$. On the other hand, by Lemma 4.6, $L_x^{D_{P_1}}$ is a proper anti-Kaehlerian isoparametric submanifold in $x+\tilde{D}_{P_1}(x)$. Therefore, it follows from Proposition 4.1 that $L_x^{D_{P_1}} \cap (x+V_{P_1})$ is a proper anti-Kaehlerian isoparametric submanifold in $x+V_{P_1}$. It is clear that $L_x^{D_{P_1}} \cap (x+V_{P_1}) = M_1(x)$. Hence, we obtain this statement. □

Define a distribution D_i' $(i=1,2)$ (respectively D') on M by $D_i'(x) := T_xM_i(x)$ (respectively $D'(x) := T_xM'(x)$) $(x \in M)$.

LEMMA 4.8. (i) *The distributions D_i' $(i=1,2)$ are totally geodesic.*
(ii) *The distribution D' is the orthogonal direct sum of D_1' and D_2'.*

PROOF. Applying Proposition 4.1 to $L_x^{D_{P_i}} \subset \tilde{D}_{P_i}(x)$, $M_i(x)$ is totally geodesic in $L_x^{D_{P_i}}$. Also, by Lemma 4.4, $L_x^{D_{P_i}}$ is totally geodesic in M. Hence, $M_i(x)$ is totally geodesic in M. This implies that D_i' is totally geodesic. Clearly we have $\dim V_{P_i} - \dim D_i' = \dim P_i$ $(i=1,2)$ and $\dim V' - \dim D' = \mathrm{codim}\, M = \dim P_1 + \dim P_2$. According to Proposition 4.2, V' is the orthogonal direct sum of V_{P_1} and V_{P_2}. From these facts, we have $\dim D' = \dim D_1' + \dim D_2'$ and furthermore $D' = D_1' \oplus D_2'$ (orthogonal direct sum). □

For simplicity, we denote $M_i(o)$ $(i=1,2)$ by M_i. Denote by ι' the immersion of M' into V' and by ι_i that of M_i into V_{P_i} $(i=1,2)$. Then we have the following proposition.

PROPOSITION 4.9. (i) *There exist an isometry $\tilde{F}$ of $V_{P_1}\times V_{P_2}$ onto V' and an isometry F of $M_1\times M_2$ onto M' satisfying $\tilde{F}\circ(\iota_1\times\iota_2) = \iota'\circ F$.*
(ii) *M_i is proper anti-Kaehlerian isoparametric in V_{P_i} $(i=1,2)$.*

We prepare the following lemma to show this proposition.

LEMMA 4.10. *Let γ be a curve in M_1 and β_s be a one-parameter family of geodesics in M' with $\beta_s(0)=\gamma(s)$, $\dot{\beta}_0(0) \perp M_1$ and $\nabla'_{\dot{\gamma}(s)}\dot{\beta}_s(0)|_{s=0} = 0$, where ∇' is the Levi-Civita connection of M'. Then we have $(\partial/\partial s)\beta_s(t)|_{s=0} \in D_1'(\beta_0(t))$.*

PROOF. From Lemma 4.8, we can show this statement by imitating the proof of Lemma 3.9 of [4]. □

From this lemma, we have the following fact.

LEMMA 4.11. *For every $x_1 \in M_1$ and every $x_2 \in M_2$, we have $M_1(x_2)\cap M_2(x_1) \neq \emptyset$.*

PROOF. From Lemma 4.10, we can show this statement by imitating the proof of Lemma 3.10 of [4]. □

For $x_1 \in M_1 \subset V_{P_1} \subset V'$, we define an isometry F_{x_1} of V' by $F_{x_1}(u) := u + x_1$ $(u \in V')$.

LEMMA 4.12. (i) *For $x_i \in M_i$ $(i=1,2)$, $M_1(x_2)\cap M_2(x_1) = \{F_{x_1}(x_2)\}$ holds.*

(ii) *This isometry F_{x_1} maps M_2 isometrically onto $M_2(x_1)$.*

PROOF. From Lemma 4.11, we can show these statements by imitating the proof of Corollary 3.11 of [4]. □

By using this lemma, we prove Proposition 4.9.

PROOF OF PROPOSITION 4.9. Define an isometry $\tilde{F}$ of $V_{P_1} \times V_{P_2}$ onto V' by $\tilde{F}(u_1, u_2) := u_1 + u_2$ ($(u_1, u_2) \in V_{P_1} \times V_{P_2}$). From (ii) of Lemma 4.12, we have

$$\begin{aligned}\tilde{F}(M_1 \times M_2) &= \bigcup_{x_1 \in M_1} \tilde{F}(\{x_1\} \times M_2) = \bigcup_{x_1 \in M_1} F_{x_1}(M_2) \\ &= \bigcup_{x_1 \in M_1} M_2(x_1) \subset M'.\end{aligned}$$

Furthermore, it follows from the completenesses of M_i ($i = 1, 2$) that $\tilde{F}(M_1 \times M_2) = M'$. This implies the statement (i). The statement (ii) is shown by imitating the proof of Proposition 4.1(iii). □

Now we prove Theorem 1.

PROOF OF THEOREM 1. From Propositions 4.1 and 4.9, it follows that there exist an isometry $\tilde{F}$ of $V_{P_1} \times V_{P_2} \times V_0$ onto V and an isometry F of the anti-Kaehlerian product manifold $M_1 \times M_2 \times V_0$ onto M satisfying $\tilde{F} \circ (\iota_1 \times \iota_2 \times \mathrm{id}_{V_0}) = \iota \circ F$. Thus, M is regarded as an (extrinsic) product of the proper anti-Kaehlerian isoparametric submanifolds M_1 (in V_{P_1}) and $M_2 \times V_0$ (in $V_{P_2} \times V_0$). This completes the proof of Theorem 1. □

5. Proof of Theorem 2. In this section, we prove Theorem 2. Let M be a proper complex equifocal submanifold in a symmetric space G/K of non-compact type and $M^{\mathbf{c}}$ be the extrinsic complexification of M, where we note that $M^{\mathbf{c}}$ is an anti-Kaehlerian equifocal submanifold in the anti-Kaehlerian symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$ associated with G/K. Let $\phi^{\mathbf{c}} : H^0([0,1], \mathfrak{g}^{\mathbf{c}}) \to G^{\mathbf{c}}$ be the parallel transport map for $G^{\mathbf{c}}$ and $\pi^{\mathbf{c}} : G^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ be the natural projection. See [8] for the definitions of $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ and $\phi^{\mathbf{c}}$. Note that $\phi^{\mathbf{c}}$ and $\pi^{\mathbf{c}}$ are anti-Kaehlerian submersions. Set $\tilde{\phi}^{\mathbf{c}} := \pi^{\mathbf{c}} \circ \phi^{\mathbf{c}}$. Let W be the complex Coxeter group associated with M. As M is proper complex equifocal, $\tilde{\phi}^{\mathbf{c}-1}(M^{\mathbf{c}})$ is a proper anti-Kaehlerian isoparametric submanifold and it extends to a complete submanifold by Theorem 1 of [8]. Denote the complete extension by the same symbol $\tilde{\phi}^{\mathbf{c}-1}(M^{\mathbf{c}})$. Hence, $M^{\mathbf{c}}$ also extends to a complete anti-Kaehlerian equifocal submanifold, which we denote by the same symbol $M^{\mathbf{c}}$. If M is decomposed into an extrinsic product of two proper complex equifocal submanifolds, then $M^{\mathbf{c}}$ is decomposed into an extrinsic product of two proper anti-Kaehlerian equifocal submanifolds. Hence, $\tilde{\phi}^{\mathbf{c}-1}(M^{\mathbf{c}})$ is decomposed into an extrinsic product of two proper anti-Kaehlerian isoparametric submanifolds, that is, W is decomposable. In the sequel, we prove the converse. Assume that W is decomposable. For simplicity, we set $\tilde{M}^{\mathbf{c}} := \tilde{\phi}^{\mathbf{c}-1}(M^{\mathbf{c}})$ and $V := H^0([0,1], \mathfrak{g}^{\mathbf{c}})$. Without loss of generality, we may assume that $\tilde{M}^{\mathbf{c}}$ contains the zero element $\hat{0}$ of V. Denote by J the complex structure of V. According to Lemma 3.8, there exist two J-invariant linear subspaces P_1 ($\neq 0$) and P_2 ($\neq 0$) of $T^{\perp}_{\hat{0}}\tilde{M}^{\mathbf{c}}$ such that $T^{\perp}_{\hat{0}}\tilde{M}^{\mathbf{c}} = P_1 \oplus P_2$ (orthogonal direct sum), $P_1 \cup P_2$ contains all complex curvature normals of $\tilde{M}^{\mathbf{c}}$ at $\hat{0}$ and that P_i ($i = 1, 2$) contain at least one complex curvature normal of $\tilde{M}^{\mathbf{c}}$ at $\hat{0}$. Let $\tilde{P}_i$ ($i = 1, 2$) be $\nabla^{\perp}$-parallel subbundle of $T^{\perp}\tilde{M}^{\mathbf{c}}$ with $\tilde{P}_i(\hat{0}) = P_i$. Set $V_{P_i} := \overline{\mathrm{Span}_J \bigcup_{\tilde{x}\in\tilde{M}^{\mathbf{c}}} \tilde{P}_i(\tilde{x})}$ ($i = 1, 2$), $V' := \overline{\mathrm{Span}_J \bigcup_{\tilde{x}\in\tilde{M}^{\mathbf{c}}} T^{\perp}_{\tilde{x}}\tilde{M}^{\mathbf{c}}}$ and $V_0 := (V')^{\perp}$. According to Proposition 4.2, we have $V = V_{P_1} \oplus V_{P_2} \oplus V_0$ (orthogonal direct sum), which we write as $V = V_{P_1} \times V_{P_2} \times V_0$. Set $\tilde{M}^{\mathbf{c}\prime}(\tilde{x}) := \tilde{M}^{\mathbf{c}} \cap (\tilde{x} + V')$ and $\tilde{M}^{\mathbf{c}}_i(\tilde{x}) := \tilde{M}^{\mathbf{c}} \cap (\tilde{x} + V_{P_i})$, where $\tilde{x} \in \tilde{M}^{\mathbf{c}}$. For simplicity, we denote $\tilde{M}^{\mathbf{c}\prime}(\hat{0})$ (respectively $\tilde{M}^{\mathbf{c}}_i(\hat{0})$) by $\tilde{M}^{\mathbf{c}\prime}$ (respectively $\tilde{M}^{\mathbf{c}}_i$). According to the proof of Theorem 1 in Section 4, there exists an isometry F of $\tilde{M}^{\mathbf{c}}_1 \times \tilde{M}^{\mathbf{c}}_2 \times V_0$ onto $\tilde{M}^{\mathbf{c}}$ satisfying $\tilde{\iota} \circ F = \tilde{\iota}_1 \times \tilde{\iota}_2 \times \mathrm{id}_{V_0}$, where $\tilde{\iota}$ is the immersion of $\tilde{M}^{\mathbf{c}}$ into V and $\tilde{\iota}_i$ ($i = 1, 2$) is that of $\tilde{M}^{\mathbf{c}}_i$ into V_{P_i}. Note that $F(\tilde{M}^{\mathbf{c}}_1 \times \tilde{M}^{\mathbf{c}}_2 \times \{\hat{0}\}) = \tilde{M}^{\mathbf{c}\prime}$. For simplicity, we set $M^{\mathbf{c}*} := \phi^{\mathbf{c}}(\tilde{M}^{\mathbf{c}})$. Set $P^*_i := \phi^{\mathbf{c}}_{*\hat{0}} P_i$ ($i = 1, 2$). Let $\tilde{P}^*_i$ ($i = 1, 2$) be the $\nabla^{\perp *}$-parallel subbundle of $T^{\perp}M^{\mathbf{c}*}$ with $\tilde{P}^*_i(e) = P^*_i$, where $\nabla^{\perp *}$ is the normal connection of $M^{\mathbf{c}*}$. Define ideals $\mathfrak{g}^{\mathbf{c}\prime}$ and $\mathfrak{g}^{\mathbf{c}}_i$ ($i = 1, 2$) of $\mathfrak{g}^{\mathbf{c}}$ by

$$\mathfrak{g}^{\mathbf{c}\prime} := \mathrm{Span}_{\mathbf{c}} \bigcup_{x^*\in M^{\mathbf{c}*}} \{g_{0*}v(x^*)^{-1}_* g^{-1}_{0*} \mid v \in T^{\perp}_{x^*}M^{\mathbf{c}*},\ g_0 \in G^{\mathbf{c}}\}$$

and

$$\mathfrak{g}^{\mathbf{c}}_i := \mathrm{Span}_{\mathbf{c}} \bigcup_{x^*\in M^{\mathbf{c}*}} \{g_{0*}v(x^*)^{-1}_* g^{-1}_{0*} \mid v \in \tilde{P}^*_i(x^*),\ g_0 \in G^{\mathbf{c}}\}\,.$$

Also, set $\mathfrak{g}^{\mathbf{c}}_0 := (\mathfrak{g}^{\mathbf{c}\prime})^{\perp}$, which is also an ideal of $\mathfrak{g}^{\mathbf{c}}$. Let $G^{\mathbf{c}\prime}$, $G^{\mathbf{c}}_0$ and $G^{\mathbf{c}}_i$ ($i = 1, 2$) be the connected Lie subgroups of $G^{\mathbf{c}}$ whose Lie algebras are $\mathfrak{g}^{\mathbf{c}\prime}$, $\mathfrak{g}^{\mathbf{c}}_0$ and $\mathfrak{g}^{\mathbf{c}}_i$ ($i = 1, 2$), respectively. As $G^{\mathbf{c}}$ is simply connected and $\mathfrak{g}^{\mathbf{c}}$, $\mathfrak{g}^{\mathbf{c}}_0$ and $\mathfrak{g}^{\mathbf{c}}_i$ ($i = 1, 2$) are ideals of $\mathfrak{g}^{\mathbf{c}}$, we have $G^{\mathbf{c}} = G^{\mathbf{c}\prime} \times G^{\mathbf{c}}_0$ and $G^{\mathbf{c}\prime} = G^{\mathbf{c}}_1 \times G^{\mathbf{c}}_2$. First we prepare the following lemma.

LEMMA 5.1. *We have* $V' \subset H^0([0,1], \mathfrak{g}^{\mathbf{c}\prime})$ *and* $V_{P_i} \subset H^0([0,1], \mathfrak{g}^{\mathbf{c}}_i)$ $(i = 1, 2)$.

PROOF. Let $v \in T^{\perp}_{x^*}M^{\mathbf{c}*}$ and $\tilde{x} \in \phi^{\mathbf{c}-1}(x^*)$. By the fact (v) in [8, Section 6], we can express as $\tilde{x} = g * \hat{0}$ in terms of some $g \in P(G^{\mathbf{c}}, G^{\mathbf{c}} \times e)$, where $P(G^{\mathbf{c}}, G^{\mathbf{c}} \times e) := \{\bar{g} \in H^1([0,1], G^{\mathbf{c}}) \mid \bar{g}(1) = e\}$. We can show that the horizontal lift $v^L_{\tilde{x}}$ of v to $\tilde{x}$ is equal to $g_* v(x^*)^{-1}_* g^{-1}_*$, where we identify $T_{\tilde{x}}H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ with $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$. Hence, as $T^{\perp}_{\tilde{x}}\tilde{M}^{\mathbf{c}}$ is the horizontal lift $(T^{\perp}_{x^*}M^{\mathbf{c}*})^L_{\tilde{x}}$ of $T^{\perp}_{x^*}M^{\mathbf{c}*}$ to $\tilde{x}$, we have

$$V' = \overline{\mathrm{Span}_J \bigcup_{x^*\in M^{\mathbf{c}*}} \{g_* v(x^*)^{-1}_* g^{-1}_* \mid g \in P(G^{\mathbf{c}}, G^{\mathbf{c}} \times e),\ v \in T^{\perp}_{x^*}M^{\mathbf{c}*}\}}\,,$$

which implies that $V' \subset H^0([0,1], \mathfrak{g}^{\mathbf{c}\prime})$. Similarly, as $\tilde{P}_i(\tilde{x})$ is the horizontal lift $\tilde{P}^*_i(x^*)^L_{\tilde{x}}$ of $\tilde{P}^*_i(x^*)$ to $\tilde{x}$, we have

$$V_{P_i} = \overline{\mathrm{Span}_J \bigcup_{x^*\in M^{\mathbf{c}*}} \{g_* v(x^*)^{-1}_* g^{-1}_* \mid g \in P(G^{\mathbf{c}}, G^{\mathbf{c}} \times e), v \in \tilde{P}^*_i(x^*)\}}$$

($i = 1, 2$), which implies that $V_{P_i} \subset H^0([0,1], \mathfrak{g}^{\mathbf{c}}_i)$. □

REMARK 5.1. We cannot conclude whether Lemma 3.3 of [2] is true because the curve $\alpha\circ\lambda$ in its proof does not necessarily belong to V_0 (i.e. the statement $\int_0^1\phi(\lambda(t))\,dt=0$ in the proof cannot follow from the assumption for α). Similarly, we cannot conclude whether $H^0([0,1],\mathfrak{g}^{\mathbf{c}\prime})=V'$ is true.

Let $\phi^{\mathbf{c}\prime}:H^0([0,1],\mathfrak{g}^{\mathbf{c}\prime})\to G^{\mathbf{c}\prime}$ (respectively $\phi_0^{\mathbf{c}}:H^0([0,1],\mathfrak{g}_0^{\mathbf{c}})\to G_0^{\mathbf{c}}$) be the parallel transport map for $G^{\mathbf{c}\prime}$ (respectively $G_0^{\mathbf{c}}$). It is clear that $\phi^{\mathbf{c}}\circ\tilde F=\phi^{\mathbf{c}\prime}\times\phi_0^{\mathbf{c}}$, where $\tilde F$ is an isometry of $H^0([0,1],\mathfrak{g}^{\mathbf{c}\prime})\times H^0([0,1],\mathfrak{g}_0^{\mathbf{c}})$ onto $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ defined by $\tilde F(u',u_0)=u'+u_0$ $((u',u_0)\in H^0([0,1],\mathfrak{g}^{\mathbf{c}\prime})\times H^0([0,1],\mathfrak{g}_0^{\mathbf{c}}))$. From $\hat 0\in\tilde M^{\mathbf{c}}$, we have $e\in M^{\mathbf{c}*}$, where e is the identity element of $G^{\mathbf{c}}$. Set $M^{\mathbf{c}*\prime}:=M^{\mathbf{c}*}\cap G^{\mathbf{c}\prime}$, which is regarded as an immersed submanifold in $G^{\mathbf{c}\prime}$. Denote by $\iota^{*\prime}$ the immersion of $M^{\mathbf{c}*\prime}$ into $G^{\mathbf{c}\prime}$ and by ι^* that of $M^{\mathbf{c}*}$ into $G^{\mathbf{c}}$.

PROPOSITION 5.2. (i) *There exists an isometry F of the anti-Kaehlerian product manifold $M^{\mathbf{c}*\prime}\times G_0^{\mathbf{c}}$ onto $M^{\mathbf{c}*}$ satisfying $\iota^*\circ F=\iota^{*\prime}\times\mathrm{id}_{G_0^{\mathbf{c}}}$.*

(ii) *$M^{\mathbf{c}*\prime}$ is proper anti-Kaehlerian equifocal in $G^{\mathbf{c}\prime}$.*

PROOF. As $V'\subset H^0([0,1],\mathfrak{g}^{\mathbf{c}\prime})$ by Lemma 5.1 and $V=V'\oplus V_0=H^0([0,1],\mathfrak{g}^{\mathbf{c}\prime})\oplus H^0([0,1],\mathfrak{g}_0^{\mathbf{c}})$ (orthogonal direct sum), we have $H^0([0,1],\mathfrak{g}_0^{\mathbf{c}})\subset V_0$. Let V_0' be the orthogonal complement of $H^0([0,1],\mathfrak{g}_0^{\mathbf{c}})$ in V_0. Clearly we have $H^0([0,1],\mathfrak{g}^{\mathbf{c}\prime})=V'\oplus V_0'$ (orthogonal direct sum). According to (i) of Proposition 4.1, the submanifold $\tilde M^{\mathbf{c}}$ is regarded as the anti-Kaehlerian product submanifold $\tilde M^{\mathbf{c}\prime}\times V_0$. From these facts, we have

$$\begin{aligned}M^{\mathbf{c}*}&=\phi^{\mathbf{c}}(\tilde M^{\mathbf{c}})=\phi^{\mathbf{c}}(\tilde M^{\mathbf{c}\prime}\times V_0)\\&=(\phi^{\mathbf{c}\prime}\times\phi_0^{\mathbf{c}})(\tilde M^{\mathbf{c}\prime}\times V_0'\times H^0([0,1],\mathfrak{g}_0^{\mathbf{c}}))\\&=\phi^{\mathbf{c}\prime}(\tilde M^{\mathbf{c}\prime}\times V_0')\times G_0^{\mathbf{c}}\\&=\phi^{\mathbf{c}\prime}(\tilde M^{\mathbf{c}}\cap H^0([0,1],\mathfrak{g}^{\mathbf{c}\prime}))\times G_0^{\mathbf{c}}=M^{\mathbf{c}*\prime}\times G_0^{\mathbf{c}}\,.\end{aligned}$$

This implies the statement (i). According to (iii) of Proposition 4.1, $\tilde M^{\mathbf{c}\prime}$ is proper anti-Kaehlerian isoparametric in V' and hence $\tilde M^{\mathbf{c}\prime}\times V_0'$ is proper anti-Kaehlerian isoparametric in $H^0([0,1],\mathfrak{g}^{\mathbf{c}\prime})$. On the other hand, it is clear that $\tilde M^{\mathbf{c}\prime}\times V_0'=\phi^{\mathbf{c}\prime\,-1}(M^{\mathbf{c}*\prime})$. Therefore, it follows from Proposition 4 of [8] and its proof that $M^{\mathbf{c}*\prime}$ is proper anti-Kaehlerian equifocal in $G^{\mathbf{c}\prime}$. □

We can show the following lemma by imitating the proof of Lemma 3.7 of [2].

LEMMA 5.3. *We have $\mathfrak{g}_1^{\mathbf{c}}\perp\mathfrak{g}_2^{\mathbf{c}}$ and hence $H^0([0,1],\mathfrak{g}^{\mathbf{c}\prime})=H^0([0,1],\mathfrak{g}_1^{\mathbf{c}})\oplus H^0([0,1],\mathfrak{g}_2^{\mathbf{c}})$ (orthogonal direct sum).*

PROOF. First we show $\mathfrak{g}_1^{\mathbf{c}}\perp\mathfrak{g}_2^{\mathbf{c}}$. Let $g_{i*}v_i(x_i^*)_*^{-1}g_{i*}^{-1}\in\mathfrak{g}_i^{\mathbf{c}}$ $(i=1,2)$, where $x_i^*\in M^{\mathbf{c}*}$, $g_i\in G^{\mathbf{c}}$ and $v_i\in\tilde P_i^*(x_i^*)$ $(i=1,2)$. We have only to show $\langle g_{1*}v_1(x_1^*)_*^{-1}g_{1*}^{-1},\ g_{2*}v_2(x_2^*)_*^{-1}g_{2*}^{-1}\rangle=0$. Suppose that $\langle g_{1*}v_1(x_1^*)_*^{-1}g_{1*}^{-1},\ g_{2*}v_2(x_2^*)_*^{-1}g_{2*}^{-1}\rangle\neq 0$. Take $\tilde g_i^0\in P(G^{\mathbf{c}},e\times G^{\mathbf{c}})$ with $\phi^{\mathbf{c}}(\tilde g_i^0*\hat 0)(=\tilde g_i^{0-1}(1))=x_i^*$ and $\tilde g_i^0(1/2)=g_i$ $(i=1,2)$. Set

$$\psi(t):=\langle\tilde g_1^0(t)_*v_1(x_1^*)_*^{-1}\tilde g_1^0(t)_*^{-1},\ \tilde g_2^0(t)_*v_2(x_2^*)_*^{-1}\tilde g_2^0(t)_*^{-1}\rangle$$

($t \in [0,1]$). As $\tilde{g}^0_{i*} v_i (x_i^*)_*^{-1} \tilde{g}^{0-1}_{i*} = (v_i)^L_{\tilde{g}^0_i * \hat{0}} \in \tilde{P}_i(\tilde{g}^0_i * \hat{0})$ ($i = 1, 2$) and $\tilde{P}_1(\tilde{g}^0_1 * \hat{0}) \perp \tilde{P}_2(\tilde{g}_2 * \hat{0})$ by Proposition 4.2, we have $\int_0^1 \psi(t)dt = 0$. There exists $\varepsilon > 0$ such that $\psi(t)\psi(1/2) > 0$ for all $t \in [1/2-\varepsilon, 1/2+\varepsilon]$ because of $\psi(1/2) \neq 0$. For simplicity, set $t_1 = 1/2 - \varepsilon$ and $t_2 = 1/2 + \varepsilon$. Define a function λ over $[0, 1]$ by

$$\lambda(t) := \begin{cases} 3t_2 t & (0 \le t \le 1/3) \\ 2t_2 - t_1 - 3(t_2 - t_1)t & (1/3 \le t \le 2/3) \\ t_1 - 2t_2 + 3t_2 t & (2/3 \le t \le 1)\,. \end{cases}$$

Then we have

$$\int_0^1 \psi(\lambda(t))dt = \frac{1}{3t_2}\int_0^1 \psi(t)dt + \frac{2t_2 - t_1}{3t_2(t_2 - t_1)}\int_{t_1}^{t_2} \psi(t)dt \neq 0.$$

On the other hand, we have

$$\begin{aligned} \int_0^1 \psi(\lambda(t))dt &= \langle (\tilde{g}^0_1 \circ \lambda)_* v_1 (x_1^*)_*^{-1} (\tilde{g}^0_1 \circ \lambda)_*^{-1},\ (\tilde{g}^0_2 \circ \lambda)_* v_2 (x_2^*)_*^{-1} (\tilde{g}^0_2 \circ \lambda)_*^{-1} \rangle_0 \\ &= \langle (v_1)^L_{(\tilde{g}^0_1 \circ \lambda) * \hat{0}},\ (v_2)^L_{(\tilde{g}^0_1 \circ \lambda) * \hat{0}} \rangle_0 = 0 \end{aligned}$$

because of $\tilde{P}_1((\tilde{g}^0_1 \circ \lambda) * \hat{0}) \perp \tilde{P}_2((\tilde{g}^0_2 \circ \lambda) * \hat{0})$, where we note that $\phi^{\mathbf{c}}((\tilde{g}^0_i \circ \lambda) * \hat{0}) = (\tilde{g}^0_i \circ \lambda)(1)^{-1} = \tilde{g}^0_i(1)^{-1} = x_i^*$ and hence $(v_i)^L_{(\tilde{g}^0_i \circ \lambda) * \hat{0}}$ is defined. Thus, a contradiction arises. Hence, we obtain $\langle g_{1*} v_1 (x_1^*)_*^{-1} g_{1*}^{-1},\ g_{2*} v_2 (x_2^*)_*^{-1} g_{2*}^{-1} \rangle = 0$. Thus, $\mathfrak{g}_1^{\mathbf{c}} \perp \mathfrak{g}_2^{\mathbf{c}}$ is shown. Furthermore, as $T^{\perp}_{x^*} M^{\mathbf{c}*} = \tilde{P}_1^*(x^*) \oplus \tilde{P}_2^*(x^*)$, we have $\mathfrak{g}^{\mathbf{c}\prime} = \mathfrak{g}_1^{\mathbf{c}} \oplus \mathfrak{g}_2^{\mathbf{c}}$ (orthogonal direct sum) and hence $H^0([0,1], \mathfrak{g}^{\mathbf{c}\prime}) = H^0([0,1], \mathfrak{g}_1^{\mathbf{c}}) \oplus H^0([0,1], \mathfrak{g}_2^{\mathbf{c}})$ (orthogonal direct sum). □

Let $\phi_i^{\mathbf{c}} : H^0([0,1], \mathfrak{g}_i^{\mathbf{c}}) \to G_i^{\mathbf{c}}$ ($i = 1, 2$) be the parallel transport map for $G_i^{\mathbf{c}}$. Set $M_i^{\mathbf{c}*} := M^{\mathbf{c}*} \cap G_i^{\mathbf{c}}$ ($i = 1, 2$), which is regarded as an immersed submanifold in $G_i^{\mathbf{c}}$. Denote by ι_i^* the immersion of $M_i^{\mathbf{c}*}$ into $G_i^{\mathbf{c}}$.

PROPOSITION 5.4. (i) *There exists an isometry F of the anti-Kaehlerian product manifold $M_1^{\mathbf{c}*} \times M_2^{\mathbf{c}*}$ onto $M^{\mathbf{c}*\prime}$ satisfying $\iota^{*\prime} \circ F = \iota_1^* \times \iota_2^*$.*

(ii) *$M_i^{\mathbf{c}*}$ is proper anti-Kaehlerian equifocal in $G_i^{\mathbf{c}}$ ($i = 1, 2$).*

PROOF. Let V_i' ($i = 1, 2$) be the orthogonal complement of V_{P_i} in $H^0([0,1], \mathfrak{g}_i^{\mathbf{c}})$. From Lemma 5.3, we have $\phi^{\mathbf{c}\prime} = \phi_1^{\mathbf{c}} \times \phi_2^{\mathbf{c}}$. Also, from the proof of Proposition 5.2, we have $M^{\mathbf{c}*\prime} = \phi^{\mathbf{c}\prime}(\tilde{M}^{\mathbf{c}\prime} \times V_0')$, where V_0' is as in the proof of Proposition 5.2. It is clear that $V_0' = V_1' \oplus V_2'$ (orthogonal direct sum). Also, according to Proposition 4.9(i), the submanifold $\tilde{M}^{\mathbf{c}\prime}$

is regarded as the anti-Kaehlerian product submanifold $\tilde{M}_1^{\mathbf{c}} \times \tilde{M}_2^{\mathbf{c}}$. From these facts, we have

$$\begin{aligned} M^{\mathbf{c}*\prime} &= (\phi_1^{\mathbf{c}} \times \phi_2^{\mathbf{c}})(\tilde{M}_1^{\mathbf{c}} \times \tilde{M}_2^{\mathbf{c}} \times V_0') \\ &= (\phi_1^{\mathbf{c}} \times \phi_2^{\mathbf{c}})((\tilde{M}_1^{\mathbf{c}} \times V_1') \times (\tilde{M}_2^{\mathbf{c}} \times V_2')) \\ &= \phi_1^{\mathbf{c}}(\tilde{M}_1^{\mathbf{c}} \times V_1') \times \phi_2^{\mathbf{c}}(\tilde{M}_2^{\mathbf{c}} \times V_2') \\ &= \phi_1^{\mathbf{c}}(\tilde{M}^{\mathbf{c}} \cap H^0([0,1], \mathfrak{g}_1^{\mathbf{c}})) \times \phi_2^{\mathbf{c}}(\tilde{M}^{\mathbf{c}} \cap H^0([0,1], \mathfrak{g}_2^{\mathbf{c}})) \\ &= M_1^{\mathbf{c}*} \times M_2^{\mathbf{c}*}. \end{aligned}$$

This implies the statement (i). According to Proposition 4.9(ii), $\tilde{M}_i^{\mathbf{c}}$ is proper anti-Kaehlerian isoparametric in V_{P_i} and hence $\tilde{M}_i^{\mathbf{c}} \times V_i'$ is proper anti-Kaehlerian isoparametric in $H^0([0,1], \mathfrak{g}_i^{\mathbf{c}})$. On the other hand, it is clear that $\tilde{M}_i^{\mathbf{c}} \times V_i' = \phi_i^{\mathbf{c}-1}(M_i^{\mathbf{c}*})$. Therefore, it follows from Proposition 4 of [8] and its proof that $M_i^{\mathbf{c}*}$ is proper anti-Kaehlerian equifocal in $G_i^{\mathbf{c}}$. □

We have the following splitting theorem for $M^{\mathbf{c}*}$ from Propositions 5.2 and 5.4.

THEOREM 5.5. *There exists an isometry F of the anti-Kaehlerian product manifold $M_1^{\mathbf{c}*} \times M_2^{\mathbf{c}*} \times G_0^{\mathbf{c}}$ onto $M^{\mathbf{c}*}$ satisfying $\iota^* \circ F = \iota_1^* \times \iota_2^* \times \mathrm{id}_{G_0^{\mathbf{c}}}$.*

Next we prove a splitting theorem for $M^{\mathbf{c}}$. Let $s : G \to G$ be the involution of G such that the set of all fixed points of s is equal to K and set $\theta := s_{*e} (: \mathfrak{g} \to \mathfrak{g})$. Also, let $\theta^{\mathbf{c}} : \mathfrak{g}^{\mathbf{c}} \to \mathfrak{g}^{\mathbf{c}}$ be the complexification of θ. Then it is clear that $(\mathfrak{g}^{\mathbf{c}}, \theta^{\mathbf{c}})$ is the orthogonal symmetric Lie algebra associated with $G^{\mathbf{c}}/K^{\mathbf{c}}$. First we show the following lemma by imitating the argument in [2, Section 4].

LEMMA 5.6. *We have $\theta^{\mathbf{c}}(\mathfrak{g}_i^{\mathbf{c}}) = \mathfrak{g}_i^{\mathbf{c}}$ $(i = 0, 1, 2)$.*

PROOF. Let $\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r$ be the simple ideal decomposition of $\mathfrak{g}$. Then it is clear that $\mathfrak{g}^{\mathbf{c}} = \mathfrak{h}_1^{\mathbf{c}} \oplus \cdots \oplus \mathfrak{h}_r^{\mathbf{c}}$ is the simple ideal decomposition of $\mathfrak{g}^{\mathbf{c}}$. As $\mathfrak{g}_i^{\mathbf{c}}$ $(i = 0, 1, 2)$ are ideals of $\mathfrak{g}^{\mathbf{c}}$, we can express as $\mathfrak{g}_i^{\mathbf{c}} = \mathfrak{h}_{i_1}^{\mathbf{c}} \oplus \cdots \oplus \mathfrak{h}_{i_{m_i}}^{\mathbf{c}}$ $(i = 0, 1, 2)$. Let $(\mathfrak{g}, \theta) = (\mathfrak{i}_1, \theta_1) \times \cdots \times (\mathfrak{i}_l, \theta_l)$ be the irreducible orthogonal symmetric Lie algebra decomposition of $(\mathfrak{g}, \theta)$, where $\theta_j = \theta|_{\mathfrak{i}_j}$ $(j = 1, \ldots, l)$. Then it is clear that $(\mathfrak{g}^{\mathbf{c}}, \theta^{\mathbf{c}}) = (\mathfrak{i}_1^{\mathbf{c}}, \theta_1^{\mathbf{c}}) \times \cdots \times (\mathfrak{i}_l^{\mathbf{c}}, \theta_l^{\mathbf{c}})$ is the orthogonal symmetric Lie algebra decomposition of $(\mathfrak{g}^{\mathbf{c}}, \theta^{\mathbf{c}})$. For each $(\mathfrak{i}_j, \theta_j)$, one of the following holds:

(I) $\mathfrak{i}_j = \mathfrak{h}_{j'}$ for some $j' \in \{1, \ldots, r\}$; or

(II) $\mathfrak{i}_j = \mathfrak{h}_{j'} \oplus \mathfrak{h}_{j''}$ for some $j', j'' \in \{1, \ldots, r\}$ and $\theta_j(\mathfrak{h}_{j'}) = \mathfrak{h}_{j''}$;

(see [6]). Suppose that $\theta^{\mathbf{c}}(\mathfrak{g}_1^{\mathbf{c}}) \neq \mathfrak{g}_1^{\mathbf{c}}$. Then there exists $(k_0, j_1, j_2) \in \{1, \ldots, l\} \times \{1_1, \ldots, 1_{m_1}\} \times (\{0_1, \ldots, 0_{m_0}\} \cup \{2_1, \ldots, 2_{m_2}\})$ satisfying $\mathfrak{i}_{k_0}^{\mathbf{c}} = \mathfrak{h}_{j_1}^{\mathbf{c}} \oplus \mathfrak{h}_{j_2}^{\mathbf{c}}$. Clearly we have $\{X + \theta^{\mathbf{c}}(X) \mid X \in \mathfrak{h}_{j_1}^{\mathbf{c}}\} \subset \mathfrak{f}^{\mathbf{c}}$. Also, from $M^{\mathbf{c}*} = \pi^{\mathbf{c}-1}(M^{\mathbf{c}})$, we have $\mathfrak{f}^{\mathbf{c}} \subset T_e M^{\mathbf{c}*}$. Hence, for each $X \in \mathfrak{h}_{j_1}^{\mathbf{c}}$, we have

$$X + \theta^{\mathbf{c}}(X) \in T_e M^{\mathbf{c}*} = T_e M_1^{\mathbf{c}*} \oplus T_e M_2^{\mathbf{c}*} \oplus \mathfrak{g}_0^{\mathbf{c}},$$

that is, $X \in T_e M_1^{\mathbf{c}*}$ and $\theta^{\mathbf{c}}(X) \in T_e M_2^{\mathbf{c}*} \oplus \mathfrak{g}_0^{\mathbf{c}}$. Thus, we have $\mathfrak{h}_{j_1}^{\mathbf{c}} \subset T_e M_1^{\mathbf{c}*}$ and $\mathfrak{h}_{j_2}^{\mathbf{c}} \subset T_e M_2^{\mathbf{c}*} \oplus \mathfrak{g}_0^{\mathbf{c}}$. Therefore, we have $\mathfrak{i}_{k_0}^{\mathbf{c}} \subset T_e M^{\mathbf{c}*}$. Next we show that $g_{0*}\mathfrak{i}_{k_0}^{\mathbf{c}} \subset T_{g_0} M^{\mathbf{c}*}$ for each

$g_0 \in M^{\mathbf{c}*}$. We denote the quantities for $g_0^{-1}M^{\mathbf{c}*}$ corresponding to $\mathfrak{g}_i^{\mathbf{c}}$ $(i = 0, 1, 2)$ (defined for $M^{\mathbf{c}*}$) by $\hat{\mathfrak{g}}_i^{\mathbf{c}}$ $(= 0, 1, 2)$. Then we have

$$\begin{aligned}\hat{\mathfrak{g}}_i^{\mathbf{c}} &= \mathrm{Span}_{\mathbf{c}} \bigcup_{x^* \in g_0^{-1}M^{\mathbf{c}*}} \{g_{1*}v(x^*)_*^{-1}g_{1*}^{-1} \mid v \in g_{0*}^{-1}\tilde{P}_i^*(g_0x^*),\ g_1 \in G^{\mathbf{c}}\} \\ &= \mathrm{Span}_{\mathbf{c}} \bigcup_{x^* \in g_0^{-1}M^{\mathbf{c}*}} \{g_{1*}(g_{0*}^{-1}v)(g_0x^*)_*^{-1}g_{0*}g_{1*}^{-1} \mid v \in \tilde{P}_i^*(g_0x^*),\ g_1 \in G^{\mathbf{c}}\} \\ &= \mathrm{Span}_{\mathbf{c}} \bigcup_{x^* \in M^{\mathbf{c}*}} \{(g_1g_0^{-1})_*v(x^*)_*^{-1}(g_1g_0^{-1})_*^{-1} \mid v \in \tilde{P}_i^*(x^*),\ g_1 \in G^{\mathbf{c}}\} = \mathfrak{g}_i^{\mathbf{c}}\end{aligned}$$

$(i = 1, 2)$. Hence, we also have $\hat{\mathfrak{g}}_0^{\mathbf{c}} = \mathfrak{g}_0^{\mathbf{c}}$. Therefore, we can show $\mathfrak{i}_{k_0}^{\mathbf{c}} \subset T_e(g_0^{-1}M^{\mathbf{c}*})$ in a similar manner to $\mathfrak{i}_{k_0}^{\mathbf{c}} \subset T_eM^{\mathbf{c}*}$. That is, we have $g_{0*}\mathfrak{i}_{k_0}^{\mathbf{c}} \subset T_{g_0}M^{\mathbf{c}*}$. Let $I_j^{\mathbf{c}}$ $(j = 1, \dots, l)$ be the connected Lie subgroup of $G^{\mathbf{c}}$ whose Lie algebra is $\mathfrak{i}_j^{\mathbf{c}}$. We have $G^{\mathbf{c}} = I_1^{\mathbf{c}} \times \cdots \times I_l^{\mathbf{c}}$. For simplicity, we express as $G^{\mathbf{c}} = I_{k_0}^{\mathbf{c}} \times H$, where $H := I_1^{\mathbf{c}} \times \cdots \times I_{k_0-1}^{\mathbf{c}} \times I_{k_0+1}^{\mathbf{c}} \times \cdots \times I_l^{\mathbf{c}}$. As $T_{g_0}g_0I_{k_0}^{\mathbf{c}} = g_{0*}\mathfrak{i}_{k_0}^{\mathbf{c}} \subset T_{g_0}M^{\mathbf{c}*}$, we have $M^{\mathbf{c}*} = \bigcup_{g_0 \in M^{\mathbf{c}*}} g_0I_{k_0}^{\mathbf{c}}$. That is $M^{\mathbf{c}*}$ is expressed as $M^{\mathbf{c}*} = \bigcup_{g_0 \in M^{\mathbf{c}*} \cap H}(I_{k_0}^{\mathbf{c}} \times \{g_0\})$. This fact deduces $I_{k_0}^{\mathbf{c}} \subset G_0^{\mathbf{c}}$, that is, $\mathfrak{i}_{k_0}^{\mathbf{c}} \subset \mathfrak{g}_0^{\mathbf{c}}$, which contradicts $\mathfrak{i}_{k_0}^{\mathbf{c}} \cap \mathfrak{g}_1^{\mathbf{c}} = \mathfrak{h}_{j_1}^{\mathbf{c}} \neq \{0\}$. Therefore, we obtain $\theta^{\mathbf{c}}(\mathfrak{g}_1^{\mathbf{c}}) = \mathfrak{g}_1^{\mathbf{c}}$. Similarly, we can obtain $\theta^{\mathbf{c}}(\mathfrak{g}_2^{\mathbf{c}}) = \mathfrak{g}_2^{\mathbf{c}}$. Hence, we also have $\theta^{\mathbf{c}}(\mathfrak{g}_0^{\mathbf{c}}) = \mathfrak{g}_0^{\mathbf{c}}$. □

Let $\mathfrak{f}_i^{\mathbf{c}}$ $(i = 0, 1, 2)$ be the eigenspace of $\theta|_{\mathfrak{g}_i^{\mathbf{c}}}$ for 1, where we note that $\theta|_{\mathfrak{g}_i^{\mathbf{c}}}$ is an involution of $\mathfrak{g}_i^{\mathbf{c}}$ by Lemma 5.6. Let $K_i^{\mathbf{c}}$ $(i = 0, 1, 2)$ be the connected Lie subgroup of $G^{\mathbf{c}}$ whose Lie algebra is $\mathfrak{f}_i^{\mathbf{c}}$. Let $\mathfrak{g}_i := \mathfrak{g}_i^{\mathbf{c}} \cap \mathfrak{g}$ $(i = 0, 1, 2)$ and G_i $(i = 0, 1, 2)$ be the connected Lie subgroup of G whose Lie algebra is $\mathfrak{g}_i$. We can show $(\mathfrak{g}_i)^{\mathbf{c}} = \mathfrak{g}_i^{\mathbf{c}}$ $(i = 0, 1, 2)$. It follows from this fact and $\theta^{\mathbf{c}}(\mathfrak{g}_i^{\mathbf{c}}) = \mathfrak{g}_i^{\mathbf{c}}$ that $\theta(\mathfrak{g}_i) = \mathfrak{g}_i$ $(i = 0, 1, 2)$. Let $\mathfrak{f}_i$ $(i = 0, 1, 2)$ be the eigenspace of $\theta|_{\mathfrak{g}_i}$ for 1 and K_i be the connected Lie subgroup of G whose Lie algebra is $\mathfrak{f}_i$. It is shown that $G_i^{\mathbf{c}}/K_i^{\mathbf{c}}$ $(i = 0, 1, 2)$ is the anti-Kaehlerian symmetric space associated with G_i/K_i, $G^{\mathbf{c}}/K^{\mathbf{c}} = G_1^{\mathbf{c}}/K_1^{\mathbf{c}} \times G_2^{\mathbf{c}}/K_2^{\mathbf{c}} \times G_0^{\mathbf{c}}/K_0^{\mathbf{c}}$ and that $G/K = G_1/K_1 \times G_2/K_2 \times G_0/K_0$. Regard $G_i^{\mathbf{c}}/K_i^{\mathbf{c}}$ (respectively G_i/K_i) $(i = 0, 1, 2)$ as totally geodesic submanifolds in $G^{\mathbf{c}}/K^{\mathbf{c}}$ (respectively G/K) through $eK^{\mathbf{c}}$ (respectively eK). Set $M_i^{\mathbf{c}} := M^{\mathbf{c}} \cap G_i^{\mathbf{c}}/K_i^{\mathbf{c}}$ $(i = 1, 2)$, which is regarded as an immersed submanifold in $G_i^{\mathbf{c}}/K_i^{\mathbf{c}}$. Denote by ι_i the immersion of $M_i^{\mathbf{c}}$ into $G_i^{\mathbf{c}}/K_i^{\mathbf{c}}$ and by ι that of $M^{\mathbf{c}}$ into $G^{\mathbf{c}}/K^{\mathbf{c}}$. We have the following splitting theorem for $M^{\mathbf{c}}$ from Theorem 5.5.

THEOREM 5.7. (i) *There exists an isometry F of the anti-Kaehlerian product manifold $M_1^{\mathbf{c}} \times M_2^{\mathbf{c}} \times G_0^{\mathbf{c}}/K_0^{\mathbf{c}}$ onto $M^{\mathbf{c}}$ satisfying $\iota \circ F = \iota_1 \times \iota_2 \times \mathrm{id}_{G_0^{\mathbf{c}}/K_0^{\mathbf{c}}}$.*

(ii) *$M_i^{\mathbf{c}}$ is proper anti-Kaehlerian equifocal in $G_i^{\mathbf{c}}/K_i^{\mathbf{c}}$ $(i = 1, 2)$.*

PROOF. Denote by $\pi_i^{\mathbf{c}}$ $(i = 0, 1, 2)$ the natural projection of $G_i^{\mathbf{c}}$ onto $G_i^{\mathbf{c}}/K_i^{\mathbf{c}}$. Clearly we have $\pi_i^{\mathbf{c}-1}(M_i^{\mathbf{c}}) = M_i^{\mathbf{c}*}$ $(i = 1, 2)$. As $M^{\mathbf{c}*}$ is identified with the anti-Kaehlerian product submanifold $M_1^{\mathbf{c}*} \times M_2^{\mathbf{c}*} \times G_0^{\mathbf{c}}$ by Theorem 5.5, we have

$$\begin{aligned}M^{\mathbf{c}} &= \pi^{\mathbf{c}}(M^{\mathbf{c}*}) = (\pi_1^{\mathbf{c}} \times \pi_2^{\mathbf{c}} \times \pi_0^{\mathbf{c}})(M_1^{\mathbf{c}*} \times M_2^{\mathbf{c}*} \times G_0^{\mathbf{c}}) \\ &= \pi_1^{\mathbf{c}}(M_1^{\mathbf{c}*}) \times \pi_2^{\mathbf{c}}(M_2^{\mathbf{c}*}) \times G_0^{\mathbf{c}}/K_0^{\mathbf{c}} = M_1^{\mathbf{c}} \times M_2^{\mathbf{c}} \times G_0^{\mathbf{c}}/K_0^{\mathbf{c}},\end{aligned}$$

which implies the statement (i). As $M_i^{\mathbf{c}*}$ is proper anti-Kaehlerian equifocal in $G_i^{\mathbf{c}}$ by (ii) of Proposition 5.4 and $M_i^{\mathbf{c}*} = \pi_i^{\mathbf{c}-1}(M_i^{\mathbf{c}})$, it follows from Proposition 4 of [8] and its proof that $M_i^{\mathbf{c}}$ is proper anti-Kaehlerian equifocal in $G_i^{\mathbf{c}}/K_i^{\mathbf{c}}$. □

Set $M_i := M \cap G_i/K_i$ $(i = 1, 2)$, which is regarded as an immersed submanifold in G_i/K_i. Denote by $\bar{\iota}_i$ the immersion of M_i into G_i/K_i and by $\bar{\iota}$ that of M into G/K.

PROOF OF THEOREM 2. Let $\iota_{G/K}$ be the natural immersion of G/K into $G^{\mathbf{c}}/K^{\mathbf{c}}$ and ι_{G_i/K_i} $(i = 1, 2)$ be that of G_i/K_i into $G_i^{\mathbf{c}}/K_i^{\mathbf{c}}$ $(i = 0, 1, 2)$. Clearly we have $\iota_{G/K} = \prod_{i=0}^{2} \iota_{G_i/K_i}$. As $M^{\mathbf{c}}$ is identified with the anti-Kaehlerian product submanifold $M_1^{\mathbf{c}} \times M_2^{\mathbf{c}} \times G_0^{\mathbf{c}}/K_0^{\mathbf{c}}$ by Theorem 5.7, we have

$$\iota_{G/K}^{-1}(M^{\mathbf{c}}) = \iota_{G_1/K_1}^{-1}(M_1^{\mathbf{c}}) \times \iota_{G_2/K_2}^{-1}(M_2^{\mathbf{c}}) \times G_0/K_0 \,.$$

Let M_i' $(i = 1, 2)$ be the maximal connected open submanifold of $\iota_{G_i/K_i}^{-1}(M_i^{\mathbf{c}})$ containing eK. As M is the maximal connected open submanifold of $\iota_{G/K}^{-1}(M^{\mathbf{c}})$ containing eK, we have $M = M_1' \times M_2' \times G_0/K_0$. This fact implies $M_i' = M_i$ $(i = 1, 2)$. Therefore, it follows that there exists an isometry F of the Riemannian product manifold $M_1 \times M_2 \times G_0/K_0$ onto M satisfying $\bar{\iota} \circ F = \bar{\iota}_1 \times \bar{\iota}_2 \times \mathrm{id}_{G_0/K_0}$. As $M_i^{\mathbf{c}}$ is proper anti-Kaehlerian equifocal in $G_i^{\mathbf{c}}/K_i^{\mathbf{c}}$ $(i = 1, 2)$ by Theorem 5.7(ii), it follows from Theorem 6 of [8] and its proof that M_i is proper complex equifocal in G_i/K_i $(i = 1, 2)$. Thus, M is decomposed into the extrinsic product of two proper complex equifocal submanifolds M_1 (in G_1/K_1) and $M_2 \times G_0/K_0$ (in $G_2/K_2 \times G_0/K_0$). □

6. The complex Coxeter groups of the principal orbits of actions of Hermann type. In this section, we recall examples of proper complex equifocal submanifolds given in [9] and describe explicitly the generators of the complex Coxeter groups associated with them. Let G/K be a symmetric space of non-compact type and H be the subgroup of G consisting of all fixed points of an involution σ of G. Note that the H-action on G/K is conjugate to the dual action of a Hermann action on the compact dual G^*/K of G/K. Hence, we call such an action on G/K an *action of Hermann type*. Denote by θ the Cartan involution associated with G/K. We may assume that $\sigma \circ \theta = \theta \circ \sigma$ by replacing H to a suitable conjugate group if necessary. Then the orbit *HeK* is totally geodesic (see [9, Lemma 4.2]). Let $\mathfrak{p}$ be the eigenspace of θ_{*e} for -1. In [9], we showed the following fact.

FACT 2. *The principal orbits of the H-action on G/K are curvature adapted and proper complex equifocal.*

Now we describe explicitly the generators of the complex Coxeter group associated with the principal orbit. Let $H(\exp Z)K$ $(Z \in \mathfrak{p})$ be a principal orbit of the H-action. Denote this orbit by M and its shape tensor by A. For simplicity, set $g := \exp Z$. There exists an r-dimensional abelian subspace $\mathfrak{t}$ of $\mathfrak{p}' := T_{eK}^{\perp} HeK (\subset \mathfrak{p})$ containing Z, where r is the cohomogeneity of the H-action. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{t}$ and $\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Delta_+} \mathfrak{p}_\alpha$ be the root space decomposition with respect to $\mathfrak{a}$. As *HeK* has Lie triple systematic normal bundle and M is a partial tube over HeK (see [9, Lemma 4.2]), we have

$$T_{gK}M = \left(\bigoplus_{\alpha\in\Delta_+\cup\{0\}}\{\tilde{X}_Z \mid X \in \mathfrak{p}_\alpha \cap T_{eK}HeK\}\right) \oplus \left(\bigoplus_{\beta\in\Delta_+} g_*(\mathfrak{p}_\beta \cap \mathfrak{p}')\right), \tag{6.1}$$

where $\tilde{X}_Z$ is the horizontal lift of X to Z and $\mathfrak{p}_0 = \mathfrak{a}$. For simplicity, we set $H_\alpha := \{\tilde{X}_Z \mid X \in \mathfrak{p}_\alpha \cap T_{eK}HeK\}$ $(\alpha \in \Delta_+ \cup \{0\})$ and $V_\beta := g_*(\mathfrak{p}_\beta \cap \mathfrak{p}')$ $(\beta \in \Delta_+)$. Furthermore, as HeK is totally geodesic, it follows from Corollary 3.2 of [9] that

$$A_v\tilde{X}_Z = -\alpha(g_*^{-1}v)\tanh\alpha(Z)\tilde{X}_Z \quad (\tilde{X}_Z \in H_\alpha,\ v \in T^\perp_{gK}M)\,. \tag{6.2}$$

Let L be the group of all fixed points of $\sigma\circ\theta$. Then we can show that $L/H\cap K$ is a symmetric space, $\mathfrak{p}'$ is regarded as $T_{e(H\cap K)}L/H\cap K$ and that $\Delta'_+ := \{\alpha|_\mathfrak{t} \mid \alpha \in \Delta_+\}$ is regarded as a positive root system with respect to a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{p}' = T_{e(H\cap K)}L/H\cap K$. As $M\cap\exp^\perp(\mathfrak{p}')$ is catched as a principal orbit of the isotropy action of $L/H\cap K$, we have

$$A_vY = \frac{\beta(g_*^{-1}v)}{\tanh\beta(Z)}Y \quad (Y \in V_\beta, v \in T^\perp_{gK}M) \tag{6.3}$$

in terms of Proposition 3.1(i) in [9], where we note $\beta(Z) \neq 0$ because $H(\exp Z)K$ is a principal orbit, that is, Z is a regular element of the linear isotropy action of $L/H\cap K$. On the other hand, we have

$$g_*^{-1}T^\perp_{gK}M = \mathfrak{t}(\subset\mathfrak{a})\,,\quad g_*^{-1}H_\alpha = \mathfrak{p}_\alpha \cap T_{eK}HeK \quad \text{and} \quad g_*^{-1}V_\beta \subset \mathfrak{p}_\beta\cap\mathfrak{p}'\,.$$

These facts together with (6.2) and (6.3) imply that

$$(D^{\mathrm{co}}_{zv} - zD^{\mathrm{si}}_{zv}\circ A_v)(\tilde{X}_Z) = (\cosh(z\alpha(g_*^{-1}v)) + \sinh(z\alpha(g_*^{-1}v))\tanh\alpha(Z))\tilde{X}_Z(\tilde{X}_Z \in H_\alpha)\,,$$
$$(D^{\mathrm{co}}_{zv} - zD^{\mathrm{si}}_{zv}\circ A_v)(Y) = \left(\cosh(z\beta(g_*^{-1}v)) - \frac{\sinh(z\beta(g_*^{-1}v))}{\tanh\beta(Z)}\right)Y \quad (Y \in V_\beta)\,.$$

According to these relations and (6.1), the set of all complex focal radii along γ_v is given by

$$\begin{aligned}&\left\{\frac{1}{\alpha(g_*^{-1}v)}\left(-\alpha(Z)+\left(j+\frac{1}{2}\right)\pi\sqrt{-1}\right)\ \middle|\ j\in\mathbf{Z},\ \alpha\in\Delta_H\setminus\Delta_v\right\}\\ &\bigcup\left\{\frac{1}{\beta(g_*^{-1}v)}(\beta(Z)+j\pi\sqrt{-1})\ \middle|\ j\in\mathbf{Z},\ \beta\in\Delta_V\setminus\Delta_v\right\},\end{aligned} \tag{6.4}$$

where $\Delta_H := \{\alpha\in\Delta_+ \mid \mathfrak{p}_\alpha\cap T_{eK}HeK \neq \{0\}\}$, $\Delta_V := \{\alpha\in\Delta_+ \mid \mathfrak{p}_\alpha\cap\mathfrak{p}' \neq \{0\}\}$ and $\Delta_v := \{\alpha\in\Delta_+ \mid \alpha(g_*^{-1}v) = 0\}$. Denote by $\tilde{A}$ the shape tensor of $(\pi\circ\phi)^{-1}(M)$, where ϕ is the parallel transport map for G and π is the natural projection of G onto G/K. Then, according to [8, Theorem 1], it follows from (6.4) that the J-spectrum $\mathrm{Spec}_J\tilde{A}^{\mathbf{c}}_{v^L}$ of $\tilde{A}^{\mathbf{c}}_{v^L}$ (where v^L is the horizontal lift of v) is given by

$$\begin{aligned}\mathrm{Spec}_J\tilde{A}^{\mathbf{c}}_{v^L} = &\left\{\frac{\alpha(g_*^{-1}v)}{-\alpha(Z)+(j+1/2)\pi\sqrt{-1}}\ \middle|\ j\in\mathbf{Z},\ \alpha\in\Delta_H\setminus\Delta_v\right\}\\ &\bigcup\left\{\frac{\beta(g_*^{-1}v)}{\beta(Z)+j\pi\sqrt{-1}}\ \middle|\ j\in\mathbf{Z},\ \beta\in\Delta_V\setminus\Delta_v\right\}.\end{aligned}$$

Set

$$\tilde{\alpha}_j^{\mathrm{H}} := \frac{\alpha^{\mathbf{c}}|_{\mathfrak{t}^{\mathbf{c}}}}{-\alpha(Z) + (j+1/2)\pi\sqrt{-1}} \quad (\alpha \in \Delta_H)$$

and

$$\tilde{\beta}_j^{\mathrm{V}} := \frac{\beta^{\mathbf{c}}|_{\mathfrak{t}^{\mathbf{c}}}}{\beta(Z) + j\pi\sqrt{-1}} \quad (\beta \in \Delta_V)\,.$$

The complex Coxeter group associated with M is isomorphic to the group generated by the complex reflections (of order two) with respect to the complex hyperplanes $l^{\mathrm{H}}_{\alpha,j} := (\tilde{\alpha}_j^{\mathrm{H}})^{-1}(1)$ $(j \in \boldsymbol{Z},\ \alpha \in \Delta_H)$ and $l^{\mathrm{V}}_{\beta,j} := (\tilde{\beta}_j^{\mathrm{V}})^{-1}(1)$ $(j \in \boldsymbol{Z},\ \beta \in \Delta_V)$ in $\mathfrak{t}^{\mathbf{c}}$. These complex hyperplanes are described as

$$\begin{aligned} l^{\mathrm{H}}_{\alpha,j} &= (\alpha^{\mathbf{c}}|_{\mathfrak{t}^{\mathbf{c}}})^{-1}(-\alpha(Z) + (j+1/2)\pi\sqrt{-1})\,, \\ l^{\mathrm{V}}_{\beta,j} &= (\beta^{\mathbf{c}}|_{\mathfrak{t}^{\mathbf{c}}})^{-1}(\beta(Z) + j\pi\sqrt{-1})\,. \end{aligned} \tag{6.5}$$

Thus, we can describe explicitly the generators of the complex Coxeter groups associated with principal orbits of the H-action in terms of the positive root system of the associated symmetric space $L/H \cap K$.

REMARK 6.1. (i) The complex hyperplanes $l^{\mathrm{H}}_{\alpha,j}$ $(j \in \boldsymbol{Z})$ are parallel and so are $l^{\mathrm{V}}_{\beta,j}$ $(j \in \boldsymbol{Z})$. Also, for $\alpha \in \Delta_H \cap \Delta_V$, $l^{\mathrm{H}}_{\alpha,j}$ and $l^{\mathrm{V}}_{\alpha,j}$ are parallel.

(ii) If $H = K$, then the complex Coxeter group associated with M is generated by the complex reflections of order two with respect to $l^{\mathrm{V}}_{\beta,j}$ $(j \in \boldsymbol{Z},\ \beta \in \Delta_V)$ because *HeK* consists of one point. The complex hyperplane $l^{\mathrm{V}}_{\beta,j}$ is described as

$$l^{\mathrm{V}}_{\beta,j} = (\beta^{\mathbf{c}})^{-1}(\beta(Z) + j\pi\sqrt{-1}) \tag{6.6}$$

because of $\mathfrak{t} = \mathfrak{a}$.

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